Positividade de Schur e Tendências Recentes em Combinatória Algébrica

François Bergeron, LACIM

APPETIZERS

Partições

PARTITIONS

$\mu \vdash n \qquad |\mu| = n$

$$\boxed{\begin{gathered} n = \mu_1 + \mu_2 + \dots + \mu_k \\ \mu_i \geq \mu_{i+1} > 0 \end{gathered}}$$

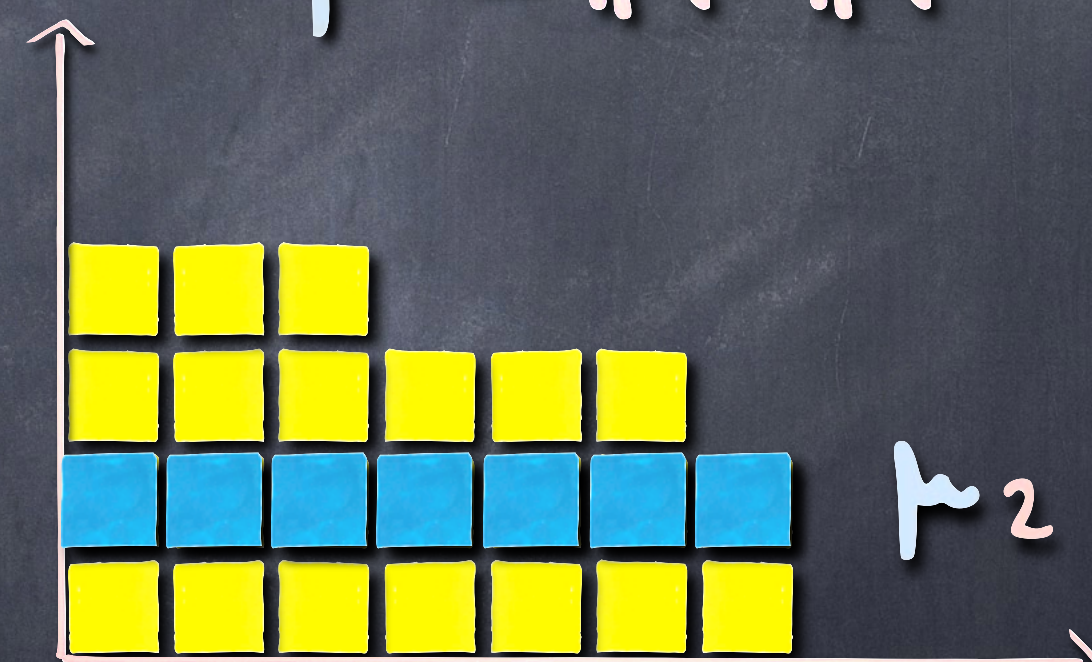
$\mu = \mu_1 \mu_2 \cdots \mu_k$

$\mu = 1^{d_1} 2^{d_2} \cdots n^{d_n}$

μ HAS d_i PARTS OF SIZE i

$\ell(\mu) = k = d_1 + d_2 + \dots + d_n$

$\mu \subset \mathbb{N} \times \mathbb{N}$



FERRERS DIAGRAM

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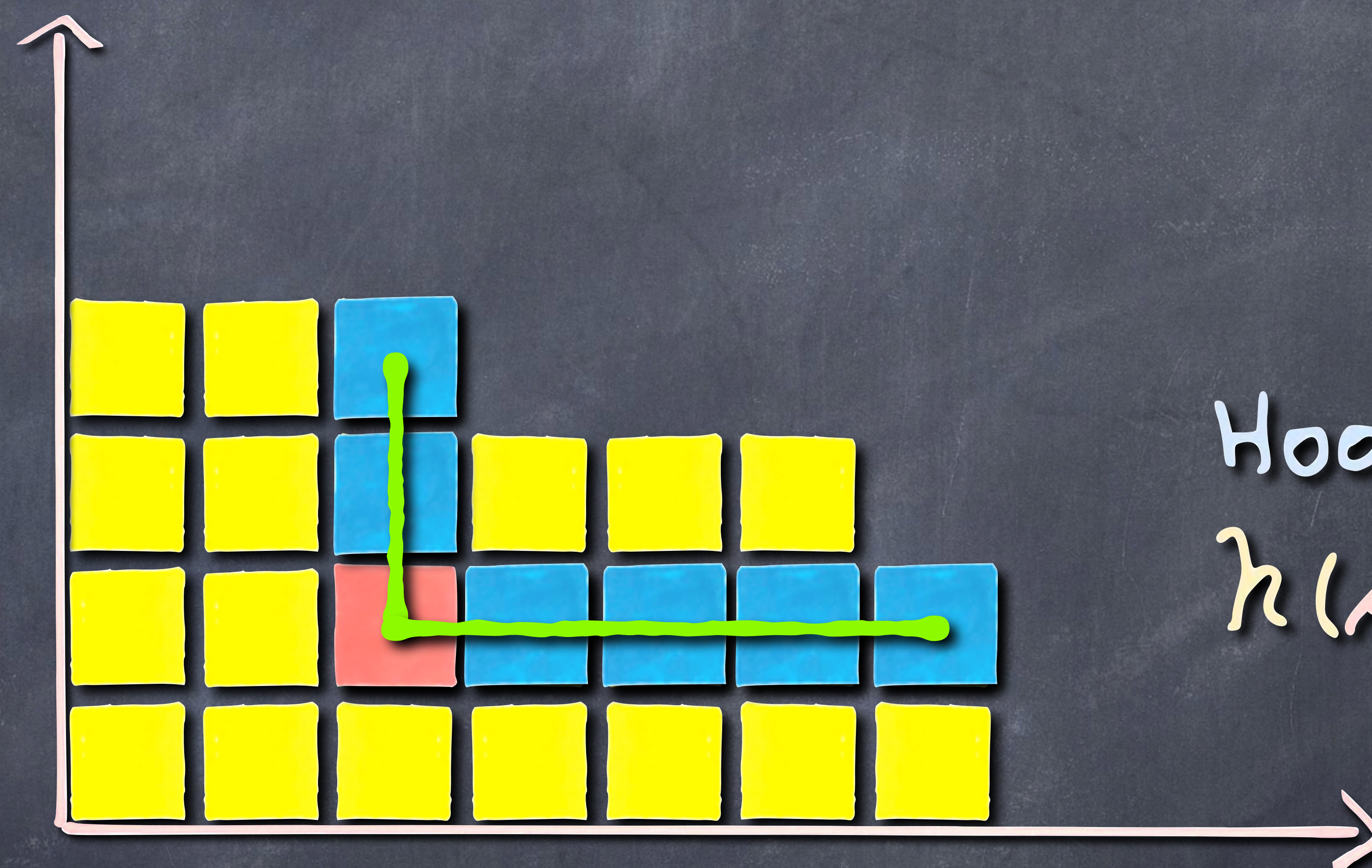
μ Partition

CELL

$c \in \mu$

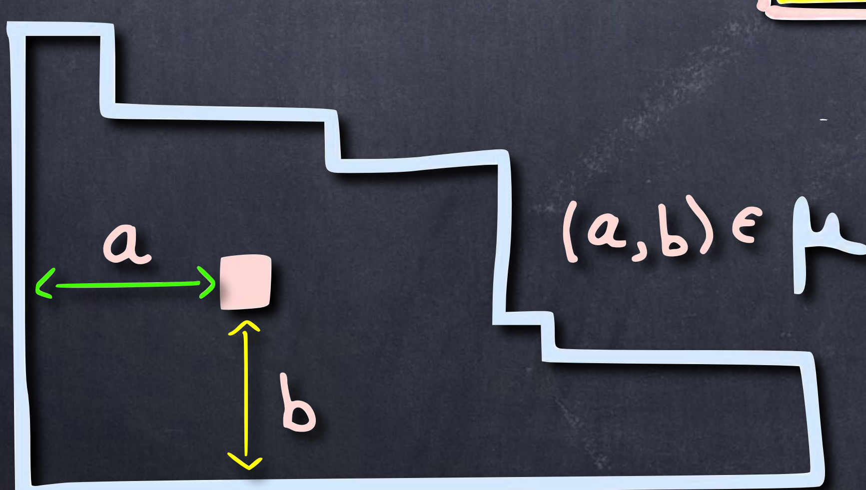
$c = (a,b)$

$\in \mathbb{N}\times\mathbb{N}$



Hook

$h(c)$



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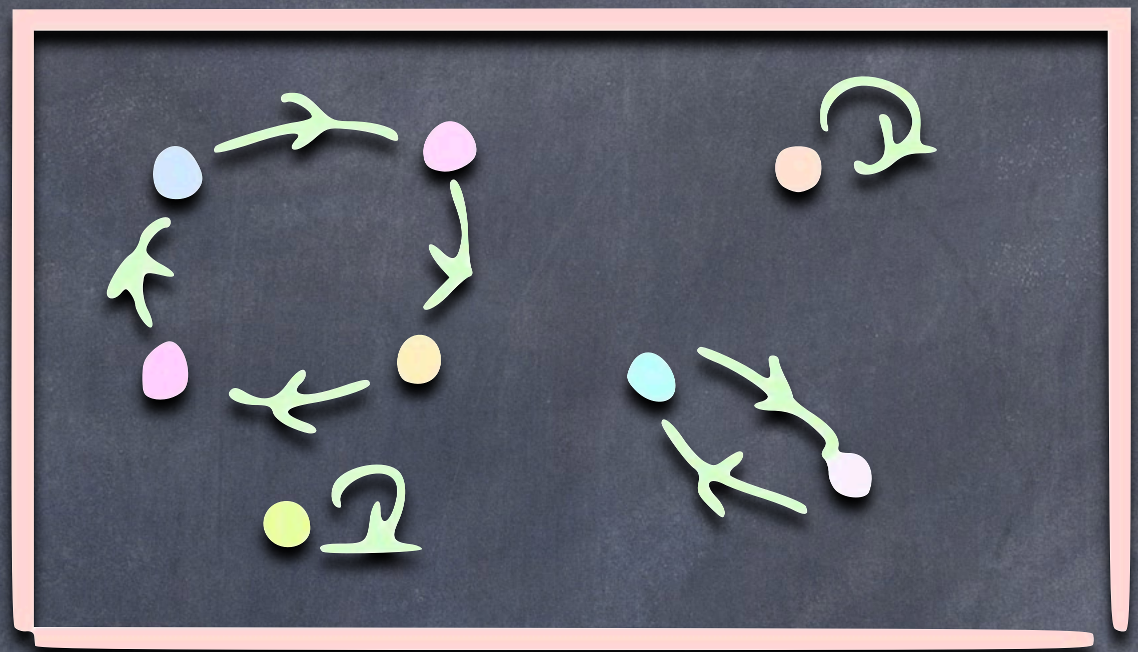
$\lambda(\sigma)$ CYCLE STRUCTURE OF σ PERMUTATION

σ



$$\lambda(\sigma) = 4211$$

$$\exists\theta \quad \sigma = \theta^{-1}\tau\theta \qquad \text{iff} \qquad \lambda(\sigma) = \lambda(\tau)$$

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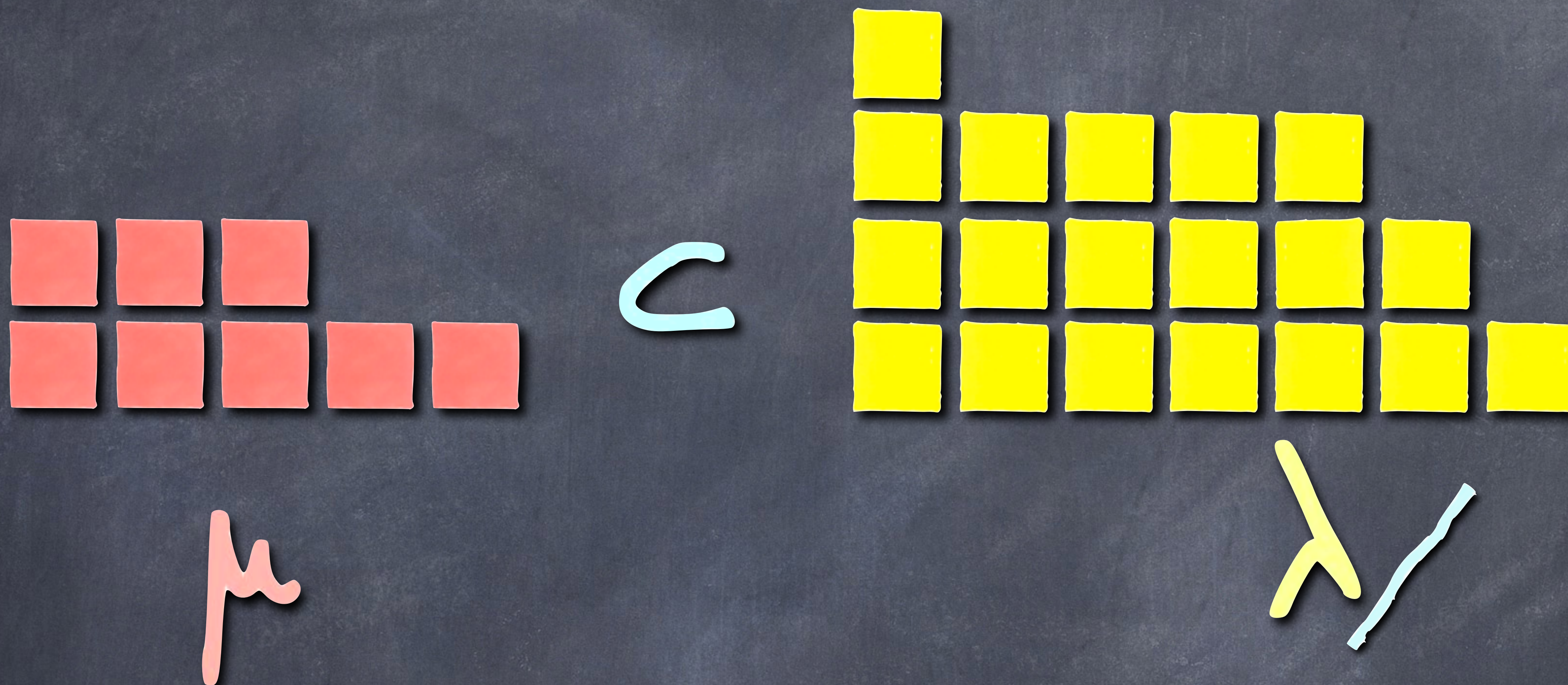
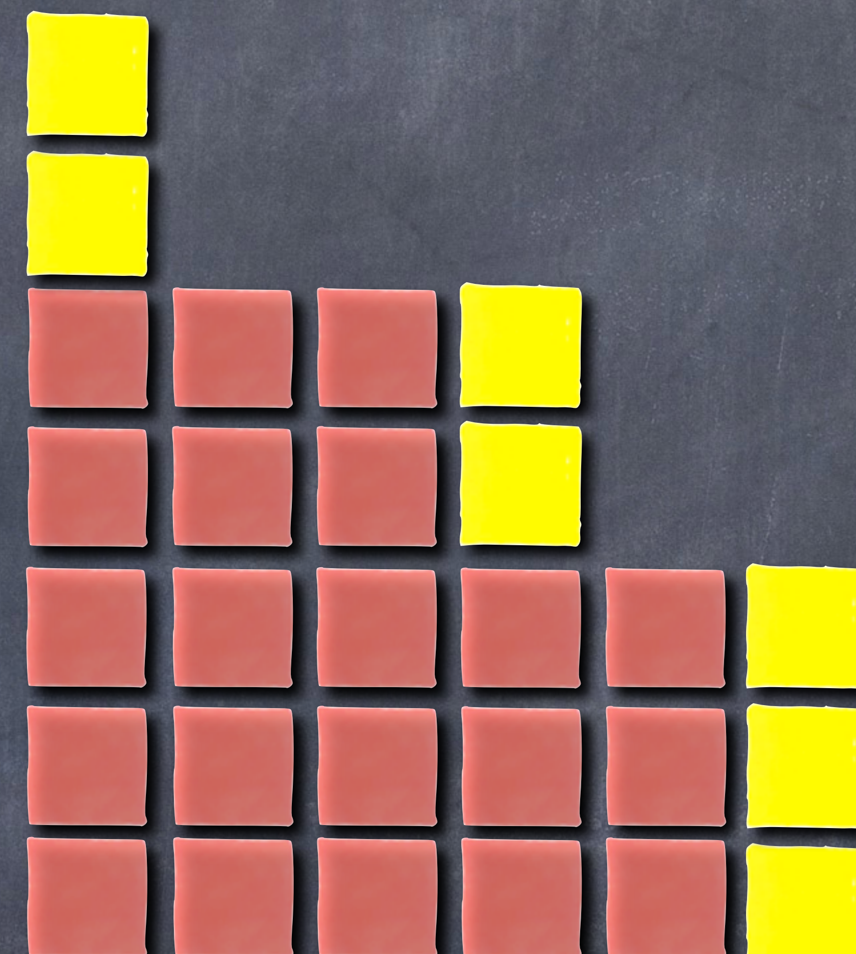
CONTAINMENT ORDER



SKEW SHAPE

$$\left(\mu + 1^{n}\right)/\mu$$

$n = 7$



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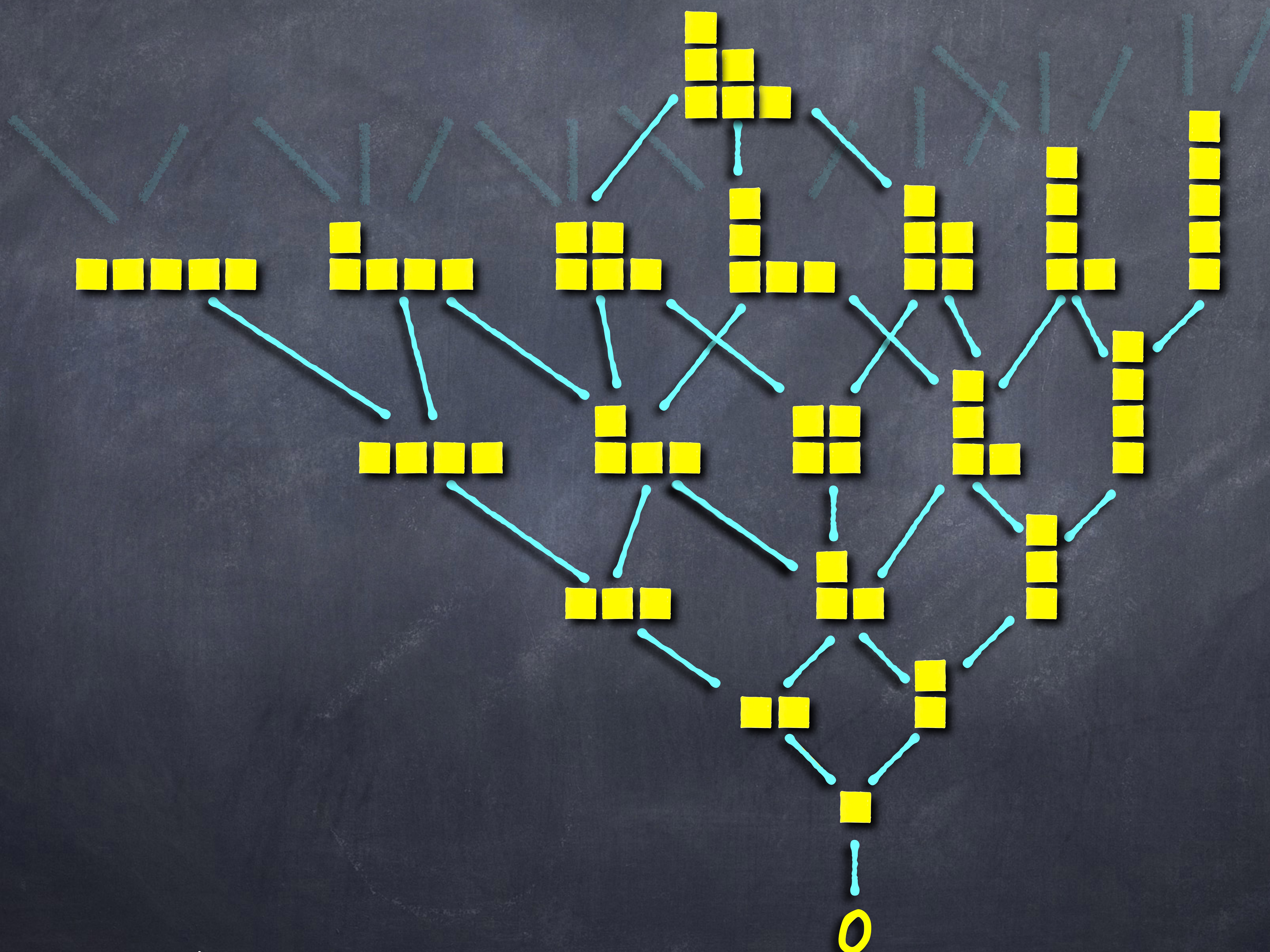
YOUNG LATTICE



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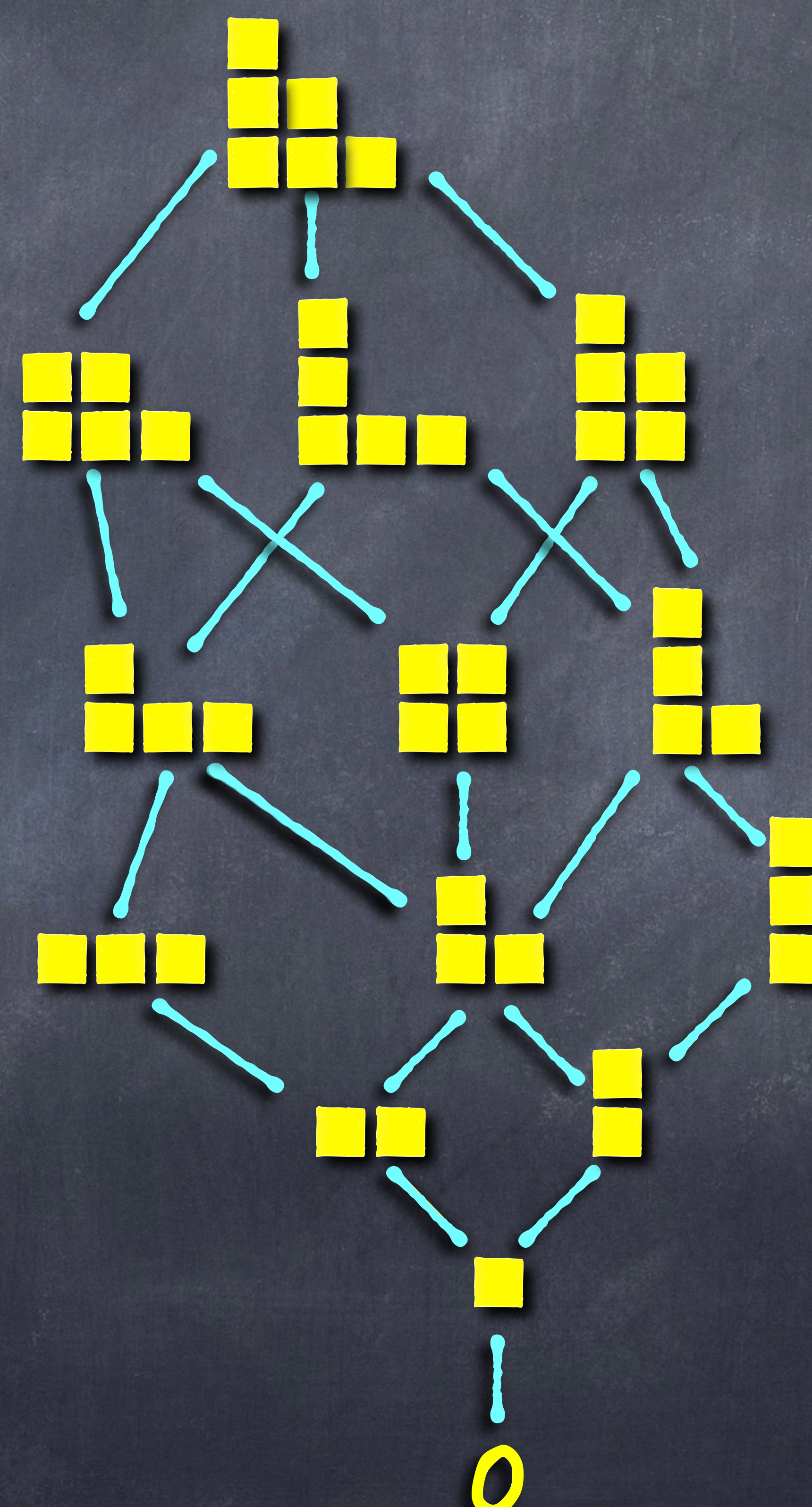
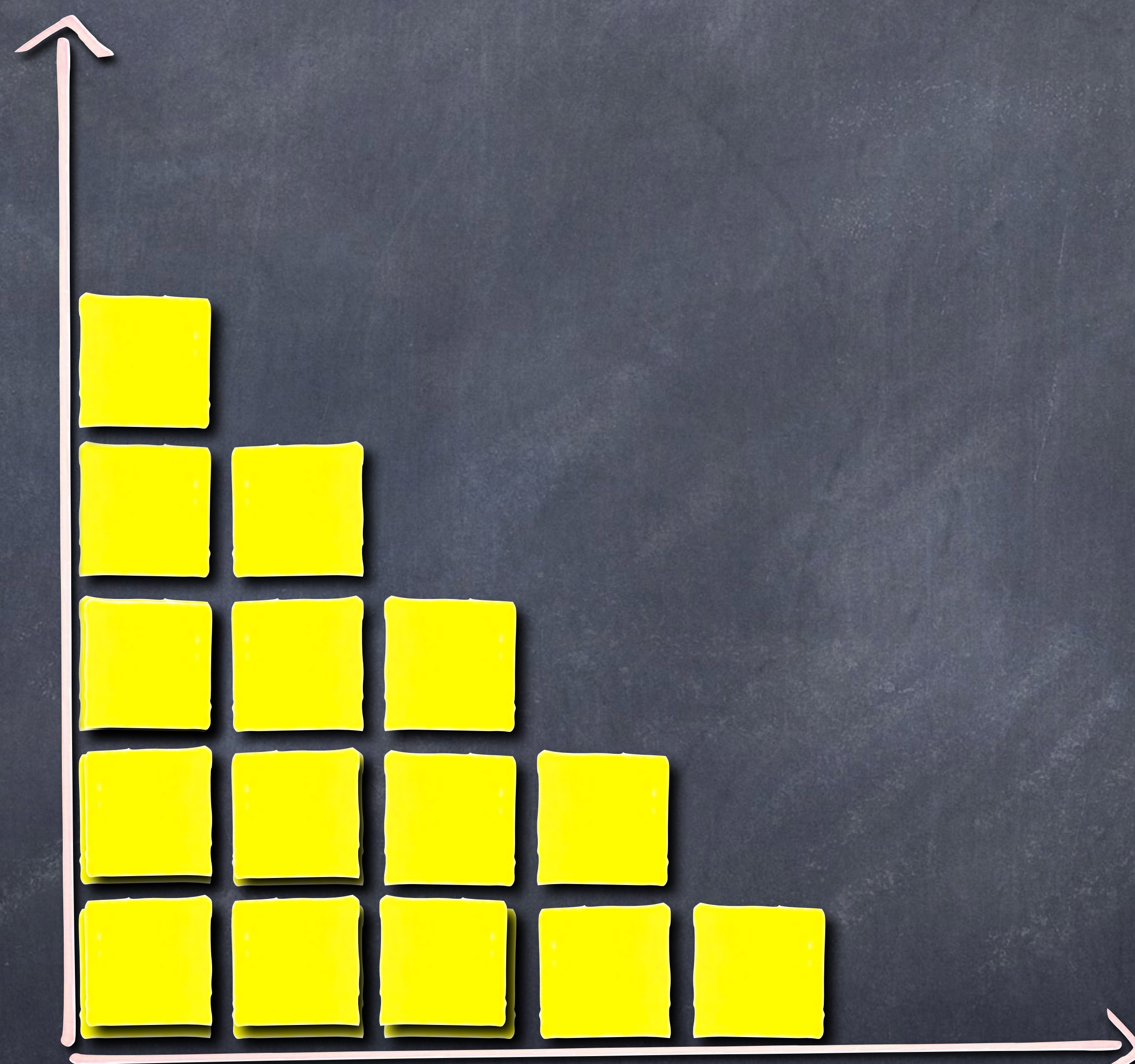
INTERESTING SUBLATTICES



n-STAIRCASE

$$\delta_n := (n-1, n-2, \ldots, 2, 1)$$

$n = 6$



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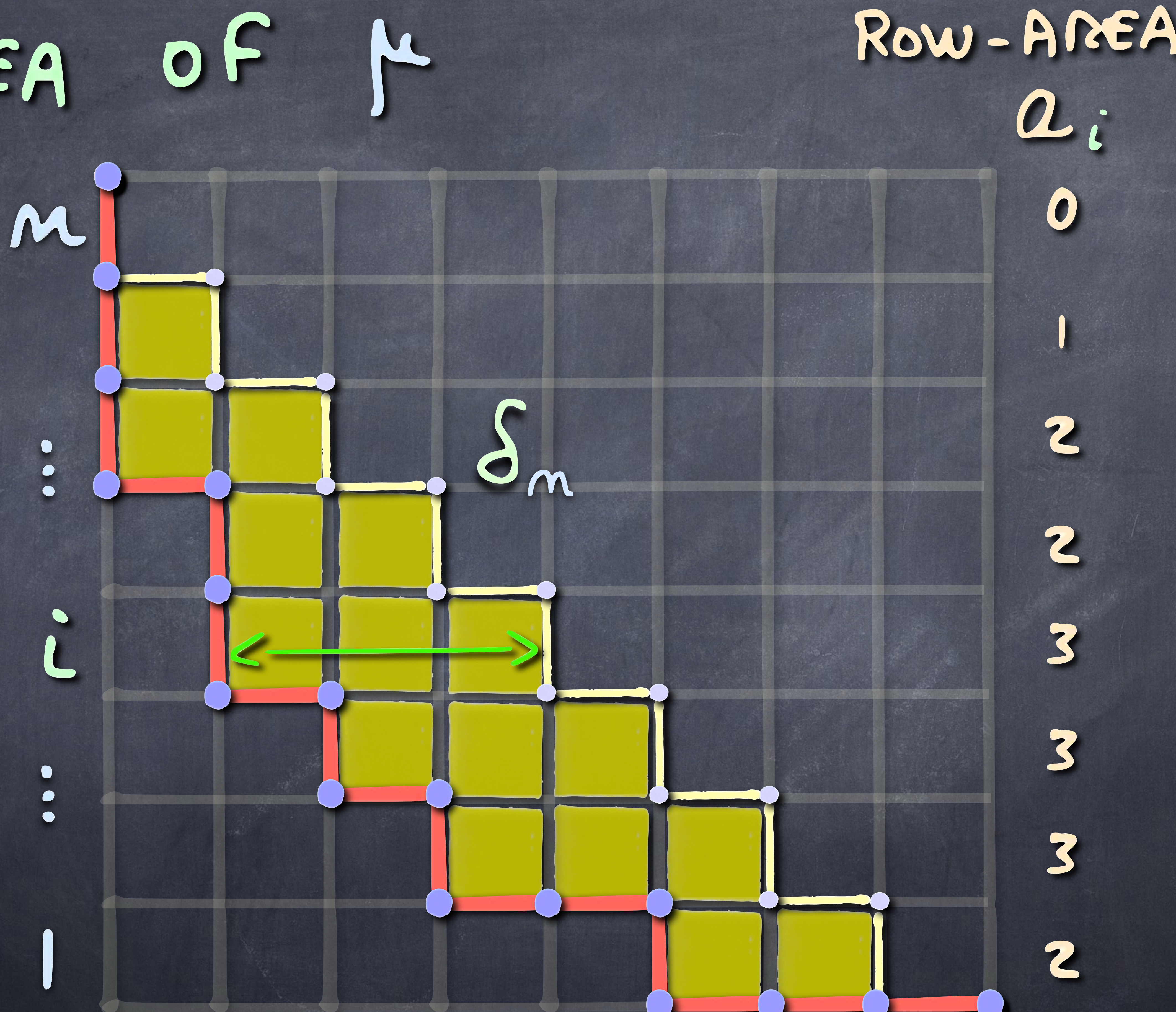
q-Catalan

$$\mathcal{C}_n(q) = \sum_{\mu \subseteq \delta_n} q^{\binom{n}{2} - |\mu|}$$

$$q^6 + q^5 + 2\,q^4 + 3\,q^3 + 3\,q^2 + 3q + 1$$

AREA OF μ



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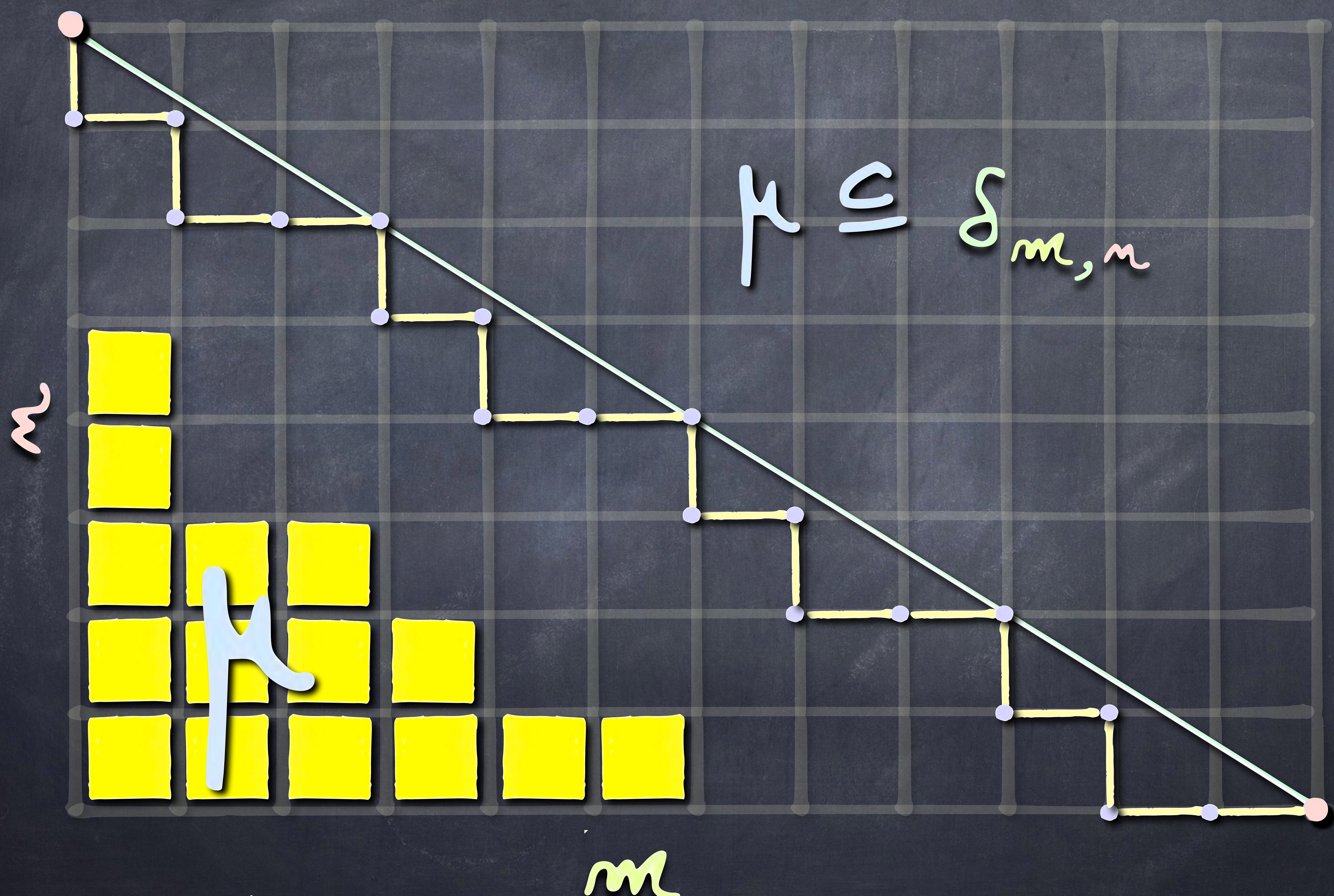

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(m,n)-q-Catalan

Area(μ)

$$\boxed{\mathcal{C}_{m,n}(q) = \sum_{\mu \subseteq \delta_{m,n}} q^{|\delta_{m,n}|-|\mu|}}$$

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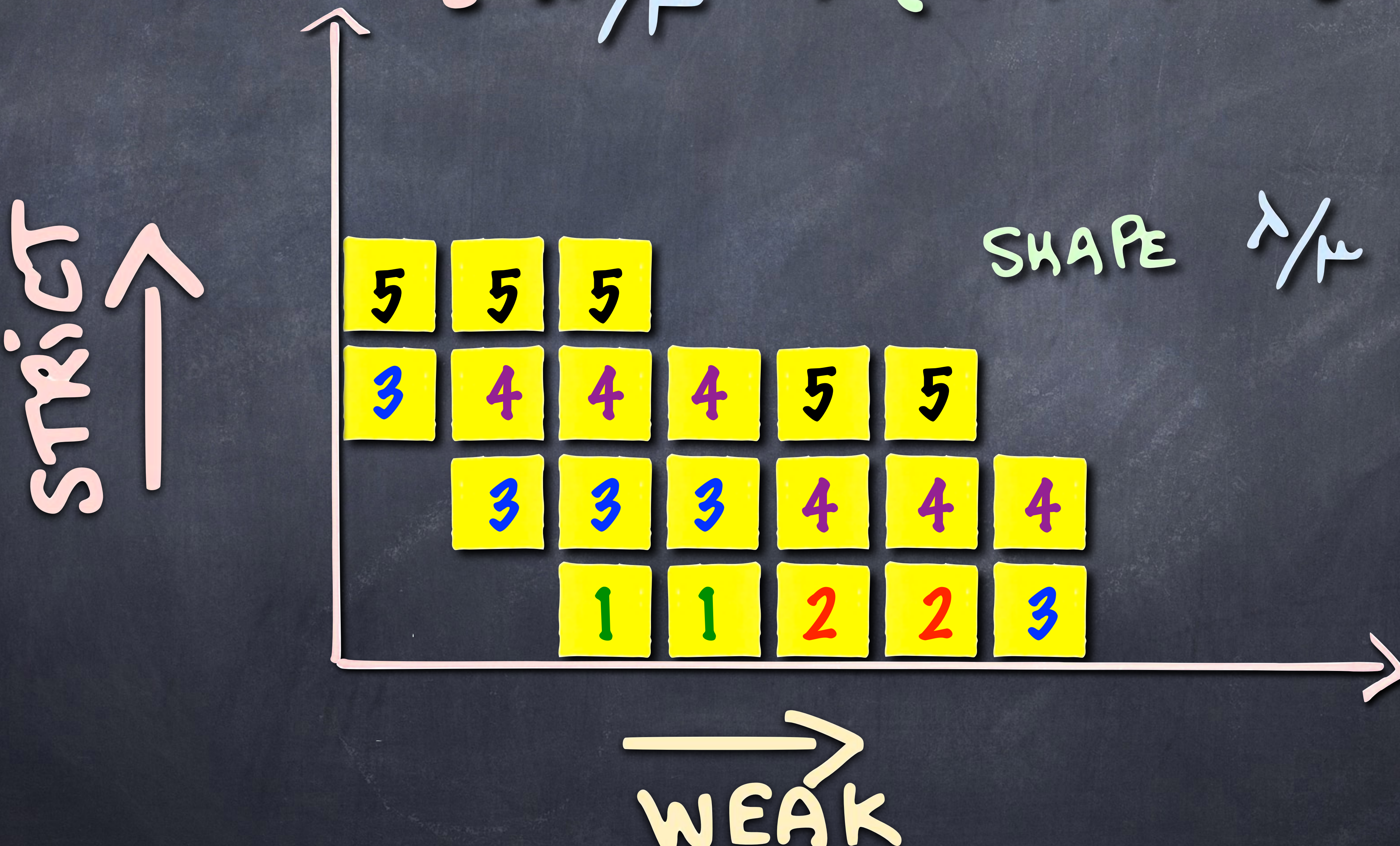
Tableaux

SEMI-STANDARD YOUNG TABLEAUX

$$\tau : \lambda/\mu \longrightarrow \{1, 2, \ldots, N\}$$



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COUNTING SEMI-STANDARD TABLEAUX

(AKA SSYT)

THE NUMBER OF SSYT OF SHAPE μ WITH VALUE IN $\{1, 2, \ldots, N\}$ IS EQUAL TO:

$$\boxed{\prod_{(i,j)\in\mu} \frac{N+i-j}{h_{ij}}}$$

THIS IS A POLYNOMIAL IN N

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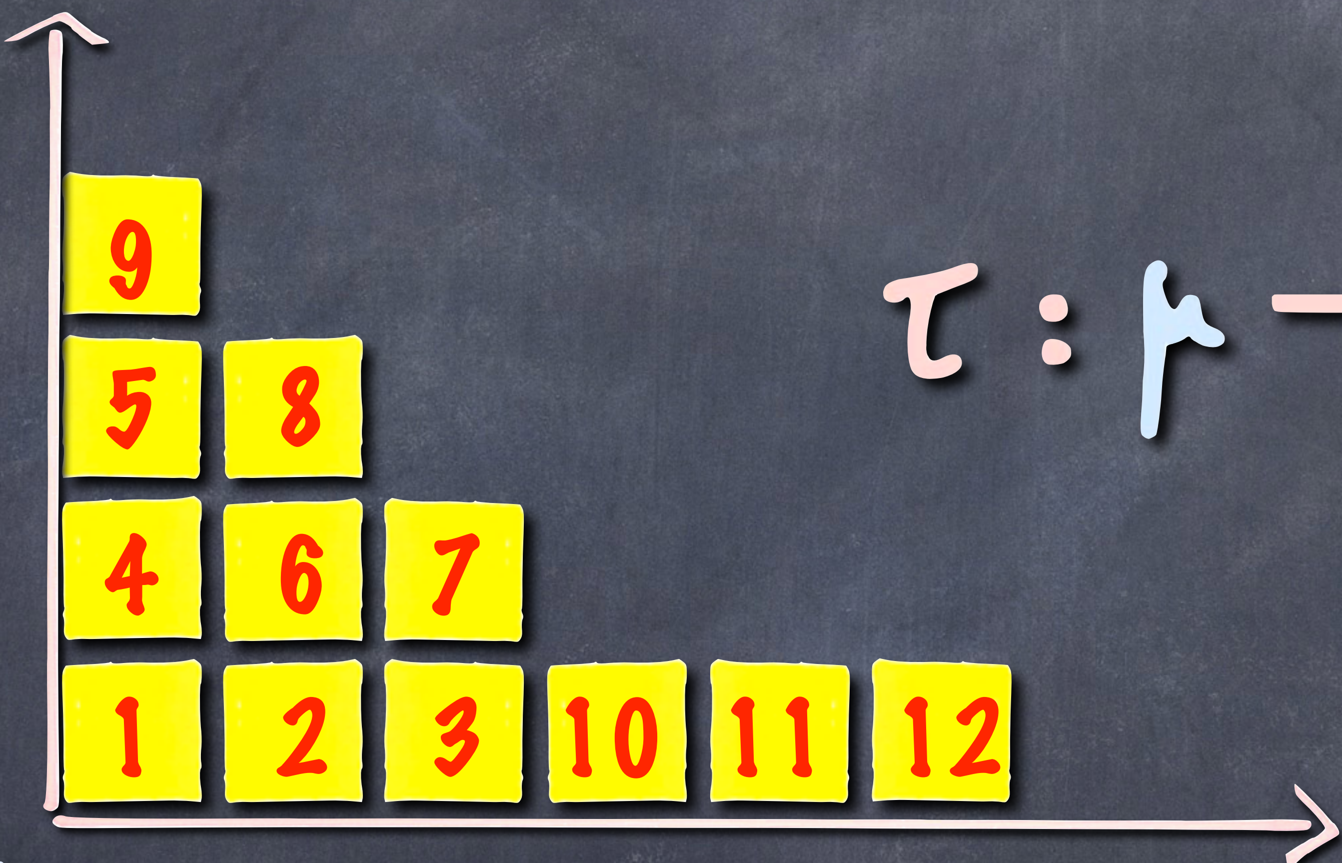
STANDARD YOUNG TABLEAUX (AKA SYT)

μ



$\tau : \mu \xrightarrow{\sim} \{1, 2, \ldots, |\mu|\}$

BIJECTIVE

$f^{\lambda/\mu}$:= NUMBER OF SYT OF SHAPE λ/μ

$$f^{\mu} = \frac{n!}{\prod_{\kappa \in \mu} h(\kappa)}$$

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$$\boxed{\sum_{\mu \subseteq \delta_n} f^{(\mu + 1^n)/\mu} = (n+1)^{n-1}}$$

PARKING FUNCTIONS

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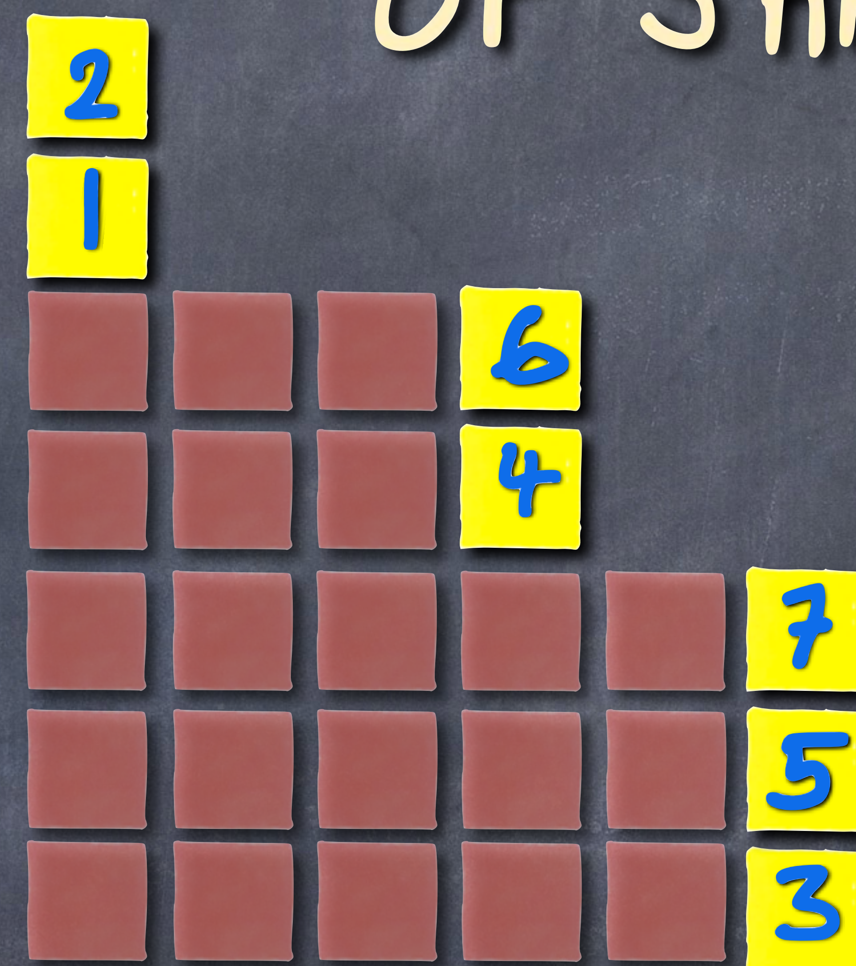
Parking Functions

of shape μ

$$(\mu + 1^n)/\mu$$



An Exercise

$$\sum_{\mu \subseteq \delta_{m,n}} f^{(\mu + 1^n)/\mu} = \;?$$

(m, n)-Parking Functions

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Funções Simétricas

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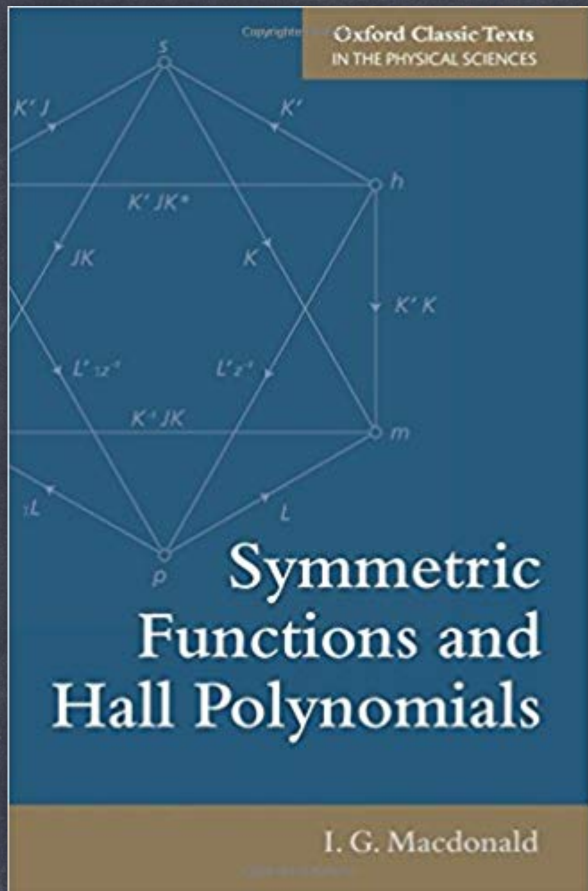

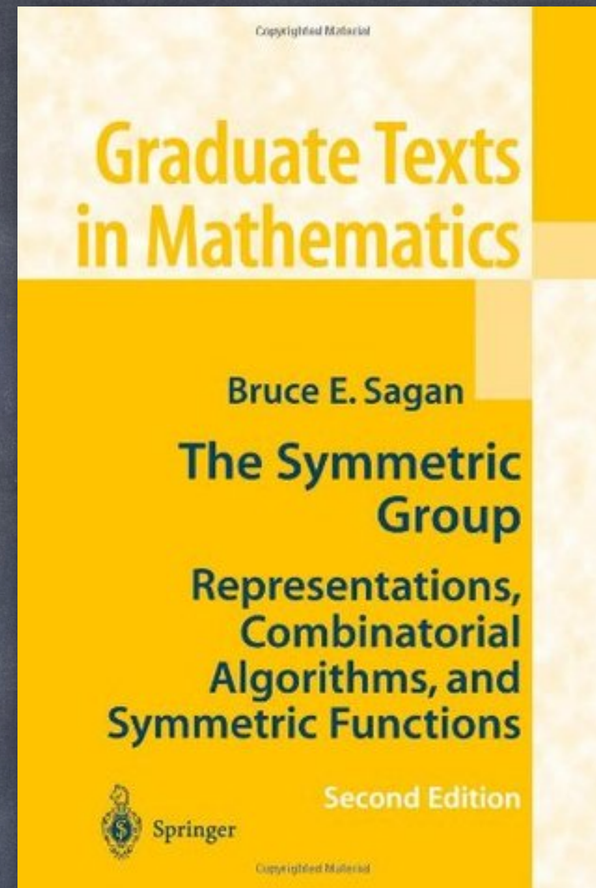

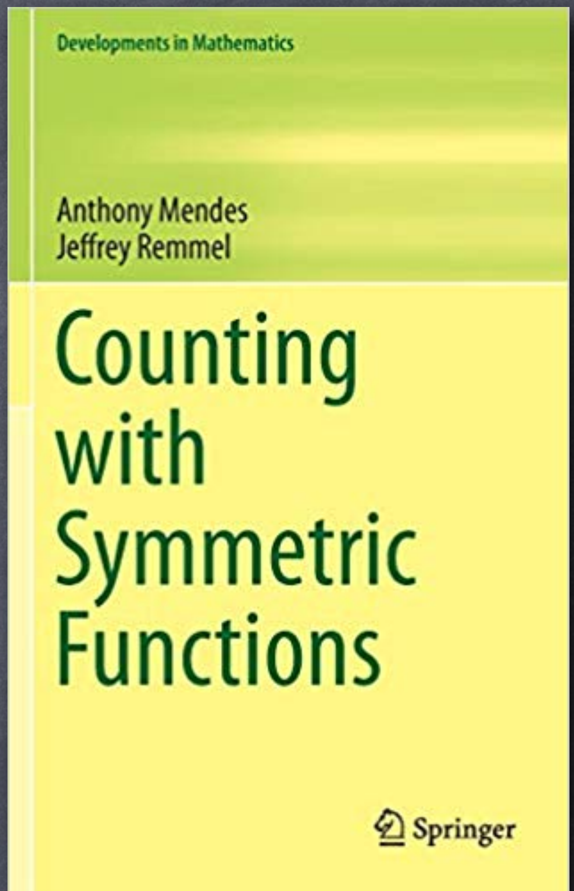

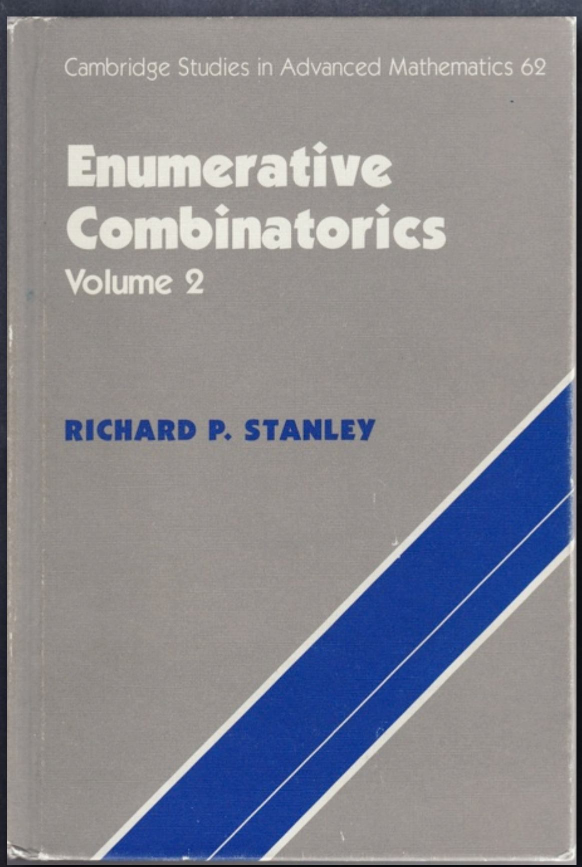

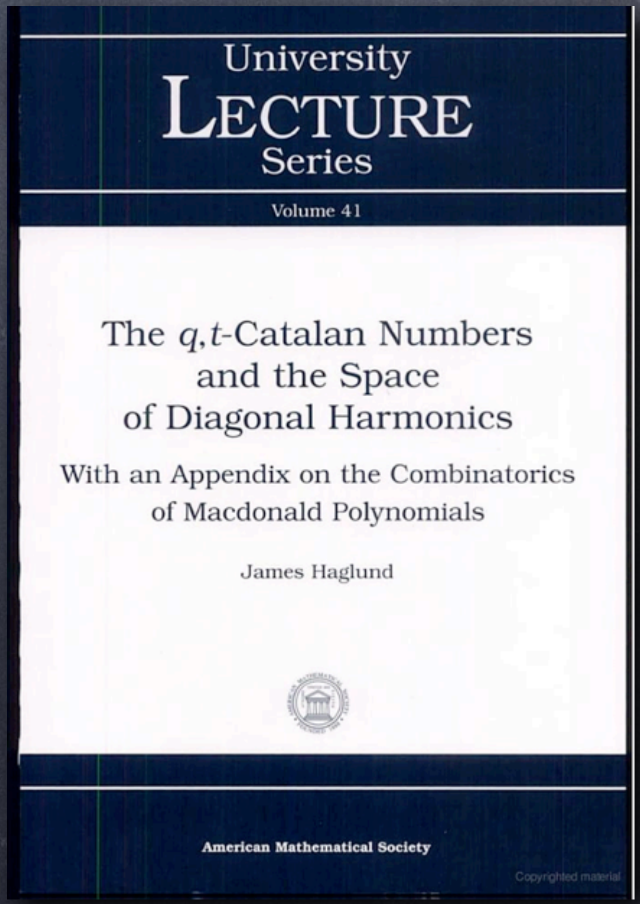

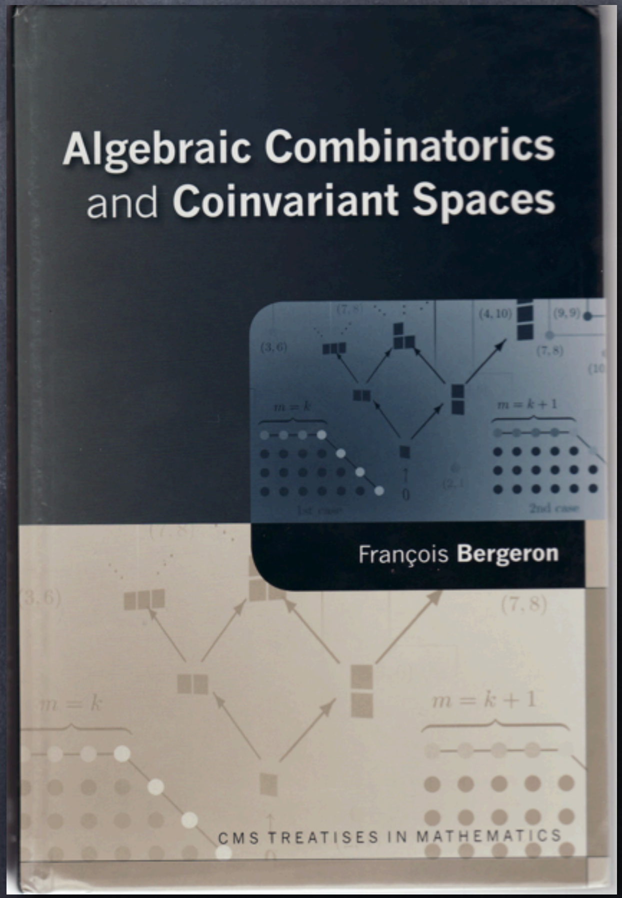

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$$m_{21}(x,y,z) = x^2y + x^2z + y^2z + xy^2 + xz^2 + yz^2$$

$$p_{21}(x,y,z) = (x^2+y^2+z^2)\,(x+y+z)$$

$$e_{21}(x,y,z) = (xy+xz+yz)\,(x+y+z)$$

$$h_{21}(x,y,z) = (x^2+y^2+z^2+xy+xz+yz)\,(x+y+z)$$

ALL TERMS ARE OF SAME DEGREE

HOMOGENOUS

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WHERE

$$h_n(Z) := \sum_{\mu \vdash n} m_\mu(Z)$$

$$e_n(Z) := m_{\underbrace{11\cdots 1}_{n}}(Z)$$

$$p_n(Z) := m_n(Z)$$

$$Z = z_1, z_2, z_3, \ldots$$

FORMULAS WITH NO "VARIABLES"

$$p_{21} = m_{21} + m_3$$

$$e_{21} = 3\,m_{111} + m_{21}$$

$$h_{21} = 3\,m_{111} + 2\,m_{21} + m_3$$

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Λ Ring of symmetric functions is graded by degree

$$\Lambda = \bigoplus_{d \geq 0} \Lambda_d$$

Degree d homogenous component

Basis of Λ_d indexed by partitions of d

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Bases of Λ_d indexed by partitions μ of d

- Monomial

$$m_\mu$$

- Complete homogenous

$$h_\mu := h_{\mu_1} h_{\mu_2} \cdots h_{\mu_k}$$

- Elementary

$$e_\mu := e_{\mu_1} e_{\mu_2} \cdots e_{\mu_k}$$

- Power sum

$$p_\mu := p_{\mu_1} p_{\mu_2} \cdots p_{\mu_k}$$

SCALAR PRODUCT

$$\langle m_\mu, h_\lambda \rangle = \begin{cases} 1 & \text{if } \lambda = \mu \\ 0 & \text{if } \lambda \neq \mu \end{cases}$$

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ADJOINT OPERATOR

$$\langle f \cdot g, h \rangle = \langle g, f^{\perp} h \rangle$$

THE ω INVOLUTION

LINEAR MULTIPLICATIVE

SELF-ADJOINT OPERATOR

$$\omega(p_k) = (-1)^{k-1} p_k$$

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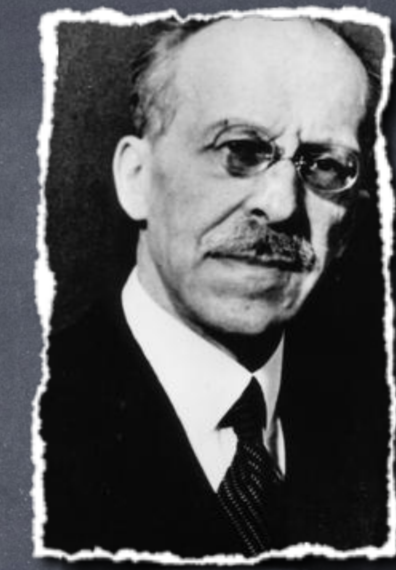
Schur Functions



Issai Schur
(1875–1941)

$$s_\mu(\mathbf{z}) := \sum_{\tau \in SSYT(\mu)} \mathbf{z}_\tau$$

$$\mathbf{z}_\tau := \prod_{c \in \mu} z_{\tau(c)} \qquad \mathbf{z} = z_1, z_2, z_3, \ldots$$

$$\langle s_\mu, s_\lambda \rangle = \begin{cases} 1 & \text{if } \lambda = \mu \\ 0 & \text{if } \lambda \neq \mu \end{cases}$$

$$\omega(s_\mu) = s_{\mu'}$$

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Skew Schur Functions

$$s_{\lambda/\mu}(\mathbf{z}) := \sum_{\tau \in SSYT(\lambda/\mu)} \mathbf{z}_\tau$$

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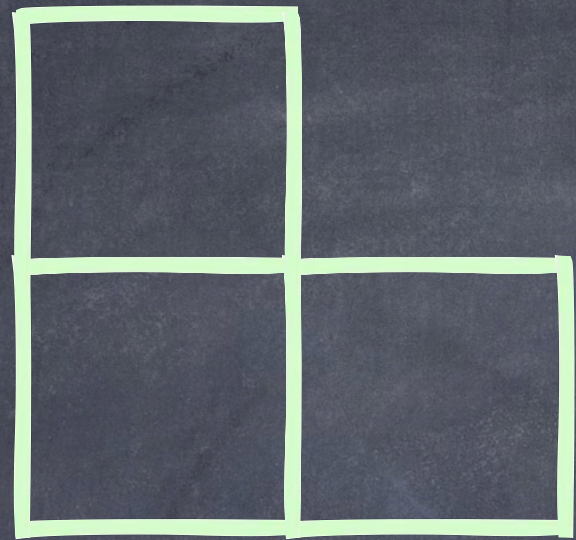

$$x < y < z$$

$$s_{21}(x,y,z) = x^2y + x^2z + y^2z + xy^2 + xz^2 + yz^2 + 2xyz$$

$$s_{21} = m_{21} + 2\,m_{111}$$

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$\boxed{\text{KOSTKA NUMBERS}}$

$$s_\lambda(z) := \sum_{\mu \vdash n} K_{\lambda\mu}\, m_\mu(z)$$

$K_{\lambda\mu}$:= # OF SEMI-STANDARD TABLEAUX OF SHAPE λ AND CONTENT μ

KOSTKA MATRIX

$$\mathcal{K}_5 = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 2 & 2 & 3 & 4 \\ 0 & 0 & 1 & 1 & 2 & 3 & 5 \\ 0 & 0 & 0 & 1 & 1 & 3 & 6 \\ 0 & 0 & 0 & 0 & 1 & 2 & 5 \\ 0 & 0 & 0 & 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

$\mu \vdash 5$ 5, 41, 32, 311, 221, 2111, 11111

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$c_{\mu\nu}^{\lambda}$

LITTLEWOOD-RICHARDSON COEFFICIENTS

$$s_\mu s_\nu = \sum_\lambda c_{\mu\nu}^{\lambda} s_\lambda$$

$$c_{\mu\nu}^{\lambda} \in \mathbb{N}$$

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Littlewood-Richardson coefficients

$c^{\lambda}_{\mu\nu}$

$$s_3\, s_3 = s_{33} + s_{42} + s_{51} + s_6$$

$$s_3\, s_{21} = s_{321} + s_{411} + s_{42} + s_{51}$$

$$s_3\, s_{111} = s_{3111} + s_{411}$$

$$s_{21}\, s_{21} = s_{2211} + s_{222} + s_{3111} + 2s_{321} + s_{33} + s_{411} + s_{42}$$

$$s_{21}\, s_{111} = s_{21111} + s_{2211} + s_{3111} + s_{321}$$

$$s_{111}\, s_{111} = s_{111111} + s_{21111} + s_{2211} + s_{222}$$

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$c^{\lambda}_{\mu\nu}$

LITTLEWOOD-RICHARDSON COEFFICIENTS

$$s^{\perp}_{\nu}\, s_{\lambda} = s_{\lambda/\nu} = \sum_{\mu} c^{\lambda}_{\mu\nu}\, s_{\mu}$$

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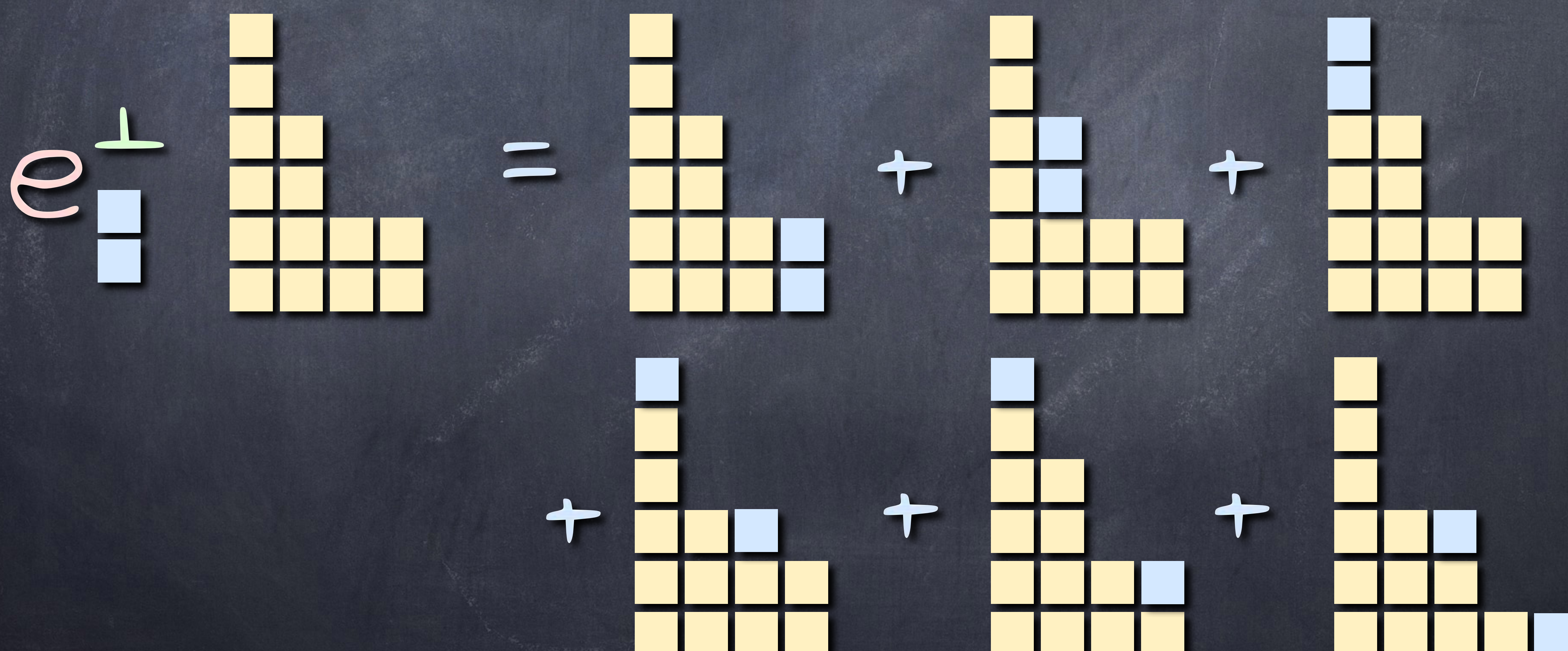
(Dual) Pieri Formula

$$e_k^{\perp}\, s_\lambda = \sum_{\mu \subset_k \lambda} s_\mu$$



PLETHYSM $s_\mu \circ s_\lambda$

Λ - RING CALCULATIONS

$$= s_\mu[s_\lambda]$$

RULES OF Λ-RING CALCULATIONS

$$(f+g)[\quad] = f[\quad] + g[\quad]$$

$$(f\cdot g)[\quad] = f[\quad]\cdot g[\quad]$$

$$p_k[X\,Y] = p_k[X]\;p_k[Y]$$

$$p_k[X/Y] = p_k[X] \div p_k[Y]$$

$$p_k[X\pm Y] = p_k[X] \pm p_k[Y]$$

$$p_k[x] = x^k \qquad\qquad p_k[cte] = cte$$

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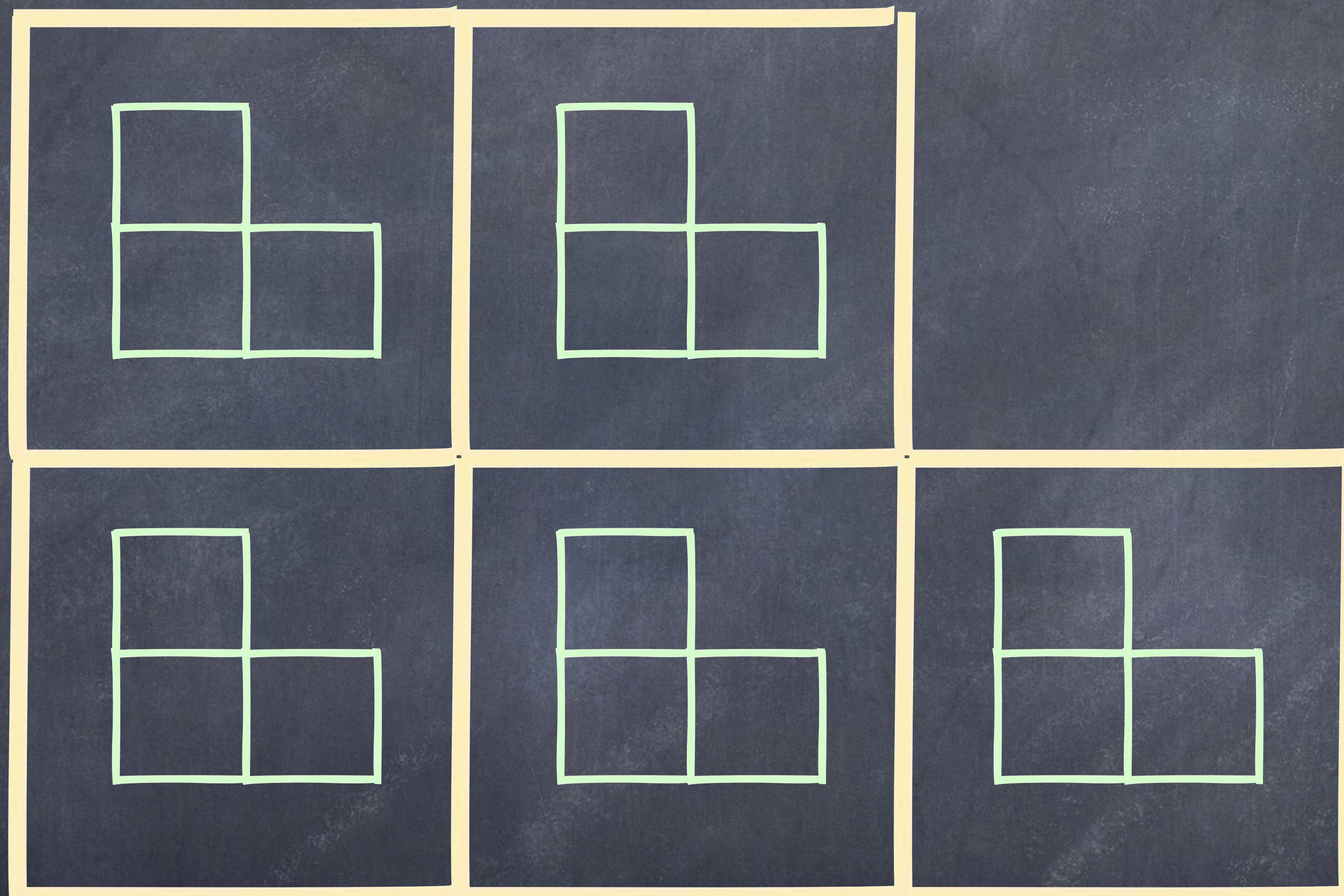
$$\Delta_{32} \circ \Delta_{21}$$



$$m_{6522} \qquad \underbrace{111111}_{6}\,\underbrace{22222}_{5}\,\underbrace{33}_{2}\,\underbrace{44}_{2}$$

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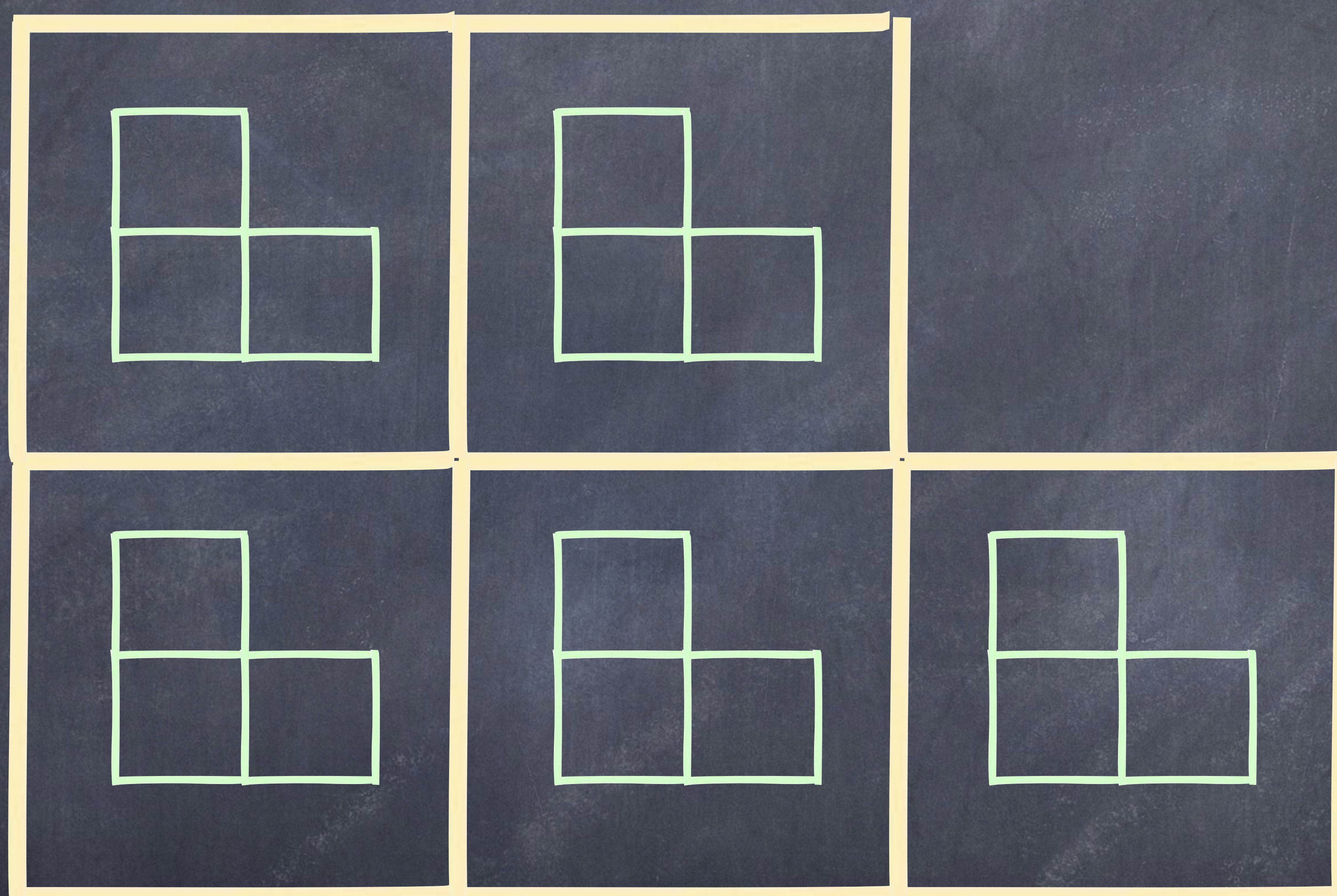
$$s_{32} \circ s_{21} = \cdots + 514\, m_{6522} + \cdots$$



$$\underbrace{1\,1\,1\,1\,1\,1}_{6}\,\underbrace{2\,2\,2\,2\,2}_{5}\,\underbrace{3\,3}_{2}\,\underbrace{4\,4}_{2}$$

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EXAMPLE:

$$\Delta_4 \circ \Delta_3 = \Delta_{12} + \Delta_{10,2} + \Delta_{93}$$
$$+ \Delta_{84} + \Delta_{822} + \Delta_{741} + \Delta_{732}$$
$$+ \Delta_{66} + \Delta_{642} + \Delta_{6222}$$
$$+ \Delta_{5421} + \Delta_{444}$$

Plethysm

Hilbert scheme

Coinvariant spaces

DAHA

Parking functions

Bosons

Group harmonics

Atomic states

Calogero-Sutherland

Fermions

Torus knots

Quasisymmetric functions

Diagonal invariants

Chemistry

Flag variety cohomology

Elliptic Hall-algebra

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SCHUR POSITIVIDADE

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Schur Positivity

$$F(\mathbf{z}) = \sum_{\lambda} a_\lambda\, s_\lambda(\mathbf{z})$$

$$a_\lambda(q) \in \mathbb{N}[q,t]$$

integer coefficient polynomial

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SCHUR POSITIVITY

$\boxed{\text{THM}}$ IF f AND g ARE SCHUR POSITIVE, THEN SO ARE

$$f \cdot g \quad \text{AND} \quad f \circ g$$

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Schur-Positivity is Rare

Among Positive coefficient Homogeneous degree d symmetric functions

$d = 6$

Proportion of Schur-Positive $\quad 1/1027458432000$

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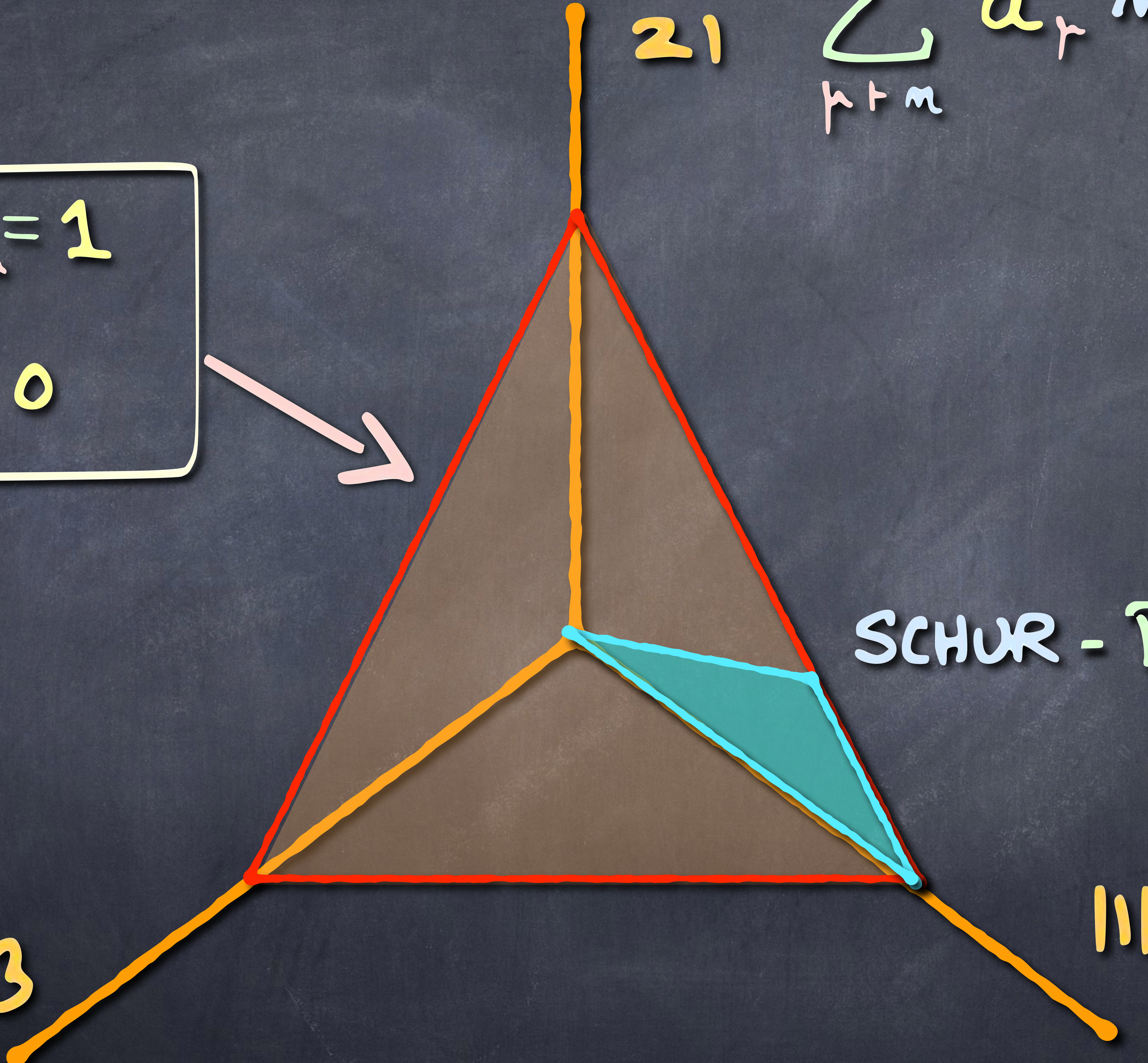
$$\sum_{\mu \vdash n} a_\mu m_\mu$$

$$\sum_\mu a_\mu = 1$$

$$a_\mu \geq 0$$



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SCHUR-POSITIVITY IS RARE


F.B.


Vic Reiner


Rebecca Patrias

THM THE PROBABILITY THAT A MONOMIAL POSITIVE SYMMETRIC FUNCTION IS SCHUR POSITIVE IS:

$$\prod_{\mu \vdash d} \Big(\sum_{\lambda} K_{\lambda\mu} \Big)^{-1}$$

$$\begin{pmatrix} 1+1+1+1+1+1+1 \\ 0 \quad 1 \quad 1 \quad 2 \quad 2 \quad 3 \quad 4 \\ 0 \quad 0 \quad 1 \quad 1 \quad 2 \quad 3 \quad 5 \\ 0 \quad 0 \quad 0 \quad 1 \quad 1 \quad 3 \quad 6 \\ 0 \quad 0 \quad 0 \quad 0 \quad 1 \quad 2 \quad 5 \\ 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 1 \quad 4 \\ 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 1 \end{pmatrix} = \begin{matrix} 7 \\ \times \\ 13 \\ \times \\ 12 \\ \times \\ 11 \\ \times \\ 8 \\ \times \\ 5 \\ \times \\ 1 \\ \hline 480480 \end{matrix}$$

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To Be
Continued