Proseminar Real Analysis

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Please see the lecture notes for further details.

1. Let $X := \mathbb{N}$ and define the set function
$$\mu(A) := \lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} \chi_A(n) \in [0,1]$$
on the collection $\mathcal{S}$ of all sets for which the above limit exists. Show that this collection is closed under disjoint unions and complements but not under intersections. Show that there is an extension to the σ-algebra generated by $\mathcal{S}$ (i.e. to $\mathfrak{P}(\mathbb{N})$) which is additive but no extension which is σ-additive. (Hints: To show that $\mathcal{S}$ is not closed under intersections take a set of even numbers $A_1 \notin \mathcal{S}$ and let A_2 be the missing even numbers. Then let $A = A_1 \cup A_2 = 2\mathbb{N}$ and $B = A_1 \cup \tilde{A}_2$, where $\tilde{A}_2 = A_2 + 1$. To obtain an extension to $\mathfrak{P}(\mathbb{N})$ consider χ_A, $A \in \mathcal{S}$, as vectors in $\ell^\infty(\mathbb{N})$ and μ as a linear functional such that you can apply the Hahn–Banach theorem.)

2. Show that $\mathcal{A} := \{A \subseteq X | A \text{ or } X \setminus A \text{ is finite}\}$ is an algebra (with X some fixed set). Show that $\Sigma := \{A \subseteq X | A \text{ or } X \setminus A \text{ is countable}\}$ is a σ-algebra. (Hint: To verify closedness under unions consider the cases where all sets are finite and where one set has finite complement.)

3. Take some set X and $\Sigma := \{A \subseteq X \mid A \text{ or } X \setminus A \text{ is countable}\}$. Show that
$$\nu(A) := \begin{cases} 0, & \text{if } A \text{ is countable,} \\ 1, & \text{else.} \end{cases}$$
is a measure

4. Show that every null set satisfies the Carathéodory condition (1.13). Hence the measure constructed in Theorem 1.9 is complete.

5. (**preimage σ-algebra**) Let $\mathcal{S} \subseteq \mathfrak{P}(Y)$. Show that $f^{-1}(\mathcal{S}) := \{f^{-1}(A) \mid A \in \mathcal{S}\}$ is a σ-algebra if $\mathcal{S}$ is. Conclude that $f^{-1}(\Sigma_Y)$ is the smallest σ-algebra on X for which f is measurable.

6. Let (X, Σ_X), (Y, Σ_Y) be two measurable spaces and $X = \dot{\bigcup}_{n\in\mathbb{N}} X_n$ a partition into measurable sets $X_n \in \Sigma_X$. Show that $f : X \to Y$ is measurable if and only if $f_n := f|_{X_n}$ is measurable with respect to the trace algebra $\Sigma_n := \Sigma_X|_{X_n}$ for all $n \in \mathbb{N}$.

7. Let $I \subseteq \mathbb{R}$ be some interval. Show that a monotone function $f : I \to \mathbb{R}$ is Borel.

8. Show that if $f \in C(\mathbb{R}^n, \mathbb{R}^n)$ is injective, then f^{-1} is Borel. (Hint: Show that $S := \{A \subseteq \mathbb{R}^n \mid f(A) \in \mathfrak{B}^n\}$ is a σ-algebra containing $\mathfrak{B}^n$.)

9. Show that for any measurable function f there exists a sequence of simple functions s_n such that $|s_n| \le |f|$ and $s_n \to f$ pointwise. If f is bounded, then the convergence will be uniform. (Hint: Split f into a linear combination of four nonnegative functions and use (2.6).)

10. Suppose $g_n \le f_n \le h_n$ are sequences of real-valued measurable functions converging to $g \le f \le h$, respectively. Show that if g, h are integrable with $\lim_{n\to\infty} \int g_n d\mu = \int g\, d\mu$ and $\lim_{n\to\infty} \int h_n d\mu = \int h\, d\mu$, then f is integrable and $\lim_{n\to\infty} \int f_n d\mu = \int f\, d\mu$.

    Moreover, suppose f_n, g_n are sequences of measurable functions with $|f_n| \le g_n$ converging to f, g, respectively. Show that if g is integrable with $\lim_{n\to\infty} \int g_n d\mu = \int g\, d\mu$, then f is integrable and $\lim_{n\to\infty} \int f_n d\mu = \int f\, d\mu$.

11. Consider
$$m(x) = \begin{cases} 0, & x < 0, \\ \frac{x}{2}, & 0 \le x < 1, \\ 1 & 1 \le x, \end{cases}$$
and let μ be the associated measure. Compute $\int_{\mathbb{R}} x\, d\mu(x)$.

12. Let $\mu(A) < \infty$ and f be a real-valued integrable function satisfying $f(x) \le M$. Show that
$$\int_A f\, d\mu \le M\mu(A)$$
with equality if and only if $f(x) = M$ for a.e. $x \in A$.

13. Let $X \subseteq \mathbb{R}$ be open, Y be some measure space, and $f : X \times Y \to \mathbb{C}$. Suppose $y \mapsto f(x, y)$ is measurable for every x and $x \mapsto f(x, y)$ is continuous for every y. Show that
$$F(x) := \int_Y f(x, y)\, d\mu(y)$$
is continuous if there is an integrable function $g(y)$ such that $|f(x, y)| \le g(y)$.

14. Let $X \subseteq \mathbb{R}$ be open, Y be some measure space, and $f : X \times Y \to \mathbb{C}$. Suppose $y \mapsto f(x, y)$ is integrable for all x and $x \mapsto f(x, y)$ is differentiable for a.e. y. Show that
$$F(x) := \int_Y f(x, y)\, d\mu(y)$$
is differentiable if there is an integrable function $g(y)$ such that $|\frac{\partial}{\partial x} f(x, y)| \le g(y)$. Moreover, $y \mapsto \frac{\partial}{\partial x} f(x, y)$ is measurable and
$$F'(x) = \int_Y \frac{\partial}{\partial x} f(x, y)\, d\mu(y)$$
in this case.

15. Let $P : \mathbb{R}^n \to \mathbb{C}$ be a nonzero polynomial. Show that $N := \{x \in \mathbb{R}^n \mid P(x) = 0\}$ is a Borel set of Lebesgue zero. (Hint: Induction using Fubini.)

16. Let $U \subseteq \mathbb{C}$ be a domain, Y be some measure space, and $f : U \times Y \to \mathbb{C}$. Suppose f is measurable and $z \mapsto f(z, y)$ is holomorphic for every y. Show that
$$F(z) := \int_Y f(z, y)\, d\mu(y)$$
is holomorphic if for every compact subset $V \subset U$ there is an integrable function $g(y)$ such that $|f(z, y)| \le g(y)$, $z \in V$. (Hint: Use Fubini and Morera's theorem from complex analysis.)

17. Let λ be Lebesgue measure on $\mathbb{R}$, and let f be a strictly increasing function with $\lim_{x\to\pm\infty} f(x) = \pm\infty$. Show that
$$f_\star\lambda = d(f^{-1}),$$
where f^{-1} is the inverse of f extended to all of $\mathbb{R}$ by setting $f^{-1}(y) = x$ for $y \in [f(x-), f(x+)]$ (note that f^{-1} is continuous).
Moreover, if $f \in C^1(\mathbb{R})$ with $f' > 0$, then
$$f_\star\lambda = \frac{1}{f'(f^{-1})} d\lambda.$$

18. Show that $\Gamma(\frac{1}{2}) = \sqrt{\pi}$. Moreover, show
$$\Gamma(n + \frac{1}{2}) = \frac{(2n)!}{4^n n!}\sqrt{\pi}$$
(Hint: Use the change of coordinates $x = t^2$ and then use $\int_{\mathbb{R}^n} \mathrm{e}^{-|x|^2} d^n x = \pi^{n/2}$.)

19. Show
$$\int_{-\infty}^{\infty} \frac{\mathrm{e}^{zx}}{a + \mathrm{e}^x} dx = \int_0^{\infty} \frac{y^{z-1}}{a + y} dy = \frac{\pi a^{z-1}}{\sin(z\pi)}, \quad 0 < \mathrm{Re}(z) < 1, a \in \mathbb{C}\backslash(-\infty, 0].$$
(Hint: First reduce it to the case $a = 1$. Then, use a contour consisting of the straight lines connecting the points $-R$, R, $R + 2\pi\mathrm{i}$, $-R + 2\pi\mathrm{i}$. Evaluate the contour integral using the residue theorem and let $R \to \infty$. Show that the contributions from the vertical lines vanish in the limit and relate the integrals along the horizontal lines.)

20. Show that the **Beta function** satisfies
$$B(u, v) := \int_0^1 t^{u-1}(1 - t)^{v-1} dt = \frac{\Gamma(u)\Gamma(v)}{\Gamma(u + v)}, \quad \mathrm{Re}(u) > 0,\ \mathrm{Re}(v) > 0.$$
A few other common forms are
$$\begin{aligned} B(u, v) &= 2\int_0^{\pi/2} \sin(\theta)^{2u-1} \cos(\theta)^{2v-1} d\theta \\ &= \int_0^{\infty} \frac{s^{u-1}}{(1 + s)^{u+v}} ds = 2^{-u-v+1} \int_{-1}^{1} (1 + s)^{u-1}(1 - s)^{v-1} ds \\ &= n\int_0^1 s^{nu-1}(1 - s^n)^{v-1} ds, \qquad n > 0. \end{aligned}$$

Use this to establish **Euler's reflection formula**

$$\Gamma(z)\Gamma(1-z) = \frac{\pi}{\sin(\pi z)}$$

and **Legendre's duplication formula**

$$\Gamma(2z) = \frac{2^{2z-1}}{\sqrt{\pi}}\Gamma(z)\Gamma(z+\frac{1}{2}).$$

Conclude that the Gamma function has no zeros on $\mathbb{C}$.

(Hint: Start with $\Gamma(u)\Gamma(v)$ and make a change of variables $x = ts$, $y = t(1-s)$. For the reflection formula evaluate $B(z, 1-z)$ using Problem 19. For the duplication formula relate $B(z,z)$ and $B(\frac{1}{2}, z)$.)

21. Show **Stirling's formula**

$$\Gamma(x) = \int_0^\infty t^{x-1}\mathrm{e}^{-t}dt = \frac{x^x}{\mathrm{e}^x}\left(\sqrt{\frac{2\pi}{x}} + O(x^{-3/2})\right), \qquad x \to \infty,$$

for $x > 0$. (Hint: The maximum of $t^x\mathrm{e}^{-t}$ occurs at $t = x$ and this suggests a change of variables $t = sx$ which gives

$$\Gamma(x) = \frac{x^x}{\mathrm{e}^x}\int_0^\infty \left(s\mathrm{e}^{1-s}\right)^x \frac{ds}{s}.$$

Now observe that the main contribution is from the maximum at $s = 1$. Split the integral into a neighborhood of $s = 1$ and the rest. The rest does not contribute asymptotically. To determine the contribution from the neighborhood make a change of variables $s = r(s)$ such that $s\mathrm{e}^{1-s} = \mathrm{e}^{-r^2/2}$.)

22. Verify the Gauss–Green theorem (by computing both integrals) in the case $u(x) = x$ and $U = B_1(0) \subset \mathbb{R}^n$.

23. Consider $X := (0,1)$ with Lebesgue measure. Is it true that

$$\bigcap_{1\le p<\infty} L^p(0,1) = L^\infty(0,1)?$$

24. Construct a function $f \in L^p(0,1)$ which has a singularity at every rational number in $[0,1]$ (such that the essential supremum is infinite on every open subinterval). (Hint: Start with the function $f_0(x) = |x|^{-\alpha}$ which has a single singularity at 0, then $f_j(x) = f_0(x - x_j)$ has a singularity at x_j.)

25. Show that $\mu(\{x \mid |f(x)| > \|f\|_\infty\}) = 0$.

26. Show the **generalized Hölder's inequality**:

$$\|f\,g\|_r \le \|f\|_p\|g\|_q, \qquad \frac{1}{p} + \frac{1}{q} = \frac{1}{r}.$$

Here we can allow $p, q, r \in (0, \infty]$ but of course $\|.\|_p$ will only be a norm for $p \ge 1$.

27. Show the iterated Hölder's inequality:

$$\|f_1 \cdots f_m\|_r \le \prod_{j=1}^m \|f_j\|_{p_j}, \qquad \frac{1}{p_1} + \cdots + \frac{1}{p_m} = \frac{1}{r}.$$

Again with $p_j, r \in (0,\infty]$ as in the previous problem.

28. Show that if $f \in L^{p_0} \cap L^{p_1}$ for some $p_0 < p_1$ then $f \in L^p$ for every $p \in [p_0, p_1]$ and we have the **Lyapunov inequality**

$$\|f\|_p \le \|f\|_{p_0}^{1-\theta} \|f\|_{p_1}^{\theta},$$

where $\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}$, $\theta \in (0,1)$. (Hint: Generalized Hölder inequality.)

29. **Hardy inequality**. Show that the integral operator

$$(Kf)(x) := \frac{1}{x} \int_0^x f(y) dy$$

is bounded in $L^p(0,\infty)$ for $1 < p \le \infty$:

$$\|Kf\|_p \le \frac{p}{p-1} \|f\|_p.$$

(Hint: Note $(Kf)(x) = \int_0^1 f(sx) ds$ and apply Minkowski's integral inequality.)

30. Let $f \in L^p(X)$, $X \subseteq \mathbb{R}^n$ open, $1 \le p < \infty$ and show that $T_a f \to f$ in L^p as $a \to 0$. (Hint: Start with $f \in C_c(X)$ and use Theorem 3.18.)

31. Let $f \in L^p(\mathbb{R}^n)$ and $g \in L^q(\mathbb{R}^n)$ with $\frac{1}{p} + \frac{1}{q} = 1$. Show that $f * g \in C_b(\mathbb{R}^n)$ with

$$\|f * g\|_\infty \le \|f\|_p \|g\|_q.$$

Show that for $1 < p < \infty$ or $p = 1$ and $g \in C_0(\mathbb{R}^n)$ we even have $f * g \in C_0(\mathbb{R}^n)$. (Hint: Problem 30.)

32. Let ϕ_ε be a symmetric approximate identity on $\mathbb{R}$, that is, $\phi_\varepsilon(x) = \phi_\varepsilon(-x)$. Show that for every bounded measurable function f we have

$$\lim_{\varepsilon \downarrow 0} (\phi_\varepsilon * f)(x) = \frac{f(x+) + f(x-)}{2},$$

at every point x where both right/left sided limits $f(x\pm) := \lim_{\varepsilon\downarrow 0} f(x \pm \varepsilon)$ exist.

Show that the same conclusion holds for integrable functions f if ϕ_ε has compact support, $\operatorname{supp}(\phi_\varepsilon) \subseteq [-s, s]$, and for every $r > 0$ we have $\lim_{\varepsilon\downarrow 0} \sup_{r \le |x| \le s} |\phi_\varepsilon(x)| = 0$.

33. **Schur test**. Let $K(x,y)$ be given and suppose there are positive measurable functions $a(x)$ and $b(y)$ such that

$$\|K(x,.)b(.)\|_{L^1(Y,d\nu)} \le C_1 a(x), \qquad \|a(.)K(.,y)\|_{L^1(X,d\mu)} \le C_2 b(y).$$

Then the operator $K : L^2(Y, d\nu) \to L^2(X, d\mu)$, defined by

$$(Kf)(x) := \int_Y K(x,y) f(y) d\nu(y),$$

for μ-almost every x is bounded with $\|K\| \le \sqrt{C_1 C_2}$. Show that the adjoint operator is given by

$$(K^* f)(x) = \int_X K(y,x)^* f(y) d\mu(y), \qquad f \in L^2(X, d\mu).$$

(Hint: Estimate $|(Kf)(x)|^2 \le |\int_Y |K(x,y)|^{1/2} b(y) |K(x,y)|^{1/2} \frac{|f(y)|}{b(y)} d\nu(y)|^2$ using Cauchy–Schwarz and integrate the result with respect to x.)

34. Let K be a self-adjoint integral operator with continuous kernel satisfying the estimate $|K(x,y)| \le k(x)k(y)$ with $k \in L^2(X, d\mu)$. Show that the conclusion from Mercer's theorem still hold if K has only a finite number of negative eigenvalues.

35. Suppose ν is an inner regular measures. Show that $\nu \ll \mu$ if and only if $\mu(K) = 0$ implies $\nu(K) = 0$ for every compact set.

36. Suppose $\nu(A) \le C\mu(A)$ for all $A \in \Sigma$. Then $d\nu = f\, d\mu$ with $0 \le f \le C$ a.e.

37. Show that $M := \{x \mid 0 < (\overline{D}\mu)(x)\}$ is a minimal support of μ.

38. **Markov (also Chebyshev) inequality**. For $f \in L^1(\mathbb{R}^n)$ and $\alpha > 0$ show
$$|\{x \in A \mid |f(x)| \ge \alpha\}| \le \frac{1}{\alpha} \int_A |f(x)| d^n x.$$
Somewhat more general, assume $g(x) \ge 0$ is nondecreasing and $g(\alpha) > 0$. Then
$$\mu(\{x \in A \mid f(x) \ge \alpha\}) \le \frac{1}{g(\alpha)} \int_A g \circ f \, d\mu.$$

39. Let $f \in L^p_{loc}(\mathbb{R}^n)$, $1 \le p < \infty$, then for a.e. $x \in \mathbb{R}^n$ we have
$$\lim_{r \downarrow 0} \frac{1}{|B_r(x)|} \int_{B_r(x)} |f(y) - f(x)|^p d^n y = 0.$$
The same conclusion hols if the balls are replaced by sets $A_j(x)$ which shrink nicely to x.

40. Prove the polar decomposition from Corollary 4.19 (Hint: Use the complex Radon–Nikodym theorem to get existence of h. Then show that $1 - |h|$ vanishes a.e.)

41. Consider $f_j(x) := x^j \cos(\pi/x)$ for $j \in \mathbb{N}$. Show that $f_j \in C[0,1]$ if we set $f_j(0) = 0$. Show that f_j is of bounded variation for $j \ge 2$ but not for $j = 1$.

42. Let $\alpha \in (0,1)$ and $\beta > 1$ with $\alpha\beta \le 1$. Set $M := \sum_{k=1}^{\infty} k^{-\beta}$, $x_0 := 0$, and $x_n := M^{-1}\sum_{k=1}^{n} k^{-\beta}$. Then we can define a function on $[0,1]$ as follows: Set $g(0) := 0$, $g(x_n) := n^{-\beta}$, and
$$g(x) := c_n\,|x - t_n x_n - (1-t_n)x_{n+1}|\,, \qquad x \in [x_n, x_{n+1}],$$
where c_n and $t_n \in [0,1]$ are chosen such that g is (Lipschitz) continuous. Show that $f = g^{\alpha}$ is Hölder continuous of exponent α but not of bounded variation. (Hint: What is the variation on each subinterval?)

43. Show that if $f, g \in BV[a,b]$ then so is $f\,g$ and
$$V_a^b(f\,g) \le V_a^b(f)\sup|g| + V_a^b(g)\sup|f|.$$
Hence, together with the norm $\|f\|_{BV} := \|f\|_\infty + V_a^b(f)$ the space $BV[a,b]$ is a Banach algebra.

44. Show that if $f \in AC[a,b]$ and $f' \in L^p(a,b)$, $p > 1$, then f is Hölder continuous:
$$|f(x) - f(y)| \le \|f'\|_p |x-y|^{1-\frac{1}{p}}.$$
Show that the function $f(x) = -\log(x)^{-1}$ is absolutely continuous but not Hölder continuous on $[0, \frac{1}{2}]$.

45. Consider $f(x) := x^2\sin(\frac{\pi}{x^2})$ on $[0,1]$ (here $f(0) = 0$). Show that f is differentiable everywhere and compute its derivative. Show that its derivative is *not* integrable. In particular, this function is not absolutely continuous and the fundamental theorem of calculus does not hold for this function.

46. Show that if $f, g \in AC[a,b]$ then so is $f\,g$ and the **product rule** $(f\,g)' = f'g + f\,g'$ holds. Conclude that $AC[a,b]$ is a closed subalgebra of the Banach algebra $BV[a,b]$. (Hint: Integration by parts. For the second part use Problem 43 and the formula for the total variation of an a.c. function.)

47. Let $f \in L^p(\mathbb{R})$ and $g \in L^q(\mathbb{R})$ with $\frac{1}{p} + \frac{1}{q} = 1$. Show that
$$\lim_{n\to\infty} \int_{\mathbb{R}} f(x+n)g(x)dx = 0$$
for $1 < p < \infty$. In other words, the sequence $f(.+n)$ converges weakly to 0.

48. Let $f \in L^p(0,1)$ and $g \in L^q(0,1)$ with $\frac{1}{p} + \frac{1}{q} = 1$. Extend f to all of $\mathbb{R}$ by periodicity, $f(x+1) = f(x)$, and denote by $\overline{f} := \int_0^1 f(x)dx$ its average. Show that
$$\lim_{n\to\infty} \int_0^1 f(n\cdot x)g(x)dx = \overline{f}\,\overline{g}$$
for $1 \le p < \infty$. In other words, the sequence $f(n\,\cdot)$ converges weakly to its average. (Hint: Establish the claim in case g is the characteristic function of an open interval.)

49. Let X be a locally compact metric space. Show that for every compact set K there is a sequence $f_n \in C_c(X)$ with $0 \le f_n \le 1$ and $f_n \downarrow \chi_K$. (Hint: Urysohn's lemma.)

50. Show that $\mathcal{S}(\mathbb{R}^n) \subset L^p(\mathbb{R}^n)$. (Hint: If $f \in \mathcal{S}(\mathbb{R}^n)$, then $|f(x)| \le C_m \prod_{j=1}^n (1+x_j^2)^{-m}$ for every m.)

51. Compute the Fourier transform of the following functions $f : \mathbb{R} \to \mathbb{C}$:

    (i) $f(x) = \chi_{(-1,1)}(x)$. (ii) $f(x) = \frac{\mathrm{e}^{-k|x|}}{k}$, $\mathrm{Re}(k) > 0$.

52. Show that
$$\psi_n(x) := H_n(x)\mathrm{e}^{-\frac{x^2}{2}} \in \mathcal{S}(\mathbb{R}),$$
where $H_n(x)$ is the **Hermite polynomial** of degree n given by
$$H_n(x) := \mathrm{e}^{\frac{x^2}{2}} \left(x - \frac{d}{dx}\right)^n \mathrm{e}^{-\frac{x^2}{2}},$$
are eigenfunctions of the Fourier transform: $\hat{\psi}_n(p) = (-\mathrm{i})^n \psi_n(p)$.

53. Prove the **Poisson summation formula**
$$\sum_{n\in\mathbb{Z}} f(n)\mathrm{e}^{-\mathrm{i}nx} = \sqrt{2\pi} \sum_{m\in\mathbb{Z}} \hat{f}(x + 2\pi m),$$
where $f \in L^1(\mathbb{R})$ satisfies $|f(x)| + |\hat{f}(x)| \le \frac{C}{(1+|x|)^\alpha}$ for some $\alpha > 1$. (Hint: Compute the Fourier coefficients of the right-hand side. To this end observe that the integrals over $[-\pi, \pi]$ give a tiling of $\mathbb{R}$ when m runs through all values in $\mathbb{Z}$.)

54. Show
$$\int_0^\infty \frac{\sin(x)^2}{x^2} dx = \frac{\pi}{2}.$$
Furthermore, use integration by parts to compute the **Dirichlet integral**
$$\lim_{R\to\infty} \int_0^R \frac{\sin(x)}{x} dx = \frac{\pi}{2}.$$
(Hint: Problem 51 (i).)

55. (Wiener) Suppose $f \in L^2(\mathbb{R}^n)$. Then the set $\{f(x + a) \mid a \in \mathbb{R}^n\}$ is total in $L^2(\mathbb{R}^n)$ if and only if $\hat{f}(p) \neq 0$ a.e. (Hint: Use Lemma 9.2 and the fact that a subspace is total if and only if its orthogonal complement is zero.)

56. Let C be a positive constant and let $f_j : \mathbb{R}^n \to \mathbb{R}$ be two measurable functions such that $f_j(x) \ge 1$ with $\lim_{|x|\to\infty} f_j(x) = \infty$ for $j = 0, 1$. Then the set $\{f \in L^2(\mathbb{R}^n) \mid \|f_0 f\|_2 + \|f_1 \hat{f}\|_2 \le C\}$ is compact in $L^2(\mathbb{R}^n)$. (Hint: Example 3.3.)

57. Suppose $f(x)\mathrm{e}^{k|x|} \in L^1(\mathbb{R})$ for some $k > 0$. Then $\hat{f}(p)$ has an analytic extension to the strip $|\mathrm{Im}(p)| < k$.

58. Consider the Hilbert space $L^2(\mathbb{R}, w(x)dx)$ corresponding to a nonnegative weight w. The corresponding orthogonal polynomials are uniquely defined up to normalization by applying Gram–Schmidt to the monomials.

    Suppose $|w(x)| \le C\mathrm{e}^{-k|x|}$ for some $C, k > 0$. Show that the orthogonal polynomials are dense in $L^2(\mathbb{R}, w(x)dx)$. (Hint: It suffices to show that $\int f(x)x^j w(x)dx = 0$ for all $j \in \mathbb{N}_0$ implies $f = 0$. Consider the Fourier transform of $f(x)w(x)$ and note that it has an analytic extension by Problem 57. Hence this Fourier transform will be zero if, e.g., all derivatives at $p = 0$ are zero.)

59. Show that $J_\nu(z)$ is a solution of the Bessel differential equation
$$z^2 u'' + zu' + (z^2 - \nu^2)u = 0.$$
    Prove the following properties of the Bessel functions.
    (a) $(z^{\pm\nu} J_\nu(z))' = \pm z^{\pm\nu} J_{\nu\mp 1}(z)$.
    (b) $J_{\nu-1}(z) + J_{\nu+1}(z) = \frac{2\nu}{z} J_\nu(z)$.
    (c) $J_{\nu-1}(z) - J_{\nu+1}(z) = 2J_\nu'(z)$.

60. Show that for $n = 3$ the fundamental solution of the Poisson equation satisfies
$$(\Phi * \chi_{B_1(0)})(x) = \begin{cases} \frac{1}{3|x|}, & |x| \ge 1, \\ \frac{3-|x|^2}{6}, & |x| \le 1. \end{cases}$$
    (Hint: Observe that the result depends only on $|x|$. Then choose $x = (0, 0, R)$ and evaluate the integral using spherical coordinates.)

61. Show the following integral representations for Bessel functions:
$$\begin{aligned} J_\nu(z) &= \frac{1}{\sqrt{\pi}\Gamma(\nu + \frac{1}{2})} \left(\frac{z}{2}\right)^\nu \int_0^\pi \mathrm{e}^{\pm\mathrm{i}z\cos(\theta)} \sin(\theta)^{2\nu} d\theta \\ &= \frac{1}{\sqrt{\pi}\Gamma(\nu + \frac{1}{2})} \left(\frac{z}{2}\right)^\nu \int_0^\pi \cos(z\cos(\theta)) \sin(\theta)^{2\nu} d\theta \\ &= \frac{2}{\sqrt{\pi}\Gamma(\nu + \frac{1}{2})} \left(\frac{z}{2}\right)^\nu \int_0^{\pi/2} \cos(z\cos(\theta)) \sin(\theta)^{2\nu} d\theta \\ &= \frac{2}{\sqrt{\pi}\Gamma(\nu + \frac{1}{2})} \left(\frac{z}{2}\right)^\nu \int_0^{\pi/2} \cos(z\sin(\theta)) \cos(\theta)^{2\nu} d\theta, \end{aligned}$$
    for $\mathrm{Re}(\nu) > -\frac{1}{2}$ and $z \in \mathbb{C}$. (Hint: Start with the third integral, replace the outer cosine by its power series and use Problem 20.)

62. Verify the expression for $J_{-1/2}(z)$ and conclude that
$$\tilde{\mathcal{H}}_{-1/2}(f)(r) = \frac{1}{\pi} \int_0^\infty \cos(rs) f(s) ds = \hat{f}(r)$$
    if we extend f to all of $\mathbb{R}$ such that $f(r) = f(-r)$. (Hint: (Problem 18).)

63. Show that m is an L^2 multiplier if and only if $m \in L^\infty(\mathbb{R}^n)$.

64. Suppose $f \in L^2(\mathbb{R}^n)$ show that $\varepsilon^{-1}(f(x+e_j\varepsilon) - f(x)) \to g_j(x)$ in L^2 if and only if $p_j \hat{f}(p) \in L^2$, where e_j is the unit vector into the j'th coordinate direction. Moreover, show $g_j = \partial_j f$ if $f \in H^1(\mathbb{R}^n)$.