MULTIPLE HYPERGEOMETRIC SERIES – APPELL SERIES AND BEYOND

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Abstract. This survey article (which will appear as a chapter in the book "Computer Algebra in Quantum Field Theory: Integration, Summation and Special Functions", Springer-Verlag) provides a small collection of basic material on multiple hypergeometric series of Appell-type and of more general series of related type.

1. Introduction

Hypergeometric series and its various generalizations, in particular such involving *multiple* series, appear in various branches of mathematics and its applications. This survey article features a small collection of selected material on multiple hypergeometric series of *Appell*-type and of more general series of closely related type.

These types of series appear very naturally in quantum field theory, in particular in the computation of analytic expressions for Feynman integrals (for which we kindly refer to other relevent chapters in this volume). Such integrals can be obtained and computed in different ways – which may lead to identities for Appell series (see e.g. M.A. Shpot [30]). On the other hand, the application of known relations for Appell series may lead to simplifications, help to solve problems or lead to more insight in quantum field theory. Therefore it is of importance that people working in this area have a basic understanding of the existing theory for such series[1]. This survey is meant to provide a very digestible, easy introduction to Appell-type series. Besides of recalling some known results including various useful identities satisfied by the series, some of the standard mathematical techniques which are used to prove and derive these identities are illustrated. We highlight some of the most fundamental properties and relations for Appell hypergeometric series and further give hints of similar relations for the series which are (slightly) beyond the hierarchy of Appell series. All the series we consider admit very explicit series and integral representations.

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[1]Researchers working with Feynman integrals who are in demand of effective manipulation of Appell-type series including differential reductions and ϵ-expansions may find HYPERDIRE (located at `https://sites.google.com/site/loopcalculations/`) useful, which is a set of Wolfram Mathematica based programs for differential reduction of Horn-type hypergeometric functions, see V. Bytev et al. [9].

To warn the reader: There exist various different types of multivariate hypergeometric series which are not covered in this survey. In particular, here we do not treat multiple hypergeometric series associated with *root systems* [16, 20, 25, 29], hypergeometric series of *matrix argument* [15], and other types of multivariate hypergeometric series such as those which mainly appear in the study of *orthogonal polynomials* of severable variables (often also associated with root systems) [11, 22].

A very important extension of Appell-type series which is just beyond the scope of this basic survey article are the multivariate hypergeometric functions considered by I.M. Gelfand, M.M. Kapranov, and A.V. Zelevinsky [14], developed in the late 1980's. These A-hypergeometric functions (or GKZ-hypergeometric functions) are fundamental objects in the theory of integrable systems as they are the holonomic solutions of a (certain) A-hypergeometric system of partial differential equations. Natural questions regarding algebraic solutions and monodromy for A-hypergeometric functions have been recently addressed by F. Beukers [4, 5].

For *basic* (or *q-series*) analogues of Appell functions, see G. Gasper and M. Rahman's text [13, Ch. 10].

2. Appell series

Appell series are a natural two-variable extension of hypergeometric series. They are treated with detail in Érdelyi et al. [12], the classical reference for special functions.

In the following, we follow to great extent the expositions from the classical texts of W.N. Bailey [3], and L.J. Slater [31] (both contain a great amount of material on hypergeometric series).

For convenience, we use the Pochhammer symbol notation for the shifted factorial,

$$(a)_n := \begin{cases} a(a+1)\dots(a+n-1) & \text{if } n=1,2,\dots, \\ 1 & \text{if } n=0. \end{cases} \tag{1a}$$

Accordingly, we have

$$(a)_n = \frac{\Gamma(a+n)}{\Gamma(a)} \tag{1b}$$

which is used as a definition for the shifted factorial in case n is not necessarily a nonnegative integer.

The goal is to generalize the Gauß hypergeometric function

$${}_2F_1\left(\begin{matrix} a, & b \\ & c \end{matrix}; x\right) = \sum_{n\geq 0} \frac{(a)_n\,(b)_n}{n!\,(c)_n} x^n$$

to a double series depending on two variables.

The easiest is to consider the simple product

$$ {}_2F_1\left(\begin{matrix}a,\ b\\ c\end{matrix};x\right)\,{}_2F_1\left(\begin{matrix}a',\ b'\\ c'\end{matrix};y\right)=\sum_{m\geq 0}\sum_{n\geq 0}\frac{(a)_m\,(a')_n\,(b)_m\,(b')_n}{m!\,n!\;(c)_m\,(c')_n}x^m y^n, $$

where on the right-hand side the indices m, n appear uncoupled.

To consider a genuine double series instead (which does not factor into a simple product of two series), we now deliberately choose to replace one or more of the three products $(a)_m\,(a')_n$, $(b)_m\,(b')_n$, $(c)_m\,(c')_n$ by products of coupled type $(a)_{m+n}$ (other choices such as $(a)_{m-n}$ or $(a)_{2m-n}$, etc., instead, may be sensible as well; they lead to Horn-type series, see Subsection 3.1).

There are five different possibilities, one of which by application of the binomial theorem gives the series

$$ \sum_{m\geq 0}\sum_{n\geq 0}\frac{(a)_{m+n}\,(b)_{m+m}}{m!\,n!\;(c)_{m+n}}x^m y^n={}_2F_1\left(\begin{matrix}a,\ b\\ c\end{matrix};x+y\right), $$

i.e., an ordinary hypergeometric series.

The other four remaining possibilities are classified as F_1-, F_2-, F_3-, and F_4-series (cf. P. Appell [1] and P. Appell & M.-J. Kampé de Fériet [2]):

$$ F_1\big(a;b,b';c;x,y\big):=\sum_{m\geq 0}\sum_{n\geq 0}\frac{(a)_{m+n}\,(b)_m\,(b')_n}{m!\,n!\;(c)_{m+n}}x^m y^n,\qquad |x|,|y|<1. \tag{2a} $$

$$ F_2\big(a;b,b';c,c';x,y\big):=\sum_{m\geq 0}\sum_{n\geq 0}\frac{(a)_{m+n}\,(b)_m\,(b')_n}{m!\,n!\;(c)_m\,(c')_n}x^m y^n,\qquad |x|+|y|<1. \tag{2b} $$

$$ F_3\big(a,a';b,b';c;x,y\big):=\sum_{m\geq 0}\sum_{n\geq 0}\frac{(a)_m\,(a')_n\,(b)_m\,(b')_n}{m!\,n!\;(c)_{m+n}}x^m y^n,\qquad |x|,|y|<1. \tag{2c} $$

$$ F_4\big(a;b;c,c';x,y\big):=\sum_{m\geq 0}\sum_{n\geq 0}\frac{(a)_{m+n}\,(b)_{m+n}}{m!\,n!\;(c)_m\,(c')_n}x^m y^n,\qquad |x|^{\frac{1}{2}}+|y|^{\frac{1}{2}}<1. \tag{2d} $$

One immediately observes the following simple identities:

$$ F_1\big(a;b,b';c;x,y\big)=\sum_{m\geq 0}\frac{(a)_m\,(b)_m}{m!\,(c)_m}x^m\,{}_2F_1\left(\begin{matrix}a+m,\ b'\\ c+m\end{matrix};y\right). \tag{3} $$

$$ F_1\big(a;b,b';c;x,0\big)=F_2\big(a;b,b';c,c';x,0\big)=F_3\big(a,a';b,b';c;x,0\big) \tag{4a} $$

$$ =F_4\big(a;b;c,c';x,0\big)={}_2F_1\left(\begin{matrix}a,\ b\\ c\end{matrix};x\right). \tag{4b} $$

$$ F_1\big(a;b,0;c;x,y\big)=F_2\big(a;b,0;c,c';x,y\big)=F_3\big(a,a';b,0;c;x,y\big) \tag{5a} $$

$$ ={}_2F_1\left(\begin{matrix}a,\ b\\ c\end{matrix};x\right). \tag{5b} $$

Using ideas of N.Ja. Vilenkin [32], W. Miller, Jr. [24] has given a Lie theoretic interpretation of the Appell functions F_1. In particular, he showed that $sl(5,\mathbb{C})$ is the dynamical symmetry algebra for the F_1.

2.1. **Contiguous relations and recursions.** All contiguous relations for the F_1 function can be derived from these four relations:

$$\begin{aligned}(a-b-b')\,F_1\big(a;b,b';c;x,y\big) - a\,F_1\big(a+1;b,b';c;x,y\big)\\ +b\,F_1\big(a;b+1,b';c;x,y\big) + b'\,F_1\big(a;b,b'+1;c;x,y\big) = 0,\end{aligned} \tag{6a}$$

$$\begin{aligned}c\,F_1\big(a;b,b';c;x,y\big) - (c-a)\,F_1\big(a;b,b';c+1;x,y\big)\\ -a\,F_1\big(a+1;b,b';c+1;x,y\big) = 0,\end{aligned} \tag{6b}$$

$$\begin{aligned}c\,F_1\big(a;b,b';c;x,y\big) + c(x-1)\,F_1\big(a;b+1,b';c;x,y\big)\\ -(c-a)x\,F_1\big(a;b+1,b';c+1;x,y\big) = 0,\end{aligned} \tag{6c}$$

$$\begin{aligned}c\,F_1\big(a;b,b';c;x,y\big) + c(y-1)\,F_1\big(a;b,b'+1;c;x,y\big)\\ -(c-a)y\,F_1\big(a;b,b'+1;c+1;x,y\big) = 0.\end{aligned} \tag{6d}$$

Similar sets of relations exist for the other Appell functions, see R.G. Buschman [8].

Recently, X. Wang [35] has used contiguous relations and induction to derive various recursion formulae for all the Appell functions F_1, F_2, F_3, F_4. (Some of the recursions for F_2 were previously given by S.B. Opps, N. Saad and H.M. Srivastava [26].) For $n=1$ these recursions reduce to equivalent forms of the known contiguous relations.

In particular, for F_1 we have

$$\begin{aligned}F_1\big(a+n;b,b';c;x,y\big) = F_1\big(a;b,b';c;x,y\big) + \frac{bx}{c}\sum_{k=1}^{n} F_1\big(a+k;b+1,b';c+1;x,y\big)\\ + \frac{b'y}{c}\sum_{k=1}^{n} F_1\big(a+k;b,b'+1;c+1;x,y\big),\end{aligned} \tag{7a}$$

$$\begin{aligned}F_1\big(a-n;b,b';c;x,y\big) = F_1\big(a;b,b';c;x,y\big) - \frac{bx}{c}\sum_{k=1}^{n-1} F_1\big(a-k;b+1,b';c+1;x,y\big)\\ - \frac{b'y}{c}\sum_{k=1}^{n-1} F_1\big(a-k;b,b'+1;c+1;x,y\big),\end{aligned} \tag{7b}$$

$$F_1\big(a;b+n,b';c;x,y\big) = F_1\big(a;b,b';c;x,y\big) + \frac{ax}{c}\sum_{k=1}^{n} F_1\big(a+1;b+k,b';c+1;x,y\big), \tag{7c}$$

$$F_1\big(a;b-n,b';c;x,y\big) = F_1\big(a;b,b';c;x,y\big) - \frac{ax}{c}\sum_{k=1}^{n-1} F_1\big(a+1;b-k,b';c+1;x,y\big), \tag{7d}$$

$$F_1\big(a;b,b';c-n;x,y\big) = F_1\big(a;b,b';c;x,y\big) + abx\sum_{k=1}^{n}\frac{F_1\big(a+1;b+1,b';c-k+2;x,y\big)}{(c-k)(c-k+1)} + ab'y\sum_{k=1}^{n}\frac{F_1\big(a+1;b,b'+1;c-k+2;x,y\big)}{(c-k)(c-k+1)}. \tag{7e}$$

For F_2 we have

$$F_2\big(a+n;b,b';c,c';x,y\big) = F_2\big(a;b,b';c,c';x,y\big) + \frac{bx}{c}\sum_{k=1}^{n}F_2\big(a+k;b+1,b';c+1,c';x,y\big) + \frac{b'y}{c'}\sum_{k=1}^{n}F_2\big(a+k;b,b'+1;c,c'+1;x,y\big), \tag{8a}$$

$$F_2\big(a-n;b,b';c,c';x,y\big) = F_2\big(a;b,b';c,c';x,y\big) - \frac{bx}{c}\sum_{k=1}^{n-1}F_2\big(a-k;b+1,b';c+1,c';x,y\big) - \frac{b'y}{c'}\sum_{k=1}^{n-1}F_2\big(a+k;b,b'+1;c,c'+1;x,y\big), \tag{8b}$$

$$F_2\big(a;b+n,b';c,c';x,y\big) = F_2\big(a;b,b';c,c';x,y\big) + \frac{ax}{c}\sum_{k=1}^{n}F_2\big(a+1;b+k,b';c+1,c';x,y\big), \tag{8c}$$

$$F_2\big(a;b-n,b';c,c';x,y\big) = F_2\big(a;b,b';c,c';x,y\big) - \frac{ax}{c}\sum_{k=1}^{n-1}F_2\big(a+1;b-k,b';c+1,c';x,y\big), \tag{8d}$$

$$F_2\big(a;b,b';c-n,c';x,y\big) = F_2\big(a;b,b';c,c';x,y\big) + abx\sum_{k=1}^{n}\frac{F_2\big(a+1;b+1,b';c-k+2,c';x,y\big)}{(c-k)(c-k+1)}. \tag{8e}$$

For F_3 we have

$$F_3\big(a+n,a';b,b';c;x,y\big) = F_3\big(a,a';b,b';c;x,y\big) + \frac{bx}{c}\sum_{k=1}^{n}F_3\big(a+k,a';b+1,b';c+1;x,y\big), \tag{9a}$$

$$F_3\big(a-n,a';b,b';c;x,y\big) = F_3\big(a,a';b,b';c;x,y\big)$$

$$- \frac{bx}{c}\sum_{k=1}^{n-1} F_3\big(a-k,a';b+1,b';c+1;x,y\big), \tag{9b}$$

$$\begin{aligned} F_3\big(a,a';b,b';c-n;x,y\big) &= F_3\big(a,a';b,b';c;x,y\big) \\ &\quad + abx\sum_{k=1}^{n} \frac{F_3\big(a+1,a';b+1,b';c-k+2;x,y\big)}{(c-k)(c-k+1)} \\ &\quad + a'b'y\sum_{k=1}^{n} \frac{F_3\big(a,a'+1;b,b'+1;c-k+2;x,y\big)}{(c-k)(c-k+1)}. \end{aligned} \tag{9c}$$

Finally, for F_4 we have

$$\begin{aligned} F_4\big(a+n;b;c,c';x,y\big) &= F_4\big(a;b;c,c';x,y\big) \\ &\quad + \frac{bx}{c}\sum_{k=1}^{n} F_4\big(a+k;b+1;c+1,c';x,y\big) \\ &\quad + \frac{by}{c'}\sum_{k=1}^{n} F_4\big(a+k;b+1;c,c'+1;x,y\big), \end{aligned} \tag{10a}$$

$$\begin{aligned} F_4\big(a-n;b;c,c';x,y\big) &= F_4\big(a;b;c,c';x,y\big) \\ &\quad - \frac{bx}{c}\sum_{k=1}^{n-1} F_4\big(a-k;b+1;c+1,c';x,y\big) \\ &\quad - \frac{by}{c'}\sum_{k=1}^{n-1} F_4\big(a-k;b+1;c,c'+1;x,y\big), \end{aligned} \tag{10b}$$

$$\begin{aligned} F_4\big(a;b;c-n,c';x,y\big) &= F_4\big(a;b;c,c';x,y\big) \\ &\quad + abx\sum_{k=1}^{n-1} \frac{F_4\big(a+1;b+1;c-k+1,c';x,y\big)}{(c-k)(c-k-1)}. \end{aligned} \tag{10c}$$

Most of these recursions can be extended to elegant recursions involving more terms. For instance,

$$\begin{aligned} F_1\big(a+n;b,b';c;x,y\big) &= \sum_{i=0}^{n}\sum_{k=0}^{n-i}\binom{n}{i}\binom{n-i}{k}\frac{(b)_i(b')_k}{(c)_{k+i}} \\ &\quad \times x^i y^j F_1\big(a+i+k;b+i,b'+k;c+i+k;x,y\big), \end{aligned} \tag{11a}$$

$$\begin{aligned} F_1\big(a-n;b,b';c;x,y\big) &= \sum_{i=0}^{n}\sum_{k=0}^{n-i}\binom{n}{i}\binom{n-i}{k}\frac{(b)_i(b')_k}{(c)_{k+i}} \\ &\quad \times (-x)^i(-y)^j F_1\big(a;b+i,b'+k;c+i+k;x,y\big), \end{aligned} \tag{11b}$$

$$F_1\big(a;b+n,b';c;x,y\big) = \sum_{k=0}^{n}\binom{n}{k}\frac{(a)_k}{(c)_k}x^k F_1\big(a+k;b+k,b';c+k;x,y\big), \tag{11c}$$

$$F_1\big(a;b-n,b';c;x,y\big) = \sum_{k=0}^{n}\binom{n}{k}\frac{(a)_k}{(c)_k}(-x)^k F_1\big(a+k;b,b';c+k;x,y\big), \tag{11d}$$

or

$$F_4\big(a;b;c-n,c';x,y\big) = \sum_{k=0}^{n}\binom{n}{k}\frac{(a)_k(b)_k}{(c)_k(c-n)_k}x^k F_4\big(a+k;b+k;c+k,c';x,y\big). \tag{12}$$

2.2. **Partial differential equations.** Let

$$z = F_1\big(a;b,b';c;x,y\big) = \sum_{m\ge0}\sum_{n\ge0}A_{m,n}x^m y^n.$$

Then

$$A_{m+1,n} = \frac{(a+m+n)(b+m)}{(1+m)(c+m+n)}A_{m,n},$$

and

$$A_{m,n+1} = \frac{(a+m+n)(b'+n)}{(1+n)(c+m+n)}A_{m,n}.$$

Denoting the partial differential operators by

$$\theta = x\frac{\partial}{\partial x} \qquad \text{and} \qquad \phi = y\frac{\partial}{\partial y},$$

we readily see that $z = F_1$ satisfies the partial differential equations

$$\big[(\theta+\phi+a)(\theta+b) - \frac{1}{x}\theta(\theta+\phi+c-1)\big]z = 0, \tag{13a}$$

$$\big[(\theta+\phi+a)(\phi+b') - \frac{1}{x}\phi(\theta+\phi+c-1)\big]z = 0. \tag{13b}$$

Now let

$$p = \frac{\partial z}{\partial x}, \quad q = \frac{\partial z}{\partial y}, \quad r = \frac{\partial z}{\partial x}\frac{\partial z}{\partial y}, \quad s = \frac{\partial z^2}{\partial x^2}, \quad t = \frac{\partial z^2}{\partial y^2}.$$

Then $z = F_1$ satisfies the partial differential equations

$$x(1-x)r + y(1-x)s + [c-(a+b+1)x]p - byq - abz = 0, \tag{14a}$$

$$y(1-y)t + x(1-y)s + [c-(a+b'+1)y]q - b'xp - ab'z = 0. \tag{14b}$$

Similarly, $z = F_2$ satisfies the partial differential equations

$$x(1-x)r - xys + [c-(a+b+1)x]p - byq - abz = 0, \tag{15a}$$

$$y(1-y)t - xys + [c'-(a+b'+1)y]q - b'xp - ab'z = 0. \tag{15b}$$

Similarly, $z = F_3$ satisfies the partial differential equations

$$x(1-x)r + ys + [c-(a+b+1)x]p - abz = 0, \tag{16a}$$

$$y(1-y)t + xs + [c-(a'+b'+1)y]q - a'b'z = 0. \tag{16b}$$

Finally, $z = F_4$ satisfies the partial differential equations

$$x(1-x)r - y^2t - 2xys + cp - (a+b+1)(xp+yq) - abz = 0, \tag{17a}$$

$$y(1-y)t - x^2r - 2xys + c'q - (a+b+1)(xp+yq) - abz = 0. \tag{17b}$$

2.3. **Integral representations.** Integral representations for Appell series are very useful. Substitution of variables in theses integrals lead to equivalent integrals. This provides an effective and easy method to derive transformation formulae for Appell series, see Subsection 2.4.

Consider the integral

$$I = \iint u^{b-1}v^{b'-1}(1-u-v)^{c-b-b'-1}(1-ux-vy)^{-a}\,\mathrm{d}u\,\mathrm{d}v,$$

taken over the triangular region $u \geq 0,\ v \geq 0,\ u+v \leq 1$. (We implicitly assume suitable conditions of the parameters a, b, b', c such that the integral is well-defined and converges.)

Now, provided $|vy/(1-ux)| < 1$, we have, by binomial expansion,

$$\begin{aligned}(1-ux-vy)^{-a} &= (1-ux)^{-a}\sum_{m\geq 0}\frac{(a)_m}{m!}\left(\frac{vy}{1-ux}\right)^m\\ &= \sum_{m\geq 0}\frac{(a)_m}{m!}v^m y^m (1-ux)^{-a-m}\\ &= \sum_{m\geq 0}\frac{(a)_m}{m!}v^m y^m \sum_{n\geq 0}\frac{(a+m)_n}{n!}u^n x^n.\end{aligned}$$

Thus,

$$\begin{aligned}I &= \sum_{m\geq 0}\sum_{n\geq 0}\frac{(a)_{m+n}}{m!n!}x^n y^m \iint u^{b-1+n}v^{b'-1+m}(1-u-v)^{c-b-b'-1}\,\mathrm{d}u\,\mathrm{d}v\\ &= \sum_{m\geq 0}\sum_{n\geq 0}\frac{(a)_{m+n}}{m!n!}x^n y^m\,\Gamma\left[\begin{matrix}b+n, b'+m, c-b-b'\\ c+m+n\end{matrix}\right],\end{aligned}$$

which yields

$$I = \Gamma\left[\begin{matrix}b, b', c-b-b'\\ c\end{matrix}\right] F_1\big(a; b, b'; c; x, y\big). \tag{18}$$

While I is a double integral, a *single integral* for F_1 even exists, see (22).

Similarly,

$$\begin{aligned}&\int_0^1\int_0^1 u^{b-1}v^{b'-1}(1-u)^{c-b'-1}(1-v)^{c'-b'-1}(1-ux-vy)^{-a}\,\mathrm{d}u\,\mathrm{d}v\\ &\qquad = \Gamma\left[\begin{matrix}b, b', c-b, c'-b'\\ c,\ c'\end{matrix}\right] F_2\big(a; b, b'; c, c'; x, y\big),\end{aligned} \tag{19}$$

and

$$\iint u^{b-1}v^{b'-1}(1-u-v)^{c-b-b'-1}(1-ux)^{-a}(1-vy)^{-a'}\,\mathrm{d}u\,\mathrm{d}v$$
$$= \Gamma\begin{bmatrix} b, b', c-b-b' \\ c' \end{bmatrix} F_3\big(a, a'; b, b'; c'; x, y\big), \tag{20}$$

the last integral taken over the triangular region $u \geq 0$, $v \geq 0$, $u + v \leq 1$.

The double integral for F_4 is more complicated:

$$\int_0^1\int_0^1 u^{a-1}v^{b-1}(1-u)^{c-a-1}(1-v)^{c'-b-1}(1-ux)^{-b}(1-vy)^{-a}$$
$$\times\left(1-\frac{uvxy}{(1-ux)(1-vy)}\right)^{c+c'-a-b-1}\mathrm{d}u\,\mathrm{d}v$$
$$= \Gamma\begin{bmatrix} a, b, c-a, c'-b \\ c,\ c' \end{bmatrix} F_4\big(a; b; c, c'; x(1-y), y(1-x)\big). \tag{21}$$

In 1881, É. Picard [27] discovered a single integral for F_1. Let

$$I' = \int_0^1 u^{a-1}(1-u)^{c-a-1}(1-ux)^{-b}(1-uy)^{-b'}\,\mathrm{d}u,$$

where $\Re c > \Re a > 0$. Then

$$I' = \sum_{m\geq 0}\sum_{n\geq 0}\int_0^1 u^{a-1}(1-u)^{c-a-1}\frac{(b)_m}{m!}u^m x^m\frac{(b')_n}{n!}u^n y^n\,\mathrm{d}u$$
$$= \sum_{m\geq 0}\sum_{n\geq 0}\frac{(b)_m(b')_n}{m!n!}x^m y^n\int_0^1 u^{a+m+n-1}(1-u)^{c-a-1}\,\mathrm{d}u$$
$$= \sum_{m\geq 0}\sum_{n\geq 0}\frac{(b)_m(b')_n}{m!n!}x^m y^n\,\Gamma\begin{bmatrix} a+m+n, c-a \\ c+m+n \end{bmatrix},$$

hence

$$I' = \Gamma\begin{bmatrix} a, c-a \\ c \end{bmatrix} F_1\big(a; b, b'; c; x, y\big). \tag{22}$$

2.3.1. *Incomplete elliptic integrals.* As immediate consequences of (22), it follows that the incomplete elliptic integrals F and E and the complete elliptic integral Π can all be expressed in terms of special cases of the Appell F_1 function:

$$F(\phi, k) := \int_0^\phi \frac{\mathrm{d}\theta}{\sqrt{1-k^2\sin^2\theta}}$$
$$= \sin\phi\ F_1\left(\frac{1}{2}; \frac{1}{2}, \frac{1}{2}; \frac{3}{2}; \sin^2\phi, k^2\sin^2\phi\right), \qquad |\Re\phi| < \frac{\pi}{2}, \tag{23a}$$

$$E(\phi, k) := \int_0^\phi \sqrt{1-k^2\sin^2\theta}\,\mathrm{d}\theta$$

$$= \sin\phi\, F_1\left(\frac{1}{2};\frac{1}{2},-\frac{1}{2};\frac{3}{2};\sin^2\phi,k^2\sin^2\phi\right), \qquad |\Re\phi|<\frac{\pi}{2}, \tag{23b}$$

$$\Pi(n,k) := \int_0^{\pi/2}\frac{\mathrm{d}\theta}{(1-n\sin^2\theta)\sqrt{1-k^2\sin^2\theta}} = \frac{\pi}{2}\,F_1\left(\frac{1}{2};1,\frac{1}{2};1;n,k^2\right). \tag{23c}$$

2.4. **Transformations.** In the single integral for the F_1 series,

$$F_1\big(a;b,b';c;x,y\big) = \Gamma\left[\begin{matrix}c\\ a,c-a\end{matrix}\right]\int_0^1 u^{a-1}(1-u)^{c-a-1}(1-ux)^{-b}(1-uy)^{-b'}\,\mathrm{d}u,$$

one may use the substitution of variables $u=1-v$ to prove

$$F_1\big(a;b,b';c;x,y\big) = (1-x)^{-b}(1-y)^{-b'}F_1\left(c-a;b,b';c;\frac{x}{x-1},\frac{y}{y-1}\right). \tag{24}$$

For $b'=0$ this reduces to the well-known *Pfaff–Kummer transformation* for the ${}_2F_1$:

$${}_2F_1\left(\begin{matrix}a,\,b\\ c\end{matrix};x\right) = (1-x)^{-b}\,{}_2F_1\left(\begin{matrix}c-a,\,b\\ c\end{matrix};\frac{x}{x-1}\right).$$

Similarly, the substitution of variables $u=\frac{v}{1-x+vx}$ can be used to prove

$$F_1\big(a;b,b';c;x,y\big) = (1-x)^{-a}F_1\left(a;-b-b'+c,b';c;\frac{x}{x-1},\frac{y-x}{1-x}\right). \tag{25}$$

For $b'=0$ this reduces again to the Pfaff–Kummer transformation for the ${}_2F_1$ series.

On the other hand, if $c=b+b'$, then

$$F_1\big(a;b,b';b+b';x,y\big) = (1-x)^{-a}\,{}_2F_1\left(\begin{matrix}a,\,b'\\ b+b'\end{matrix};\frac{y-x}{1-x}\right) \tag{26a}$$

$$= (1-y)^{-a}\,{}_2F_1\left(\begin{matrix}a,\,b\\ b+b'\end{matrix};\frac{x-y}{1-y}\right). \tag{26b}$$

Similarly,

$$F_1\big(a;b,b';c;x,y\big) = (1-y)^{-a}F_1\left(a;b,c-b-b';c;\frac{x-y}{1-y},\frac{y}{y-1}\right), \tag{27}$$

$$F_1\big(a;b,b';c;x,y\big) = (1-x)^{c-a-b}(1-y)^{-b'}F_1\left(c-a;c-b-b',b';c;x,\frac{x-y}{1-y}\right), \tag{28}$$

$$F_1\big(a;b,b';c;x,y\big) = (1-x)^{-b}(1-y)^{c-a-b'}F_1\left(c-a;b,c-b-b';c;\frac{y-x}{1-x},y\right). \tag{29}$$

Further,

$$F_2\big(a;b,b';c,c';x,y\big) = (1-x)^{-a}F_2\left(a;c-b,b';c,c';\frac{x}{x-1},\frac{y}{1-x}\right), \tag{30}$$

$$F_2\big(a;b,b';c,c';x,y\big) = (1-y)^{-a} F_2\bigg(a;b,c'-b';c,c';\frac{x}{1-y},\frac{y}{y-1}\bigg), \tag{31}$$

$$F_2\big(a;b,b';c,c';x,y\big) = (1-x-y)^{-a} F_2\bigg(a;c-a,c'-b';c,c';\tfrac{x}{x+y-1},\tfrac{y}{x+y-1}\bigg). \tag{32}$$

Also *quadratic transformations* are known for Appell functions, see B.C. Carlson [10].

2.5. **Reduction formulae.** The transformations of Subsection 2.4 readily imply the following reduction formulae (typically a double series being reduced to a single series):

- $y = x$ in F_1:

$$F_1\big(a;b,b';c;x,x\big) = (1-x)^{c-a-b-b'}\,{}_2F_1\left(\begin{matrix}c-a,\,c-b-b'\\ c\end{matrix};x\right). \tag{33a}$$

By Euler's transformation this is

$$F_1\big(a;b,b';c;x,x\big) = {}_2F_1\left(\begin{matrix}a,\,b+b'\\ c\end{matrix};x\right). \tag{33b}$$

- $c = b + b'$ in F_1:

$$F_1\big(a;b,b';b+b';x,y\big) = (1-y)^{-a}\,{}_2F_1\left(\begin{matrix}a,\,b\\ b+b'\end{matrix};\frac{x-y}{1-y}\right). \tag{34}$$

- $c = b$ in F_2:

$$F_2\big(a;b,b';b,c';x,y\big) = (1-x)^{-a}\,{}_2F_1\left(\begin{matrix}a,\,b'\\ c'\end{matrix};\frac{y}{1-x}\right). \tag{35}$$

- $y = 1$ in F_1:
  Since

$$F_1\big(a;b,b';c;x,y\big) = \sum_{m\ge 0}\frac{(a)_m\,(b)_m}{m!\,(c)_m}x^m\,{}_2F_1\left(\begin{matrix}a+m,\,b'\\ c+m\end{matrix};y\right)$$

and

$${}_2F_1\left(\begin{matrix}a,\,b\\ c\end{matrix};1\right) = \Gamma\left[\begin{matrix}c,\,c-a-b\\ c-a,\,c-b\end{matrix}\right], \qquad \Re(c-a-b)>0,$$

we have

$$F_1\big(a;b,b';c;x,1\big) = \Gamma\left[\begin{matrix}c,\,c-a-b'\\ c-a,\,c-b'\end{matrix}\right]\,{}_2F_1\left(\begin{matrix}a,\,b\\ c-b'\end{matrix};x\right), \tag{36}$$

for $\Re(c-a-b')>0$.

- An $F_1 \leftrightarrow F_3$ transformation:
  Since

$$F_1\big(a;b,b';c;x,y\big) = \sum_{m\ge 0}\frac{(a)_m\,(b)_m}{m!\,(c)_m}x^m\,{}_2F_1\left(\begin{matrix}a+m,\,b'\\ c+m\end{matrix};y\right)$$

and

$${}_2F_1\left(\begin{matrix}a,\,b\\ c\end{matrix};y\right) = (1-y)^{-b}\,{}_2F_1\left(\begin{matrix}c-a,\,b\\ c\end{matrix};\frac{y}{y-1}\right),$$

we have

$$F_1\big(a;b,b';c;x,y\big)=(1-y)^{-b'}\sum_{m\geq 0}\frac{(a)_m\,(b)_m}{m!\,(c)_m}x^m\,{}_2F_1\left(\begin{matrix}c-a,\,b'\\ c+m\end{matrix};\frac{y}{y-1}\right)$$
$$=(1-y)^{-b'}F_3\left(a,c-a;b,b';c;x,\frac{y}{y-1}\right). \tag{37}$$

Hence, any F_1 function can be expressed in terms of an F_3 function. The converse is only true when $c=a+a'$.

- $a'=c-a$ and $b'=c-b$ in F_3:

Since by Equation (34) the F_1 function reduces to an ordinary ${}_2F_1$ function when $c=b+b'$, we have

$$F_3\left(a,c-a;b,c-b;c;x,\frac{y}{y-1}\right)=(1-x)^{-a}(1-y)^{c-b}\,{}_2F_1\left(\begin{matrix}a,\,c-b\\ c\end{matrix};\frac{y-x}{1-x}\right). \tag{38}$$

- $c'=a$ in F_2:

$$F_2\big(a;b,b';c,a;x,y\big)=(1-y)^{-b'}F_1\left(b;a-b',b';c;x,\frac{x}{1-y}\right). \tag{39}$$

Conversely, any F_1 function can be expressed in terms of an F_2 function where $c'=a$.

If further $c=a$, then

$$F_2\big(a;b,b';a,a;x,y\big)=(1-x)^{-b}(1-y)^{-b'}\,{}_2F_1\left(\begin{matrix}b,\,b'\\ a\end{matrix};\frac{xy}{(1-x)(1-y)}\right). \tag{40}$$

2.6. **An expansion of an F_4 series.** In 1940 and 1941, J.L. Burchnall and T.W. Chaundy [6, 7] gave the following expansion of an F_4 series in terms of products of two hypergeometric ${}_2F_1$ series:

$$F_4\big(a;b;c,c';x(1-y),y(1-x)\big)$$
$$=\sum_{m\geq 0}\frac{(a)_m\,(b)_m\,(1+a+b-c-c')_m}{m!\,(c)_m\,(c')_m}x^m y^m$$
$$\times\,{}_2F_1\left(\begin{matrix}a+m,\,b+m\\ c+m\end{matrix};x\right)\,{}_2F_1\left(\begin{matrix}a+m,\,b+m\\ c'+m\end{matrix};y\right). \tag{41}$$

This expansion has applications to classical orthogonal polynomials. It can also be used to deduce the double integral representation for F_4. Various special cases are interesting enough to state separately:

- $c'=1+a+b-c$ in F_4:

We have the product formula

$$F_4\big(a;b;c,1+a+b-c;x(1-y),y(1-x)\big)=\,{}_2F_1\left(\begin{matrix}a,\,b\\ c\end{matrix};x\right)\,{}_2F_1\left(\begin{matrix}a,\,b\\ c'\end{matrix};y\right). \tag{42}$$

- $c'=b$ in F_4:

Here we have the reduction formula

$$F_4\big(a;b;c,b;x(1-y),y(1-x)\big) \\ = (1-x)^{-a}(1-y)^{-a}\,F_1\left(a;1+a-c,c-b;c;\frac{xy}{(1-x)(1-y)},\frac{x}{x-1}\right). \qquad (43)$$

- $c'=b$ and $c=a$ in F_4:
  Further specialization of (43) gives the quite attractive summation formula

$$F_4\big(a;b;a,b;x(1-y),y(1-x)\big) = (1-x)^{1-b}(1-y)^{1-a}(1-x-y)^{-1}. \qquad (44a)$$

Written out in explicit terms, this is

$$\sum_{m\geq 0}\sum_{n\geq 0}\frac{(a)_{m+n}\,(b)_{m+n}}{m!\,n!\,(a)_m\,(b)_n}x^m(1-y)^m y^n(1-x)^n = \frac{(1-x)^{1-b}(1-y)^{1-a}}{(1-x-y)}. \qquad (44b)$$

For $y=0$ this reduces to I. Newton's binomial expansion formula

$${}_1F_0\left(\begin{matrix}b\\ -\end{matrix};x\right) = (1-x)^{-b}.$$

3. Related series and extensions of Appell series

3.1. **Horn functions.** In 1931, Jacob Horn [17] studied convergent bivariate hypergeometric functions $\sum_{m,n} f_{m,n}x^m y^n$ with certain (degree and other) restrictions on the two ratios of consecutive terms

$$\frac{f_{m+1,n}}{f_{m,n}}, \qquad \frac{f_{m,n+1}}{f_{m,n}}.$$

He arrived at a complete set of 34 different functions among which are the Appell functions F_1, F_2, F_3, F_4.

They include series such as

$$G_1(a,b,b';x,y) := \sum_{m\geq 0}\sum_{n\geq 0}\frac{(a)_{m+n}(b)_{n-m}(b')_{m-n}}{m!n!}x^m y^n, \qquad (45)$$

$$H_3(a,b,c;x,y) := \sum_{m\geq 0}\sum_{n\geq 0}\frac{(a)_{2m+n}(b)_n}{(c)_{m+n}\,m!n!}x^m y^n, \qquad (46)$$

and

$$H_7(a,b,b',c;x,y) := \sum_{m\geq 0}\sum_{n\geq 0}\frac{(a)_{2m-n}(b)_n(b')_n}{(c)_m\,m!n!}x^m y^n. \qquad (47)$$

3.2. **Kampé de Fériet series.** In 1937, J. Kampé de Fériet [18] introduced the following bivariate extension of the generalized hypergeometric series:

$$F^{p:q}_{r:s}\left(\begin{matrix}a_1,\dots,a_p:b_1,b'_1;\dots;b_q,b'_q;\\ c_1,\dots,c_r:d_1,d'_1;\dots;d_s,d'_s;\end{matrix}x,y\right)$$
$$=\sum_{m\geq 0}\sum_{n\geq 0}\frac{(a_1)_{m+n}\dots(a_p)_{m+n}}{(c_1)_{m+n}\dots(c_r)_{m+n}}\frac{(b_1)_m(b'_1)_n\dots(b_q)_m(b'_q)_n}{(d_1)_m(d'_1)_n\dots(d_s)_m(d'_s)_n}\frac{x^m y^n}{m!n!}. \quad (48)$$

Numerous identities exist for special instances of such series. For illustration, we list three summation formulae.

- P.W. Karlsson [19], 1994:

$$F^{0:3}_{1:1}\left(\begin{matrix}-:a,d-a;b,d-b;c,-c;\\ d:e,d+e-a-b-c;\end{matrix}1,1\right)=\Gamma\begin{bmatrix}e,\ e+d-a-b-c\\ e-c,\ e+d-a-b\end{bmatrix}, \quad (49)$$

where $\Re(e)>0$ and $\Re(d+e-a-b-c)>0$.

- S.N. Pitre and J. Van der Jeugt [28], 1996:

$$F^{0:3}_{1:1}\left(\begin{matrix}-:a,d-a;b,d-b;c,d-c;\\ d:e,d+e-a-b-c;\end{matrix}1,1\right)=\Gamma\begin{bmatrix}e,e+d-a-b-c,e-d\\ e-a,\ e-b,\ e-c\end{bmatrix}, \quad (50)$$

where $\Re(e-d)>0$ and $\Re(d+e-a-b-c)>0$. Further

$$F^{0:3}_{1:1}\left(\begin{matrix}-:a,d-a;b,d-b;c,e-c-1;\\ d:e,d+e-a-b-c;\end{matrix}1,1\right)$$
$$=\Gamma\begin{bmatrix}1-a,\ 1-b,\ e,\ e-d,\ d+e-a-b-c\\ 1-d,\ e-a,\ e-b,\ e-c,\ 1+d-a-b\end{bmatrix}, \quad (51)$$

where $\Re(d+e-a-b-c)>0$, and $d-a$ or $d-b$ is a negative integer.

3.3. **Lauricella series.** In 1893, G. Lauricella [21] investigated properties of the following four series $F_A^{(n)}$, $F_B^{(n)}$, $F_C^{(n)}$, $F_D^{(n)}$, of n variables:

$$F_A^{(n)}\big(a;b_1,\dots,b_n;c_1,\dots,c_n;x_1,\dots,x_n\big)$$
$$=\sum_{m_1\geq 0}\cdots\sum_{m_n\geq 0}\frac{(a)_{m_1+\cdots+m_n}(b_1)_{m_1}\dots(b_n)_{m_n}}{(c_1)_{m_1}\dots(c_n)_{m_n}\,m_1!\dots m_n!}x_1^{m_1}\dots x_n^{m_n}, \quad (52)$$

where $|x_1|+\cdots+|x_n|<1$.

$$F_B^{(n)}\big(a_1,\dots,a_n;b_1,\dots,b_n;c;x_1,\dots,x_n\big)$$
$$=\sum_{m_1\geq 0}\cdots\sum_{m_n\geq 0}\frac{(a_1)_{m_1}\dots(a_n)_{m_n}(b_1)_{m_1}\dots(b_n)_{m_n}}{(c)_{m_1+\cdots+m_n}\,m_1!\dots m_n!}x_1^{m_1}\dots x_n^{m_n}, \quad (53)$$

where $|x_1|,\dots,|x_n|<1$.

$$F_C^{(n)}\big(a;b;c_1,\dots,c_n;x_1,\dots,x_n\big)$$

$$= \sum_{m_1\ge 0} \cdots \sum_{m_n\ge 0} \frac{(a)_{m_1+\cdots+m_n}\,(b)_{m_1+\cdots+m_n}}{(c_1)_{m_1}\ldots(c_n)_{m_n}\,m_1!\ldots m_n!} x_1^{m_1}\ldots x_n^{m_n}, \tag{54}$$

where $|x_1|^{\frac{1}{2}} + \cdots + |x_n|^{\frac{1}{2}} < 1$.

$$F_D^{(n)}\big(a; b_1, \ldots, b_n; c; x_1, \ldots, x_n\big) = \sum_{m_1\ge 0} \cdots \sum_{m_n\ge 0} \frac{(a)_{m_1+\cdots+m_n}\,(b_1)_{m_1}\ldots(b_n)_{m_n}}{(c)_{m_1+\cdots+m_n}\,m_1!\ldots m_n!} x_1^{m_1}\ldots x_n^{m_n}, \tag{55}$$

where $|x_1|, \ldots, |x_n| < 1$.

Certainly, we have

$$F_A^{(2)} = F_2, \qquad F_B^{(2)} = F_3, \qquad F_C^{(2)} = F_4, \qquad F_D^{(2)} = F_1.$$

Many properties for Lauricella functions, such as integral representations and partial differential equations, are given by Appell and Kampé de Fériet [2]. From the vast amount of material, we single out the following integral representation of the Lauricella $F_D^{(n)}$ series as a specific example.

3.3.1. *Integral representation of $F_D^{(n)}$.* The formula

$$F_D^{(n)}\big(a; b_1, \ldots, b_n; c; x_1, \ldots, x_n\big) = \Gamma\begin{bmatrix} c \\ a, c-a \end{bmatrix} \int_0^1 u^{a-1}(1-u)^{c-a-1}(1-ux_1)^{-b_1}\ldots(1-ux_n)^{-b_n}\,\mathrm{d}u, \tag{56}$$

where $\Re c > \Re a > 0$, is very useful for deriving relations for F_D series. It can be easily verified by Taylor expansion of the integrand, followed by termwise integration.

3.3.2. *Group theoretic interpretations.* A group theoretic interpretation of the Lauricella $F_A^{(n)}$ functions corresponding to the most degenerate principal series representations of $\mathrm{SL}(n, \mathbb{R})$ was given by N.Ja. Vilenkin [33] (see also [34, Sec. 16.3.4]). Similarly, W. Miller, Jr. [23] has shown that the Lauricella $F_D^{(n)}$ functions transform as basis vectors corresponding to irreducible representations of the Lie algebra $sl(n+3, \mathbb{C})$ (by which he generalized his previous observation in [24] for the $n = 2$ case, corresponding to the Appell functions F_1).

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