POSITIVIDADE DE SCHUR E TENDÊNCIAS RECENTES EM COMBINATÓRIA ALGÉBRICA

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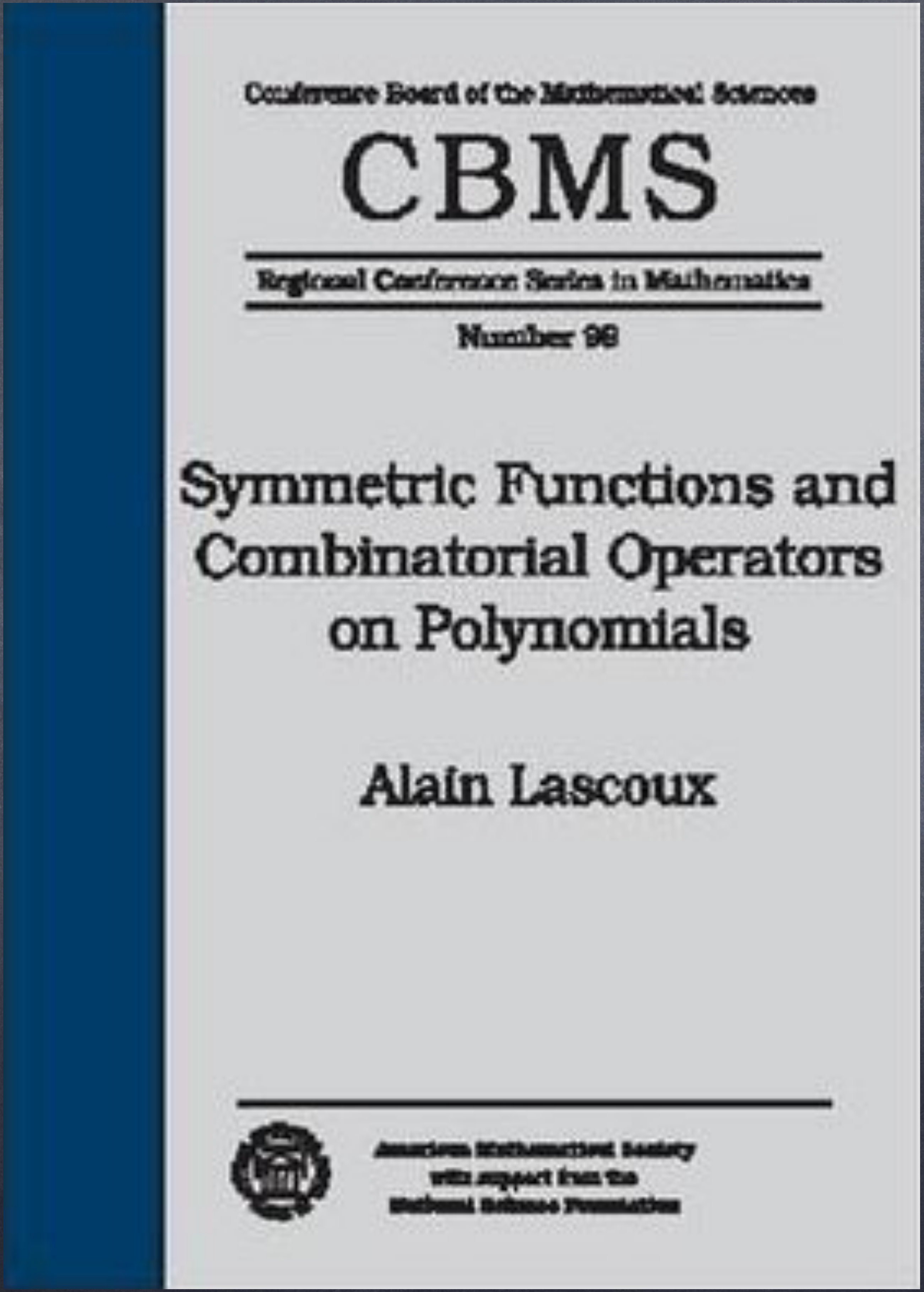
Main Courses

SCHUR - POSITIVITY
AND
e - POSITIVITY



Conference Board of the Mathematical Sciences

CBMS

Regional Conference Series in Mathematics

Number 99

Symmetric Functions and Combinatorial Operators on Polynomials

Alain Lascoux

MONOMIAL-POSITIVITY

$$F(z) = \sum_{\lambda} a_{\lambda} m_{\lambda}(Z)$$

$a_{\lambda}(q) \in \mathbb{N}[q,t]$

INTEGER COEFFICIENT POLYNOMIAL

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SCHUR - POSITIVITY

$$F(z) = \sum_{\lambda} a_\lambda\, e_\lambda(Z)$$

$a_\lambda(q) \in \mathbb{N}[q,t]$

INTEGER COEFFICIENT
POLYNOMIAL

e-POSITIVITY

$$F(z) = \sum_{\lambda} a_\lambda\, e_\lambda(Z)$$

$a_\lambda(q) \in \mathbb{N}[q,t]$

INTEGER COEFFICIENT POLYNOMIAL

e-POSITIVITY $\Longrightarrow$ SCHUR-POSITIVITY

$\Downarrow$

MONOMIAL-POSITIVITY

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MONOMIAL-POSITIVITY IS STICKY

THM IF f AND g ARE MONOMIAL-POSITIVE, THEN SO ARE

$$f \cdot g, \quad \text{AND} \quad f \circ g$$

SCHUR - POSITIVITY is STICKY

THM IF f AND g ARE SCHUR - POSITIVE, THEN SO ARE

$$f \cdot g\,,\quad f^{\perp} g\,,\quad \text{AND}\quad f \circ g$$

e-Positivity is less sticky

THM If f and g are e-positive, then so are

$$f \cdot g, \quad f^{\perp} g, \quad \text{and} \quad \xcancel{f \circ g}$$

SCHUR-POSITIVE is RARE

F.B.

Vic
Reiner

Rebecca
Patrias

THM The probability that a monomial-positive symmetric function is Schur-positive is:

$$\prod_{\mu \vdash d} \left(\sum_{\lambda} K_{\lambda\mu} \right)^{-1}$$

SCHUR-POSITIVE is RARE

F.B.

Vic Reiner

Rebecca Patrias

THM The probability that a Schur-positive symmetric function is e-positive is:

$$\prod_{\mu \vdash d} \left(\sum_{\lambda} K_{\lambda\mu} \right)^{-1}$$

$$\begin{pmatrix} 1 + & 1 + & 1 + & 1 + & 1 + & 1 + & 1 \\ 0 & 1 & 1 & 2 & 2 & 3 & 4 \\ 0 & 0 & 1 & 1 & 2 & 3 & 5 \\ 0 & 0 & 0 & 1 & 1 & 3 & 6 \\ 0 & 0 & 0 & 0 & 1 & 2 & 5 \\ 0 & 0 & 0 & 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} = \begin{matrix} 7 \\ \times \\ 13 \\ \times \\ 12 \\ \times \\ 11 \\ \times \\ 8 \\ \times \\ 5 \\ \times \\ 1 \\ \hline 480480 \end{matrix}$$

SCHUR - POSITIVITY

e-positivity

$$\begin{pmatrix}
1 & 1 & 1 & 1 & 1 & 1 & 1 \\
\overset{+}{0} & 1 & 1 & 2 & 2 & 3 & 4 \\
\overset{+}{0} & 0 & 1 & 1 & 2 & 3 & 5 \\
\overset{+}{0} & 0 & 0 & 1 & 1 & 3 & 6 \\
\overset{+}{0} & 0 & 0 & 0 & 1 & 2 & 5 \\
\overset{+}{0} & 0 & 0 & 0 & 0 & 1 & 4 \\
\overset{+}{0} & 0 & 0 & 0 & 0 & 0 & 1
\end{pmatrix}$$

$$1 \times 2 \times 3 \times 5 \times 7 \times 13 \times 26 = 70980$$

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WHERE DO WE ENCOUNTER SCHUR POSITIVITY ?

- REPRESENTATION THEORY OF $\mathbb{S}_n$
- REPRESENTATION THEORY OF GL_k
- ALGEBRAIC GEOMETRY
- COMBINATORICS
- GEOMETRIC COMPLEXITY THEORY

REPRESENTATION THEORY OF $\mathbb{S}_n$

$$\mathbb{S}_n \times \mathcal{V} \xrightarrow{\rho} \mathcal{V}$$

FROBENIUS CHARACTERISTIC

$$\mathcal{V}(\mathbf{z}) := \frac{1}{n!} \sum_{\sigma} \mathrm{TRACE}(\rho(\sigma))\, p_{\lambda(\sigma)}(\mathbf{z})$$

$\lambda(\sigma)$ CYCLE STRUCTURE OF σ

$$\lambda(\sigma) = 1^{d_1} 2^{d_2} \cdots n^{d_n}$$

σ HAS d_i CYCLES OF SIZE i

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FROBENIUS CHARACTERISTIC

$$\mathcal{V}(\mathbf{z}) := \frac{1}{n!} \sum_{\sigma} \mathrm{TRACE}(\rho(\sigma))\, p_{\lambda(\sigma)}(\mathbf{z})$$

- $\mathcal{W}$ IRREDUCIBLE IFF $\mathcal{W}_\mu(\mathbf{z}) = s_\mu$ FOR SOME PARTITION μ

THERE IS A NATURAL INDEXING

- $(\mathcal{V}_1 \oplus \mathcal{V}_2)(\mathbf{z}) = \mathcal{V}_1(\mathbf{z}) + \mathcal{V}_2(\mathbf{z})$
- $(\mathcal{V}_1 \otimes \mathcal{V}_2)(\mathbf{z}) = \mathcal{V}_1(\mathbf{z}) \cdot \mathcal{V}_2(\mathbf{z})$

LITTLEWOOD-RICHARDSON COEFFICIENTS $c_{\mu\nu}^{\lambda} \in \mathbb{N}$

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DECOMPOSITION INTO IRREDUCIBLES

$$\mathcal{V} = \bigoplus_{\lambda \vdash n} c_\lambda \mathcal{W}_\lambda$$

$\mathcal{W}_\lambda$ IRREDUCIBLE REPRESENTATIONS OF $\mathbb{S}_n$

$$\mathcal{V}(\mathbf{z}) = \sum_{\lambda \vdash n} c_\lambda \Delta_\lambda(\mathbf{z})$$

MAKES THE CALCULATION OF THE c_λ ALGORITHMIC.

A TOY EXAMPLE

$$\mathcal{V} := \mathbb{Q}\{x_1y_1, x_1y_2, x_2y_1, x_2y_2\}$$

$$\mathbb{S}_2 := \{\mathrm{Id}, (1,2)\}$$

$$\sigma(x_iy_j) := x_{\sigma(i)}y_{\sigma(j)}$$

MATRIX $M_{\mathrm{Id}} := \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$ $M_{(1,2)} := \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$

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$$M_{\mathrm{Id}} := \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \qquad M_{(1,2)} := \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$$

$$\mathcal{V}(Z) = \frac{1}{2}\left(\mathrm{TRACE}(M_{\mathrm{Id}})\, p_{11} + \mathrm{TRACE}(M_{(1,2)})\, p_2\right)$$

$$\mathcal{V}(Z) = \frac{1}{2}\left(4 \cdot p_{11} + 0 \cdot p_2 \right)$$

$$\mathcal{V}(Z) = 2\,\Delta_{11} + 2\Delta_2$$

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$$\mathcal{V}(\mathbf{Z}) = 2\,\Delta_{11} + 2\,\Delta_{2}$$

$$\mathcal{V} = \mathbb{Q}\{x_1y_1 - x_2y_2\} \oplus \mathbb{Q}\{x_1y_2 - x_2y_1\}$$
$$\oplus\, \mathbb{Q}\{x_1y_1 + x_2y_2\} \oplus \mathbb{Q}\{x_1y_2 + x_2y_1\}$$

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• REPRESENTATION THEORY OF GL_k

Φ POLYNOMIAL FUNCTOR

VECTOR SPACE

V

OF DIMENSION k

LINEAR TRANSFORMATION

$T: V \xrightarrow{\sim} V$

VECTOR SPACE

$\Phi(V)$

LINEAR TRANSFORMATION

$\Phi(T): \Phi(V) \longrightarrow \Phi(V)$

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MATRIX FORMULATION

$T = (t_{ij})$ $\quad\quad$ $\Phi(T) = (\varphi_{k\ell})$

$\varphi_{k\ell}$ IS A POLYNOMIAL IN THE t_{ij}

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CHARACTER OF Φ

$$\Phi(Q) := \text{TRACE}\ \Phi\begin{pmatrix} q_1 & & & 0 \\ & q_2 & & \\ & & \ddots & \\ 0 & & & q_k \end{pmatrix}$$

VARIABLES $Q = q_1, q_2, \ldots, q_k$

- $\Phi(Q)$ IS A SYMMETRIC FUNCTION
- Φ IS IRREDUCIBLE IFF
  $\Phi(Q)$ IS A SCHUR FUNCTION

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A TOY EXAMPLE GL_2

$$\mathcal{V} := \mathbb{Q}\{x_1, x_2, y_1, y_2\}$$

$$T : \mathcal{V} \longrightarrow \mathcal{V}$$

$$\begin{pmatrix} x_1 & x_2 \\ y_1 & y_2 \end{pmatrix} \rightsquigarrow \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x_1 & x_2 \\ y_1 & y_2 \end{pmatrix}$$

$\Phi(\mathcal{V})$ POLYNOMIALS OF DEGREE 2 IN x_i, y_j

$\Phi(\mathcal{V})$ POLYNOMIALS OF DEGREE 2 IN x_i, y_j

$$\Phi(\mathcal{V}) = \mathbb{Q}\{x_1^2, x_2^2, x_1x_2, y_1^2, y_2^2, y_1y_2, x_1y_1, x_1y_2, x_2y_1, x_2y_2\}$$

$$\Phi(q,t) = \text{TRACE}\ \Phi\begin{pmatrix} q & 0 \\ 0 & t \end{pmatrix}$$

$$x_i \xmapsto{\Phi} q\,x_i \qquad y_j \xmapsto{\Phi} t\,y_j$$

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$$\Phi(q,t) = \text{TRACE}\ \Phi\begin{pmatrix} q & 0 \\ 0 & t \end{pmatrix}$$

$$\Phi(V) = \mathbb{Q}\{\underbrace{x_1^2, x_2^2, x_1x_2}, y_1^2, y_2^2, y_1y_2, x_1y_1, x_1y_2, x_2y_1, x_2y_2\}$$

$$\Phi(q,t) = 3\,q^2 + 3\,t^2 + 4\,q\,t$$

$$\Phi(q,t) = 3\,\Delta_2(q,t) + \Delta_{11}(q,t)$$

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$$\Phi(q,t) = 3\Delta_2(q,t) + \Delta_{11}(q,t)$$

$$\begin{aligned}\Phi(\mathcal{V}) = \mathbb{Q}\{x_1^2, x_1y_1, y_1^2\} \oplus \mathbb{Q}\{x_2^2, x_2y_2, y_2^2\} \\ \oplus\, \mathbb{Q}\{x_1x_2, x_1y_2+x_2y_1, y_1y_2\} \\ \oplus\, \mathbb{Q}\{x_1y_2-x_2y_1\}\end{aligned}$$

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$$\boxed{GL_k \times \mathbb{S}_n}$$

$$\mathbb{Q}\{x_1^2, x_2^2, x_1x_2, y_1^2, y_2^2, y_1y_2, x_1y_1, x_1y_2, x_2y_1, x_2y_2\}$$

$$\begin{pmatrix} x_1 & x_2 \\ y_1 & y_2 \end{pmatrix} \rightsquigarrow \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x_1 & x_2 \\ y_1 & y_2 \end{pmatrix} \sigma$$

$$\mathcal{V}(\mathbf{Q};\mathbf{z}) := \frac{1}{n!} \sum_{\sigma} \mathrm{TRACE}(\mathbf{Q}\cdot(-)\cdot\sigma)\, p_{\lambda(\sigma)}(\mathbf{z})$$

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$\mathbb{GL}_k \times \mathbb{S}_n$

$$\mathbb{Q}\{x_1^2, x_2^2, x_1x_2, y_1^2, y_2^2, y_1y_2, x_1y_1, x_1y_2, x_2y_1, x_2y_2\}$$

$$\begin{pmatrix} x_1 & x_2 \\ y_1 & y_2 \end{pmatrix} \rightsquigarrow \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x_1 & x_2 \\ y_1 & y_2 \end{pmatrix} \sigma$$

$$\begin{pmatrix} x_1 & x_2 \\ y_1 & y_2 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} x_1 & x_2 \\ y_1 & y_2 \end{pmatrix}$$

$$
\begin{aligned}
\Phi(V) = {} & \mathbb{Q}\{x_1^2+x_2^2,\, x_1y_1+x_2y_2,\, y_1^2+y_2^2\} \\
& \oplus \mathbb{Q}\{x_1x_2,\, x_1y_2+x_2y_1,\, y_1y_2\} \\
& \oplus \mathbb{Q}\{x_1^2-x_2^2,\, x_1y_1-x_2y_2,\, y_1^2-y_2^2\} \\
& \oplus \mathbb{Q}\{x_1y_2-x_2y_1\}
\end{aligned}
$$

$$
\begin{aligned}
\Phi(q,t;\mathbf{Z}) = {} & 2(q^2+qt+t^2)\Delta_2(\mathbf{Z}) \\
& +(q^2+qt+t^2)\Delta_{11}(\mathbf{Z}) \\
& +qt\,\Delta_{11}(\mathbf{Z})
\end{aligned}
$$

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$$\begin{aligned}\Phi(q,t;Z) &= 2(q^2+q\,t+t^2)\,s_2(Z)\\ &\quad+(q^2+q\,t+t^2)\,s_{11}(Z)\\ &\quad+q\,t\;s_{11}(Z)\end{aligned}$$

$$\begin{aligned}\Phi(q,t;Z) &= 2\;s_2(q,t)\,s_2(Z)\\ &\quad+s_2(q,t)\,s_{11}(Z)+s_{11}(q,t)\,s_{11}(Z)\end{aligned}$$

$$\boxed{\Phi = 2\,s_2\otimes s_2+s_2\otimes s_{11}+s_{11}\otimes s_{11}}$$

IAN G. MACDONALD

COMBINATORIAL MACDONAL POLYNOMIALS

EXAMPLES

$$H_2(q,t;z) = \Delta_2 + q\,\Delta_{11}$$

$$H_{11}(q,t;z) = \Delta_2 + t\,\Delta_{11}$$

$$H_3(q,t;z) = \Delta_3 + (q+q^2)\,\Delta_{21} + q^3\,\Delta_{111}$$

$$H_{21}(q,t;z) = \Delta_3 + (q+t\,)\,\Delta_{21} + q\,t\,\Delta_{111}$$

$$H_3(q,t;z) = \Delta_3 + (t+t^2)\,\Delta_{21} + t^3\,\Delta_{111}$$

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$H_n(x;q,t)$

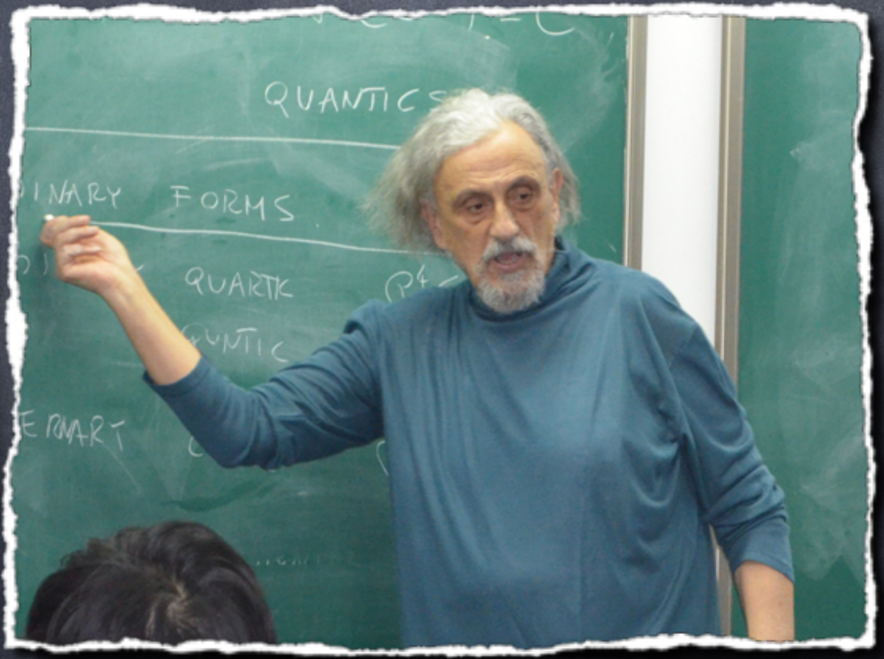
GRADED CHARACTER OF THE

- COINVARIANT RING OF $\mathfrak{S}_n$
- COHOMOLOGY RING OF THE FULL FLAG MANIFOLD
- MODULE OF $\mathfrak{S}_n$-HARMONIC POLYNOMIALS

$H_\mu(x;0,t)$

GRADED CHARACTER OF THE COHOMOLOGY RING OF SPRINGER VARIETIES

GARSIA



PROCESI

$H_\mu(q,t;z)$ SCHUR - POSITIVE

$$\begin{pmatrix} 1 & q^3+q^2+q & q^4+q^2 & q^5+q^4+q^3 & q^6 \\ 1 & q^2+q+t & q^2+qt & q^3+q^2t+qt & q^3t \\ 1 & qt+q+t & q^2+t^2 & q^2t+qt^2+qt & q^2t^2 \\ 1 & t^2+q+t & qt+t^2 & qt^2+t^3+qt & qt^3 \\ 1 & t^3+t^2+t & t^4+t^2 & t^5+t^4+t^3 & t^6 \end{pmatrix}$$

PROOF

VANISHING THEOREMS AND CHARACTER FORMULAS
FOR THE HILBERT SCHEME
OF POINTS IN THE PLANE (INVENT. MATH.)

HAIMAN

2002

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$\boxed{\text{THM}}$ (HAIMAN 2002)

$$\boxed{H_\mu(q,t;Z) = \mathcal{M}_\mu(q,t;Z)}$$

$\mathcal{M}_\mu$ SMALLEST MODULE CONTAINING

$$V_\mu := \det\left(x_i^a y_i^b\right)_{\substack{1\le i\le n\\ (a,b)\in\mu}}$$

CLOSED UNDER

- PARTIAL DERIVATIVES

GARSIA

HAIMAN

1994

DIAGONAL COINVARIANT SPACE

GARSIA

HAIMAN

1994

ACTION OF $GL_2 \times \mathbb{S}_n$ ON POLYNOMIALS IN THE VARIABLES X

$$\overset{\sigma}{\circlearrowleft}$$

$$T \circlearrowright X = \begin{pmatrix} x_1, & x_2, & \cdots, & x_n \\ y_1, & y_2, & \cdots, & y_n \end{pmatrix}$$

$$F(X) \longmapsto F(T \cdot X \cdot \sigma)$$

$$\in \mathbb{Q}[X]$$

$\mathcal{E}_n$ DIAGONAL COINVARIANT SPACE

$$\mathcal{E}_n := \mathbb{Q}[X]/I_n$$

$I_n :=$ IDEAL GENERATED BY CONSTANT TERM FREE DIAGONAL INVARIANTS (*)

(*) GENERATED BY $p_{k,j} := \sum_{i=1}^{n} x_i^k y_i^j$

GRADED BY x,y-DEGREES

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ξ_m CHARACTER

$$\xi_1(q,t;Z) = s_1$$

$$\xi_2(q,t;Z) = s_2 + (q+t)s_{11}$$

$$\xi_3(q,t;Z) = s_3 + (q^2+qt+t^2+q+t)s_{21} + (\underbrace{q^3+q^2t+qt^2+t^3+qt}_{q,t\text{ - CATALAN}})s_{111}$$

CONJECTURE IN 1994

$$\dim(\xi_m) = (m+1)^{m-1}$$

$$\xi_1 = 1 \otimes \Delta_1$$

$$\xi_2(q,t;z) = \Delta_2 + (q+t)\Delta_{11}$$

$$\xi_3 = 1 \otimes \Delta_3 + (\Delta_1 + \Delta_2) \otimes \Delta_{21} + (\Delta_{11} + \Delta_3) \otimes \Delta_{111}$$

$$\begin{aligned}\xi_4 = 1 \otimes \Delta_4 &+ (\Delta_1 + \Delta_2 + \Delta_3) \otimes \Delta_{31} \\ &+ (\Delta_{21} + \Delta_2 + \Delta_4) \otimes \Delta_{22} \\ &+ (\Delta_{11} + \Delta_{21} + \Delta_{31} + \Delta_3 + \Delta_4 + \Delta_5) \otimes \Delta_{211} \\ &+ (\underbrace{\Delta_{31} + \Delta_{41} + \Delta_6}_{q,t\text{-CATALAN}}) \otimes \Delta_{1111}\end{aligned}$$

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A "CONCRETE" RÉALISATION

SMALLEST MODULE CONTAINING

$$V_n := \prod_{1 \leq i < j \leq n} (x_i - x_j)$$

CLOSED UNDER

- PARTIAL DERIVATIVES
- POLARIZATION $\sum_{i=1}^{n} y_i \partial x_i^k$

POLARIZATION

POLARIZATION OPERATORS

$$\sum_{i=1}^{n} y_i \partial x_i^k$$

POLARIZATION

$$\begin{tikzcd}
x_1^2 + x_2^2 + \cdots + x_m^2 \arrow[d, rightsquigarrow, "\sum_{i=1}^{m} y_i \partial x_i"] \\
2(x_1 y_1 + x_2 y_2 + \cdots + x_m y_m) \arrow[d, rightsquigarrow, "\sum_{i=1}^{m} y_i \partial x_i"] \\
2(y_1^2 + y_2^2 + \cdots + y_m^2)
\end{tikzcd}$$

POLARIZATION

$$x_1^2 + x_2^2 + \dots + x_n^2$$

$$\Big\downarrow \; \sum_{i=1}^{n} y_i \partial x_i^2$$

$$2(y_1 + y_2 + \dots + y_n)$$

$$\Big\downarrow \; \sum_{i=1}^{n} x_i \partial y_i$$

$$2(x_1 + x_2 + \dots + x_n)$$

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A TOY EXAMPLE $\mathcal{E}_2$

$$V_2 = x_2 - x_1 \; \xrightarrow{\sum_{i=1}^{n} y_i \partial x_i} \; y_2 - y_1$$

$$\mathcal{E}_2 = \underset{\underset{s_2}{\circlearrowright}}{\overset{1}{\mathbb{Q}\{1\}}} \oplus \mathbb{Q}\{\underset{\underset{s_{11}}{\circlearrowright}}{\overset{q}{x_2 - x_1}}, \underset{\underset{s_{11}}{\circlearrowright}}{\overset{t}{y_2 - y_1}}\}$$

$$\mathcal{E}_2(q,t) = 1 \otimes s_2 + (q \otimes t)\, s_{11}$$

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∇ NABLA OPERATOR

$$\nabla H_\mu := \Big(\prod_{(a,b)\in\mu} q^a t^b\Big) H_\mu$$

F.B.

GARSIA

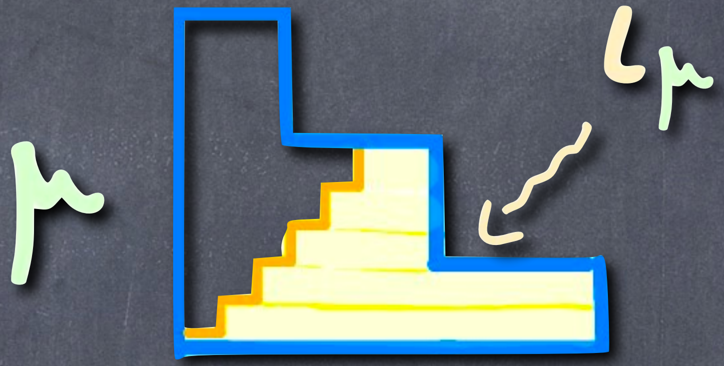
1994

THM (HAIMAN 2002)

$$\boxed{\xi_n(q,t;z) = \nabla(e_n)(q,t;z)}$$

SCHUR - POSITIVITY CONJECTURE (F.B. 1994)

$$\nabla\left(\left(\frac{-1}{qt}\right)^{\ell_\mu}\Delta_\mu\right)$$



$$\begin{pmatrix}
0 & 1 & s_1 & s_1+s_2 & s_{11}+s_3 \\
0 & s_1 & s_2 & s_{11}+s_2+s_3 & s_{21}+s_4 \\
0 & s_{11} & 0 & s_{21} & s_{22} \\
0 & s_2 & s_{11}+s_3 & s_{21}+s_3+s_4 & s_{31}+s_5 \\
1 & s_1+s_2+s_3 & s_2+s_{21}+s_4 & s_{11}+s_{21}+s_3+s_{31}+s_4+s_5 & s_{31}+s_{41}+s_6
\end{pmatrix}$$

STILL OPEN EXCEPT FOR μ HOOKS

STILL OPEN EXCEPT FOR μ HOOKS

FOLLOWS FROM

COMPOSITIONAL

(m, n)-SHUFFLE

THEOREM

MELLIT

2016

+

LLT-POLYNOMIAL

$\mathbb{L}_\mu(t;z)$

IS

SCHUR-POSITIVE

HAIMAN

GROJNOVSKI

2007

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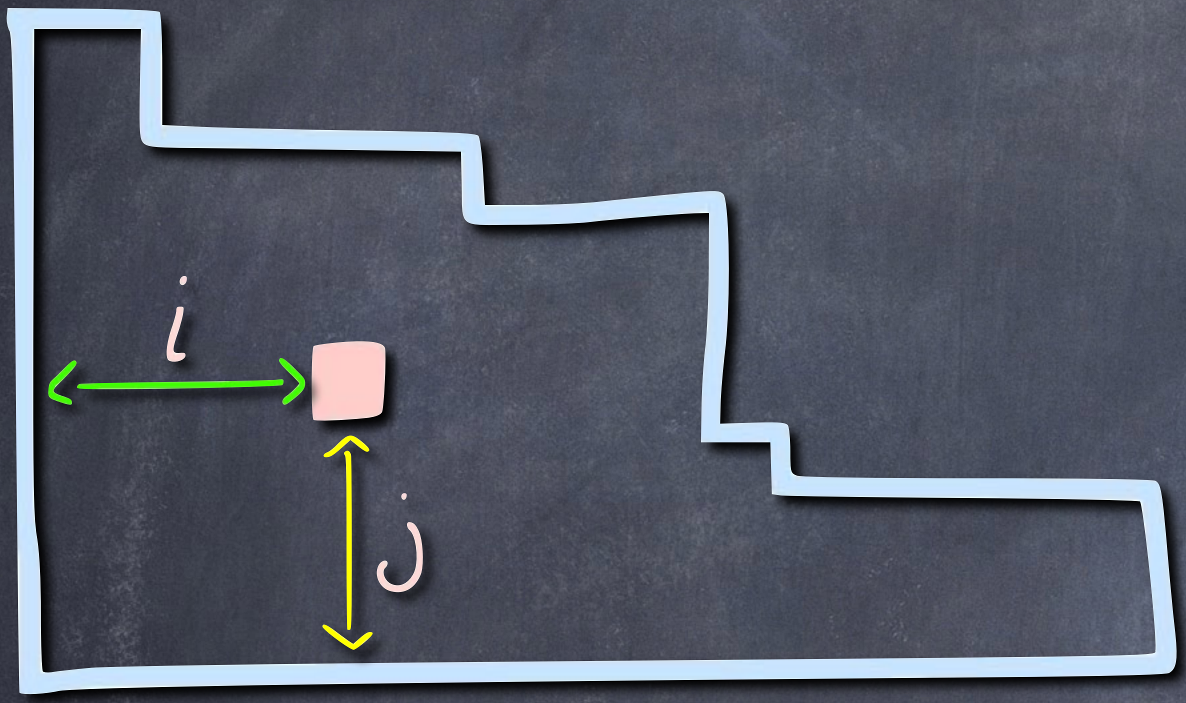
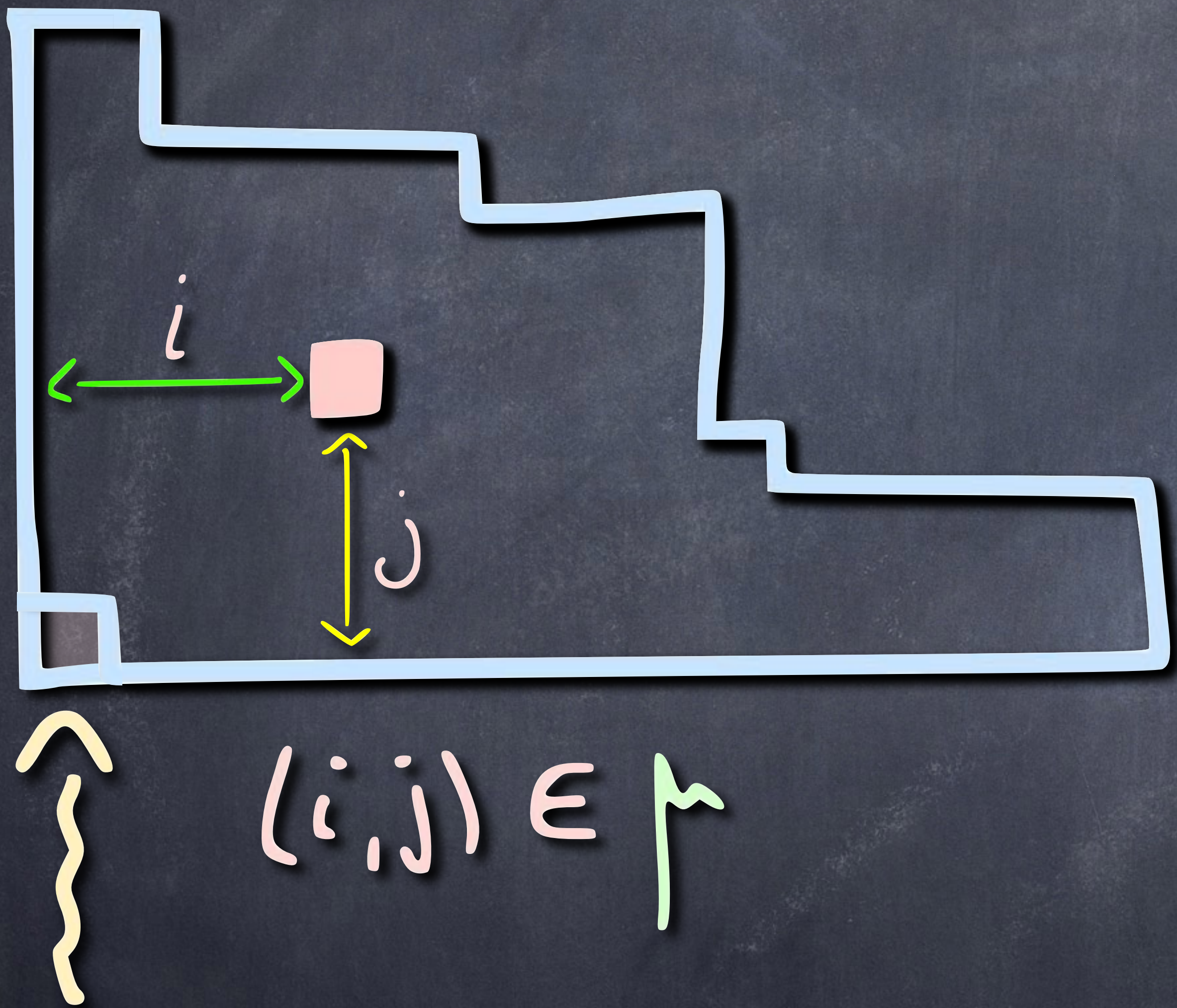
Δ_f OPERATORS WITH MACDONALD POLYNOMIALS AS JOINT EIGENFUNCTIONS



$(i,j) \in \mu$

$$\Delta_f H_\mu = \overbrace{f(\ldots, q^i t^j, \ldots)}^{\text{EIGENVALUE}}_{(i,j)\in\mu} H_\mu$$

Δ_f OPERATORS WITH MACDONALD POLYNOMIALS AS JOINT EIGENFUNCTIONS



$$\Delta_f H_\mu = \overbrace{f(\ldots, q^i t^j, \ldots)}^{\text{EIGENVALUE}}{}_{(i,j)\in\mu} H_\mu$$

$$\Delta'_f H_\mu = \overbrace{f(\ldots, q^i t^j, \ldots)}^{\text{EIGENVALUE}}{}_{\substack{(i,j)\in\mu \\ (i,j)\neq(0,0)}} H_\mu$$

THE Δ-CONJECTURE

HAGLUND

$$\Delta'_{e_k}(e_n) = \sum_{\mu \leq \delta_n} \Big(\sum_{J} q^{(J\,|\,a)} \Big) \mathbb{L}_\mu(t;z)$$

$$\mathrm{DESC}(\mu) \subseteq J \subseteq \{1,2,\dots,n\}$$

$$\#J = k$$

$$\mathbb{L}_\mu(t;z) = \sum_{\tau \in \mathrm{SSYT}((\mu+1^n)/\mu)} t^{\mathrm{Dinv}(\tau)} z_\tau$$

$$(J\,|\,a) = \sum_{i \in J} a_i$$

$$(k = n-1) \Rightarrow (J\,|\,a) = \mathrm{AREA}(\mu)$$

GENERIC
MODULES OF DIAGONAL HARMONIC POLYNOMIALS

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ACTION OF $GL_\infty \times \mathbb{S}_n$ ON POLYNOMIALS IN THE VARIABLES

$$\begin{pmatrix} x_1 & x_2 & \cdots & x_n \\ y_1 & y_2 & \cdots & y_n \\ \vdots & \vdots & \ddots & \vdots \\ z_1 & z_2 & \cdots & z_n \\ \vdots & \vdots & \ddots & \vdots \end{pmatrix}$$

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$f(x) \longmapsto f(x\cdot\sigma)$ Action of $\mathbb{S}_n$

$$X = \overset{\overset{\sigma}{\circlearrowright}}{\begin{pmatrix} x_1 & x_2 & \cdots & x_n \\ y_1 & y_2 & \cdots & y_n \\ \vdots & \vdots & \ddots & \vdots \\ z_1 & z_2 & \cdots & z_n \\ \vdots & \vdots & \ddots & \vdots \end{pmatrix}}$$

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ACTION OF GL_∞

$$f(X) \longmapsto f(T\cdot X)$$

$$T \circlearrowright \begin{pmatrix} x_1 & x_2 & \cdots & x_n \\ y_1 & y_2 & \cdots & y_n \\ \vdots & \vdots & \ddots & \vdots \\ z_1 & z_2 & \cdots & z_n \\ \vdots & \vdots & \ddots & \vdots \end{pmatrix}$$

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POLARIZATION

POLARIZATION OPERATORS

$$\sum_{i=1}^{n} y_i \partial x_i^k$$

GENERIC CHARACTER ξ_n

IRRED. FOR GL_∞ - ACTION

↓

$$\xi_n = \sum_{\mu \vdash n} \sum_{\lambda} c_{\lambda\mu} (\Delta_\lambda \otimes \Delta_\mu),$$

↑

IRRED. FOR $\mathbb{S}_n$ - ACTION

$c_{\lambda\mu} \in \mathbb{N}$

A TOY EXAMPLE $\mathfrak{E}_2$ $\quad V_2 = x_2 - x_1$

$$\boxed{GL_\infty\text{-ACTION}}$$

$$\begin{array}{ccccccc}
 & 1 & & q_1 & q_2 & & q_k \\
\mathfrak{E}_2 = & \mathbb{Q}\{1\} & \oplus\ \mathbb{Q}\{ & x_2 - x_1, & y_2 - y_1, & \cdots, & z_2 - z_1, \cdots\} \\
 & \circlearrowright & & \circlearrowright & \circlearrowright & & \circlearrowright \\
 & s_2 & & s_{11} & s_{11} & & s_{11}
\end{array}$$

$$\boxed{S_n\text{-ACTION}}$$

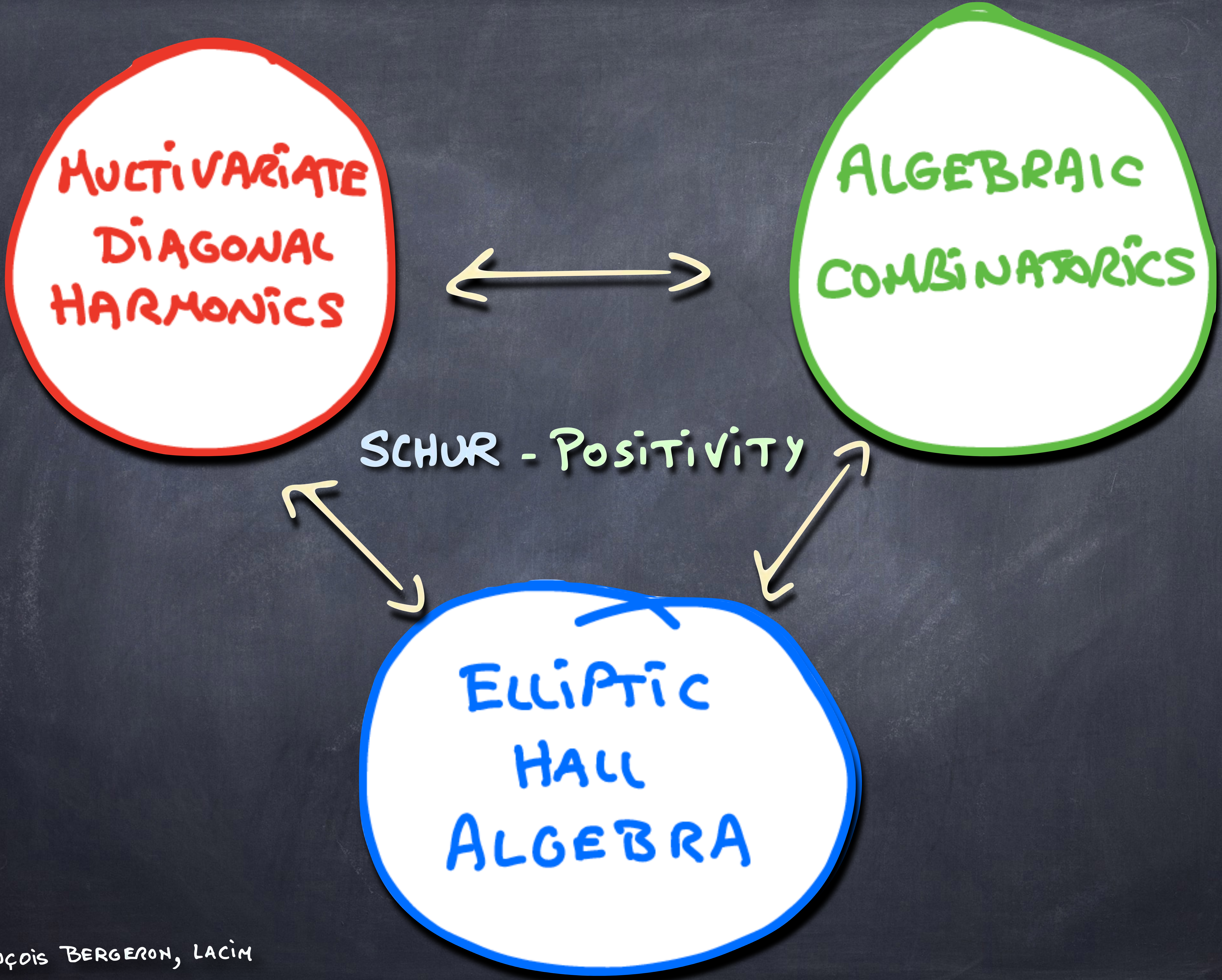
$$\mathfrak{E}_2 = 1 \otimes s_2 + (q_1 + q_2 + \cdots + q_k + \cdots) \otimes s_{11}$$

$$= 1 \otimes s_2 + s_1 \otimes s_{11}$$

$$
\begin{aligned}
\xi_4 = 1 \otimes \Delta_4 &+ (\Delta_1 + \Delta_2 + \Delta_3) \otimes \Delta_{31} \\
&+ (\Delta_{21} + \Delta_2 + \Delta_4) \otimes \Delta_{22} \\
&+ (\Delta_{11} + \Delta_{21} + \Delta_{31} + \Delta_3 + \Delta_4 + \Delta_5) \otimes \Delta_{211} \\
&+ \boxed{(\Delta_{111} +}\, \underbrace{\Delta_{31} + \Delta_{41} + \Delta_6}_{q,t\text{-CATALAN}}) \otimes \Delta_{1111}
\end{aligned}
$$

$$\Delta_{111}(q,t) = 0$$

$$\left((e_k^{\perp} \otimes \mathrm{Id})\, \xi_n\right)(q, t; z) = \Delta'_{e_{n-1-k}}\, e_n$$



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To Be Continued