COMPACTNESS OF THE CANONICAL SOLUTION OPERATOR TO $\overline{\partial}$ RESTRICTED TO BERGMAN SPACES

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In this paper we characterize compactness of the canonical solution operator to $\overline{\partial}$ restricted to radial symmetric Bergman spaces in one complex variable. The necessary and sufficient condition is expressed in terms of the Bergman space norms of the monomials z^n.

1 Introduction

Let Ω denote a bounded domain in $\mathbb{C}^n$ and consider the canonical solution operator to $\overline{\partial}$

$$S : A^2_{(0,1)}(\Omega) \longrightarrow L^2(\Omega)$$

which has the properties $\overline{\partial} S(g) = g$ and $S(g) \perp A^2(\Omega)$, here $A^2_{(0,1)}(\Omega)$ denotes the space of all $(0,1)$-forms $g = \sum_{j=1}^n g_j(z)\, d\overline{z}_j$ with holomorphic coefficients g_j belonging to the Bergman space

$$A^2(\Omega) = \{f : \Omega \longrightarrow \mathbb{C} \ holomorphic \ : \ \int_{\mathbb{C}^n} |f(z)|^2 \, d\lambda(z) < \infty\},$$

here λ denotes the Lebesgue measure in $\mathbb{C}^n$. In [6] the following result is shown

Proposition 1 *The canonical solution operator*

$$S : A^2_{(0,1)}(\Omega) \longrightarrow L^2(\Omega)$$

has the form

$$S(g)(z) = \int_\Omega B(z,w) < g(w), z-w > \, d\lambda(w),$$

where B denotes the Bergman kernel of Ω and

$$< g(w), z-w >= \sum_{j=1}^n g_j(w)(\overline{z}_j - \overline{w}_j),$$

for $z = (z_1, \ldots, z_n)$ and $w = (w_1, \ldots, w_n)$.

In the sequel the Hilbert Schmidt property of the canonical solution operator is investigated: it is shown that the canonical solution operator for $A^2(\mathbb{D})$, where $\mathbb{D}$ is the unit disc in $\mathbb{C}$, has the Hilbert Scmidt property, but the canonical solution operator for $A^2(\mathbb{B}^n)$, where $\mathbb{B}^n$ is the unit ball in $\mathbb{C}^n$, $n > 1$, fails to be Hilbert Schmidt, for further details see [6] .

Here we consider Bergman spaces $A^2(D, d\mu)$, where D is a disc in $\mathbb{C}$ and $d\mu(z) = m(z)d\lambda(z)$, m being a radial symmetric weight function. We suppose that the monomials $\{z^n\}$, $n \in \mathbb{N}_0$ constitute an orthogonal basis in $A^2(D, d\mu)$.

Let

$$c_n^2 = \int_D |z|^{2n} \, d\mu(z).$$

We solve the $\overline{\partial}$-equation $\overline{\partial} u = g$, where $g \in A^2(D, d\mu)$ and obtain a necessary and sufficient condition for the canonical solution operator to $\overline{\partial}$ restricted to $A^2(D, d\mu)$ to be a compact operator. The canonical solution operator S has the property $S(g) \perp A^2(D, d\mu)$. This condition is expressed in terms of the sequence $(c_n)_n$ defined above.

For this purpose we use the fact that the canonical solution operator $S : A^2(D, d\mu) \longrightarrow L^2(D, d\mu)$ can be expressed in the form

$$S(g)(z) = \int_D B(z, w) g(w) (z - w)^- d\mu(w),$$

where B denotes the Bergman kernel of $A^2(D, d\mu)$ and $g \in A^2(D, d\mu)$ (see [7]).

The adjoint operator $S^* : L^2(D, d\mu) \longrightarrow A^2(D, d\mu)$ can be written in the form

$$S^*(f)(w) = \int_D B(w, z) f(z) (z - w) d\mu(z),$$

where $f \in L^2(D, d\mu)$. We will show that $S^*S : A^2(D, d\mu) \longrightarrow A^2(D, d\mu)$ is a diagonal operator and will then use the fact that S^*S is compact if and only if S is compact.

It is pointed out in [4] that in the proof that compactness of the solution operator for $\overline{\partial}$ on $(0,1)$-forms implies that the boundary of Ω does not contain any analytic variety of dimension grearter than or equal to 1, it is only used that there is a compact solution operator to $\overline{\partial}$ on the $(0,1)$-forms with holomorphic coefficients. In this case compactness of the solution operator restricted to $(0,1)$-forms with holomorphic coefficients implies already compactness of the solution operator on general $(0,1)$-forms.

A similar situation appears in [10] where the Toeplitz C^* -algebra $\mathcal{T}(\Omega)$ is considered and the relation between the structure of $\mathcal{T}(\Omega)$ and the $\overline{\partial}$-Neumann problem is discussed (see [10] , Corollary 4.6).

The question of compactness of the $\overline{\partial}$-Neumann operator is of interest for various reasons ,see [5] for an excellent survey.

In [6] and [7] we investigated the Hilbert Schmidt property of the canonical solution operator to $\overline{\partial}$ the weaker property, of compactness appears to be more difficult in the case of several variables where S^*S fails to be a diagonal operator.

The canonical solution operator to $\overline{\partial}$ restricted to $(0,1)$-forms with holomorphic coefficients can also be interpreted as the Hankel operator

$$H_{\overline{z}}(g) = (I - P)(\overline{z}g),$$

where $P : L^2(\Omega) \longrightarrow A^2(\Omega)$ denotes the Bergman projection. See [1], [2], [3], [8], [9], [11] and [13] for details.

2 Compact solution operators.

Proposition 2 *Let Ω be a bounded domain in $\mathbb{C}^n$. The adjoint operator*

$$S^* : L^2(\Omega) \longrightarrow A^2_{(0,1)}(\Omega)$$

of the canonical solution operator S to $\overline{\partial}$ restricted to the Bergman space can be written in the form

$$S^*g(w) = \sum_{j=1}^n \int_\Omega (z_j - w_j) B(w,z) g(z)\, d\lambda(z)$$

where $B(w,z)$ denotes the Bergman kernel of Ω and $g \in L^2(\Omega)$.

Proof. Let $f(z) = \sum_{j=1}^n f_j(z)\, d\overline{z}_j$ be a (0,1)-form with coefficents $f_j \in A^2(\Omega),\ j = 1, \dots n$ and let $g \in L^2(\Omega)$. Then

$$(S(f), g) = \int_\Omega \int_\Omega B(z,w) \sum_{j=1}^n f_j(w)(\overline{z}_j - \overline{w}_j)\overline{g(z)}\, d\lambda(z)\, d\lambda(w)$$

$$= \int_\Omega \sum_{j=1}^n f_j(w) \left(\int_\Omega B(w,z)(z_j - w_j) g(z)\, d\lambda(z) \right)^- d\lambda(w),$$

which implies the result. $\square$

By analogous methods we obtain

Proposition 3 *The adjoint operator $S^* : L^2(D, d\mu) \longrightarrow A^2(D, d\mu)$ can be written in the form*

$$S^*(f)(w) = \int_D B(w,z) f(z)(z-w) d\mu(z),$$

where $f \in L^2(D, d\mu)$. Here $B(w,z)$ denotes the Bergman kernel of $A^2(D, d\mu)$.

From [7] we know that for the orthonormal basis $\{u_n(z) = z^n/c_n\}$ we have

$$S(u_n)(z) = \overline{z} u_n(z) - \frac{c_n z^{n-1}}{c_{n-1}^2}$$

and that the Bergman kernel can be written in the form

$$B(z,w) = \sum_{k=0}^{\infty} \frac{z^k \overline{w}^k}{c_k^2},$$

and the expansion of the Bergman kernel converges uniformly on compact subsets. Now we are able to show that

Proposition 4 *For the orthonormal basis $\{u_n(z) = z^n/c_n\}$ we have*

$$S^*S(u_n)(w) = \left(\frac{c_{n+1}^2}{c_n^2} - \frac{c_n^2}{c_{n-1}^2} \right) u_n(w) \quad , n = 1, 2, \ldots$$

and

$$S^*S(u_0)(w) = \frac{c_1^2}{c_0^2}\, u_0(w).$$

Proof. From Proposition 3 it follows that

$$\begin{aligned} S^*S(u_n)(w) &= \int_D B(w,z)(z-w) \left(\frac{\overline{z} z^n}{c_n} - \frac{c_n z^{n-1}}{c_{n-1}^2} \right) d\mu(z) \\ &= \int_D \sum_{k=0}^{\infty} \frac{w^k \overline{z}^k}{c_k^2} (z-w) \left(\frac{\overline{z} z^n}{c_n} - \frac{c_n z^{n-1}}{c_{n-1}^2} \right) d\mu(z) \end{aligned}$$

This integral is computed in two steps: first the multiplication by z

$$\int_D \sum_{k=0}^{\infty} \frac{w^k \overline{z}^k}{c_k^2} \left(\frac{\overline{z} z^{n+1}}{c_n} - \frac{c_n z^n}{c_{n-1}^2} \right) d\mu(z)$$

$$
\begin{aligned}
&= \int_D \frac{z^{n+1}}{c_n} \sum_{k=0}^{\infty} \frac{w^k \overline{z}^{k+1}}{c_k^2} \, d\mu(z) - \frac{c_n}{c_{n-1}^2} \int_D z^n \sum_{k=0}^{\infty} \frac{w^k \overline{z}^k}{c_k^2} \, d\mu(z) \\
&= \frac{w^n}{c_n^3} \int_D |z|^{2n+2} \, d\mu(z) - \frac{w^n}{c_{n-1}^2 c_n^2} \int_D |z|^{2n} \, d\mu(z) \\
&= \left(\frac{c_{n+1}^2}{c_n^3} - \frac{c_n}{c_{n-1}^2} \right) \; w^n
\end{aligned}
$$

And now the multiplication by w

$$
w \int_D \sum_{k=0}^{\infty} \frac{w^k \overline{z}^k}{c_k^2} \left(\frac{\overline{z} z^n}{c_n} - \frac{c_n z^{n-1}}{c_{n-1}^2} \right) \, d\mu(z)
$$

$$
\begin{aligned}
&= w \int_D \frac{z^n}{c_n} \sum_{k=0}^{\infty} \frac{w^k \overline{z}^{k+1}}{c_k^2} \, d\mu(z) - w \int_D \frac{c_n z^{n-1}}{c_{n-1}^2} \sum_{k=0}^{\infty} \frac{w^k \overline{z}^k}{c_k^2} \, d\mu(z) \\
&= w \left(\frac{c_n w^{n-1}}{c_{n-1}^2} - \frac{c_n w^{n-1}}{c_{n-1}^2} \right) \\
&= 0,
\end{aligned}
$$

which implies that

$$
S^* S(u_n)(w) = \left(\frac{c_{n+1}^2}{c_n^2} - \frac{c_n^2}{c_{n-1}^2} \right) u_n(w) \quad , n = 1, 2, \ldots ,
$$

the case $n = 0$ follows from an analogous computation. □

With the help of the last result it is not difficult to characterize compactness of the canonical solution operator on $A^2(D, d\mu)$

Proposition 5 *The canonical solution operator*

$$
S : A^2(D, d\mu) \longrightarrow L^2(D, d\mu)
$$

is compact if and only if

$$
\lim_{n \to \infty} \left(\frac{c_{n+1}^2}{c_n^2} - \frac{c_n^2}{c_{n-1}^2} \right) = 0.
$$

Proof. Proposition 4 says that $S^* S$ is a diagonal operator with respect to the orthonormal basis $\{u_n(z) = z^n / c_n\}$ of $A^2(D, d\mu)$. Therefore it is easily seen that S^*S is compact if and only if

$$\lim_{n\to\infty}\left(\frac{c_{n+1}^2}{c_n^2}-\frac{c_n^2}{c_{n-1}^2}\right)=0.$$

Now the conclusion follows, since S^*S is compact if and only if S is compact (see for instance [12]). $\square$

Remarks.

a) If D is the unit disc in $\mathbb{C}$ and $m(z)=(1-|z|^2)^\alpha$, where $\alpha\geq 0$, then the condition appearing in Proposition 5 is always satisfied. The solution operator to $\overline{\partial}$ is even Hilbert Schmidt (see [7]).

The condition is violated in certain weighted spaces of entire functions, but there

an additional difficulty appears (see Remark c).

b) It turns out that in the case of several variables the operator S^*S does not behave like a diagonal operator, the images $S^*S(u_n)(w)$ are in this case much more complicated.

c) In our reasoning it is important to suppose that the domains involved are bounded, since we consider multiplication by $\overline{z}$. The case of weighted Bergman spaces of entire functions will be treated in a forthcoming paper.

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