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Mateusz Piorkowski, Msc

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In
memory
of

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Table of Contents

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Abstract

The goal of this thesis is to present some new results and techniques related to the nonlinear steepest descent method for Riemann–Hilbert problems. The first application deals with the long-time asymptotics of solutions to the Korteweg–de Vries equation with steplike initial data. We discuss an ill-posedness of the associated matrix problem and construct the necessary singular global parametrix solution in order to perform the asymptotic analysis. This construction is then further explicated by formulating the global parametrix problem as a scalar Riemann–Hilbert problem on the complex torus. This approach reduces the explicit construction of the global parametrix solution to finding quasiperiodic meromorphic functions with prescribed poles. We also present an alternative solution to the aforementioned ill-posedness, which does not rely on the existence of a matrix solution and uses a vector solution instead. This approach is based on Fredholm theory of singular integral operators of Cauchy-type.

The second part of the thesis deals with local parametrix problems, which comprise another class of auxiliary Riemann–Hilbert problems frequently appearing in the context of the nonlinear steepest descent method. We discuss a novel approach, which granted certain a priori estimates, makes these local problems superfluous. As an application we study the Plancherel–Rotach asymptotics for a class of orthogonal polynomials on a finite interval. The corresponding weight functions lack certain analytic continuation properties, which makes the usual formulation of the local parametrix problems impossible. However, using our new technique we are able to perform the nonlinear steepest descent analysis rigorously, without relying on the formulation or solvability of any local parametrix problems.

Zusammenfassung

Das Ziel der vorliegenden Arbeit ist die Präsentation einiger neuen Resultate und Verfahren im Zusammenhang mit der Deift–Zhou Methode für Riemann–Hilbert Probleme. Die erste Anwendung beschäftigt sich mit der langzeit Asymptotik von Lösungen der Korteweg–de Vries Gleichung mit stufenartiger Anfangsbedingung. Wir diskutieren das dazugehörige Matrix Problem und konstruieren die notwendigerweise singuläre globale Parametrix Lösung. Diese Konstruktion wird näher erläutert, indem das dazugehörige globale Parametrix Problem als ein Riemann–Hilbert Problem auf dem Torus formuliert wird. Diese Vorgehensweise erlaubt es uns die explizite Konstruktion der globalen Parametrix Lösung auf das Finden von meromorphen quasiperiodischen Funktion mit vorgegebenen Polen zurückzuführen. Wir präsentieren auch einen alternativen Zugang bei der auf eine Matrix Lösung zu gänze verzichtet wird. Diese Methodik basiert auf der Fredholm Theorie für singuläre Integraloperatoren.

Der zweite Teil dieser Dissertation behandelt lokale Parametrix Probleme, welche eine weitere Klasse von Riemann–Hilbert Problemen darstellen, die in Anwendungen häufiger vorkommen. Wir diskutieren eine neue Angehensweise bei der, mit Hilfe von a priori Schätzungen, diese Probleme überflüssig gemacht werden. Als Anwendung behandeln wir die Plancherel–Rotach Asymptotik für eine bestimmte Klasse von orthogonalen Polynomen auf einem endlichem Intervall. Die dazugehörige Gewichtsfunktion kann dabei bestimmte Regularitätsbedingungen nicht erfüllen, was dazu führt, dass die gängige Formuliereng eines lokalen Parametrix Problems unmöglich wird. Mit unserer neuen Methodik können wir jedoch diese Schwierigkeit umgehen und die Deift–Zhou Analyse rigoros angewenden, ohne das lokale Parametrix Problem zu formulieren.

1. Introduction

1.1. **Riemann–Hilbert problems.** The term *Riemann–Hilbert problem* (R-H) was originally reserved for Hilbert's twenty-first problem on his famous list of unsolved mathematical problems, published at the turn of the centuries [47].[1] In a nutshell, it asks whether any given monodromy group of the fundamental group of a punctured Riemann sphere can be realized as the monodromy group of a Fuchsian linear system. This problem has a fascinating history. A variation of Hilbert's problem was solved affirmatively by Plemelj [70] in 1908 and believed to be equivalent to the original version. However in 1989, Bolibruch [9], [10] found a counterexample implying a negative solution to Hilbert's twenty-first problem. We will not go into more details on the early history of R-H problems or the monodromy problem for Fuchsian systems, referring instead to [6]. Let us just mention that the Sokhotski–Plemelj formula for singular Cauchy integrals, first shown by Sokhotski [76] and rediscovered by Plemelj during his study of Hilbert's twenty-first problem, remains of central importance in R-H theory to this day.

The crucial development for us will be the nonlinear steepest descent method for R-H problems, formulated by Deift and Zhou in their seminal paper [22] on the modified Korteweg de–Vries equation (mKdV). Since then, it has developed into an established method with applications in various fields, among others integrable wave equations ([15], [21], [22], [23]), random matrix theory ([8], [14], [16], [18], [58]) and Painlevé transcendents ([24], [35], [38, Ch. 8]). The general approach can be summarized in three steps as follows:

I. The original R-H problem is deformed so that the oscillatory terms in the jump matrices become exponentially decaying.

II. The exponentially decaying terms are ignored leading to an explicitly solvable matrix-valued global parametrix R-H problem.

III. In the vicinity of stationary points of the phase function, local parametrix R-H problems are solved explicitly.

The global and local parametrix solutions glued together lead to a uniform approximation of the original R-H problem, which implies an asymptotic formula for the quantity of interest.

The present thesis contains applications of the nonlinear steepest descent method, for which one of the above steps becomes problematic. The first three papers [30], [66], [68] deal with a R-H problem related to the Korteweg–de Vries (KdV) for which the associated global parametrix problem, which will be referred to as the *model problem*, is not always solvable. In particular, step II can only be partially performed. Two strategies around this issue are presented in the two papers [30] and [66], with [68] offering an alternative point of view by reformulating the model problem as a scalar-valued R-H problem on the torus.

The final paper [67] deals with instances where step III concerning local parametrix problems cannot be performed. Instead, it presents an alternative approach based on a priori estimates, such that the local parametrix problems can be skipped altogether. We also apply our method to obtain improved estimates on the Plancherel–Rotach asymptotics of orthogonal polynomials with a new class of weight functions.

In the following we give a short account of the two aforementioned topics, for which we use the nonlinear steepest descent method in this thesis: the KdV equation and Plancherel–Rotach asymptotics.

1.2. **The Korteweg–de Vries equation.** This nonlinear wave equation given by

$$q_t(x,t) = 6q(x,t)q_x(x,t) - q_{xxx}(x,t), \quad (x,t) \in \mathbb{R} \times \mathbb{R}_+,$$

was first described by Boussinesq [11] and independently by Lord Rayleigh [71] in the 1870s to model shallow water waves and solitons. The latter were first experimentally observed by Russell in 1834, who gave a detailed account in [72]. The equation is named after Korteweg

[1] Riemann's contributions are less documented and more speculative, see [6, p. 7]

and de–Vries [54] who rediscovered and analysed it in 1895. For a more in-depth account of the early history of the KdV equation see [51].

Interest in the KdV equation grew after Zabusky and Kruskal [56] observed using numerical simulations[2] that nonlinearly interacting KdV solitons pass through each other without changing their shape. Gardner et al. [39] explained this phenomenon by introducing a new method to solve the KdV equation with fast decaying initial data via the scattering transform. The idea is to regard the initial data as a potential of a one-dimensional Schrödinger operator. Then, the direct scattering transform linearizes the KdV equation, meaning that the KdV flow translates to a linear evolution of the scattering data. As this method is analogous to solving linear differential equation via the Fourier transform, the scattering transform is also known as the *nonlinear* Fourier transform. For an account of the scattering transform for one-dimensional Schrödinger operators see [20].

Zakharov and Faddeev [33] noted that the KdV equation can be understood as an infinite dimensional completely integrable Hamiltonian systems with the scattering transform playing the role of a canonical transformation mapping physical coordinates to action-angle variables. In particular, the KdV equation has an infinite number of conserved quantities (see [40] for a simple proof using the Miura transform). Moreover, Lax [60] rewrote the KdV equation using a Lax pair leading to the discovery of an infinite hierarchy of KdV equations. Subsequently, Zakharov and Shabat showed in [75] that the Lax pair approach can also be used for the nonlinear Schrödinger (NLS) equation by introducing the Zakharov–Shabat system. More PDEs have been found to be integrable since, many of which are special cases of the AKNS system [1]. Further reading on this topic can be found in [4], [7], [32], [42], [59] and references therein.

While integrability is of great interest by itself, the natural question emerged whether the scattering transform can be used to compute the long-time asymptotics of solution to the KdV equation and other integrable equation. The first results in this direction were obtain by Ablowitz and Newell [2], Manakov [61] and Shabat [73] in 1973. Based on the work of Zakharov and Manakov [62], Ablowitz and Segur [3] determined the leading asymptotics for solutions of the KdV, mKdV and sine-Gordon equations with rapidly decaying initial data and no solitons. These asymptotics were extended for the KdV equation to all orders by Buslaev and Sukhanov [12].

R-H theory entered the picture with the work of Its [48], who reformulated the aforementioned approach by Manakov [61] as a R-H problem. He obtained the long-time asymptotics for solutions of the NLS equation with decaying initial data by solving explicitly a model isomonodromy problem involving the parabolic cylindrical functions. However, this method leads in the case of other integrable equations to monodromy problems related to the classical Painlevé transcendents (see [50]), for which no explicit solutions of the associated isomonodromy problem were known. The next breakthrough came from the work of Deift and Zhou who introduced in [22] the nonlinear steepest descent method, which analogously to the classical steepest descent method allows for the systematic extraction of leading asymtptotics from oscillatory R-H problems. Shortly afterwards, these methods were applied to the KdV equation in the collisionless shock region with decaying initial data [21]. The associated R-H formulation of the KdV equation was obtained by Shabat [74]. For a more detailed survey on the early developments of R-H techniques applied to integrable PDEs see [15].

A detailed discussion of the KdV asymptotics in the soliton and similarity regions using the nonlinear steepest descent method is given by Grunert and Teschl in [44]. The authors observed that the associated vector R-H problem can have nonunique solutions. To guarantee uniqueness an additional symmetry condition on the R-H solution was introduced. Note that the issue of uniqueness is more subtle for the KdV equation than for most other integrable PDEs, because the associated R-H problem is vector valued instead of matrix valued. This feature plays an important role in the first three papers of this thesis [30], [66], [68].

More recently, there has been interest in the long-time asymptotics for solutions of the KdV equation with steplike initial data. Because of Galilean invariance, it suffices to consider

[2] A related phenomenon was already observed by Fermi et al. [34] and led to the Fermi-Pasta-Ulam-Tsingou problem which can be explained by the reversible interactions of KdV solitons observed by Zabusky and Kruskal.

the two cases

$$\lim_{x\to-\infty} q_0(x) = \pm c^2, \quad c > 0,$$

where q_0 denotes the initial data which decays at $+\infty$. Here, the $+$ sign corresponds to a rarefaction wave analyzed in [5] and the $-$ sign to a shock wave analyzed in [27] and [28]. For the necessary scattering theory for steplike potentials see [29], [31]. The interesting case for us will be that of the shock wave in the transition region [27, Sect. 4], also called the *elliptic wave region*. That solutions indeed have asymptotically the form of a modulated elliptic wave was first shown by Gurevich and Pitaevskii [45], [46] for the case of a shock discontinuity initial data and further analyzed with the help of the inverse scattering transform by Khruslov [52], [53]. The analogous problem for the mKdV equation was studied through the nonlinear steepest descent method in [55]. Recently, modulated elliptic waves appeared in the asymptotic analysis of a KdV soliton gas in [43], where the authors encountered the same ill-posedness of the global parametrix problem as the one we describe in [30].

In [30], [66] and [68] we take a closer look at the elliptic wave region in the KdV shock case. While the expected asymptotics were announced in [27], the rigorous justification was given in [30]. The papers [66] and [68] are follow-up work motivated by the aforementioned ill-posedness of the global parametrix problem.

1.3. **Plancherel–Rotach asymptotics.** Plancherel–Rotach asymptotics refer to asymptotics of orthogonal polynomials as the degree n goes to infinity, scaled by the largest zero (see [13], [19], [77]).[3] They have been first studied for Hermite polynomials by Plancherel and Rotach in [69] and since then extended to a wide class of orthogonal polynomials, see for example [8], [17], [18], [57]. The case of the orthogonality measure having support on the unit interval or unit circle has been studied in great generality by Bernstein and Szegő, and can be found in Szegő's book on orthogonal polynomials [77, Ch. 12].

The R-H formulation for orthogonal polynomials was introduced by Fokas et al. in [36], [37] in the early 90s. Some of its first applications to the study of large degree asymptotics of orthogonal polynomials were done by Bleher and Its [8] and Deift et al. [17], [18]. Studying such asymptotics was motivated by the Wigner–Dyson–Mehta universality conjecture in random matrix theory in genera (see [26], [64], [78]). It states that the local eigenvalue statistics are determined solely by the symmetry class of the underlying matrix ensemble. Here, one needs to distinguish between the three classes of orthogonal, unitary and symplectic ensembles which display increasingly strong eigenvalue repulsion, as pointed out by Dyson in [26].[4] For a standard reference on the Wigner–Dyson–Mehta conjecture and random matrix theory in general consult [65].

The connection to orthogonal polynomials was discovered by Gaudin and Mehta [41], [63]. They showed that eigenvalue correlations can be characterized through the Christoffel–Darboux formula for orthogonal polynomials. This implies that questions regarding eigenvalue universality can be reduced to Plancherel–Rotach asymptotics of orthogonal polynomials. A self-contained exposition of this method can be found in Deift's book on orthogonal polynomials and random matrix theory [14].

The key paper for our purposes is by Kuijlaar et al. [57]. There, the authors derive via the nonlinear steepest descent method a series expansions in the degree n for orthogonal polynomials and related quantities. The associated orthogonality measure is an analytic perturbation of the classical Jacobi weight function. The R-H analysis is based on [18] where the case of exponential weight functions was studied, while the global parametrix solution is constructed in terms of the Szegő function [77, Ch. 10]. Interestingly, while the Szegő functions determines the leading term in the series expansion of the polynomials, all the other terms are explicitly computable from the local parametrix solution given in terms of Bessel functions. In a follow up work [58], these results are used to show bulk and edge universality for the corresponding unitary ensemble.

The final paper presented in this thesis follows the nonlinear steepest descent analysis performed in [57]. The main difference is the class of admissible weight functions, which

[3] In our case of the orthogonality measure having finite support, there is no need for scaling.

[4] This classification has been further extended to so-called 'β ensembles', see [25] and references therein.

has implications for the associated local parametrix problems and the aforementioned series expansion. While bulk universality remains unchanged, the choice of weight functions has a strong effect on edge universality.

1.4. **Outline of the thesis.** As already mentioned, the main focus of the thesis is on R-H problems that have to be explicitly solved in the framework of the nonlinear steepest descent method. Indeed, the first three papers ([30], [66], [68]) concerning the KdV equation will deal with the global parametrix problem, also referred to as the *model problem*. The final paper is focused on local parametrix problems.

The four papers are summarized in more detail below and listed as chapters of the present thesis. Note that [30] has been co-authored by Iryna Egorova and [30], [68] by Gerald Teschl. The underlying mathematical theory is based on the work of countless researchers over the past few decades. Thus, it is unavoidable that some important contributions will be overlooked, for which the author sincerely apologizes.

1. **Asymptotics of the Korteweg–de Vries shock waves via the Riemann–Hilbert approach**,
(joint with I. Egorova and G. Teschl), *in preparation*.
The first paper deals with the long-time asymptotics of solutions to the KdV equation with steplike initial data in the elliptic wave region and justifies rigorously Theorem 5.1 in [27] (see also [28]). As explained in Section 3, the main obstacle in a straightforward application of the nonlinear steepest descent method is an ill-posedness of the matrix model problem. In fact, while the holomorphic vector solution containing the KdV asymptotics is constructed explicitly in Section 5, a second linearly independent holomorphic solution fails to exist for certain arbitrary large times t. The specific conditions under which such ill-posedness occurs can be found in Theorem 3.3.
The existence of a holomorphic matrix valued solution to the model problem is necessary for matching with the local parametrix solutions around the stationary points of the phase function and thus is an integral part of the nonlinear steepest descent method. Similarly as in [43], we construct in Section 6 a second linearly independent meromorphic solution to the vector R-H model problem with a simple pole at the origin. It turns out that due to certain symmetry conditions imposed on the R-H solution, the pole becomes removable in the final analysis involving a small-norm R-H problem. This allows us to extract asymptotics for solutions of the KdV equation with steplike initial data in the elliptic wave region, uniformly as $t \to \infty$. As announced in [28], the KdV solutions converge to a modulated elliptic wave related to a genus one compact Riemann (see [45], [46], [49]).

2. **Parametrix problem for the Korteweg–de Vries equation with steplike initial data**,
submitted.
The second paper deals also with the R-H analysis of solutions to the KdV equation with steplike initial data in the elliptic wave region. The ill-posedness of the model matrix R-H problem discussed in the first paper [30] raises the natural question whether instead of the matrix solution a vector solution would suffice. This is further supported by the fact, that the symmetric vector solution already contains the desired asymptotic information.
To address this question we use the singular integral formulation for R-H problems, summarized in Section 4. However, instead of showing invertibility of the associated singular integral operators via a Neumann series, we use the fact that these operators are Fredholm of index 0 (see [79]). In particular, unique solvability of such a singular integral equation implies invertibility of the underlying operator. This insight allows us in Section 5 to directly approximate the exact R-H solution by the vector model and parametrix solutions glued together. Hence, we avoid the formulation of a small-norm R-H problem, which requires the construction of

matrix-valued model and parametrix solutions. In the appendix we also state an abstract result, generalizing the approach presented in the main text.

3. A scalar Riemann–Hilbert problem on the torus: Applications to the KdV equation,
(joint with G. Teschl), *in preparation.*
This is a brief article motivated by the first paper [30] on the KdV equation with steplike initial data. In it we present an alternative approach to constructing meromorphic vector solutions to the model problem, by formulating it as a scalar R-H problem on the complex torus. As the scalar jump function is just a phase, meromorphic solutions to this R-H problem correspond bijectively to quasiperiodic meromorphic functions. As finding these is a classical problem, we can easily write down all possible meromorphic solutions in terms of ratios of Jacobi theta functions and characterize them by a simple condition on their divisors.

This approach also allows us to deduce the form and uniqueness of the vector model solution and simplifies the construction of singular matrix solutions.

4. Riemann–Hilbert theory without local parametrix problems: Applications to orthogonal polynomials,
submitted.
The final paper illustrates a new method for avoiding local parametrix problems in the framework of the nonlinear steepest descent method. It is motivated by the observation that the explicit construction of local parametrix solutions is often required for rigorous analysis, but does not contribute to the leading asymptotics (see Section 6 in the first paper [30] for an explicit example). Central to our approach are a priori estimates for the exact solution of the R-H problem that one tries to approximate. The abstract framework is laid out in Section 2 and the general result is summarized in Theorem 3.1.

We also demonstrate our method on a specific problem, namely the Plancherel–Rotach asymptotics of orthogonal polynomials on the interval $[-1, 1]$. This part is based on the paper [57] where the nonlinear steepest descent analysis is performed for the R-H problem associated to a wide class of orthogonal polynomials. There, the authors solve the local parametrix problems explicitly in terms of Bessel functions.

In our application the weight functions are not required to have an analytic continuation around the end points ± 1. In particular, the local parametrix problem cannot be formulated as in [57]. However, the a priori estimates needed for our approach can be obtained, and thus we can nonetheless rigorously perform the nonlinear steepest descent analysis, without solving any local parametrix problems. The resulting error estimates in the Plancherel-Rotach asymptotics can be found in Theorem 4.4. We also comment on possible relations to edge universality in random matrix theory.

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ASYMPTOTICS OF THE KORTEWEG–DE VRIES SHOCK WAVES VIA THE RIEMANN–HILBERT APPROACH

IRYNA EGOROVA, MATEUSZ PIORKOWSKI, AND GERALD TESCHL

Abstract. This paper discusses some general aspects and techniques associated with the long-time asymptotics of steplike solutions of the Korteweg–de Vries (KdV) equation via vector Riemann–Hilbert problems. We also elaborate on an ill-posedness of the matrix Riemann–Hilbert problem for the KdV case in the class of matrices with L^2 integrable singularities. Furthermore, we refine the asymptotics for the shock wave in the Whitham zone derived previously and rigorously justify it for a more general class of initial data. In particular, we clarify the influence of resonances and of the discrete spectrum on the main asymptotical term.

1. Introduction

The nonlinear steepest descent (NSD) analysis for oscillatory Riemann–Hilbert (R-H) problems is a versatile tool in asymptotic analysis. This procedure naturally starts from a reformulation of the original scattering problem as a R-H factorization problem. In most cases this will be a matrix R-H problem as these are typically more convenient to analyze. Indeed, the fact that a nonsingular solution can be used to cancel jumps on certain parts of the contour is a crucial trick which lies at the heart of the theory. However, for some problems, most prominently the Korteweg–de Vries equation

$$q_t(x,t) = 6q(x,t)q_x(x,t) - q_{xxx}(x,t), \quad (x,t) \in \mathbb{R} \times \mathbb{R}_+, \tag{1.1}$$

it turned out that a vector R-H is the right choice. This is related to the fact that even in the simplest case of a single soliton there is a nontrivial solution of the associated vanishing problem (see [18]). However, this is in contradiction to the classical uniqueness result for matrix R-H problems and shows that the matrix problem cannot have a solution in this situation. The remedy, as pointed out in [18], is to work with the vector R-H problem and impose an additional symmetry condition to retain uniqueness.

Next, recall that the asymptotic analysis of such a R-H problem usually consists of three steps: The first step deforms the problem in such a way that the leading asymptotic contribution is revealed. In the second step the parts of the jump which are expected not to contribute to the leading asymptotics are dropped, yielding a model problem which then needs to be solved explicitly. In most cases, it is possible to find a matrix solution to this model problem and hence the final step, namely showing that the solution of the model problem indeed asymptotically

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approximates the solution of the original R-H problem, can be performed using the well-established tools for matrix problems. However, for model problems leading to explicit solutions in terms of Jacobi theta functions, finding a nonsingular[1] matrix solution is not always possible (see [3, Thm. 5.6], [2, Sect. 3], [16, Sect. 3])

The main purpose of the present note is to study in depth such a R-H problem coming from the KdV equation, having only singular matrix model solutions for certain exceptional values of the parameters x and t. Indeed for these values, the initial and the model problems do not have invertible bounded matrix solutions with admissible L^2-integrable singularities in the points of discontinuity of the contour. We will refer to this feature as the ill-possedness of matrix R-H problems for the KdV equation.

The specific example that we will consider is the R-H problem associated with the long-time asymptotical behaviour of the shock waves for the KdV equation. The shock wave is the solution to the initial value problem (1.1), with initial data $q(x,0) = q(x)$ satisfying:

$$\begin{cases} q(x) \to 0, & \text{as} \quad x \to +\infty, \\ q(x) \to -c^2, & \text{as} \quad x \to -\infty, \quad c > 0. \end{cases} \tag{1.2}$$

We recall that the asymptotic behavior of the shock wave was first described on a physical level of rigor in the pioneering work of Gurevich and Pitayevskii [19], [20]. By applying the Whitham approach to the pure step initial data ($q(x) = 0$ for $x > 0$ and $q(x) = -c^2$ for $x \le 0$), the authors derived the leading asymptotics in terms of a modulated elliptic wave. For arbitrary steplike initial data (1.2) the analogous asymptotic term was calculated in [11] and [13] using the NSD method. In particular, it was shown that in the elliptic zone $-6c^2 t < x < 4c^2 t$ the shock wave is expected to be close to a modulated one gap solution of the KdV equation in the limit of $t \to \infty$. However, this has not been rigorously justified till now.

The main result of this paper is the completion of the asymptotic analysis for the shock wave in the Whitham zone, in the framework of the standard NSD method. Even though the inverse scattering transform for the KdV equation is given in terms of a vector R-H problem, the NSD approach involves building a matrix solution to the model R-H problem in order to match it with the local parametrix solutions. Since the nonsingular matrix model solution does not exist for certain arbitrary large pairs x and t, we will instead use a singular matrix model solution which, despite its singular behaviour, can be used to bound the error term in the asymptotics, as shown [16, Sect. 3]. Note that for decaying initial data or rarefaction waves, meaning $q(x) \to 0$ as $x \to +\infty$ and $q(x) \to c^2$ as $x \to -\infty$, the nonsingular matrix model solutions always exist (see [1], [18]).

As for the shock wave case, to characterize the pairs (x,t) for which the nonsingular matrix model solution fails to exist, we must recall the trace formula for a finite gap KdV solution. Denote by $\xi = \frac{x}{12t}$ the slowly varying parameter and consider values (x,t) satisfying

$$\xi \in \mathcal{I}_\varepsilon := [-\frac{c^2}{2} + \varepsilon, \frac{c^2}{3} - \varepsilon], \tag{1.3}$$

for an arbitrary small $\varepsilon > 0$. Then, as is shown in [19], [11], there exists a smooth monotonously increasing positive function $a = a(\xi)$ such that $a(-\frac{c^2}{2}) = 0$ and

[1] In what follows *nonsingular* refers to the R-H matrix solution being invertible with at most L^2-integrable singularities on either side of the jump contour

$a(\frac{c^2}{3}) = c$. This function characterizes the Whitham zone of the modulated elliptic wave $q^{\mathrm{mod}}(x,t,\xi)$, which is the periodic one gap solution of the KdV equation on the ray $\xi = \mathrm{const}$. This one gap solution is associated with the spectrum

$$\mathfrak{G}(\xi) := [-c^2, -a^2(\xi)] \cup \mathbb{R}_+, \tag{1.4}$$

and with the initial Dirichlet divisor $(\lambda(0,0,\xi), \pm)$ defined via the scattering data of the potential (1.2) by the formulas (5.16) and (4.6) later. Let $\lambda(x,t,\xi) \in [-a^2(\xi), 0\,]$ be the solution of the Dubrovin equations ([24, Ch. 12]) corresponding to the initial value $(\lambda(0,0,\xi), \pm)$. Then the trace formula reads

$$q^{\mathrm{mod}}(x,t,\xi) = -c^2 - a(\xi)^2 - 2\lambda(x,t,\xi).$$

We will show (see Remark 5.2) that the set of local minima of $q^{\mathrm{mod}}(x,t,\xi)$:

$$\mathcal{O}(\xi) = \{(x,t):\ \lambda(x,t,\xi) = 0\},$$

coincides with the set of points where the associated matrix model problem has no nonsingular solution. Evidently, these pairs (x,t) appear for each $\xi \in I_\varepsilon$ and for arbitrary large t.

In turn, the circumstances which lead to the ill-possedness of the initial R-H problem associated with the shock wave for certain (arbitrary large) points (x,t) are the following. Let $\phi(k,x,t)$ be the right Jost solution to the underlying spectral equation of the problem (1.1)–(1.2):

$$L(t)y = -\frac{d^2}{dx^2}y + q(x,t)y = k^2 y, \tag{1.5}$$

normalized as

$$\lim_{x\to+\infty} \mathrm{e}^{-\mathrm{i}kx}\phi(k,x,t) = 1. \tag{1.6}$$

In Section 3 we show that if $\phi(0,x,t) = 0$ for a pair (x,t), then the nonsingular matrix solution for the initial R-H problem does not exist. In connection with this observation an additional spectral problem appears: to find conditions which would guarantee that the right Jost solution associated with the shock wave is nonzero at the edge of the continuous spectrum for sufficiently large x and t with $(x,t) \in \mathcal{D}_\varepsilon$, where

$$\mathcal{D}_\varepsilon := \{(x,t) \in \mathbb{R}\times\mathbb{R}_+:\ \frac{x}{12t} \in \mathcal{I}_\varepsilon\}. \tag{1.7}$$

It should be noted that the same condition $\phi(0,x,t) = 0$ leads to the ill-posedness of the matrix R-H problem in the decaying case $q(x,t) \to 0$, $x \to \pm\infty$. In this case, assuming then the discrete spectrum is absent, the Jost solutions are positive below the spectrum (cf. [15, Corollary 2.4]) and hence also at the boundary of the spectrum $k = 0$ by continuity (note that the zeros of a nontrivial solution of a Sturm–Liouville equation must always be simple). Thus a nonsingular matrix solution always exists in this situation (this also follows from [27, Theorem 9.3]). However, in the presence of the discrete spectrum this is no longer true.

Our main result is the following

Theorem 1.1. *Let $q(x,t)$ be the unique solution of the initial value problem (1.1)–(1.2) with the initial data satisfying*

$$\int_0^{+\infty} \mathrm{e}^{\eta x}(|q(x)| + |q(-x) + c^2|)dx < \infty, \qquad x^6 q^{(i)}(x) \in L^1(\mathbb{R}), \quad i = 1, ..., 4, \tag{1.8}$$

for some positive $\eta > 0$. For any $\xi \in \mathcal{I}_\varepsilon$ with $\varepsilon > 0$ (see (1.3)), let $a = a(\xi) \in (0, c)$ be defined implicitly by

$$\int_0^{\mathrm{i}a} \left(k^2 + \xi + \frac{c^2 - a^2}{2}\right) \sqrt{\frac{k^2 + a^2}{k^2 + c^2}}\, dk = 0. \tag{1.9}$$

Let $p_0 = p_0(\xi)$ be the point on the two-sheeted Riemann surface associated with $\mathfrak{G}(\xi)$ (see (1.4)), uniquely defined via the Jacobi inversion problem

$$\int_{-a^2}^{p_0} \frac{d\lambda}{\sqrt{\lambda(\lambda + c^2)(\lambda + a^2)}} = \mathrm{i}\Delta(\xi), \tag{1.10}$$

with

$$\Delta(\xi) = \frac{\int_{\mathrm{i}a}^{\mathrm{i}c} \frac{2\log\left|T(s)\prod_{j=1}^N \frac{s - \mathrm{i}\kappa_j}{s + \mathrm{i}\kappa_j}\right| + \log\left|\frac{s + \mathrm{i}c\ell}{s}\right|}{|(s^2+c^2)(s^2+a^2)|^{1/2}} ds}{\int_0^{\mathrm{i}a} \left((s^2 + c^2)(s^2 + a^2)\right)^{-1/2} ds} - \frac{\pi\ell}{2}, \tag{1.11}$$

where:

- *$T(k)$ is the right transmission coefficient for the initial data (1.8);*
- *$-\kappa_1^2 < \ldots < -\kappa_N^2$ is the discrete spectrum of the problem;*
- *$\ell = -1$ if the initial data has a resonance at the point $\mathrm{i}c$, and $\ell = 1$ in the general (nonresonant) case.*

Let $q^{\mathrm{mod}}(x, t, \xi)$ be the periodic (one gap) solution to the KdV equation associated with the spectrum $\mathfrak{G}(\xi)$ and the initial Dirichlet divisor $p_0 = (\lambda(0, 0, \xi), \pm)$. Then for all $x \to \infty$, $t \to +\infty$ such that $(x, t) \in \mathcal{D}_\varepsilon$, the following asymptotics is valid uniformly with respect to $\xi \in \mathcal{I}_\varepsilon$:

$$q(x, t) = q^{\mathrm{mod}}(x, t, \xi) + O(t^{-1}). \tag{1.12}$$

Formula (1.12) is obtained in the framework of a standard NSD approach applied to a vector R-H problem. It includes some transformations (conjugations and deformations) which lead to an equivalent R-H problem with the jump matrix asymptotically close, as $t \to \infty$, to an exactly solvable model R-H problem except in small vicinities of two extreme points $\pm\mathrm{i}a(\xi)$. The approach also involves the construction of a proper matrix model solution and an associated matrix solution of the local parametrix problems. However, when performing this analysis in the KdV steplike case, it is essential to take into account some specific features of the vector R-H problems. Note that unlike the matrix R-H problem, the proof of uniqueness for a vector R-H problem is typically more sophisticated and depends on particular properties of the jump matrix and of the contour, as well as on the class of admissible singularities for the solution. That is why it seems important for us to perform NSD deformations and conjugations in a way that does not affect this uniqueness. To this end, in each transformation we impose additional symmetry assumptions on the contour, on the jump matrix and on the solution itself, including the model problem solution (see Remark 3.2).

The initial R-H problem solution $(m_1(k, x, t), m_2(k, x, t))$ is unique (see Theorem 2.1) and satisfies the aforementioned symmetry assumption. This symmetry requirement implies a symmetry of the "error vector", which in turn, allows us to apply a new formula

$$q(x, t) = \lim_{k \to \infty} 2k^2 \left(m_1(k, x, t) m_2(k, x, t) - 1\right) \tag{1.13}$$

for computing the leading term of the asymptotics, and this simplifies essentially the final asymptotical analysis.

Note that the traditional formula which connects the potential $q(x,t)$ with the solution of the initial R-H problem *(i)–(iii)* is the following one:

$$\frac{\partial}{\partial x}\lim_{k\to\infty} 2\mathrm{i}k(m_1(k,x,t)-1) = q(x,t). \tag{1.14}$$

Formula (1.13) not only avoids the necessity to justify the differentiation in (1.14) in an asymptotical expansion but also allows to extract the asymptotics from the model vector R-H solution in a shorter and more transparent way (see Section 4) compared to [11], [13] and [16].

2. Well-posedness of the initial (meromorphic) vector R-H problem

In this section we recall the statement of the initial R-H vector problem for the KdV shock wave (see [11]) and prove its well-posedness. Note in the present study we weaken the decay conditions on the initial data compared to [11], where it is assumed that

$$|q(x)| + |q(-x)+c^2| = O(\mathrm{e}^{-(c+\eta)x}), \quad x\to+\infty, \quad \eta>0.$$

We choose the still quite restrictive condition (1.8) to avoid complications with the analytical continuation of the scattering data in the framework of the NSD method. However, (1.8) also guarantees the existence of the unique classical solution $q(x,t)$ for the Cauchy problem (1.1)–(1.2) (cf. [14, 17]) satisfying

$$\int_0^{+\infty} |x|(|q(x,t)| + |q(-x,t)+c^2|)dx < \infty, \qquad t\in\mathbb{R}. \tag{2.1}$$

In turn, this means that the use of the inverse scattering transform for the formulation of the respective R-H problem is well grounded.

We start with recalling some well known facts of the scattering theory for the step-like Schrödinger operator (1.5) with emphasis on analytical properties of the scattering data due to (1.8) and with a detailed description of the influence of resonance on them.

The spectrum of the operator (1.5) with potential (2.1) consists of an absolutely continuous part $[-c^2,\infty)$ plus a finite number of eigenvalues $-\kappa_j^2\in(-\infty,-c^2)$, $1\le j\le N$ enumerated as in Theorem 1.1.

Let $\phi(k,x,t)$ be the right Jost solution of (1.5) satisfying (1.6) and let $\phi_1(k,x,t)$ be the Jost solution asymptotically close to the free exponent associated with the left background:

$$\lim_{x\to-\infty}\mathrm{e}^{\mathrm{i}k_1x}\phi_1(k,x,t)=1, \qquad k_1:=\sqrt{k^2+c^2}. \tag{2.2}$$

Here $k_1>0$ for $k\in[0,\mathrm{i}c)_r$. The last notation denotes the right side of the cut along the interval $[0,\mathrm{i}c]$. Accordingly, $k_1<0$ for $k\in[0,\mathrm{i}c)_l$, the left side of the cut. The left Jost solution admits the usual representation via the transformation operator:

$$\phi_1(k,x,t)=\mathrm{e}^{-\mathrm{i}k_1x}+\int_{-\infty}^x K_1(x,y,t)\mathrm{e}^{-\mathrm{i}k_1y}dy,$$

where $K_1(x,y,t)$ is a real-valued function with

$$|K_1(x,y,t)| \le C \int_{-\infty}^{\frac{x+y}{2}} |q(s,t)+c^2| ds. \tag{2.3}$$

Note that the function $\phi(k,x,t)$ is a holomorphic function of k in $\mathbb{C}^+ := \{k \in \mathbb{C}\colon \operatorname{Im} k > 0\}$ and continuous up to the real axis. It is real-valued for $k \in [0, \mathrm{i}c]$, and does not have a discontinuity on this interval. As for the function $\phi_1(k,x,t)$, it is holomorphic in the domain $\mathbb{C}^+ \setminus (0, \mathrm{i}c]$ and continuous up to the boundary, where $[\phi_1(k,x,t)]_r = \overline{[\phi_1(k,x,t)]_l}$ for $k \in [0, \mathrm{i}c]$.

We observe that condition (1.8) together with (2.3) imply that for $t = 0$ the second left Jost solution:

$$\breve{\phi}_1(k,x,0) = \mathrm{e}^{\mathrm{i}k_1 x} + \int_{-\infty}^{x} K_1(x,y,0)\mathrm{e}^{\mathrm{i}k_1 y} dy,$$

defined for $k_1 \in \mathbb{R}$, where $\breve{\phi}_1 = \overline{\phi_1}$, admits an analytical continuation into the domain

$$\mathcal{V} = \{k : \operatorname{Re} k_1 \in [-c, c], \quad 0 < \operatorname{Im} k_1(k) < \eta\}.$$

Note that $\mathcal{V}$ is a neighbourhood of the interval $[\mathrm{i}c, 0)$. Then the limiting values satisfy

$$[\breve{\phi}_1(k,x,0)]_r = [\phi_1(k,x,0)]_l, \quad [\breve{\phi}_1(k,x,0)]_l = [\phi_1(k,x,0)]_r, \quad \text{for } k \in [0, \mathrm{i}c]. \tag{2.4}$$

For $k \in \mathcal{V}$ introduce two Wronskians :

$$W(k) = \phi_1(k,x,0)\phi'(k,x,0) - \phi_1'(k,x,0)\phi(k,x,0);$$
$$\breve{W}(k) = \breve{\phi}_1(k,x,0)\phi'(k,x,0) - \breve{\phi}_1'(k,x,0)\phi(k,x,0),$$

where $f' = \frac{\partial}{\partial x} f$. Then by (2.4)

$$[W(k)]_r = [\breve{W}(k)]_l = [\overline{W(k)}]_l = [\overline{\breve{W}(k)}]_r. \tag{2.5}$$

The Wronskian $W(k)$ of the Jost solutions is in fact a holomorphic function in $\mathbb{C}^+ \setminus (0, \mathrm{i}c]$ with simple zeros at points $\mathrm{i}\kappa_j$ of the discrete spectrum. It is continuous up to the boundary of the domain, with the only possible additional zero at the point $k = \mathrm{i}c$, the edge of the continuous spectrum. Unlike the case considered in [11], we admit the possible resonance at the point $\mathrm{i}c$, that is, we do not assume the condition $W(\mathrm{i}c) \neq 0$ corresponding to the nonresonant case. In the resonant case the Wronskian has a square root zero at $k = \mathrm{i}c$ (cf. [12]).

In $\mathcal{V}$ introduce also the function

$$\chi(k) := \frac{4kk_1}{W(k)\breve{W}(k)}. \tag{2.6}$$

From (2.5) it follows that its limiting values satisfy

$$[\chi(k)]_r = \mathrm{i}|\chi(k)|, \quad [\chi(k)]_l = -\mathrm{i}|\chi(k)|, \quad k \in [0, \mathrm{i}c]. \tag{2.7}$$

We also observe that

$$\chi(k) = C(k - \mathrm{i}c)^{\ell/2}(1 + o(1)), \quad C \neq 0, \quad k \to \mathrm{i}c, \tag{2.8}$$

where

$$\ell := \begin{cases} 1, & \text{if} \quad W(\mathrm{i}c) \neq 0 \quad \text{(nonresonant case)}; \\ -1, & \text{if} \quad W(\mathrm{i}c) = 0 \quad \text{(resonant case)}. \end{cases}$$

Let $R(k)$ be the right reflection coefficient of the initial data satisfying (1.8) and let

$$\gamma_j := \|\phi(\mathrm{i}\kappa_j, \cdot, 0)\|^{-2}_{L^2(\mathbb{R})}$$

be the right normalizing constants for $j = 1, ..., N$. The set

$$\{R(k),\ k \in \mathbb{R};\quad |\chi(k)|,\ k \in [0, \mathrm{i}c];\quad \mathrm{i}\kappa_j,\ \gamma_j,\quad j = 1, ..., N\}, \tag{2.9}$$

constitute the minimal set of the scattering data to reconstruct uniquely the solution of the initial value problem (1.1)–(1.2) (cf. [5], Corollary 4.4)

Next, the Jost solutions (2.2) and (1.6) are connected by the scattering relation

$$T(k,t)\phi_1(k,x,t) = \overline{\phi(k,x,t)} + R(k,t)\phi(k,x,t), \qquad k \in \mathbb{R}, \tag{2.10}$$

where $T(k,t) = \frac{2\mathrm{i}k}{W(k,t)}$ and $R(k,t)$ are the right transmission and reflection coefficients. We use the notation $T(k) = T(k,0)$ and $R(k) = R(k,0)$. Observe that

$$|T(k)|^2 = k\left[\frac{\chi(k)}{\sqrt{k^2+c^2}}\right]_{r,l}, \quad k \in [0, \mathrm{i}c]. \tag{2.11}$$

We define a vector-valued function $m(k,x,t) = (m_1(k,x,t), m_2(k,x,t))$, meromorphic in the spectral parameter $k \in \mathbb{C} \setminus (\mathbb{R} \cup [-\mathrm{i}c, \mathrm{i}c])$ for fixed x, t, as follows

$$m(k,x,t) = \begin{cases} \left(T(k,t)\phi_1(k,x,t)\mathrm{e}^{\mathrm{i}kx},\quad \phi(k,x,t)\mathrm{e}^{-\mathrm{i}kx}\right), & k \in \mathbb{C}^+ \setminus (0, \mathrm{i}c], \\ m(-k,x,t)\sigma_1, & k \in \mathbb{C}^- \setminus [-\mathrm{i}c, 0), \end{cases} \tag{2.12}$$

where $\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ is the first Pauli matrix. The vector function $m(k,x,t)$ evidently has at most simple poles at the points $\pm\mathrm{i}\kappa_j$. It is known that the following asymptotical formula, for $k \to \infty$, holds:

$$m(k,x,t) = (1,1) - \frac{1}{2\mathrm{i}k}\left(\int_x^{+\infty} q(y,t)dy\right)(-1,1) + O\left(\frac{1}{k^2}\right).$$

This expansion allows us to extract the shock wave solution using formula (1.14). However, as was mentioned in the introduction, the formula (1.13), which can be computed using the well-known asymptotic formulas for the Weyl functions, is more convenient.

Indeed, it is known that for k large enough both functions $\phi(k,x,t)$ and $\phi_1(k,x,t)$ do not vanish for all x and t. Thus,

$$\begin{aligned} m_1(k,x,t)m_2(k,x,t) &= T(k,t)\phi(k,x,t)\phi_1(k,x,t) \\ &= \frac{2\mathrm{i}k}{\frac{\phi'(k,x,t)}{\phi(k,x,t)} - \frac{\phi_1'(k,x,t)}{\phi_1(k,x,t)}} = \frac{2\mathrm{i}k}{\mathfrak{m}(k,x,t) - \mathfrak{m}_1(k,x,t)}, \end{aligned}$$

where $\mathfrak{m}$ and $\mathfrak{m}_1$ are the right and left Weyl functions corresponding to the potential $q(x,t)$. For $k \to \infty$ we have (cf. [6]):

$$\begin{aligned} \mathfrak{m}(k,x,t) &= \mathrm{i}k + \frac{q(x,t)}{2\mathrm{i}k} + \frac{f(x,t)}{4k^2} + O(k^{-3}), \\ \mathfrak{m}_1(k,x,t) &= -\mathrm{i}k - \frac{q(x,t)}{2\mathrm{i}k} + \frac{f(x,t)}{4k^2} + O(k^{-3}). \end{aligned}$$

Thus,

$$m_1(k,x,t)m_2(k,x,t) - 1 = \frac{2\mathrm{i}k}{2\mathrm{i}k + \frac{q(x,t)}{\mathrm{i}k} + O(k^{-3})} - 1 = \frac{q(x,t)}{2k^2} + O(k^{-4}),$$

which proves (1.13). □

The following existence/uniqueness result is then valid:

Theorem 2.1. *Let*

- *the potential $q(x)$ satisfy (1.2) and (1.8);*
- *the set (2.9) be its right scattering data;*
- *$\Sigma = \mathbb{R} \cup [\mathrm{i}c, -\mathrm{i}c]$ be the jump contour oriented left-to-right$\cup$ top-down;*
- *the phase function $\Phi(k) = \Phi(k, x, t)$ be defined by the formula:*

$$\Phi(k) = 4\mathrm{i}k^3 + \mathrm{i}k\frac{x}{t}, \quad k \in \mathbb{C}.$$

Then $m(k) = m(k, x, t)$ defined in (2.12) for all $(x, t) \in \mathbb{R} \times \mathbb{R}_+$ is the unique solution of the following vector Riemann–Hilbert problem:
Find a vector-valued function $m(k)$, meromorphic away from Σ, satisfying:

(i) The jump condition $m_+(k) = m_-(k)v(k)$

$$v(k) = \begin{cases} \begin{pmatrix} 1 - |R(k)|^2 & -\overline{R(k)}\mathrm{e}^{-2t\Phi(k)} \\ R(k)\mathrm{e}^{2t\Phi(k)} & 1 \end{pmatrix}, & k \in \mathbb{R}, \\[2ex] \begin{pmatrix} 1 & 0 \\ \mathrm{i}|\chi(k)|\mathrm{e}^{2t\Phi(k)} & 1 \end{pmatrix}, & k \in [\mathrm{i}c, 0], \\[2ex] \sigma_1 (v(-k))^{-1} \sigma_1, & k \in [0, -\mathrm{i}c]; \end{cases} \tag{2.13}$$

(ii) the pole conditions

$$\begin{aligned} \operatorname{Res}_{\mathrm{i}\kappa_j} m(k) &= \lim_{k \to \mathrm{i}\kappa_j} m(k) \begin{pmatrix} 0 & 0 \\ \mathrm{i}\gamma_j^2 \mathrm{e}^{t\Phi(\mathrm{i}\kappa_j)} & 0 \end{pmatrix}, \\ \operatorname{Res}_{-\mathrm{i}\kappa_j} m(k) &= \lim_{k \to -\mathrm{i}\kappa_j} m(k) \begin{pmatrix} 0 & -\mathrm{i}\gamma_j^2 \mathrm{e}^{t\Phi(\mathrm{i}\kappa_j)} \\ 0 & 0 \end{pmatrix}, \end{aligned} \tag{2.14}$$

(iii) the symmetry condition

$$m(-k) = m(k)\sigma_1, \quad k \in \mathbb{C} \setminus \Sigma, \tag{2.15}$$

(iv) and the normalization condition

$$\lim_{\kappa \to \infty} m(\mathrm{i}\kappa) = (1 \quad 1). \tag{2.16}$$

(v) In addition, the function $m(k)$ has the following behavior in a vicinity of the point $\mathrm{i}c$: If $\chi(k)$ satisfies (2.8) with $\ell = 1$ then $m(k)$ has continuous limits as k approaches $\mathrm{i}c$ from the domain $\mathbb{C} \setminus \Sigma$. If $\ell = -1$ then one has

$$\begin{aligned} m(k) &= \Big(C_1(x,t)(k - \mathrm{i}c)^{-1/2}, C_2(x,t)\Big)(1 + o(1)) \quad C_1 C_2 \neq 0; \ \text{or} \\ m(k) &= (C(x,t), 0)\,(1 + o(1)) \quad \text{as} \ \ k \to \mathrm{i}c. \end{aligned} \tag{2.17}$$

At the point $-\mathrm{i}c$ the analog of condition (2.17) holds by symmetry (2.15).

Proof. The facts that m satisfies the jump condition (2.13) and the pole conditions (2.14) are established in [11]. Note that the jump matrix on $\mathbb{R}$ also satisfies the symmetry $v(k) = \sigma_1(v(-k))^{-1}\sigma_1$. To prove uniqueness, assume first that $\tilde{m}(k)$ and $\hat{m}(k)$ are two solutions for the R-H problem *(i)–(v)*. Then $\mu(k) := \tilde{m}(k) - \hat{m}(k)$ satisfies *(i)–(iii), (v)* and instead of *(iv)* we have

$$\mu(k) = O(k^{-1}), \quad k \to \infty.$$

In $\mathbb{C}^+ \setminus (0, \mathrm{i}c]$ introduce the meromorphic function

$$F(k) = \mu_1(k)\overline{\mu_1(\overline{k})} + \mu_2(k)\overline{\mu_2(\overline{k})},$$

where $\mu_{1,2}$ are the components of μ. Then $F(k) = O(k^{-2})$ as $k \to \infty$. Note that since the exact values of the constants C_1, C_2 and C in (2.17) are not specified, they may be different for $\tilde{m}$ and $\hat{m}$. Furthermore, since $-\overline{k} = k$ for $k \in \mathrm{i}\mathbb{R}$, it follows from the symmetry condition *(iii)* that for such k, $\mu_i(\overline{k}) = \mu_j(k)$, $i \neq j$. We thus get $F(k) = O((k - \mathrm{i}c)^{-1/2})$ as $k \to \mathrm{i}c$ when $\ell = -1$. For the nonresonant case $\ell = 1$ the function $F(k)$ has continuous limits everywhere on $\mathbb{R} \cup [0, \mathrm{i}c]$. Let us denote for simplicity $F_r(k)$ and $F_l(k)$ the limiting values of F from the right and left sides of $[0, \mathrm{i}c]$, and $F_+(k)$ for the limiting values on the real axis from above. Then by the symmetry condition (2.15) we get

$$\begin{aligned} F_+(k) &= \mu_{1,+}(k)\overline{\mu_{1,-}(k)} + \mu_{2,+}(k)\overline{\mu_{2,-}(k)}, \\ F_r(k) &= \mu_{1,r}(k)\overline{\mu_{2,l}(k)} + \mu_{2,r}(k)\overline{\mu_{1,l}(k)}, \\ F_l(k) &= \mu_{1,l}(k)\overline{\mu_{2,r}(k)} + \mu_{2,l}(k)\overline{\mu_{1,r}(k)}. \end{aligned}$$

The jump condition (2.13) implies

$$F_+(k) = (1 - |R(k)|^2)|\mu_{1,-}|^2 + |\mu_{2,-}|^2 + 2\mathrm{i}\,\mathrm{Im}\left(R(k)\mathrm{e}^{2t\Phi(k)}\overline{\mu_{1,-}(k)}\mu_{2,-}(k)\right),$$

$$\begin{aligned} F_l(k) &= \mathrm{Re}\left(\mu_{1,l}(k)\overline{\mu_{2,l}(k)}\right) - \mathrm{i}|\mu_{2,l}(k)|^2|\chi(k)|\mathrm{e}^{2t\Phi(k)}, \\ F_r(k) &= \mathrm{Re}\left(\mu_{1,l}(k)\overline{\mu_{2,l}(k)}\right) + \mathrm{i}|\mu_{2,l}(k)|^2|\chi(k)|\mathrm{e}^{2t\Phi(k)}. \end{aligned} \tag{2.18}$$

Note that $\Phi(k) \in \mathbb{R}$ for $k \in \mathrm{i}\mathbb{R}$. From this and (2.18) it follows that

$$\begin{aligned} \mathrm{Re}\, F_l(k) = \mathrm{Re}\, F_r(k) &= \mathrm{Re}\left(\mu_{1,l}(k)\overline{\mu_{2,l}(k)}\right), \\ \mathrm{Im}\, F_l(k) &= -\,\mathrm{Im}\, F_r(k) \in \mathbb{R}_-. \end{aligned} \tag{2.19}$$

The pole condition (2.14) is satisfied by the vector $\mu(k)$. Alongside with the symmetry property this implies

$$\mathrm{Res}_{\mathrm{i}\kappa_j} F(k) = 2\mathrm{i}\gamma_j^2 |\mu_2(\mathrm{i}\kappa_j)|^2 \in \mathrm{i}\mathbb{R}_+.$$

Let now $\omega > c$ be arbitrary large and let $\mathcal{C}_\omega$ be the boundary of the domain $(\mathbb{C}^+ \cap \{k : \ |k| < \omega\}) \setminus (0, \mathrm{i}c]$. We treat $\mathcal{C}_\omega$ as a closed contour oriented counterclockwise. By Cauchy's theorem

$$\oint_{\mathcal{C}_\omega} F(k)dk = 2\pi\mathrm{i} \sum_{j=1}^{N} \mathrm{Res}_{\mathrm{i}\kappa_j} F(k),$$

and since $F(k) = O(k^{-2})$ as $k \to \infty$, the integral over the upper semicircle will asymptotically vanish as $\omega \to \infty$ and we get

$$\int_{\mathbb{R}} F_+(k)dk + \int_0^{\mathrm{i}c} F_l(k)dk + \int_{\mathrm{i}c}^0 F_r(k)dk + 4\pi \sum_{j=1}^{N} \gamma_j^2 |\mu_2(\mathrm{i}\kappa_j)|^2 = 0.$$

Taking into account (2.19), the real part of this integral reads

$$0 = \int_{\mathbb{R}} \big((1-|R(k)|^2)|\mu_{1,-}|^2 + |\mu_{2,-}|^2\big)\, dk + 2\int_0^c |\mu_{2,l}(\mathrm{i}s)|^2|\chi(\mathrm{i}s)|\mathrm{e}^{2t\Phi(\mathrm{i}s)}ds$$
$$+ 4\pi\sum_{j=1}^N \gamma_j^2|\mu_2(\mathrm{i}\kappa_j)|^2.$$

But $|R(k)|<1$ for $k\in\mathbb{R}\setminus\{0\}$, and therefore all summands in the last formula are non-negative. Thus, we obtain $\mu_2(\mathrm{i}\kappa_j)=0$ (which implies that $\mu_1(k)$ does not have a pole at $\mathrm{i}\kappa_j$) and

$$\mu_{2,-}(k)=0, \quad \text{for} \quad k\in\mathbb{R}; \qquad \mu_{2,l}(k)=0, \quad \text{for} \quad k\in[\mathrm{i}c,0]; \qquad \mu_{1,-}(k)=0, \quad \text{for} \quad k\in\mathbb{R}.$$

From this and (2.13) it immediately follows that $\mu_{1,+}(k)=\mu_{2,+}(k)=0$ and $\mu_{2,r}(k)=\mu_{2,l}(k)=0$ for $k\in[\mathrm{i}c,0]$. Thus, the function $\mu_2(k)$ is a holomorphic function in $\mathbb{C}$ with $\mu_2(k)\to 0$ as $k\to\infty$. By Liouville's theorem $\mu_2(k)\equiv 0$ in $\mathbb{C}$. In turn, this identity and formula (2.13) imply: $\mu_{1,r}(k)=\mu_{1,l}(k)$ for $k\in[\mathrm{i}c,0]$. Therefore, $\mu_1(k)$ is also a holomorphic function in $\mathbb{C}$ vanishing at infinity, thus $\mu_1(k)\equiv 0$. This proves uniqueness.

It remains to verify *(v)*. The case $\ell=1$ implies that the Wronskian $W(k,t)$ of the Jost solutions $\phi(k,x,t)$ and $\phi_1(k,x,t)$ does not vanish at $k=\mathrm{i}c$ for all t (cf. [13], formula (6.2)). This implies that $T(k,t)$ is bounded and continuous as $k\to\mathrm{i}c$, and the same is true for the components of the vector m.

If $\ell=-1$ then $W(\mathrm{i}c,t)=0$. Now, if, in turn $\phi(\mathrm{i}c,x,t)\neq 0$, then $\phi_1(\mathrm{i}c,x,t)\neq 0$ (otherwise the Wronskian would not have zero at $k=\mathrm{i}c$). This proves the first line of (2.17). If $\phi(\mathrm{i}c,x,t)=0$, then also $\phi_1(\mathrm{i}c,x,t)=0$. Since $W(k,t)=\tilde C(t)(k-\mathrm{i}c)^{1/2}(1+o(1))$ and $\phi_1(k,x,t)=\tilde C_1(x,t)(k-\mathrm{i}c)^{1/2}(1+o(1))$ as $k\to\mathrm{i}c$, this proves the second line of (2.17). □

Theorem 2.1 guarantees the well-posedness of the initial meromorphic vector R-H (IM R-H) problem for all $(x,t)\in\mathbb{R}\times\mathbb{R}_+$. In the domain $\mathcal{D}_\varepsilon$ given by (1.7), (1.3) where we intend to study and justify the asymptotics of its solution $m(k,x,t)$ as $k\to\infty$, the IM R-H problem admits an equivalent holomorphic statement.

3. Holomorphic statement of the initial vector R-H problem

In this section we take a closer look at the ill-posedness of the associated matrix R-H problem. Let $a(\xi)$ be defined implicitly by (1.9), then as shown in [11]:

$$0 < a(-\frac{c^2}{2}+\varepsilon) \le a(\xi) \le a(\frac{c^2}{3}-\varepsilon) < c.$$

Recall that the discrete spectrum is denoted by $-\kappa_j^2$, $j=1,\dots,N$, with $c<\kappa_N<\dots<\kappa_1$. Choose $\rho>0$ sufficiently small, such that

$$\rho < \frac{1}{4}\min\Big\{\sqrt{c^2+\eta^2}-c,\ \kappa_N-c,\ a(-\frac{c^2}{2}+\varepsilon),$$
$$c-a(\frac{c^2}{3}-\varepsilon),\ \min_{j=1,..,N}|\kappa_{j-1}-\kappa_j|,\ \rho_1\Big\}, \tag{3.1}$$

where $\eta>0$ is the decay estimate from (1.8) and $\rho_1>0$ is defined implicitly by formula (4.3) below. Note, that since $k_1=\sqrt{c^2+k^2}$ and $\eta>\sqrt{c^2+\eta^2}-c$, the reflection coefficient $R(k)$ and the function $\chi(k)$ from (2.13) are well defined in the domains

$$\begin{aligned} &\Omega_R := \{k : \rho > \operatorname{Im} k > 0\}, \quad \text{and} \\ &\Omega_\chi := \{k : \operatorname{Im} k \in (0, c+\rho),\ |\operatorname{Re} k| < \rho\} \setminus (0, \mathrm{i}c]\}, \end{aligned} \tag{3.2}$$

respectively, up to their boundaries.

Denote by

$$\mathbb{D}_j := \{k : \ |k - \mathrm{i}\kappa_j| < \rho.\}, \quad j = 1, \dots, N$$

and by

$$\mathbb{T}_j := \partial \mathbb{D}_j = \{k : \ |k - \mathrm{i}\kappa_j| = \rho\}, \quad j = 1, \dots, N$$

the small nonintersecting contours around the points of the discrete spectrum oriented counterclockwise. Let $\mathcal{C} := \{k : \operatorname{Im} k = \rho\}$ be the upper boundary of Ω_R considered as a contour oriented from left to right. We observe that with our choice of ρ (cf. (3.1))

$$\operatorname{dist}(\mathbb{T}_N, \overline{\Omega_\chi}) > 2\rho, \quad \operatorname{dist}\left(\mathrm{i}a\left(-c^2/2 + \varepsilon\right), \overline{\Omega_R}\right) > 3\rho.$$

Introduce also the functions:

$$P(k) := \prod_{j=1}^{N} \frac{k + \mathrm{i}\kappa_j}{k - \mathrm{i}\kappa_j}, \quad k \in \mathbb{C}; \quad Q(k) := \left(\frac{k - \mathrm{i}c}{k + \mathrm{i}c}\right)^{\frac{\ell}{4}}, \quad k \in \mathbb{C} \setminus [-\mathrm{i}c, \mathrm{i}c]; \tag{3.3}$$

where ℓ is as in Theorem 1.1 and $Q(\infty) = 1$.

Redefine now the solution $m(k) = m(k, x, t)$ of the IM R-H problem as follows:

$$m^{\text{ini}}(k) = \begin{cases} m(k) A_j(k) \left(P(k) Q(k)\right)^{-\sigma_3}, & k \in \mathbb{D}_j, \quad j = 1, .., N; \\ m(k) A_0(k) \left(P(k) Q(k)\right)^{-\sigma_3}, & k \in \Omega_R; \\ m(k) \left(P(k) Q(k)\right)^{-\sigma_3}, & k \in \mathbb{C}^+ \setminus (\overline{\Omega_R} \cup [0, \mathrm{i}c] \cup \cup_{j=1}^{N} \overline{\mathbb{D}_j}); \\ m^{\text{ini}}(-k) \sigma_1, & k \in \mathbb{C}^-, \end{cases} \tag{3.4}$$

where we denoted

$$A_j(k) = \begin{pmatrix} 1 & -\frac{k - \mathrm{i}\kappa_j}{\mathrm{i}\gamma_j^2 \mathrm{e}^{2t\Phi(\mathrm{i}\kappa_j)}} \\ 0 & 1 \end{pmatrix}, \quad A_0(k) = \begin{pmatrix} 1 & 0 \\ -R(k) \mathrm{e}^{t\Phi(k)} & 1 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Introduce the contours in the lower half plane: $\mathcal{C}^* := \{k : \ -k \in \mathcal{C}\}$ oriented right-to left, and $\mathbb{T}_j^* := \{k : -k \in \mathbb{T}_j\}$, $j = 1, ..., N$ oriented counterclockwise. Define the functions (cf. (3.2),(3.3)):

$$\begin{aligned} &\mathcal{R}(k) := R(k) P^{-2}(k) Q^{-2}(k), \quad k \in \mathcal{C}; \\ &X(k) := \chi(k) Q^{-2}(k) P^{-2}(k), \quad k \in \Omega_\chi. \end{aligned} \tag{3.5}$$

Note that

$$X_\pm(k) = \pm \mathrm{i} |X(k)|, \quad k \in [\mathrm{i}c, 0]. \tag{3.6}$$

Then we have the following

Lemma 3.1. *For all $(x,t) \in \mathcal{D}_\varepsilon$ the vector function $m^{\text{ini}}(k) = m^{\text{ini}}(k, x, t)$ is the unique solution of the following R-H problem:*

Find a vector-valued function, holomorphic in the domain

$$\mathbb{C} \setminus \Sigma^{\text{ini}}, \quad \Sigma^{\text{ini}} := \mathcal{C} \cup \mathcal{C}^* \cup \cup_j (\mathbb{T}_j \cup \mathbb{T}_j^*) \cup [\mathrm{i}c, -\mathrm{i}c]),$$

satisfying

- *the symmetry condition* $m^{\mathrm{ini}}(-k) = m^{\mathrm{ini}}(k)\sigma_1$, $k \in \mathbb{C} \setminus \Sigma^{\mathrm{ini}}$*;*
- *the jump condition* $m_+^{\mathrm{ini}}(k) = m_-^{\mathrm{ini}}(k) v^{\mathrm{ini}}(k)$, $k \in \Sigma^{\mathrm{ini}}$*, where*

$$v^{\mathrm{ini}}(k) = \begin{cases} \begin{pmatrix} 1 & 0 \\ \mathcal{R}(k)\mathrm{e}^{2t\Phi(k)} & 1 \end{pmatrix}, & k \in \mathcal{C}, \\ \begin{pmatrix} \exp(\frac{\mathrm{i}\ell\pi}{2}) & 0 \\ \mathrm{i}|X(k)|\mathrm{e}^{2t\Phi(k)} & \exp(\frac{-\mathrm{i}\ell\pi}{2}) \end{pmatrix}, & k \in [\mathrm{i}c, \mathrm{i}\rho], \\ \begin{pmatrix} 1 & \frac{(k-\mathrm{i}\kappa_j)P^2(k)Q^2(k)}{\mathrm{i}\gamma_j^2 \mathrm{e}^{2t\Phi(\mathrm{i}\kappa_j)}} \\ 0 & 1 \end{pmatrix}, & k \in \mathbb{T}_j, \\ \exp(\frac{\mathrm{i}\ell\pi}{2}\sigma_3), & k \in [\mathrm{i}\rho, -\mathrm{i}\rho], \\ \sigma_1 [v^{\mathrm{ini}}(-k)]^{-1}\sigma_1, & k \in [-\mathrm{i}\rho, -\mathrm{i}c], \\ \sigma_1 [v^{\mathrm{ini}}(-k)]\sigma_1, & k \in \mathcal{C}^* \cup \cup_{j=1}^N \mathbb{T}_j^*. \end{cases} \tag{3.7}$$

- *the normalizing condition* $m^{\mathrm{ini}}(k) \to (1,\ 1)$ *as* $k \to \infty$.
- *at points* $\pm\mathrm{i}c$ *it has at most fourth root singularities:* $m^{\mathrm{ini}}(k) = O(k \mp \mathrm{i}c)^{-1/4}$ *as* $k \to \pm\mathrm{i}c$.

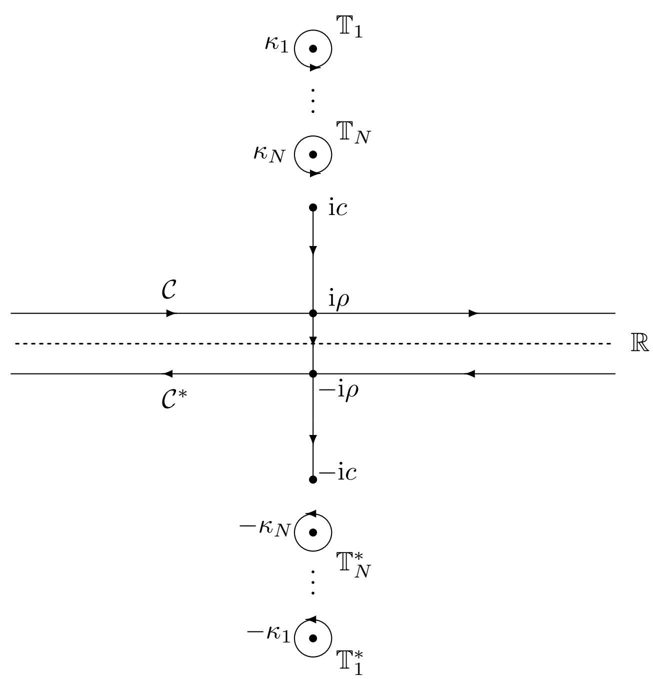

FIGURE 1. Jump contour Σ^{ini}

Proof. The proof is very similar to that one given in [11], with only one difference: we can use the identity $R_-(k) - R_+(k) + \mathrm{i}|\chi(k)| = 0$ (Lemma 3.2, [11]) on the interval $[-\mathrm{i}\rho, -\mathrm{i}\rho]$ and take into account the influence of the function $Q(k)$. In particular, we used that $Q_-(k)Q_+^{-1}(k) = \exp(\frac{\mathrm{i}\ell\pi}{2})$, and $Q_+(k)Q_-(k) = |Q^2(k)|$ for $k \in [\mathrm{i}c, -\mathrm{i}c]$. □

Since $m^{\mathrm{ini}}(k)$ is a piecewise holomorphic vector function, we call the problem stated in Lemma 3.1 the initial holomorphic (IH) R-H problem. As already mentioned in the Introduction, the symmetry condition is crucial for uniqueness and plays an essential role in the final asymptotical analysis. That is why, all transformations steps carried out to get from the initial R-H problem to an R-H problem asymptotically close to an exactly solvable model vector R-H problem, should respect the following symmetry conditions:

Hypothesis 3.2. *Each vector R-H problem should satisfy:*

- *The jump contour Σ is symmetric with respect to the map $k \mapsto -k$;*
- *On $\mathbb{C} \setminus \Sigma$ the vector solution $m(k)$ is holomorphic and satisfies $m(-k) = m(k)\sigma_1$;*
- *Let $\mathcal{L} \subset \Sigma$ be a subcontour. We denote by $\mathcal{L}^* = \{k : -k \in \mathcal{L}\} \subset \Sigma$ its inversion, if $\mathcal{L}^*$ has the orientation of the following type: when k moves in the positive direction along $\mathcal{L}$, then $-k$ moves in the positive direction along $\mathcal{L}^*$. In this case, the jump matrix $v(k)$ of the jump problem*

$$m_+(k) = m_-(k)v(k), \quad k \in \Sigma, \tag{3.8}$$

should satisfy $\det v(k) = 1$ *and the symmetry*

$$v(-k) = \sigma_1 v(k)\sigma_1, \quad k \in \mathcal{L} \cup \mathcal{L}^*. \tag{3.9}$$

If the inversion of $\mathcal{L}$ has the opposite orientation, we denote it by $(\mathcal{L}^*)^{-1}$. For example, $\mathcal{L} = [\mathrm{i}c, 0]$ and $(\mathcal{L}^*)^{-1} = [0, -\mathrm{i}c]$ are both oriented top-bottom. In this case,

$$v(-k) = \sigma_1 v(k)^{-1}\sigma_1, \quad k \in \mathcal{L} \cup (\mathcal{L}^*)^{-1}; \tag{3.10}$$

- *The vector-function $m(k)$ is continuous up to the boundary, except at the node points of the contour (the ends and self intersections of Σ, and a finite number of points of discontinuity of the jump matrix), where fourth root singularities are admissible;*
- *$m(k) \to (1, 1)$ as $k \to \infty$.*

Evidently, the IH R-H problem formulated in Lemma 3.1 satisfies all these requirements. Alongside with it, we can write down an analogous matrix R-H problem with the same jump matrix $v^{\mathrm{ini}}(k)$ given by (3.7). This can be done in two ways. Either by imposing a symmetry condition (see (3.14) below), or by the standard normalization to the unit matrix $\mathbb{I}$ at infinity. Simultaneous use of both conditions may seem excessive. In fact, we observe the following.

Let $\Sigma \subset \mathbb{C}$ be a union of finitely many smooth curves (finite or infinite) which intersect in at most a finite number of points and all intersections are transversal (this condition can of course be relaxed, but it is sufficient for the applications we have in mind). We will also require Σ to be symmetric with respect to the inversion $k \mapsto -k$.

Let now $v(k)$ be a piecewise continuous bounded matrix function on Σ satisfying (3.9) or (3.10), with $\det v(k) \equiv 1$. The points of discontinuity of the jump matrix, together with the (finite) set of boundary points $\partial\Sigma$ and the self intersection points of Σ, are denoted by $\mathcal{G}$. We assume that $0 \notin \mathcal{G}$.

Finally, let $\mathcal{H}$ be the class of 2×2 matrix functions $M(k)$ holomorphic in $\mathbb{C}\setminus\Sigma$, which have continuous limits up to the boundary $\Sigma\setminus\mathcal{G}$ and have a limit as $k\to\infty$ (avoiding Σ). At points of $\mathcal{G}$ we allow singularities of the form:

$$M(k) = O((k-\kappa)^{-1/4}), \quad \text{as} \quad k\to\kappa\in\mathcal{G}. \tag{3.11}$$

Now for an admissible $M\in\mathcal{H}(\Sigma)$ we consider the following R-H factorization problem

$$M_+(k) = M_-(k)v(k), \qquad k\in\Sigma, \tag{3.12}$$

together with the normalization condition

$$M(\infty) := \lim_{k\to\infty} M(k) = \mathbb{I} \tag{3.13}$$

and the symmetry condition

$$M(-k) = \sigma_1 M(k)\sigma_1, \qquad k\in\mathbb{C}\setminus\Sigma. \tag{3.14}$$

Theorem 3.3. *Suppose $\Sigma\subset\mathbb{C}$ is an admissible contour and $v(k)$, $k\in\Sigma$ an admissible matrix as specified above. Then the following propositions are valid:*

(a) If a solution $M\in\mathcal{H}(\Sigma)$ of (3.12) exists for which $\det M(\infty)\neq 0$, then $M(\infty)^{-1}M(k)$ solves (3.12)–(3.13), and every other solution of (3.12) is given by $\tilde M(k) = \tilde M(\infty)M(\infty)^{-1}M(k)$ in this case. Moreover, $\det M(k) = \det M(\infty)$.

(b) If (3.12) has a nonsingular, that is invertible solution from $\mathcal{H}(\Sigma)$, then every solution $M\in\mathcal{H}(\Sigma)$ of (3.12) satisfies the symmetry condition (3.14) provided $M(\infty)$ satisfies the symmetry condition. In this case M is of the form

$$M(k) = \begin{pmatrix} \alpha(k) & \beta(k)\\ \beta(-k) & \alpha(-k)\end{pmatrix}, \qquad M(\infty) = \begin{pmatrix} a & b\\ b & a\end{pmatrix}$$

with $\det M(\infty) = a^2-b^2$. If M is nonsingular then $a+b\neq 0$.

(c) Suppose (3.12) has a nonsingular solution M satisfying (3.14). Then the vector function $m(k)$

$$m(k) = \frac{1}{a+b}(1,\ 1)M(k) = \frac{1}{a+b}(\alpha(k)+\beta(-k),\, \beta(k)+\alpha(-k)).$$

solves the same jump problem $m_+(k) = m_-(k)v(k)$ and satisfies (2.15) and (2.16). Moreover, in this case m is the unique solution of this problem with admissible singularities of the type (3.11).

(d) Suppose the vector problem described in Remark 3.2 has a solution m which satisfies the condition $m_\pm(0) = (0,0)$. Then there is no invertible solution of the problem (3.12), (3.14) in $\mathcal{H}(\Sigma)$.

Proof. (a). This follows similarly as in [7, Theorem 7.18].

(b). Let $M(k)\in\mathcal{H}(\Sigma)$ be the solution of the problem (3.12)–(3.13). By *(a)* it suffices to show that M satisfies (3.14). To this end set $\tilde M(k) = \sigma_1 M(-k)\sigma_1$. Then $\tilde M(\infty) = \mathbb{I}$ and $\tilde M(k)\in\mathcal{H}$. Taking into account the symmetry of Σ, for example, (3.10), we see that

$$\begin{aligned}\tilde M_+(k) &= \sigma_1 M_-(-k)\sigma_1 = \sigma_1 M_+(-k)v^{-1}(-k)\sigma_1\\ &= \sigma_1 M_+(-k)\sigma_1\sigma_1 v^{-1}(-k)\sigma_1 = \tilde M_-(k)v(k).\end{aligned}$$

Thus $\tilde M(k)$ solves (3.12)–(3.13) and by uniqueness, $\tilde M(k)\equiv M(k)$. This proves (3.14). The rest is straightforward.

(c). By assumption we have a solution M as in *(b)* and hence one easily checks that m satisfies (3.8), as well as (2.15) and (2.16). If $\tilde{m}$ is a second solution, then as in *(a)* we see that (3.8) implies that $c = \tilde{m}(k)M^{-1}(k)$ is a constant vector. Hence by (2.16) we see $c = \frac{1}{a+b}(1,1)$.

(d). Suppose that there exists an invertible symmetric matrix $M(k)$ satisfying (3.12). Without loss of generality we can assume $M(\infty) = \mathbb{I}$ and hence by the previous item our assumption implies $m_+(0) = (\alpha_+(0) + \beta_-(0), \beta_+(0) + \alpha_-(0)) = (0,0)$. Consequently

$$M_+(0) = \begin{pmatrix} \alpha_+(0) & \beta_+(0) \\ -\alpha_+(0) & -\beta_+(0) \end{pmatrix}$$

implying $\det M(k) = \det M_+(0) = 0$. $\square$

In particular, item *(d)* implies that any technique relying on existence of a bounded nonsingular matrix solution is bound to fail at all points in the (x,t)-plane where $m_+(0) = (0,0)$ holds. Recall now that the vector function (2.12) is the unique solution of the IM R-H problem, making $m^{\mathrm{ini}}(k)$ the unique solution of the IH R-H problem. After the transformation (3.4) the point $k = 0$ became an inner point of the contour Σ^{ini}. Moreover, taking into account the scattering relation (2.10) and the fact $\phi(+0,x,t) = \phi(-0,x,t) = \overline{\phi(+0,x,t)}$, it is straightforward to check that

$$\begin{aligned} m_\pm^{\mathrm{ini}}(0,x,t) &= \Big(\overline{\phi(\pm 0,x,t)}P^{-1}(0)Q_\pm^{-1}(0),\ \phi(\pm 0,x,t)P(0)Q_\pm(0)\Big) \\ &= (-1)^N \phi(0,x,t)\left(\mathrm{e}^{\pm\frac{\mathrm{i}\ell\pi}{4}},\ \mathrm{e}^{\mp\frac{\mathrm{i}\ell\pi}{4}}\right) \end{aligned}$$

Thus, if $\phi(0,x^*,t^*) = 0$ for arbitrary large $(x^*,t^*) \in \mathcal{D}_\varepsilon$, then $m_\pm^{\mathrm{ini}}(0,x^*,t^*) = 0$ and by Theorem 3.3, *(d)* we can talk about ill-posedness of the respective matrix R-H problem. Moreover, even for the one-soliton (reflectionless, decaying) case this occurs as pointed out in the discussion after Lemma 2.5 in [18].

4. From the IH R-H problem to the model R-H problem

Now we recall briefly the conjugation and deformation steps which lead to the model problem solution in the domain $\mathcal{D}_\varepsilon$. As is shown in [11], (see also [13]) for $\xi = \frac{x}{12t} \in (-\frac{c^2}{2}, \frac{c^2}{3})$ the equality (1.9) generates an implicitly given positive smooth function $a(\xi)$, monotonously increasing such that $a(-\frac{c^2}{2}) = 0$, $a(\frac{c^2}{3}) = c$. In the domain $\mathbb{C} \setminus [\mathrm{i}c, -\mathrm{i}c]$ we introduce the function

$$g(k) := g(k,x,t) = 12\int_{\mathrm{i}c}^{k}\left(k^2 + \xi + \frac{c^2 - a^2}{2}\right)\sqrt{\frac{k^2 + a^2}{k^2 + c^2}}\,dk. \tag{4.1}$$

Here we use the standard branch of the square root with the cut along $\mathbb{R}_-$.

Lemma 4.1. *([11]). The function g possesses the following properties*

(a) $g(k) = -g(-k)$ *for* $k \in \mathbb{C} \setminus [\mathrm{i}c, -\mathrm{i}c]$;

(b) $g_-(k) + g_+(k) = 0$ *as* $k \in [\mathrm{i}c, \mathrm{i}a] \cup [-\mathrm{i}a, -\mathrm{i}c]$;

(c) $g_-(k) - g_+(k) = B$ *as* $k \in [\mathrm{i}a, -\mathrm{i}a]$, *where* $B := B(\xi) = -2g_+(\mathrm{i}a) > 0$;

(d) *the asymptotical behavior*

$$\Phi(k,\xi) - \mathrm{i}g(k,\xi) = O\left(\frac{1}{k}\right).$$

holds as $k \to \infty$.

The signature table for the imaginary part of function g is shown in the following figure:

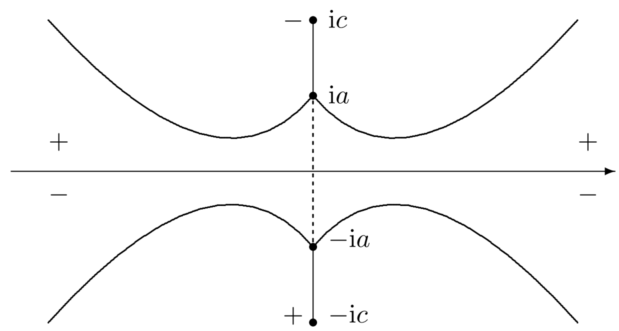

FIGURE 2. Sign of $\mathrm{Im}(g)$

STEP 1. Let $m^{\mathrm{ini}}(k)$ be the unique vector solution of the IH R-H problem. Redefine it by

$$m^{(1)}(k) := m^{\mathrm{ini}}(k)\mathrm{e}^{(\mathrm{i}tg(k)-t\Phi(k))\sigma_3}. \tag{4.2}$$

Then $m^{(1)}(k)$ is a piecewise-holomorphic function in $\mathbb{C}$ which satisfies the symmetry requirements of Remark 3.2 and solves the jump problem $m^{(1)}_+(k) = m^{(1)}_-(k)v^{(1)}(k)$ with

$$v^{(1)}(k) = \begin{cases} \begin{pmatrix} 1 & 0 \\ \mathcal{R}(k)\mathrm{e}^{2\mathrm{i}tg(k)} & 1 \end{pmatrix}, & k \in \mathcal{C}, \\ \begin{pmatrix} \mathrm{e}^{\mathrm{i}t(g_+ - g_-)+\frac{\mathrm{i}\ell\pi}{2}} & 0 \\ \mathrm{i}|X(k)|\mathrm{e}^{\mathrm{i}t(g_+ + g_-)} & \mathrm{e}^{-\mathrm{i}t(g_+ - g_-)-\frac{\mathrm{i}\ell\pi}{2}} \end{pmatrix}, & k \in [\mathrm{i}c, \mathrm{i}\rho], \\ \begin{pmatrix} 1 & h_j(k,t) \\ 0 & 1 \end{pmatrix}, & k \in \mathbb{T}_j,\ j = 1, .., N, \\ \exp((-\mathrm{i}tB + \mathrm{i}\frac{\ell\pi}{2})\sigma_3), & k \in [\mathrm{i}\rho, -\mathrm{i}\rho], \\ \sigma_1[v^{(1)}(-k)]^{-1}\sigma_1, & k \in [-\mathrm{i}\rho, -\mathrm{i}c], \\ \sigma_1[v^{(1)}(-k)]\sigma_1, & k \in \mathcal{C}^* \cup \cup_j \mathbb{T}_j^*, \end{cases}$$

where $\mathcal{R}(k)$ and $X(k)$ are given by (3.5) and

$$h_j(k,t) := h_j(k,t,\xi) = -\mathrm{i}(k - \mathrm{i}\kappa_j)P^2(k)Q^2(k)\gamma_j^{-2}\mathrm{e}^{-2t(\Phi(\mathrm{i}\kappa_j)-\Phi(k))-2tig(k)}.$$

Since $\mathrm{Im}\, g(\mathrm{i}\kappa_j) < -\delta < 0$ (cf. Figure 2), uniformly with respect to $\xi \in \mathcal{I}_\varepsilon$, we conclude that there exists $\rho_1 > 0$ such that

$$\max_{j=1,..,N} \sup_{|k-\kappa_j|\le\rho_1} (|\Phi(\mathrm{i}\kappa_j) - \Phi(k)| + \mathrm{Im}\, g(k)) < -C(\varepsilon) < 0. \tag{4.3}$$

Taking into account (3.1), we prove

Lemma 4.2. *The following estimate is valid uniformly with respect to $\xi \in \mathcal{I}_\varepsilon$* [2]

$$\max_j \sup_{k\in\mathbb{T}_j} |h_j(k,t)| = O(\mathrm{e}^{-C(\varepsilon)t}).$$

[2] by $C(\varepsilon)$ we will denote any positive constant with respect to k, ξ, x and t

Put now $b := a - \rho$. Recall that the smoothness of the initial data (1.8) up to the 6-th derivative implies that $R(k)$ for $k \in \mathbb{R}$ is a smooth function with $R(k) = O(k^{-4})$ as $k \to \pm\infty$ (see [12, Thm. 4.1]). From item **(c)** of Lemma 4.1, (3.1) and the signature table of $g(k)$ we conclude that the following proposition is valid:

Lemma 4.3. *Uniformly with respect to* $\xi \in \mathcal{I}_\varepsilon$

$$\|v^{(1)}(k) - \mathbb{I}\|_{L^\infty(\mathcal{C})} + \|v^{(1)}(k) - \mathbb{I}\|_{L^1(\mathcal{C})} = O(\mathrm{e}^{-C(\varepsilon)t}),$$

$$\|v^{(1)}(k) - \mathrm{e}^{\left(-\mathrm{i}tB + \mathrm{i}\frac{\pi\ell}{2}\right)\sigma_3}\|_{L^\infty([0,\mathrm{i}b])} = O(\mathrm{e}^{-C(\varepsilon)t}).$$

STEP 2. Our next conjugation step deals with a factorization of the jump matrix on the set $[\mathrm{i}c, \mathrm{i}a] \cup [-\mathrm{i}a, -\mathrm{i}c]$. To this end consider the following function $F(k) = F(k,\xi)$, $k \in \mathbb{C} \setminus [\mathrm{i}c, -\mathrm{i}c]$:
(4.4)

$$F(k) := \exp\left\{\frac{w(k)}{2\pi\mathrm{i}}\left(\int_{\mathrm{i}c}^{\mathrm{i}a} \frac{f(s)}{s-k}ds + \int_{-\mathrm{i}c}^{-\mathrm{i}a} \frac{f(s)}{s-k}ds - \mathrm{i}\Delta_F \int_{-\mathrm{i}a}^{\mathrm{i}a} \frac{ds}{w(s)(s-k)}\right)\right\},$$

where

$$w(k) = \sqrt{(k^2+c^2)(k^2+a^2)}, \quad k \in \mathbb{C} \setminus ([\mathrm{i}c, \mathrm{i}a] \cup [-\mathrm{i}a, -\mathrm{i}c]), \quad w(0) > 0,$$

$$f(k) := \frac{\log|X(k)|}{w_+(k)}, \quad k \in [\mathrm{i}c, -\mathrm{i}c], \tag{4.5}$$

and

$$\Delta_F = \Delta_F(\xi) := 2\mathrm{i}\int_{\mathrm{i}a}^{\mathrm{i}c} f(s)ds \left(\int_{-\mathrm{i}a}^{\mathrm{i}a} \frac{ds}{w(s)}\right)^{-1} \in \mathbb{R}. \tag{4.6}$$

Remark 4.4. *Putting together* (2.11), (3.5) *and* (3.3) *we conclude that* $\Delta = \Delta(\xi)$ *given by* (1.11) *and* Δ_F *given by* (4.6), (4.5) *are connected by*

$$\Delta = \Delta_F - \frac{\ell\pi}{2}. \tag{4.7}$$

Next, since $X(k) = \chi(k)Q^{-2}(k)P^{-2}(k)$ has bounded non-vanishing values at points $\pm\mathrm{i}c$, we get

Lemma 4.5 ([11],[26])**.** *The function* $F(k)$ *possesses the following properties:*

(1). $F(-k) = F^{-1}(k)$ *for* $k \in \mathbb{C} \setminus [\mathrm{i}c, -\mathrm{i}c]$*;*
(2). $F_+(k)F_-(k) = |X(k)|$ *for* $k \in [\mathrm{i}c, \mathrm{i}a]$*;*
(3). $F_+(k) = F_-(k)\mathrm{e}^{\mathrm{i}\Delta_F}$ *for* $k \in [\mathrm{i}a, -\mathrm{i}a]$*;*
(4). $F(k) \to 1$ *as* $k \to \infty$*;*
(5). $F_+(k)F_-(k) = (F_+(-k)F_-(-k))^{-1}$ *for* $k \in [-\mathrm{i}a, -\mathrm{i}c]$*;*
(6). $F(k)$ *has finite limits as* $k \to \pm\mathrm{i}c$*.*

Taking into account these properties and property (2.7) we observe that the matrix $v^{(1)}(k)$ can be factorized on $[\mathrm{i}c, \mathrm{i}a]$ as follows:

$$v^{(1)}(k) = G_-(k)\begin{pmatrix} 0 & \mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix} G_+(k)^{-1},$$

where (cf. (3.2), (3.5), (3.3), (2.6)):

$$G(k) = \begin{pmatrix} F^{-1}(k) & -\frac{F(k)\mathrm{e}^{-2\mathrm{i}tg(k)}}{X(k)} \\ 0 & F(k) \end{pmatrix}, \quad k \in \Omega_\chi.$$

Inside the domain Ω_χ introduce the subdomain Ω_1 surrounded by the contour Σ_1 oriented as depicted in Figure 3. Denote by Σ_1^* its inversion in $\mathbb{C}^-$. Define $m^{(2)}(k)$ as

$$m^{(2)}(k) := m^{(1)}(k) \begin{cases} G(k), & k \in \Omega_1, \\ (F(k))^{-\sigma_3}, & k \in \overline{\mathbb{C}^+} \setminus \overline{\Omega_1}, \\ m^{(2)}(-k)\sigma_1, & k \in \mathbb{C}^-. \end{cases}$$

Since $F(k) \to 1$ as $k \to \infty$, the normalization condition is preserved for $m^{(2)}(k)$. The correctness of its definition by symmetry in the lower half plane is due to properties of *(1), (2), (5)* of Lemma 4.5. Moreover, due to property *(6)*, (4.2) and Lemma 3.1, we have

$$m^{(2)}(k) = O(k \mp \mathrm{i}c)^{-1/4}, \text{ as } k \to \pm\mathrm{i}c; \quad m^{(2)}(k) = O(1), \text{ as } k \to \pm\mathrm{i}a;$$
$$m^{(2)}(k) = O(1), \text{ as } k \to \pm\mathrm{i}\rho.$$

Note that the set $\mathcal{G}^{(2)} = \{\pm\mathrm{i}c, \pm\mathrm{i}a, \pm\mathrm{i}\rho\}$ is the set of all node points of the R-H problem for $m^{(2)}(k)$. Taking into account property (**c**) of Lemma 4.1, property (3) of Lemma 4.5 and (4.7), we see that

$$\frac{F_-(k)}{F_+(k)}\mathrm{e}^{\mathrm{i}t(g_+(k)-g_-(k)+\mathrm{i}\ell\pi/2)} = \mathrm{e}^{-\mathrm{i}tB-\mathrm{i}\Delta}, \quad k \in [\mathrm{i}a, -\mathrm{i}a], \tag{4.8}$$

and therefore the jump matrix for $m^{(2)}(k)$ looks as follows
(4.9)

$$v^{(2)}(k) = \begin{cases} \begin{pmatrix} 0 & \mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix}, & k \in [\mathrm{i}c, \mathrm{i}a], \\ \begin{pmatrix} \frac{F_-}{F_+}\mathrm{e}^{\mathrm{i}t(g_+-g_-)+\mathrm{i}\ell\pi/2} & 0 \\ \frac{\mathrm{i}|X|}{F_+F_-}\mathrm{e}^{\mathrm{i}t(g_++g_-))} & \frac{F_+)}{F_-}\mathrm{e}^{\mathrm{i}t(g_--g_+)-\mathrm{i}\ell\pi/2} \end{pmatrix} & k \in [\mathrm{i}a, \mathrm{i}b] \\ \begin{pmatrix} \mathrm{e}^{-\mathrm{i}tB-\mathrm{i}\Delta} & 0 \\ 0 & \mathrm{e}^{\mathrm{i}tB+\mathrm{i}\Delta} \end{pmatrix} + \mathcal{A}(k,t), & k \in [\mathrm{i}b, 0], \\ \begin{pmatrix} 1 & -\frac{F^2(k)}{X(k)}\mathrm{e}^{-2\mathrm{i}tg(k)} \\ 0 & 1 \end{pmatrix}, & k \in \Sigma_1, \\ \begin{pmatrix} 1 & 0 \\ \mathcal{R}(k)F^{-2}(k)\mathrm{e}^{2\mathrm{i}tg(k)} & 1 \end{pmatrix}, & k \in \mathcal{C}, \\ \begin{pmatrix} 1 & F^2(k)h_j(k,t) \\ 0 & 1 \end{pmatrix}, & k \in \mathbb{T}_j, \ j = 1, .., N, \\ \sigma_1(v^{(2)}(-k))^{-1}\sigma_1, & k \in [0, -\mathrm{i}c] \\ \sigma_1 v^{(2)}(-k)\sigma_1, & k \in \Sigma_1^* \cup \mathcal{C}^* \cup \cup_j \mathbb{T}_j^*, \end{cases}$$

where the matrix

$$\mathcal{A}(k,t) = [F_-(k)]^{\sigma_3}\left(v^{(1)}(k) - \mathrm{e}^{(-\mathrm{i}tB+\mathrm{i}\frac{\ell\pi}{2})\sigma_3}\right)[F_-(k)]^{-\sigma_3}$$

is supported on $[\mathrm{i}b, \mathrm{i}\rho]$ and admits, according to Lemma 4.3, the estimate

$$\|\mathcal{A}(k,t)\|_{L^\infty([0,\mathrm{i}b])} = O(\mathrm{e}^{-C(\varepsilon)t}).$$

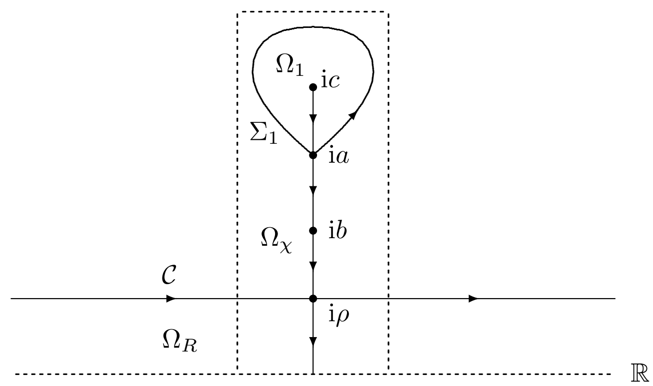

FIGURE 3. Jump contour $\Sigma^{(2)}$ in $\mathbb{C}^+$ (without the $\mathbb{T}_j$'s)

Lemma 4.3 and Lemma 4.2 together with properties *(1)* and *(4)* of Lemma 4.5 also imply

Lemma 4.6. *Uniformly with respect to* $\xi \in \mathcal{I}_\varepsilon$

$$\|v^{(2)}(k) - \mathbb{I}\|_{L^\infty(\mathcal{K})} + \|v^{(2)}(k) - \mathbb{I}\|_{L^1(\mathcal{K})} = O(\mathrm{e}^{-C(\varepsilon)t}), \ \ as \ \ t \to \infty,$$

where $\mathcal{K} = \mathcal{C} \cup \mathcal{C}^* \cup \cup_j (\mathbb{T}_j \cup \mathbb{T}_j^*)$.

Remark 4.7. *Formula* (4.8) *allows us to shorten the expression for* $v^{(2)}(k)$ *on the interval* $[\mathrm{i}a, \mathrm{i}b]$. *However, we use the form* (4.9) *of the jump matrix on* $[\mathrm{i}a, \mathrm{i}b] \cup [-\mathrm{i}b, -\mathrm{i}a]$*, because it simplifies further considerations of the local parametrix problem.*

Let $\mathcal{B}$ be a vicinity of point ia with the boundary $\partial\mathcal{B}$ satisfying

$$\frac{\rho}{2} < \operatorname{dist}(\partial\mathcal{B}, \mathrm{i}a) < 2\rho,$$

where ρ is defined by (3.1). Its precise shape will be described later in Section 7. Without loss generality one can assume that $\mathrm{i}b \in \partial\mathcal{B}$. Denote $\mathcal{B}^* = \{k : \ -k \in \mathcal{B}\}$ and the jump contour for $m^{(2)}(k)$ by

$$\Sigma^{(2)} := \mathcal{C} \cup \mathcal{C}^* \cup \Sigma_1 \cup \Sigma_1^* \cup \cup_j (\mathbb{T}_j \cup \mathbb{T}_j^*) \cup [\mathrm{i}c, -\mathrm{i}c], \tag{4.10}$$

and let

$$\Sigma_\rho = \Sigma^{(2)} \setminus (\mathcal{B} \cup \mathcal{B}^*) \tag{4.11}$$

be the part of our contour outside the small vicinities of the points $\pm\mathrm{i}a$. Put

$$v^{\mathrm{mod}}(k) = \begin{cases} \mathrm{i}\sigma_1, & k \in [\mathrm{i}c, \mathrm{i}a], \\ \mathrm{e}^{-\mathrm{i}\Lambda\sigma_3}, & k \in [\mathrm{i}a, 0], \\ \sigma_1 (v^{\mathrm{mod}}(-k))^{-1}\sigma_1, & k \in [0, -\mathrm{i}c], \\ \mathbb{I}, & k \in \Sigma^{(2)} \setminus [\mathrm{i}c, -\mathrm{i}c], \end{cases} \tag{4.12}$$

where

$$\Lambda := tB + \Delta \in \mathbb{R}. \tag{4.13}$$

The consideration above shows that uniformly with respect to $\xi \in \mathcal{I}_\varepsilon$

$$\|v^{(2)}(k) - v^{\mathrm{mod}}(k)\|_{L^\infty(\Sigma_\rho)\cap L^1(\Sigma_\rho)} = O(\mathrm{e}^{-C(\varepsilon)t}), \quad t \to \infty. \tag{4.14}$$

The matrix $v^{\mathrm{mod}}(k)$ is piecewise constant with respect to k. In the next section we study briefly the respective vector R-H problem. It was solved in [11], [13], however the uniqueness was not established there. Moreover, using the trace formula we propose here a shorter and more transparent way to compute the expansion of $m_1^{\mathrm{mod}}(k)m_2^{\mathrm{mod}}(k)$ as $k \to \infty$, which will approximate the analogous expansion for the initial R-H problem, because of

$$m_1^{(2)}(k)m_2^{(2)}(k) = m_1^{\mathrm{ini}}(k)m_2^{\mathrm{ini}}(k), \quad |\operatorname{Im} k| > \kappa_1 + \rho. \tag{4.15}$$

5. Unique solution for the vector model R-H problem

Lemma 5.1. *The following R-H problem has a unique solution: find a vector-valued function $m^{\mathrm{mod}}(k) = (m_1^{\mathrm{mod}}(k)\ m_2^{\mathrm{mod}}(k))$ holomorphic in the domain $\mathbb{C} \setminus [\mathrm{i}c, -\mathrm{i}c]$, continuous up to the boundary except for points of the set $\mathcal{G}^{\mathrm{mod}} := \{\mathrm{i}c, \mathrm{i}a, -\mathrm{i}a, -\mathrm{i}c\}$ and satisfying the jump condition:*

$$m_+^{\mathrm{mod}}(k) = m_-^{\mathrm{mod}}(k)v^{\mathrm{mod}}(k), \tag{5.1}$$

$$v^{\mathrm{mod}}(k) = \begin{cases} \begin{pmatrix} 0 & \mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix}, & k \in [\mathrm{i}c, \mathrm{i}a], \\ \begin{pmatrix} 0 & -\mathrm{i} \\ -\mathrm{i} & 0 \end{pmatrix}, & k \in [-\mathrm{i}a, -\mathrm{i}c], \\ \begin{pmatrix} \mathrm{e}^{-\mathrm{i}\Lambda} & 0 \\ 0 & \mathrm{e}^{\mathrm{i}\Lambda} \end{pmatrix}, & k \in [\mathrm{i}a, -\mathrm{i}a], \end{cases} \tag{5.2}$$

the symmetry condition

$$m^{\mathrm{mod}}(-k) = m^{\mathrm{mod}}(k) \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \tag{5.3}$$

and the normalization condition

$$\lim_{k\to\mathrm{i}\infty} m^{\mathrm{mod}}(k) = (1\ \ 1). \tag{5.4}$$

At any point $\kappa \in \mathcal{G}^{\mathrm{mod}}$ the vector function $m^{\mathrm{mod}}(k)$ can have at most a fourth root singularity: $m^{\mathrm{mod}}(k) = O((k-\kappa)^{-1/4})$, $k \to \kappa$.

Proof. To prove uniqueness, assume that m and $\hat{m}$ are two solutions of the R-H problem. Their difference $\tilde{m} = m - \hat{m}$ is a holomorphic vector in $\mathbb{C} \setminus [\mathrm{i}c, -\mathrm{i}c]$ which satisfies conditions (5.2) and (5.3) and has the following behavior

$$\tilde{m}(k) = (1, -1)\frac{\tilde{h}}{k}(1 + O(k^{-1})), \quad \text{as} \quad k \to \mathrm{i}\infty.$$

Moreover, $\tilde{m}(k) = O((k-\kappa)^{-1/4}))$ as $k \to \kappa$ for $\kappa \in \mathcal{G}^{\mathrm{mod}}$.

In $\mathbb{C} \setminus [\mathrm{i}c, -\mathrm{i}c]$, introduce a holomorphic function

$$f(k) := \tilde{m}_1(k)\overline{\tilde{m}_1(\overline{k})} + \tilde{m}_2(k)\overline{\tilde{m}_2(\overline{k})}. \tag{5.5}$$

Due to (5.3) this function is even: $f(-k) = f(k)$ and satisfies

$$f(k) = \frac{2|\tilde{h}|^2}{k^2}(1 + O(k^{-2})), \quad \text{as} \quad k \to \mathrm{i}\infty; \tag{5.6}$$

(5.7) $$f(k) = O((k-\kappa)^{-1/2})), \quad \text{as} \quad k \to \kappa, \quad \text{for} \quad \kappa \in \mathcal{G}^{\mathrm{mod}}.$$

Since $-\overline{k} = k$ for $k \in \mathrm{i}\mathbb{R}$ and taking into account (5.3), we get

$$\begin{aligned} f_+(k) &= \tilde{m}_{1,+}(k)\overline{\tilde{m}_{2,-}(k)} + \tilde{m}_{2,+}(k)\overline{\tilde{m}_{1,-}(k)}, \\ f_-(k) &= \tilde{m}_{1,-}(k)\overline{\tilde{m}_{2,+}(k)} + \tilde{m}_{2,-}(k)\overline{\tilde{m}_{1,+}(k)}, \end{aligned} \qquad k \in [\mathrm{i}c, -\mathrm{i}c].$$

By use of (5.2)

$$f_+(k) = \pm\mathrm{i}\left(|\tilde{m}_{2,-}(k)|^2 + |\tilde{m}_{1,-}(k)|^2\right) = -f_-(k) \in \mathrm{i}\mathbb{R}, \quad k \in [\pm\mathrm{i}c, \pm\mathrm{i}a],$$

(5.8)
$$f_+(k) = \mathrm{e}^{-\mathrm{i}\Lambda}\tilde{m}_{1,-}(k)\overline{\tilde{m}_{2,-}(k)} + \mathrm{e}^{\mathrm{i}\Lambda}\tilde{m}_{2,-}(k)\overline{\tilde{m}_{1,-}(k)} = f_-(k) \in \mathbb{R}, \quad k \in [\mathrm{i}a, -\mathrm{i}a].$$

Thus the function $f(k)$ has no jump on $[\mathrm{i}a, -\mathrm{i}a]$ and is the solution of the following scalar jump problem

$$f_+(k) = -f_-(k), \quad k \in [\mathrm{i}c, \mathrm{i}a] \cup [-\mathrm{i}a, -\mathrm{i}c],$$

which satisfies (5.6) and (5.7). The unique solution of this problem is given by the formula

$$f(k) = -\frac{2|\tilde{h}|^2}{\sqrt{(k^2+c^2)(k^2+a^2)}}.$$

Therefore, if $\tilde{h} \neq 0$ then $f(0) < 0$. But according to (5.5) and (5.8) we have $f_+(0) = f_-(0) \geq 0$. Thus, $\tilde{h} = 0$ and hence

$$\tilde{m}_{1,-}(k) = \tilde{m}_{1,+}(k) = \tilde{m}_{2,+}(k) = \tilde{m}_{2,-}(k) = 0, \quad k \in [\mathrm{i}c, \mathrm{i}a] \cup [-\mathrm{i}a, -\mathrm{i}c].$$

In particular, we see that the jump along $[\mathrm{i}c, \mathrm{i}a] \cup [-\mathrm{i}a, -\mathrm{i}c]$ is removable and the only solution of this problem is trivial: $\tilde{m}(k) \equiv 0$. □

Now we recall briefly how to solve problem (5.1)–(5.4) (cf. [11]). Consider the two-sheeted Riemann surface $\mathbb{X} = \mathbb{X}(\xi)$ associated with the function

$$w(k) = \sqrt{(k^2+c^2)(k^2+a^2)},$$

defined on $\mathbb{C} \setminus ([-\mathrm{i}c, -\mathrm{i}a] \cup [\mathrm{i}a, \mathrm{i}c])$ with $w(0) > 0$. The sheets of $\mathbb{X}$ are glued along the cuts $[\mathrm{i}c, \mathrm{i}a]$ and $[-\mathrm{i}a, -\mathrm{i}c]$. Points on this surface are denoted by $p = (k, \pm)$. To simplify notations we keep the notation $k = (k, +)$ for the upper sheet of $\mathbb{X}$. The canonical homology basis of cycles $\{\mathbf{a}, \mathbf{b}\}$ is chosen as follows: The **a**-cycle surrounds the points $-\mathrm{i}a, \mathrm{i}a$ starting on the upper sheet from the left side of the cut $[\mathrm{i}c, \mathrm{i}a]$ and continues on the upper sheet to the left part of $[-\mathrm{i}a, -\mathrm{i}c]$ and returns after changing sheets. The cycle **b** surrounds the points $\mathrm{i}a, \mathrm{i}c$ counterclockwise on the upper sheet. Consider the normalized holomorphic differential

(5.9) $$d\omega = \Gamma\frac{d\zeta}{w(\zeta)}, \quad \text{where } \Gamma := \left(\int_{\mathbf{a}} \frac{d\zeta}{w(\zeta)}\right)^{-1} \in \mathrm{i}\mathbb{R}_-,$$

then $\int_{\mathbf{a}} d\omega = 1$ and

(5.10) $$\tau = \tau(\xi) = \int_{\mathbf{b}} d\omega \in \mathrm{i}\mathbb{R}_+.$$

Let

$$\theta_3(z \,|\, \tau) = \sum_{n \in \mathbb{Z}} \exp\left\{(n^2\tau + 2nz)\pi\mathrm{i}\right\}, \quad z \in \mathbb{C},$$

be the Jacobi theta function. Recall that θ_3 is an even function, $\theta_3(-z\,|\,\tau) = \theta_3(z\,|\,\tau)$, and satisfies

$$\theta_3(z+n+\tau(\xi)\ell\,|\,\tau) = \theta_3(z\,|\,\tau)\exp\left\{-\pi\mathrm{i}\tau\ell^2 - 2\pi\mathrm{i}\ell z\right\} \text{ for } \ l,n\in\mathbb{Z}..$$

Furthermore, let $A(p) = \int_{\mathrm{i}c}^{p} d\omega$ be the Abel map on X. We identify the upper sheet of X with the complex plane $\mathbb{C}\setminus([\mathrm{i}c,\mathrm{i}a]\cup[-\mathrm{i}a,-\mathrm{i}c])$ with cuts, and put $(k,+)=k$. Allowing only paths of integration in $\mathbb{C}\setminus[\mathrm{i}c,-\mathrm{i}c]$ we observe that $A(k)$ is a holomorphic function in that given domain with the following properties:

- $A_+(k) = -A_-(k) \pmod 1$ for $k\in[\mathrm{i}c,\mathrm{i}a]\cup[-\mathrm{i}a,-\mathrm{i}c]$;
- $A_+(k) - A_-(k) = -\tau$ as $k\in[\mathrm{i}a,-\mathrm{i}a]$;
- $A(-k) = -A(k) + \frac{1}{2}(\mod 1)$ as $k\in\mathbb{C}\setminus[\mathrm{i}c,-\mathrm{i}c]$,
- $A_+(\mathrm{i}a) = -\frac{\tau}{2} = -A_-(\mathrm{i}a)$, $A_+(-\mathrm{i}a) = -\frac{\tau}{2}+\frac{1}{2}$, $A_-(-\mathrm{i}a) = \frac{\tau}{2}+\frac{1}{2}$.
- $A((\infty,+)) = \frac{1}{4}$; $\quad A(k) - A((\infty,+)) = -\Gamma k^{-1} + O(k^{-3})$ as $k\to\infty$.

On $\mathbb{C}\setminus[\mathrm{i}c,-\mathrm{i}c]$ introduce two functions

$$\alpha^{\Lambda}(k) = \theta_3\left(A(k) - \frac{1}{2} - \frac{\tilde\Lambda}{2}\,\Big|\,\tau\right)\theta_3\left(A(k) - \frac{\tilde\Lambda}{2}\,\Big|\,\tau\right),$$

$$\beta^{\Lambda}(k) = \theta_3\left(-A(k) - \frac{1}{2} - \frac{\tilde\Lambda}{2}\,\Big|\,\tau\right)\theta_3\left(-A(k) - \frac{\tilde\Lambda}{2}\,\Big|\,\tau\right),$$

where $\tilde\Lambda = \frac{\Lambda}{2\pi}\in\mathbb{R}$ and $A(k) = A((k,+))$ for $k\in\mathbb{C}$. The properties of the Abel integrals listed above imply that the functions $\alpha^0(k)$ and $\beta^0(k)$ have square root singularities at the points $\pm\mathrm{i}a$. Using the formula (cf. [10])

$$\theta_3\left(u\,|\,\tau\right)\,\theta_3\left(u-\frac{1}{2}\,\Big|\,\tau\right) = \theta_3\left(2u - \frac{1}{2}\,\Big|\,2\tau\right)\,\theta_3\left(\frac{1}{2}\,\Big|\,2\tau\right),$$

we can represent the functions $\alpha^{\Lambda}(k)$ and $\beta^{\Lambda}(k)$ as

$$\alpha^{\Lambda}(k) = \theta_3\left(2A(k) - \frac{1}{2} - \tilde\Lambda\,\Big|\,2\tau\right)\,\theta_3\left(\frac{1}{2}\,\Big|\,2\tau\right),$$

$$\beta^{\Lambda}(k) = \theta_3\left(-2A(k) + \frac{1}{2} - \tilde\Lambda\,\Big|\,2\tau\right)\,\theta_3\left(\frac{1}{2}\Big|\,2\tau\right).$$

Introduce the functions

$$\hat\alpha(k) := \frac{\alpha^{\Lambda}(k)}{\alpha^0(k)} = \frac{\theta_3\left(2A(k) - \frac{1}{2} - \tilde\Lambda\,\big|\,2\tau\right)}{\theta_3\left(2A(k) - \frac{1}{2}\,\big|\,2\tau\right)} \tag{5.11}$$

$$\hat\beta(k) := \frac{\beta^{\Lambda}(k)}{\beta^0(k)} = \frac{\theta_3\left(-2A(k) + \frac{1}{2} - \tilde\Lambda\,\big|\,2\tau\right)}{\theta_3\left(-2A(k) + \frac{1}{2}\,\big|\,2\tau\right)}. \tag{5.12}$$

Evidently, both functions $\hat\alpha(k)$ and $\hat\beta(k)$ have square root singularities at the points $\pm\mathrm{i}a$ if $\tilde\Lambda\notin\mathbb{Z}$. Moreover,

$$\lim_{k\to\infty}\hat\alpha(k) = \lim_{k\to\infty}\hat\beta(k) = \frac{\theta_3\left(\tilde\Lambda\,\big|\,2\tau\right)}{\theta_3\left(0\,|\,2\tau\right)}.$$

Due to the first three properties of the Abel map we get

$$\hat\alpha_+(k) = \hat\beta_-(k) \text{ and } \hat\beta_+(k) = \hat\alpha_-(k) \text{ for } k\in[\mathrm{i}c,\mathrm{i}a]\cup[-\mathrm{i}a,-\mathrm{i}c],$$

$$\hat{\alpha}_+(k) = \mathrm{e}^{-\mathrm{i}\Lambda}\hat{\alpha}_-(k) \quad \text{and} \quad \hat{\beta}_+(k) = \mathrm{e}^{\mathrm{i}\Lambda}\hat{\beta}_-(k) \text{ for } k \in [\mathrm{i}a, -\mathrm{i}a],$$
$$\hat{\alpha}(-k) = \hat{\beta}(k) \text{ for } k \in \mathbb{C} \setminus [\mathrm{i}c, -\mathrm{i}c].$$

Now introduce the function

$$\tilde{\gamma}(k) = \sqrt[4]{\frac{k^2+a^2}{k^2+c^2}}, \tag{5.13}$$

defined uniquely on the set $\mathbb{C} \setminus ([\mathrm{i}c, \mathrm{i}a] \cup [-\mathrm{i}a, -\mathrm{i}c])$ by the condition $\arg \tilde{\gamma}(0) = 0$. This function satisfies the jump conditions

$$\begin{aligned} \tilde{\gamma}_+(k) &= \mathrm{i}\tilde{\gamma}_-(k), \quad && k \in [\mathrm{i}c, \mathrm{i}a], \\ \tilde{\gamma}_+(k) &= -\mathrm{i}\tilde{\gamma}_-(k), \quad && k \in [\mathrm{i}a, -\mathrm{i}c]. \end{aligned}$$

Then the vector function

$$m^{\mathrm{mod}}(k) = \left(\tilde{\gamma}(k)\frac{\hat{\alpha}(k)}{\hat{\alpha}(\infty)}, \ \tilde{\gamma}(k)\frac{\hat{\beta}(k)}{\hat{\beta}(\infty)} \right) \tag{5.14}$$

solves problem (5.1)–(5.4).

Note that both components of the vector-valued function $m^{\mathrm{mod}}(k)$ are bounded everywhere except for small vicinities of the points of the set $\mathcal{G}^{\mathrm{mod}}$, where they have singularities of the type $(k-\kappa)^{-1/4}$, $\kappa \in \mathcal{G}^{\mathrm{mod}}$.

Remark 5.2. *We observe that*

$$\hat{\alpha}_\pm(0) = \frac{\theta_3\left(\mp\tau - 1 - \tilde{\Lambda} \,\middle|\, 2\tau\right)}{\theta_3\left(\pm\tau + 1 \,\middle|\, 2\tau\right)}, \quad \hat{\beta}_\pm(0) := \frac{\theta_3\left(\pm\tau + 1 - \tilde{\Lambda} \,\middle|\, 2\tau\right)}{\theta_3\left(\pm\tau + 1 \,\middle|\, 2\tau\right)}.$$

This means that for $\tilde{\Lambda} = \frac{1}{2} \pmod n$ we have $m_\pm^{\mathrm{mod}}(0) = (0,0)$. From Theorem 3.3 it follows then that for $\Lambda = 2\pi\tilde{\Lambda} = \pi(2n+1)$, $n \in \mathbb{Z}$ the matrix model R-H problem associated with the jump (5.2) does not have an invertible solution.

Remark 5.3. *For $\tilde{\Lambda} \in \mathbb{Z}$ we have $\hat{\alpha}(\pm\mathrm{i}a) = \hat{\beta}(\pm\mathrm{i}a) = 1$. By (5.13), (5.14), therefore:*

$$m^{\mathrm{mod}}(\pm\mathrm{i}a) = (0 \ \ 0), \quad \textit{as} \quad \Lambda = 2\pi n.$$

Thus the points (x,t) for which $\tilde{\Lambda} \in \mathbb{Z}$ are those points where the vector model solution does not have singularities at the points $\pm\mathrm{i}a$. However, the matrix model solution will have fourth order singularities at $\pm\mathrm{i}a$ for these pairs (x,t).

Recall now that we constructed the solution for the jump problem (5.2) with $\tilde{\Lambda} = \frac{\Lambda}{2\pi}$ and Λ given by formula (4.13). Due to (5.3), the asymptotic expansion of the vector components product should be the following:

$$m_1^{\mathrm{mod}}(k)m_2^{\mathrm{mod}}(k) = 1 + \frac{q^{\mathrm{mod}}(x,t,\xi)}{2k^2} + O(k^{-4}). \tag{5.15}$$

Let us show that in fact for any fixed ξ coefficient, $q^{\mathrm{mod}}(x,t,\xi)$ represents the classical one-gap solution for the KdV equation associated with the spectrum $\mathfrak{G}(\xi)$ (cf. (1.4)) and with the initial Dirichlet divisor p_0 defined uniquely by the Jacobi inversion (compare (1.10), (1.11)):

$$\int_{-a^2}^{p_0} d\hat{\omega} = \mathrm{i}\Delta, \quad p_0 = (\lambda(0,0,\xi), \pm). \tag{5.16}$$

Here $d\hat\omega$ is the normalized holomorphic Abel differential of the first kind on the elliptic Riemann surface $\mathbb{M} = \mathbb{M}(\xi)$ associated with the function

$$\mathcal{R}(\lambda, \xi) = \sqrt{\lambda(\lambda + c^2)(\lambda + a(\xi)^2)},$$

with cuts along the spectrum. Let $\hat{\mathbf{b}}$, $\hat{\mathbf{a}}$ be the canonical basis on $\mathbb{M}$, where the cycle $\hat{\mathbf{b}}$ surrounds the interval $[-c^2, -a^2]$ counterclockwise on the upper sheet and the cycle $\hat{\mathbf{a}}$ supplements $\hat{\mathbf{b}}$ by passing along the gap $[-a^2, 0]$ in the positive direction on the lower sheet and then changing the sheet. The normalization for $d\hat\omega$ is given by formula $\int_{\hat{\mathbf{a}}} d\hat\omega = 2\pi\mathrm{i}$. Denote $\int_{\hat{\mathbf{b}}} d\hat\omega = \hat\tau$. Then it is straightforward to check that $\hat\tau = 4\pi\mathrm{i}\tau$ (cf. (5.10)).

Furthermore, let $\hat A(p) := \int_\infty^p d\hat\omega$ be the associated Abel map and

$$\mathcal{K} := -\hat A(-a^2) = -\frac{\hat\tau}{2} + \pi\mathrm{i}$$

be the Riemann constant. Introduce the wave and frequency numbers $V = V(\xi)$ and $W = W(\xi)$ ([22], [25]), which are $\hat{\mathbf{b}}$ - periods of the normalized Abelian differentials of the second kind $d\Omega_1$ and $d\Omega_3$ on $\mathbb{M}$, uniquely defined by the order of the pole at infinity

$$d\Omega_1 = \frac{\mathrm{i}}{2\sqrt{\lambda}}(1 + O(\lambda^{-1}))d\lambda, \quad d\Omega_3 = -\frac{3\mathrm{i}}{2}\sqrt{\lambda}(1 + O(\lambda^{-1}))d\lambda, \quad \lambda \to \infty,$$

and by the normalization conditions $\int_{\hat{\mathbf{a}}} d\Omega_{1,3} = 0$. Thus,

$$\mathrm{i}V := \int_{\hat{\mathbf{b}}} d\Omega_1, \qquad \mathrm{i}W := \int_{\hat{\mathbf{b}}} d\Omega_3.$$

The following result is obtained in [13].

Lemma 5.4. *Let $B = B(\xi)$ be as in Lemma 4.1, (**c**) and $\Gamma = \Gamma(\xi)$ be given by (5.9). Then the following identities hold*

$$tB = Vx - 4Wt, \quad 4\pi\mathrm{i}\Gamma = -V.$$

Recall now that the one-gap solution corresponding to the spectrum $\mathfrak{G}(\xi)$ and to the initial divisor (5.16), can be expressed by the trace formula:

$$q^{\mathrm{per}}(x, t, \xi) = -c^2 - a^2 - 2\lambda(x, t, \xi), \tag{5.17}$$

where $\lambda(x, t) = \lambda(x, t, \xi) \in [-a^2, 0]$ is the projection of $p(x, t) = (\lambda(x, t), \pm) \in \mathbb{M}$, which is the unique solution of the Jacobi inversion problem

$$\int_{p_0}^{p(x,t)} d\hat\omega = \mathrm{i}(Vx - 4Wt) \pmod{2\pi\mathrm{i}}. \tag{5.18}$$

We can also represent it as

$$\int_{-a^2}^{p(x,t)} d\hat\omega = \mathrm{i}(Vx - 4Wt + \Delta).$$

Evidently $\lambda(x, t) = 0$ corresponds to the local minimum of $q^{\mathrm{per}}(x, t)$. Indeed,

$$\lambda(x, t) = 0 \quad \textit{iff} \quad \tfrac{Vx - 4Wt + \Delta}{2\pi} = \tfrac{1}{2} \pmod{\mathbb{Z}}.$$

To compare $q^{\mathrm{per}}(x, t, \xi)$ with the second term of the expansion for the product $m_1^{\mathrm{mod}}(k) m_2^{\mathrm{mod}}(k) =: p(k)$, which is given by formula (see (5.11), (5.12))

$$p(k) = \tilde\gamma^2(k) \frac{\theta_3(2A(k) - \frac12 - \tilde\Lambda)\, \theta_3(-2A(k) + \frac12 - \tilde\Lambda)\, \theta_3(0)^2}{(\theta_3(2A(k) - \frac12))^2 (\theta_3(\tilde\Lambda))^2},$$

we first prove

Lemma 5.5. *The function $p(k)$, $k\in\mathbb{C}$, admits the following representation:*

$$p(k)=\frac{k^2-\lambda(x,t)}{\sqrt{(k^2+a^2)(k^2+c^2)}}. \tag{5.19}$$

Proof. Given (5.11) and (5.12), consider the function

$$\tilde p(k)=p(k)\tilde\gamma^{-2}(k)=\frac{\hat\alpha(k)\hat\beta(k)}{\hat\alpha(\infty)\hat\beta(\infty)}.$$

By the symmetry property we have $\tilde p(-k)=\tilde p(k)$. Moreover, this function does not have jumps for $k\in[-\mathrm{i}c,\mathrm{i}c]$, and $\tilde p(k)\to 1$ as $k\to\infty$. Thus, it must be a meromorphic (in fact, rational) function of $\lambda=k^2$ in the whole complex plane. Due to (5.16) and (5.18) the function $\hat\alpha(k)\hat\beta(k)$ has the only zero, simple with respect to λ, at the point $\lambda=\lambda(x,t)$, and the only simple pole (again with respect to λ) at $\lambda=-a^2$. We conclude that

$$\tilde p(k)=\frac{\hat\alpha(k)\hat\beta(k)}{\hat\alpha(\infty)\hat\beta(\infty)}=\frac{k^2-\lambda(x,t)}{k^2+a^2},$$

which together with (5.13) implies (5.19). □

In turn, decomposing (5.19) with respect to $\frac{1}{2k^2}$ we get the same trace formula (5.17) for $q^{\mathrm{mod}}(x,t,\xi)$ in (5.15). It proves that

$$q^{\mathrm{mod}}(x,t,\xi)=q^{\mathrm{per}}(x,t,\xi). \tag{5.20}$$

Moreover, the property of the combination of theta functions in $m_1^{\mathrm{mod}}(k)m_2^{\mathrm{mod}}(k)$ to be a rational function of the spectral parameter $\lambda=k^2$ is tightly connected with the analogous property of the product of two branches of the Baker–Akhiezer function. It allows us to expect that this approach may considerably simplify the evaluation of asymptotics in the case of finite gap backgrounds.

6. The matrix model R-H problem solution and its properties

In this section we propose a proper matrix model solution with a nonintegrable singularity at the point $k=0$.

Theorem 6.1. *There exists a matrix model solution $M^{\mathrm{mod}}(k)$ of the model R-H jump problem which satisfies the following conditions:*

(1) It is holomorphic in $\mathbb{C}\setminus[\mathrm{i}c,-\mathrm{i}c]$, continuous up to the sides of the contour $[\mathrm{i}c,-\mathrm{i}c]$ except at the points $\mathcal{G}^{\mathrm{mod}}\cup\{0\}$;

(2) At points of $\mathcal{G}^{\mathrm{mod}}$ it has weak singularities, $M^{\mathrm{mod}}(k)=O((k-\kappa)^{-1/4})$ as $k\to\kappa\in\mathcal{G}^{\mathrm{mod}}$, and $M^{\mathrm{mod}}(k)=O(k^{-1})$ as $k\to 0$,[3]

(3) It possesses the symmetry condition:

$$M^{\mathrm{mod}}(-k)=\sigma_1 M^{\mathrm{mod}}(k)\sigma_1; \tag{6.1}$$

(4) It satisfies the normalization condition

$$M(k)\to\mathbb{I},\quad k\to\infty. \tag{6.2}$$

(5) $\det M^{\mathrm{mod}}(k)=1$ *for all* $k\in\mathbb{C}$*;*

[3]we can not call it a pole, because the matrix has a jump in this point

(6) The vector $m^{(2)}(k)[M^{\mathrm{mod}}(k)]^{-1}$ is a holomorphic function in a vicinity $\mathcal{O}$ of the point $k=0$;

For the proof of this theorem we will need the following

Lemma 6.2. *There exists a vector solution $\nu(k)=(\nu_1(k),\nu_2(k))$ to the model R-H problem (5.2) which satisfies the following properties:*

- *The symmetry condition $\nu_1(k)=\nu_2(-k)$, $k\in\mathbb{C}\setminus[\mathrm{i}c,-\mathrm{i}c]$;*
- *The asymptotical behavior:*

$$\nu(k)=\mathrm{i}k(-1,1)(1+O(k^{-1})),\quad k\to\infty. \tag{6.3}$$

- *The vector function $\nu(k)$ is holomorphic in $\mathbb{C}\setminus[\mathrm{i}c,-\mathrm{i}c]$, continuous up to the boundary except of points $\mathcal{G}^{\mathrm{mod}}$, where fourth root singularities are admissible.*

Proof. From Lemma 5.4 it follows that the vector ν solves the jump problem $\nu_+(k)=\nu_-(k)v^{\mathrm{mod}}(k)$ where

$$v^{\mathrm{mod}}(k)=v^{\mathrm{mod}}(k,x,t,\xi)=\begin{cases}\mathrm{i}\sigma_1, & k\in[\mathrm{i}c,\mathrm{i}a],\\ -\mathrm{i}\sigma_1, & k\in[-\mathrm{i}a,-\mathrm{i}c],\\ \mathrm{e}^{(4\mathrm{i}W(\xi)t-\mathrm{i}V(\xi)x-\mathrm{i}\Delta(\xi))\sigma_3}, & k\in[\mathrm{i}a,-\mathrm{i}a].\end{cases} \tag{6.4}$$

From formulas (2.3) and (2.6) of [13] it follows that:

$$\begin{aligned}\mathrm{i}V(\xi)\ &=Z_+(k)-Z_-(k),\ \text{for } k\in[\mathrm{i}a,-\mathrm{i}a],\\ Z(k):=Z(k,\xi)&=\mathrm{i}\int_{\mathrm{i}c}^{k}\frac{(s^2-h)ds}{\sqrt{(s^2+c^2)(s^2+a^2)}},\\ h:=&\int_{\mathrm{i}a}^{0}\frac{s^2ds}{\sqrt{(s^2+c^2)(s^2+a^2)}}\left(\int_{\mathrm{i}a}^{0}\frac{ds}{\sqrt{(s^2+c^2)(s^2+a^2)}}\right)^{-1}.\end{aligned}$$

Recall also that

$$Z_+(k)+Z_-(k)=0\pmod{2\pi\mathrm{i}},\quad k\in[\mathrm{i}c,\mathrm{i}a]\cup[-\mathrm{i}a,-\mathrm{i}c].$$

In fact

$$Z(k)=\mathrm{i}\int_{-c^2}^{k^2}\frac{\lambda-h}{2\mathcal{R}(\lambda)}d\lambda$$

is the classical quasimomentum associated with the Riemann surface $\mathbb{M}(\xi)$. Thus,

$$v^{\mathrm{mod}}(k,x,t,\xi)=\mathrm{e}^{(4\mathrm{i}W(\xi)t+(Z_+(k)-Z_-(k))x-\mathrm{i}\Delta(\xi))\sigma_3},\quad k\in[\mathrm{i}a,-\mathrm{i}a].$$

We see that the vector

$$\mathcal{S}(k):=\mathcal{S}(k,x,t,\xi)=m^{\mathrm{mod}}(k,x,t,\xi)\mathrm{e}^{-Z(k,\xi)x\,\sigma_3} \tag{6.5}$$

solves the jump problem $\mathcal{S}_+(k)=\mathcal{S}_-(k)v^{\mathcal{S}}(k)$,

$$v^{\mathcal{S}}(k)=v^{\mathcal{S}}(k,t,\xi)=\begin{cases}\mathrm{i}\sigma_1, & k\in[\mathrm{i}c,\mathrm{i}a],\\ -\mathrm{i}\sigma_1, & k\in[-\mathrm{i}a,-\mathrm{i}c],\\ \mathrm{e}^{(4\mathrm{i}W(\xi)t-\mathrm{i}\Delta(\xi))\sigma_3}, & k\in[\mathrm{i}a,-\mathrm{i}a].\end{cases}$$

Let us treat the variables x, t, ξ as independent variables. Then $\frac{\partial}{\partial x}v^{\mathcal{S}}(k) = 0$ and the vector

$$\hat{S}(k) = \frac{\partial}{\partial x}\mathcal{S}(k,x,t,\xi)$$
$$= \left(\left(\frac{\partial}{\partial x}m_1^{\mathrm{mod}}(k) - Z(k)m_1^{\mathrm{mod}}(k)\right)\mathrm{e}^{-Z(k)x}, \ \ \left(\frac{\partial}{\partial x}m_2^{\mathrm{mod}}(k) + Z(k)m_2^{\mathrm{mod}}(k)\right)\mathrm{e}^{Z(k)x}\right)$$

solves the same jump problem as $\mathcal{S}(k)$:

$$\hat{S}_+(k) = \hat{S}_-(k)v^{\mathcal{S}}(k).$$

Reversing the conjugation step (6.5) applied to the vector $\hat{S}$, we conclude that the vector

$$\begin{aligned}\nu(k) &:= \hat{S}(k)\mathrm{e}^{Z(k)x\,\sigma_3}\\ &= \left(\frac{\partial}{\partial x}m_1^{\mathrm{mod}}(k) - Z(k)m_1^{\mathrm{mod}}(k), \ \ \frac{\partial}{\partial x}m_2^{\mathrm{mod}}(k) + Z(k)m_2^{\mathrm{mod}}(k)\right),\end{aligned}$$

solves the model R-H problem (6.4), which is the same as (5.2).

Next, since $Z(k) = \mathrm{i}k(1+O(k^{-1})$ as $k \to \infty$, it is easy to see that (6.3) is fulfilled. The singularities of $\nu(k)$ at the points of $\mathcal{G}^{\mathrm{mod}}$ are the same as for $m^{\mathrm{mod}}(k)$. This follows from formulas (5.11)–(5.14) and the fact, that the differentiation $\frac{\partial}{\partial x}m^{\mathrm{mod}}(k)$ does not affect the part of the denominators in (5.14), which are responsible for singularities, for example

$$\frac{\partial}{\partial x}m_1^{\mathrm{mod}}(k) = \tilde{\gamma}(k)\frac{V(\xi)}{2\pi}\frac{\theta_3(0\,|\,2\tau)}{\theta_3(2A(k)-\frac{1}{2}\,|\,2\tau)}\ \frac{d}{d\tilde{\Lambda}}\left(\frac{\theta_3\left(2A(k)-\frac{1}{2}-\tilde{\Lambda}\,|\,2\tau\right)}{\theta_3\left(\tilde{\Lambda}\,|\,2\tau\right)}\right),$$

because $\frac{\partial\tilde{\Lambda}}{\partial x} = \frac{V(\xi)}{2\pi}$. □

Corollary 6.3. *The vector function* $\tilde{\nu}(k) := \frac{\nu(k)}{\mathrm{i}k}$ *solves the jump condition* (5.2) *and satisfies the antisymmetry condition*

$$\tilde{\nu}_1(-k) = -\tilde{\nu}_2(k), \tag{6.6}$$

with normalization

$$\tilde{\nu}(k) \to (-1 \ \ 1), \quad k \to \infty. \tag{6.7}$$

It is holomorphic outside the contour $[\mathrm{i}c, -\mathrm{i}c]$*, has fourth root singularities at points of* $\mathcal{G}^{\mathrm{mod}}$ *and a singularity* $\tilde{\nu}(k) = O(k^{-1})$ *as* $k \to 0$.

Proof of theorem 6.1. Set

$$M^{\mathrm{mod}}(k) := \frac{1}{2}\begin{pmatrix} m_1^{\mathrm{mod}}(k) - \tilde{\nu}_1(k) & m_2^{\mathrm{mod}}(k) - \tilde{\nu}_2(k)\\ m_1^{\mathrm{mod}}(k) + \tilde{\nu}_1(k) & m_2^{\mathrm{mod}}(k) + \tilde{\nu}_2(k)\end{pmatrix},$$

which evidently solves the model jump problem (5.1). Equalities (5.3) and (6.6) guarantees the structure

$$M^{\mathrm{mod}}(k) = \frac{1}{2}\begin{pmatrix}\psi_1(k) & \psi_2(k)\\ \psi_2(-k) & \psi_1(-k)\end{pmatrix},$$

and, therefore (6.1). Equality (6.2) follows from (6.7). Singularities described by item *(2)* are evident.

Let us discuss the invertibility of $M^{\mathrm{mod}}(k)$. Put $s(k) := \det M^{\mathrm{mod}}(k)$. Computing it, we get

$$s(k) = \frac{m_1(k)\nu_2(k) - \nu_1(k)m_2(k)}{2\mathrm{i}k},$$

where $\nu(k)$ is defined in Lemma 6.2. Evidently, $s(k)$ does not have jumps. It is meromorphic with the only possible pole at $k = 0$, and bounded at infinity: $\lim_{k\to\infty} s(k) = 1$. Thus, we get $s(k) = 1 + \frac{C}{k}$, where C is a constant. But due to (5.3) we know that it is even: $s(-k) = s(k)$, implying $C = 0$ and $\det M^{\mathrm{mod}}(k) \equiv 1$. This proves item *(5)*.

It remains to prove item *(6)*. We have

$$\begin{aligned}\left[M^{\mathrm{mod}}(k)\right]^{-1} &= \frac{1}{2}\begin{pmatrix} \psi_1(-k) & -\psi_2(k) \\ -\psi_2(-k) & \psi_1(k)\end{pmatrix} \\ &= \frac{1}{2}\begin{pmatrix} m_1^{\mathrm{mod}}(-k) - \tilde{\nu}_1(-k) & -m_2^{\mathrm{mod}}(k) + \tilde{\nu}_2(k) \\ -m_1^{\mathrm{mod}}(k) - \tilde{\nu}_1(k) & m_2^{\mathrm{mod}}(-k) + \tilde{\nu}_2(-k)\end{pmatrix}.\end{aligned}$$

Define $f(k) := m^{(2)}(k)[M^{\mathrm{mod}}(k)]^{-1}$, $k \in \mathcal{O}$, where $\mathcal{O}$ is a small vicinity of the point $k = 0$ with $\operatorname{diam}\mathcal{O} < \rho$ (that is, $\mathcal{O}$ is situated inside the strip between $\mathcal{C}$ and $\mathcal{C}^*$). Since $f(k)$ does not have jumps in $\mathcal{O}$, and we have the symmetry $f(-k) = f(k)\sigma_1$, it is sufficient to prove that

Lemma 6.4. *$f_1(k)$ has a removable singularity at the point $k = 0$.*

Proof. The singularity at the point $k = 0$ is at most a simple pole for f_1. To show that it is removable, it is sufficient to prove that $f_1(k) = o(k^{-1})$ from any fixed direction. As an appropriate direction we take the real positive ray $k > 0$. We use the trivial fact that if $k \to 0$ then $-k \to 0$.

To simplify notation, put $\tilde{m}(k) = m^{(2)}(k)$, $m(k) = m^{\mathrm{mod}}(k)$. Then

$$\tilde{m}_1(k) \to \tilde{m}_{1,+}(0), \quad \tilde{m}_1(-k) \to \tilde{m}_{1,-}(0), \quad \nu_1(k) \to \nu_{1,+}(0), \quad \nu_1(-k) \to \nu_{1,-}(0),$$

and

$$\tilde{m}_{1,+}(0)\nu_{1,-}(0) = \tilde{m}_{1,-}(0)\nu_{1,+}(0),$$

because the jump in $\mathcal{O}$ for the model R-H problem is diagonal, and satisfied by $m^{(2)}$ and ν. Due to symmetries,

$$\psi_2(-k) = m_1(k) + \frac{\nu_1(k)}{\mathrm{i}k},$$

thus

$$\begin{aligned} f_1(k) &= \frac{1}{2}\left(\tilde{m}_1(k)\psi_1(-k) - \tilde{m}_2(k)\psi_2(-k)\right) \\ &= \frac{1}{2\mathrm{i}k}\left(\nu_1(-k)\tilde{m}_1(k) - \nu_2(-k)\tilde{m}_2(k)\right) + O(1), \quad k \to 0.\end{aligned}$$

But

$$\begin{aligned}&\nu_1(-k)\tilde{m}_1(k) - \nu_2(-k)\tilde{m}_2(k) = \nu_1(-k)\tilde{m}_1(k) - \nu_1(k)\tilde{m}_1(-k) \to \\ &\tilde{m}_{1,+}(0)\nu_{1,-}(0) - \tilde{m}_{1,-}(0)\nu_{1,+}(0) = 0, \quad k \to 0, \quad k \in \mathbb{R}_+.\end{aligned}$$

□

Corollary 6.5. *The vector $m^{(2)}(k)[M^{\mathrm{mod}}(k)]^{-1}$ is a holomorphic function in $\mathcal{O}$.*

This proves Theorem 6.1.

7. The matrix solution of the parametrix problem

In this section we study the matrix solutions of the local R-H problems in vicinities of the points $\pm\mathrm{i}a$. Consider first the point $\mathrm{i}a$. Let $\mathcal{B}$ be a vicinity of this point as described at the end of Section 4. Introduce in it a local change of variables

$$w^{3/2}(k) = -\frac{3\mathrm{i}t}{2}(g(k) - g_\pm(\mathrm{i}a)), \quad k \in \mathcal{B}, \tag{7.1}$$

with the cut along the interval $J := [\mathrm{i}c, \mathrm{i}a] \cap \overline{\mathcal{B}}$. We observe that

$$w^{3/2}(k) = P(a)\mathrm{e}^{\frac{3\pi\mathrm{i}}{4}} t(k - \mathrm{i}a)^{3/2}(1 + O(k - \mathrm{i}a)), \quad P(a) > 0. \tag{7.2}$$

Indeed, from (4.1) and Lemma 4.1 it follows that for $\mathrm{i}s \to \mathrm{i}a \pm 0$

$$\begin{aligned}\mathrm{Re}(-\mathrm{i}g(\mathrm{i}s)) &= 12\int_{a\pm 0}^{s}\left(\frac{c^2-a^2}{2} + \xi - s^2\right)\sqrt{\frac{a+s}{c^2-s^2}}\sqrt{a-s}\,ds \\ &= -8\left(\frac{c^2-3a^2}{2} + \xi\right)\sqrt{\frac{2a}{c^2-a^2}}(a-s)^{3/2}(1+O(a-s)).\end{aligned}$$

Since $a(\xi)$ is a monotonous function with $a(\frac{c^2}{3}) = c$ and $a(-\frac{c^2}{2}) = 0$, this implies (7.2) with $P(a) > 0$. Thus, $w(k)$ is a holomorphic function in $\mathcal{B}$ with $w(\mathrm{i}a) = 0$, $w'(k) \neq 0$.

Till now we did not specify a particular shape of the boundary $\partial\mathcal{B}$ and the shape of the contour Σ_1 inside $\mathcal{B}$ (cf. Figure 3). Treating $w(k)$ as a conformal map, let us think of $\mathcal{B}$ as a preimage of a disc $\mathcal{O}$ of radius $P^{2/3}(a)\rho t^{2/3}$ centred at the origin. Since $w(k) = P_1(a)t^{2/3}(\mathrm{i}k+a)(1+o(1))$, the function $w(k)$ maps the interval $[\mathrm{i}a, \mathrm{i}c] \cap \mathcal{B}$ into the negative half axis. We can always choose the contours $\Sigma_1 \cap \mathcal{B}$ to be contained in the preimage of the rays $\arg w = \pm\frac{2\pi\mathrm{i}}{3}$.

Next, in $\mathcal{B}$ introduce the function

$$r(k) := \frac{\sqrt{X(k)}}{F(k)}\,\mathrm{e}^{\mp\frac{\mathrm{i}\pi}{4}}\,\mathrm{e}^{\frac{\mp\mathrm{i}tB}{2}}, \quad k \in \mathcal{B} \cap \{k : \pm\mathrm{Re}\,k > 0\},$$

where X and F are defined by (3.5) and (4.4) respectively, and $B = -2g_+(\mathrm{i}a)$. By (3.6) and Lemma 4.5 we conclude that

$$r_+(k) = \frac{\sqrt{|\chi(k)|}}{F_+(k)}\,\mathrm{e}^{-\frac{\mathrm{i}tB}{2}}, \quad r_-(k) = \frac{\sqrt{|\chi(k)|}}{F_-(k)}\,\mathrm{e}^{\frac{\mathrm{i}tB}{2}}, \quad k \in [\mathrm{i}c, 0] \cap \mathcal{B}.$$

Therefore,

$$r_+(k)r_-(k) = 1, \quad k \in J; \quad r_+(k) = r_-(k)\mathrm{e}^{-\mathrm{i}\Delta - \mathrm{i}tB}, \quad k \in J', \tag{7.3}$$

where we defined

$$J := [\mathrm{i}c, \mathrm{i}a] \cap \overline{\mathcal{B}}, \quad J' := [\mathrm{i}a, \mathrm{i}b] = [\mathrm{i}a, 0] \cap \overline{\mathcal{B}}.$$

Denote also

$$\mathcal{L}_1 := \Sigma_1 \cap \overline{\mathcal{B}} \cap \{\mathrm{Re}\,k \geq 0\}; \quad \mathcal{L}_2 := \Sigma_1 \cap \overline{\mathcal{B}} \cap \{\mathrm{Re}\,k \leq 0\}.$$

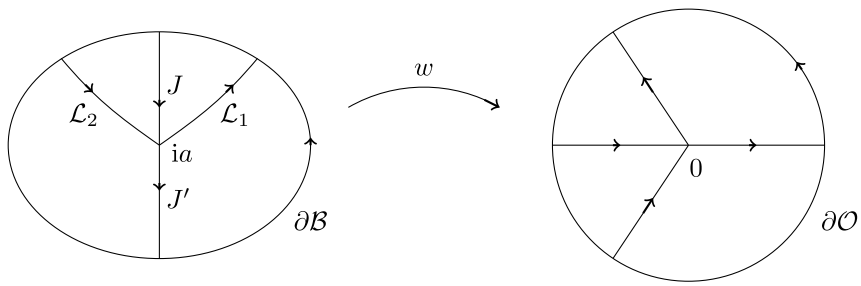

FIGURE 4. The local change of variables $w(k)$.

Recall that the vector function $m^{(2)}(k)$ satisfies the jump condition $m_+^{(2)}(k) = m_-^{(2)}(k)v^{(2)}(k)$, with the jump matrix (4.9). Redefine now $m^{(2)}(k)$ inside the domains $\mathcal{B}$ and $\mathcal{B}^*$ by the formula

$$m^{(3)}(k) = \begin{cases} m^{(2)}(k)[r(k)]^{-\sigma_3}, & k \in \mathcal{B}, \\ m^{(3)}(-k)\sigma_1, & k \in \mathcal{B}^*, \\ m^{(2)}(k), & k \in \mathbb{C} \setminus (\overline{\mathcal{B}} \cup \overline{\mathcal{B}^*}). \end{cases} \tag{7.4}$$

Using (7.3) we get $m_+^{(3)}(k) = m_-^{(3)}(k)v^{(3)}(k)$ with

$$v^{(3)}(k) = \begin{cases} \begin{pmatrix} 1 & 0 \\ \mathrm{i}e^{-4/3w(k)^{3/2}} & 1 \end{pmatrix}, & k \in J', \\ \mathrm{i}\sigma_1, & k \in J, \\ \begin{pmatrix} 1 & \mathrm{i}e^{4/3w(k)^{3/2}} \\ 0 & 1 \end{pmatrix}, & k \in \mathcal{L}_1, \\ \begin{pmatrix} 1 & -\mathrm{i}e^{4/3w(k)^{3/2}} \\ 0 & 1 \end{pmatrix}, & k \in \mathcal{L}_2, \\ r(k)^{-\sigma_3}, & k \in \partial\mathcal{B}, \\ \sigma_1[v^{(3)}(-k)]\sigma_1, & k \in \partial\mathcal{B}^* \cup \Sigma_{\mathcal{B}}^*, \\ v^{(2)}(k), & k \in \Sigma^{(2)} \setminus (\Sigma_{\mathcal{B}}^* \cup \Sigma_{\mathcal{B}}), \end{cases} \tag{7.5}$$

where $\Sigma^{(2)}$ is defined by (4.10) and

$$\Sigma_{\mathcal{B}} := J \cup J' \cup \mathcal{L}_1 \cup \mathcal{L}_2, \quad \Sigma_{\mathcal{B}}^* := \{k : -k \in \Sigma_{\mathcal{B}}\}, \tag{7.6}$$

are orientated as in Figure 7. In particular, $\partial\mathcal{B}^*$ should be oriented counterclockwise.

We observe that transformation (7.4) applied in $\mathcal{B}$ to the matrix model solution,

$$M(k) := M^{\mathrm{mod}}(k)[r(k)]^{-\sigma_3}, \quad k \in \mathcal{B} \setminus \Sigma_{\mathcal{B}}, \tag{7.7}$$

wipes out the jump along J', i.e. in $\mathcal{B}$ the matrix M satisfies the jump condition $M_+(k) = \mathrm{i}M_-(k)\sigma_1$, $k \in J$. Next by (7.1), the function $w^{1/4}(k)$ has the following jump along the interval J:

$$w_+^{1/4}(k) = w_-^{1/4}(k)\mathrm{i}, \quad k \in J.$$

Recall that $\mathcal{O} = w(\mathcal{B})$. It is now straightforward to check that the matrix

$$N(w) = \frac{1}{\sqrt{2}} \begin{pmatrix} w^{1/4} & w^{1/4} \\ -w^{-1/4} & w^{-1/4} \end{pmatrix}, \quad w \in \overline{\mathcal{O}},$$

solves the jump problem

$$N_+(w(k)) = \mathrm{i}N_-(w(k))\sigma_1, \quad k \in J.$$

Therefore, in $\mathcal{B}$ we have $M(k) = H(k)N(w(k))$, where $H(k)$ is a holomorphic matrix function in $\mathcal{B}$. Moreover, since $\det N(w) = \det[r(k)^{\sigma_3}] = 1$, we have

$$\det H(k) = \det M^{\mathrm{mod}}(k) = \det M(k). \tag{7.8}$$

According to (7.7) we get then

$$M^{\mathrm{mod}}(k) = H(k)N(w(k))r(k)^{\sigma_3}, \quad k \in \partial\mathcal{B}. \tag{7.9}$$

Next, by property (**b**) of Lemma 4.1 $w_+(k)^{3/2} = -w_-(k)^{3/2}$, $k \in J$, that is

$$v^{(3)}(k) = d_-(k)^{\sigma_3}\mathcal{S}\, d_+(k)^{-\sigma_3}, \quad k \in \mathcal{B},$$

where

$$d(k) := \tilde{d}(w(k)), \quad \tilde{d}(w) = \mathrm{e}^{2/3w^{3/2}},$$

and

$$\mathcal{S} = \begin{cases} \mathrm{i}\sigma_1, & k \in J, \\ \begin{pmatrix} 1 & 0 \\ \mathrm{i} & 1 \end{pmatrix}, & k \in J', \\ \begin{pmatrix} 1 & \mathrm{i} \\ 0 & 1 \end{pmatrix}, & k \in \mathcal{L}_1, \\ \begin{pmatrix} 1 & -\mathrm{i} \\ 0 & 1 \end{pmatrix}, & k \in \mathcal{L}_2. \end{cases}$$

Let us consider the constant matrix $\mathcal{S}$ as the jump matrix on the contour $\Gamma := w(\Sigma_{\mathcal{B}})$ (see (7.6)). Let $\mathcal{A}(w)$ be the matrix solution of the jump problem

$$\mathcal{A}_+(w) = \mathcal{A}_-(w)\mathcal{S}, \quad w \in \Gamma,$$

satisfying the boundary condition

$$\mathcal{A}(w) = N(w)\Psi(w)\tilde{d}(w)^{\sigma_3}, \quad w \in \partial\mathcal{O}, \quad t \to \infty,$$

where

$$\Psi(w) = \mathbb{I} + \frac{C}{w^{3/2}}(1 + O(w^{-3/2})), \quad w \to \infty,$$

is an invertible matrix and C is a constant matrix with respect to w, t and ξ. The solution $\mathcal{A}(w)$ can be expressed via the Airy functions and their derivatives in a standard way (see, for example, [8], [4] Chapter 3, [16] or [1]). In particular,

$$C = \frac{1}{48}\begin{pmatrix} -1 & 6 \\ -6 & 1 \end{pmatrix}, \tag{7.10}$$

and in the domain between the contours $w(J')$ and $w(\mathcal{L}_1)$ we have

$$\mathcal{A}(w) = \sqrt{2\pi}\begin{pmatrix} -y_1'(w) & \mathrm{i}y_2'(w) \\ -y_1(w) & \mathrm{i}y_2(w) \end{pmatrix},$$

where $y_1(w) = \mathrm{Ai}(w)$ and $y_2(w) = \mathrm{e}^{-\frac{2\pi\mathrm{i}}{3}}\mathrm{Ai}(\mathrm{e}^{-\frac{2\pi\mathrm{i}}{3}}w)$. The precise formula for $\mathcal{A}(w)$ in the other domains can be obtained by simple multiplication of the jump matrix $\mathcal{S}$, but it is not important for us.

Define the matrix

$$M^{\mathrm{par}}(k) := H(k)\mathcal{A}(w(k))d(k)^{-\sigma_3}, \quad k \in \mathcal{B} \setminus \Sigma_{\mathcal{B}}.$$

This matrix then solves in $\mathcal{B}$ the jump problem

$$M_+^{\mathrm{par}}(k) = M_-^{\mathrm{par}}(k)v^{(3)}(k), \quad k \in \Sigma_{\mathcal{B}} = J \cup J' \cup \mathcal{L}_1 \cup \mathcal{L}_2, \tag{7.11}$$

and satisfies for sufficiently large t the boundary condition

$$M_+^{\mathrm{par}}(k) = H(k)N(w(k))\Psi(w(k)) = M(k)\Psi(w(k)), \quad k \in \partial\mathcal{B}. \tag{7.12}$$

In $\mathcal{B}^*$ we define $M^{\mathrm{par}}(k)$ by symmetry

$$M^{\mathrm{par}}(k) = \sigma_1 M^{\mathrm{par}}(-k)\sigma_1.$$

8. Completion of asymptotical analysis

The aim of this section is to establish that the solution $m^{(3)}(k)$ is well approximated by $\begin{pmatrix}1 & 1\end{pmatrix} M^{\mathrm{par}}(k)$ inside the domain $\mathcal{B}\cup\mathcal{B}^*$, and by $\begin{pmatrix}1 & 1\end{pmatrix} M^{\mathrm{mod}}(k)$ in $\mathbb{C}\setminus(\mathcal{B}\cup\mathcal{B}^*)$. We follow the well-known approach via singular integral equations (see e.g., [9], [18], [21] Chapter 4, [23]). To simplify notation we introduce

$$\tilde{\Sigma} = \Sigma^{(2)} \cup \partial\mathcal{B} \cup \partial\mathcal{B}^*.$$

Set

$$\hat{m}(k) = m^{(3)}(k)(M^{\mathrm{as}}(k))^{-1}, \quad M^{\mathrm{as}}(k) := \begin{cases} M^{\mathrm{par}}(k), & k \in (\mathcal{B}\cup\mathcal{B}^*), \\ M^{\mathrm{mod}}(k), & k \in \mathbb{C}\setminus(\mathcal{B}\cup\mathcal{B}^*). \end{cases} \tag{8.1}$$

Formula (7.11) implies that $\hat{m}$ does not have jumps inside $\mathcal{B}\cup\mathcal{B}^*$. Moreover, from (7.4) and item *(6)* of Theorem 6.1 this vector is a holomorphic bounded function inside the strip between $\mathcal{C}$ and $\mathcal{C}^*$. Let us compute the jump of this vector on $\partial\mathcal{B}$ by use of (7.5), (7.9), (7.7) and (7.12):

$$\begin{aligned} \hat{m}_+ &= m_+^{(3)}\left(M_+^{\mathrm{par}}\right)^{-1} = m_-^{(3)} r^{-\sigma_3}\Psi^{-1}M_+^{-1} = m_-^{(3)}\left(M_-^{\mathrm{mod}}\right)^{-1} M_-^{\mathrm{mod}} r^{-\sigma_3}\Psi^{-1}M_+^{-1} \\ &= \hat{m}_- M_-^{\mathrm{mod}} r^{-\sigma_3}\Psi^{-1} r^{\sigma_3}\left(M_+^{\mathrm{mod}}\right)^{-1} = \hat{m}_- M_+ \Psi^{-1} M_+^{-1}. \end{aligned}$$

Here we took into account (7.7) and the fact that M^{mod} does not have a jump on $\partial\mathcal{B}$. Note also that both matrices $M_+(k)$ and $M^{\mathrm{mod}}(k)$ are bounded with respect to t uniformly on $\partial\mathcal{B}$.

Next, the structure of the matrix $\Psi(w(k))$ implies that

$$\Psi^{-1}(w(k)) = \mathbb{I} + \frac{\mathcal{F}(k,t)}{t(g(k) - g_\pm(\mathrm{i}a))}, \quad \|\mathcal{F}(k,t)\| \le O(1), \quad t\to\infty, \tag{8.2}$$

where the matrix norm estimate $O(1)$ is uniform with respect to k on the compact $\partial\mathcal{B}\cup\partial\mathcal{B}^*$, and uniform with respect to $\xi \in \mathcal{I}_\varepsilon$. Hence $\hat{m}(k)$ solves the jump problem

$$\hat{m}_+(k) = \hat{m}_-(k)\hat{v}(k),$$

where (cf. (4.10), (4.11)):

$$\hat{v}(k) = \begin{cases} \mathbb{I} + M(k)\frac{\mathcal{F}(k,t)}{t(g(k)-g_\pm(\mathrm{i}a))}M(k)^{-1}, & k\in\partial\mathcal{B}, \\ \sigma_1\hat{v}(-k)\sigma_1, & k \in \partial\mathcal{B}^*, \\ M_-^{\mathrm{mod}}(k)v^{(3)}(k)(M_+^{\mathrm{mod}}(k))^{-1}, & k\in\Sigma_\rho, \end{cases}$$

and satisfies the symmetry and normalization conditions:

$$\hat{m}(k) = \hat{m}(-k)\sigma_1, \qquad \hat{m}\to(1,\ 1), \quad k\to\infty.$$

Abbreviate $W(k) = \hat{v}(k) - \mathbb{I}$. Recall the estimate (4.14). Hence

$$
(8.3)\qquad W(k) = \begin{cases} \frac{1}{t(g(k)-g_\pm(\mathrm{i}a))} M_+(k)\,\mathcal{F}(k,t)\,M_+^{-1}(k), & k\in\partial\mathcal{B},\\ \sigma_1 W(-k)\sigma_1, & k\in\partial\mathcal{B}^*,\\ M_-^{\mathrm{mod}}(k)(v^{(3)}(k)-v^{\mathrm{mod}}(k))(M_+^{\mathrm{mod}}(k))^{-1}, & k\in\Sigma_\rho\setminus[\mathrm{i}\rho,-\mathrm{i}\rho],\\ 0 & k\in[\mathrm{i}\rho,-\mathrm{i}\rho], \end{cases}
$$

where we treat $v^{\mathrm{mod}}(k)$ as in (4.12). Thus the error vector $\hat{m}(k)$ has jumps on the contour

$$\hat{\Sigma} = \Sigma_\rho \cup \partial\mathcal{B}\cup\partial\mathcal{B}^*\setminus[\mathrm{i}\rho,-\mathrm{i}\rho]$$

only. This contour does not pass in the vicinities of the singular points $\mathrm{i}a, -\mathrm{i}a$ and 0. We observe that for all $(x,t)\in\mathcal{D}_\varepsilon$ the matrix $W(k)$ is continuous on any smooth part of the contour $\hat{\Sigma}$ and bounded with respect to k. Moreover, due to (4.14) and (7.5) we have

$$\|k^j(v^{(3)}(k)-v^{\mathrm{mod}}(k))\|_{L^p(\Sigma_\rho\setminus[\mathrm{i}\rho,-\mathrm{i}\rho])} = O(e^{-C(\varepsilon)t}),\quad p\in[1,\infty],\quad j=0,1,2$$

(the estimates on the higher moments will be used later). Here we took into account that the reflection coefficient $R(k)$ decays as $O(k^{-4})$ under condition (1.8). Thus, we get using (8.3) and (7.8) the estimate

Lemma 8.1. *The following estimates hold uniformly with respect to $\xi\in\mathcal{I}_\varepsilon$ and $(x,t)\in\mathcal{D}_\varepsilon$:*

$$(8.4)\qquad \|k^j W(k)\|_{L^p(\hat{\Sigma})} \le C(\varepsilon)t^{-1},\quad p\in[1,\infty],\quad j=0,1,2.$$

Now we are ready to apply the technique of singular integral equations. Since this is well known (see, for example, [9], [18], [23]) we will be brief and only list the necessary notions and estimates.

Let $\mathfrak{C}$ denote the Cauchy operator associated with $\hat{\Sigma}$:

$$(\mathfrak{C}h)(k) = \frac{1}{2\pi\mathrm{i}}\int_{\hat{\Sigma}} h(s)\frac{ds}{s-k},\qquad k\in\mathbb{C}\setminus\hat{\Sigma},$$

where $h=(h_1,h_2)\in L^2(\hat{\Sigma})$. Let $\mathfrak{C}_+ f$ and $\mathfrak{C}_- f$ be its non-tangential limiting values from the left and right sides of $\hat{\Sigma}$ respectively.

As usual, we introduce the operator $\mathfrak{C}_W : L^2(\hat{\Sigma})\cup L^\infty(\hat{\Sigma})\to L^2(\hat{\Sigma})$ by the formula $\mathfrak{C}_W f = \mathfrak{C}_-(fW)$, where W is our error matrix (8.3). Then,

$$\|\mathfrak{C}_W\|_{L^2(\hat{\Sigma})\to L^2(\hat{\Sigma})} \le C\|W\|_{L^\infty(\hat{\Sigma})} \le O(t^{-1}),$$

as well as

$$(8.5)\qquad \|(\mathbb{I}-\mathfrak{C}_W)^{-1}\|_{L^2(\hat{\Sigma})\to L^2(\hat{\Sigma})} \le \frac{1}{1-O(t^{-1})}$$

for sufficiently large t. Consequently, for $t\gg 1$, we may define a vector function

$$\mu(k) = (1,\ 1) + (\mathbb{I}-\mathfrak{C}_W)^{-1}\mathfrak{C}_W\big((1,\ 1)\big)(k).$$

Then by (8.4) and (8.5)

$$
\begin{aligned}
\|\mu(k)-(1,\ 1)\|_{L^2(\tilde{\Sigma})} &\le \|(\mathbb{I}-\mathfrak{C}_W)^{-1}\|_{L^2(\tilde{\Sigma})\to L^2(\tilde{\Sigma})}\|\mathfrak{C}_-\|_{L^2(\tilde{\Sigma})\to L^2(\tilde{\Sigma})}\|W\|_{L^\infty(\tilde{\Sigma})}\\
(8.6)\qquad &= O(t^{-1}).
\end{aligned}
$$

With the help of μ, (8.1) can be represented as

$$\hat{m}(k) = (1,\ 1) + \frac{1}{2\pi\mathrm{i}} \int_{\hat{\Sigma}} \frac{\mu(s)W(s)ds}{s-k},$$

and in virtue of (8.6) and Lemma 8.1 we obtain as $k \to +\mathrm{i}\infty$:

$$\hat{m}(k) = (1,\ 1) + \frac{1}{2\pi\mathrm{i}} \int_{\hat{\Sigma}} \frac{(1,\ 1)W(s)}{s-k} ds + E(k),$$

where

$$|E(k)| \le \frac{1}{\mathrm{Im}(k \mp \mathrm{i}c)} \|W\|_{L^2(\hat{\Sigma})} \|\mu(k) - (1,\ 1)\|_{L^2(\hat{\Sigma})} \le \frac{O(t^{-2})}{\mathrm{Im}(k \mp \mathrm{i}c)},$$

where $O(t^{-2})$ is uniformly bounded with respect to $\xi \in \mathcal{I}_\varepsilon$, $(x,t) \in \mathcal{D}_\varepsilon$ and $k \to \infty$. In the regime $\mathrm{Re}\, k = 0$, $\mathrm{Im}\, k \to +\infty$ we have

$$\frac{1}{2\pi\mathrm{i}} \int_{\hat{\Sigma}} \frac{(1,\ 1)W(s)}{k-s} ds = \frac{f_0(\xi,t)}{2\mathrm{i}kt}(1,\ -1) + \frac{f_1(\xi,t)}{2k^2 t}(1,\ 1) + O(t^{-1})O(k^{-3}) + O(t^{-2})O(k^{-1}),$$

where $f_{0,1}(\xi,t)$ are uniformly bounded for $t \to \infty$ and $\xi \in \mathcal{I}_\varepsilon$. Furthermore $O(k^{-s})$ are vector functions depending on k only and $O(t^{-s})$ are as above. Hence,

$$m^{(3)}(k) = \hat{m}(k) M^{\mathrm{mod}}(k) = m^{\mathrm{mod}}(k) + \frac{f_0(\xi,t)}{2\mathrm{i}kt}(1,\ -1)M^{\mathrm{mod}}(k) + \frac{f_1(\xi,t)}{2k^2 t} m^{\mathrm{mod}}(k) + O(t^{-1})O(k^{-3}) + O(t^{-2})O(k^{-1}).$$

Now we are in a position to apply (1.13), making use of (4.15), (5.15),(5.20), (5.17). Note that since all conjugation steps in the vicinity of ∞ involved diagonal matrices with determinant 1, we have for the solution to IVM R-H problem from Theorem 2.1:

$$m_1(k)m_2(k) = m_1^{(3)}(k)m_2^{(3)}(k) = m_1^{\mathrm{mod}}(k)m_2^{\mathrm{mod}}(k) + O(t^{-1})O(k^{-2}).$$

Here we used that the entries of $M^{\mathrm{mod}}(k)$ are uniformly bounded for $\xi \in \mathcal{I}_\varepsilon$ and that the k^{-1} term disappears by symmetry (2.15). Theorem 1.1 is proved.

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B. Verkin Institute for Low Temperature Physics, and Engineering of National Avcademy, of Sciences of Ukraine, 47,Lenin ave, 61103 Kharkiv, Ukraine
E-mail address: iraegorova@gmail.com

Faculty of Mathematics, University of Vienna, Oskar-Morgenstern-Platz 1, 1090 Wien
E-mail address: Mateusz.Piorkowski@univie.ac.at

Faculty of Mathematics, University of Vienna, Oskar-Morgenstern-Platz 1, 1090 Wien
E-mail address: Gerald.Teschl@univie.ac.at
URL: http://www.mat.univie.ac.at/~gerald/

PARAMETRIX PROBLEM FOR THE KORTEWEG–DE VRIES EQUATION WITH STEPLIKE INITIAL DATA

MATEUSZ PIORKOWSKI

Abstract. In this paper we study the asymptotics of the Korteweg–de Vries equation with steplike initial data, which leads to shock waves in the transition region between the dispersive tail and the soliton region, as $t \to \infty$. In our previous work we have obtained uniform estimates in x and t by constructing a singular global parametrix solution. However in this work, we present an alternative approach to the usual argument involving a small norm Riemann–Hilbert problem, which is based instead on Fredholm index theory for singular integral operators. In particular, we avoid the construction of a matrix-valued global parametrix solution and utilize only the symmetric vector solution, which always exists and is unique.

1. Introduction

The Korteweg de–Vries (KdV) equation is one of the most investigated nonlinear wave equations that admits a Lax pair representation and thus can be solved via scattering theory. The explicit asymptotic analysis can be performed by the Deift–Zhou nonlinear steepest descent method for Riemann–Hilbert (R-H) problems, ([3], [5], [6], [7]). It involves contour deformations and the introduction of auxiliary functions to obtain a R-H problem with jumps matrices that are either independent of the complex parameter k, or exponentially converging to the identity matrix for $t \to \infty$. Ignoring the exponentially converging part, one obtains a model problem, also referred to as the *global parametrix problem.* In most applications it can be solved explicitly with the help of special functions (in our case Jacobi theta functions), from which the relevant asymptotics can be obtained. The rigorous justification of this method is however nontrivial and leads to a local R-H problem which has to be solved around the oscillatory points (where the exponential convergence fails). The solution of this so-called *local parametrix problem* again involves the use of special functions (in our case Airy functions) and needs to converge to the model solution locally uniformly away from the oscillatory points.

The above steps for the KdV equation with steplike initial data (shock wave) in the region between the dispersive tail and the soliton region, also called *elliptic wave region* after the form of the solutions, have been already performed in [8] and are summarized in the next section. The main goal of this paper is the final part of the analysis which differs from the usual argumentation involving the construction of a model matrix solution (or equivalently two linearly independent vector solutions). A peculiar feature of this step is, that while the relevant asymptotics can be read

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off from the symmetric model vector solution, the justification of the asymptotics requires the construction of a second linearly independent model vector solution ([1], [2], [15], [21]). The reason for this is that inverting the model matrix solution results eventually in a singular integral equation of the form

$$(\mathbb{I} - \mathfrak{C}_u^{\Sigma})\phi = \mathfrak{C}_u^{\Sigma}((1 \ \ 1)) \tag{1.1}$$

where $\mathfrak{C}_u^{\Sigma}$ is a singular Cauchy-type operator depending on u, which is a matrix-valued $L^\infty(\Sigma; \mathbb{C}^{2\times 2})$-function with $\|u\|_{L^\infty(\Sigma;\mathbb{C}^{2\times 2})} \to 0$, as $t \to \infty$ (see [2, Ch. 7]). As $\|\mathfrak{C}_u^{\Sigma}\|_{L^2(\Sigma;\mathbb{C}^2)} = O(\|u\|_{L^\infty(\Sigma;\mathbb{C}^{2\times 2})})$, we can invert $\mathbb{I} - \mathfrak{C}_u^{\Sigma}$ for t large enough by writing down the Neumann series. In particular, we know that equation (1.1) has a unique solution, which can be then used to write down the corresponding unique solution of the R-H problem. As the invertibility of the singular integral operator $\mathbb{I} - \mathfrak{C}_u^{\Sigma}$ is obtained by the smallness of $\|u\|_{L^\infty(\Sigma;\mathbb{C}^{2\times 2})}$, we will refer to this approach as the *small norm R-H approach*.

The existence of a second vector-valued solution fails in the case of interest for discrete but arbitrary large times, as we have recently shown in [12]. The remedy has been the construction of a vector-valued meromorphic solution with a simple pole at the origin. Due to the inherit symmetries of the KdV R-H problem, the pole cancels in the final step of the nonlinear steepest descent analysis (see [12, Sect. 6]). However the natural question remains, whether the construction of the meromorphic model solutions is necessary. In this work we show that it is not, by proving a uniform error estimate for $t \to \infty$. Our approach avoids the construction of a model matrix solution entirely and relies instead on Fredholm index theory to argue for invertibility of the relevant singular integral operators. We shall refer to this method as the *Fredholm R-H approach*. This idea can be found in [25, Prop. 4.4] (see also [15], [18, Sect. 6] and [19]). While we concentrate on the KdV case with steplike initial data, this paper can be regarded as an introduction to this alternative method which generalizes to other problems solvable via the nonlinear steepest descent method.

The structure of the paper is as follows. Section 2 summarizes the necessary scattering theory to obtain the R-H formulation of the KdV equation. Section 3 contains a local change of variables which results in an explicitly solvable parametrix problem. The necessary theory of Fredholm integral operators with emphasis on the symmetries of our problem can be found in Section 4. The subsequent Section 5 contains the main idea of our new method: the construction of two auxiliary R-H problems and an application of Fredholm theory from the previous section to prove uniform error estimates for the approximation of the KdV solution. The discussion section contains some further comments and a short scheme for obtaining the full asymptotic expansion of the KdV solution. The two appendices contain some proof technicalities left out in the main text and a general theorem which describes the method used in this paper.

2. Preliminaries

2.1. Initial data. We consider the KdV equation (cf. [8]), given by

$$q_t(x,t) = 6q(x,t)q_x(x,t) - q_{xxx}(x,t), \quad (x,t) \in \mathbb{R} \times \mathbb{R}_+$$

with steplike initial data $q(x) = q(x,0) \in C^{11}(\mathbb{R})$, i.e.

$$\lim_{x\to\infty} q(x) = 0,$$
$$\lim_{x\to-\infty} q(x) = -c^2, \quad c > 0$$

such that

$$\int_0^{+\infty} \mathrm{e}^{C_0 x}(|q(x)| + |q(-x) + c^2|)dx < \infty, \quad C_0 > c \tag{2.1}$$

and

$$\int_{-\infty}^{\infty} (x^6 + 1)|q^{(i)}(x)|dx < \infty, \quad i = 1, ..., 11.$$

It has been shown that the above Cauchy problem has a unique solution $q(\cdot,t) \in C^3(\mathbb{R})$ (cf. [11], [13]). Existence of classical solutions has been proven under more general assumptions in [23], but we require the more restrictive condition (2.1) for analytic continuation in the frame work of the nonlinear steepest descent method. We focus on the asymptotic behaviour of solutions in the elliptic wave region given by $-6c^2t < x < 4c^2t$.

2.2. **Scattering transform.** To solve the KdV equation via the scattering transform, we need to regard the solution $q(x,t)$ as a potential of a self-adjoint Schrödinger operator:

$$L(t) = -\frac{d^2}{dx^2} + q(\,\cdot\,,t), \quad \mathfrak{D}(L) = H^2(\mathbb{R}) \subset L^2(\mathbb{R}).$$

Because of the behaviour of $q(x,t)$ for $x \to \pm\infty$, one can find unique Jost solutions $\phi(k,x,t)$, $\phi_1(k,x,t)$ of the stationary Schrödinger equation

$$L(t)\psi(k,x,t) = k^2\psi(k,x,t), \quad \mathrm{Im}(k) > 0$$

determined by

$$\lim_{x\to\infty} \mathrm{e}^{-\mathrm{i}kx}\phi(k,x,t) = 1, \quad \lim_{x\to-\infty} \mathrm{e}^{\mathrm{i}k_1 x}\phi_1(k,x,t) = 1$$

where $k_1 := \sqrt{k^2 + c^2}$ is holomporhic in $\mathbb{C}\backslash[-\mathrm{i}c, \mathrm{i}c]$ with $k_1 > 0$ for $k > 0$. We endow $[-\mathrm{i}c, \mathrm{i}c]$ with an orientation from top to bottom, hence $+$ $(-)$ denotes the limit from the right (left), e.g. $k_{1,+} = -k_{1,-}$. The Jost solutions ϕ and ϕ_1 are holomorphic in the domain $\mathbb{C}^U = \{k : \mathrm{Im}(k) > 0\}$, and $\mathbb{C}^U_c := \mathbb{C}^U \setminus (0, \mathrm{i}c]$ respectively and continuous up to the boundary. Hence, we can evaluate ϕ and ϕ_1 on the real axis, which results in the scattering relations

$$T(k,t)\phi_1(k,x,t) = \overline{\phi(k,x,t)} + R(k,t)\phi(k,x,t), \quad k \in \mathbb{R},$$
$$T_1(k,t)\phi(k,x,t) = \overline{\phi_1(k,x,t)} + R_1(k,t)\phi_1(k,x,t), \quad k_1 \in \mathbb{R},$$

where $T(k,t)$ $(T_1(k,t))$ and $R(k,t)$ $(R_1(k,t))$ are the transmission and reflection coefficients determined uniquely by the above equations. In the case of no solitons, $T(k,t)$ and $T_1(k,t)$ are holomorphic in $\mathbb{C}^U_c$ and continuous up to the boundary, while $R(k,t)$ and $R_1(k,t)$ have an analytic extension to the domain $\{k : 0 < \mathrm{Im}(k) < C_0\} \setminus (0, \mathrm{i}c]$, because of assumption (2.1). We also introduce an auxiliary function

$$\chi(k,t) := -\lim_{\varepsilon\to 0+} \overline{T(k+\varepsilon,t)}T_1(k+\varepsilon,t), \quad k \in (0, \mathrm{i}c]$$

and extend it to $[-\mathrm{i}c, 0)$ via

$$\chi(-k,t) = -\chi(k,t).$$

More properties of the above functions can be found in [8, Sect. 2], and shall be mentioned when needed. The next step involves determining a minimal scattering data, from which the potential can be reconstructed. From Theorem 2.1 below, it follows that one possible choice is given by

$$S(t) = \{R(k,t),\ k \in \mathbb{R};\ \chi(k,t),\ k \in [0, \mathrm{i}c]\}.$$

Here, $S(t)$ denotes the scattering data of the solution $q(x,t)$ of the KdV equation, which evolves linearly from the scattering data $S(0)$ of the initial data $q(x)$ via

$$R(k,t) = R(k,0)\mathrm{e}^{8\mathrm{i}k^3 t} = R(k)\mathrm{e}^{8\mathrm{i}k^3 t},$$
$$\chi(k,t) = \chi(k,0)\mathrm{e}^{8\mathrm{i}k^3 t} = \chi(k)\mathrm{e}^{8\mathrm{i}k^3 t}.$$

This method effectively linearizes the KdV equation. The R-H approach is then used to perform the inverse scattering transform $S(t) \to q(x,t)$ and is outlined in the following theorem taken from [8]:

Theorem 2.1. *Let $m(k) = m(k,x,t)$ be given by*

$$m(k,x,t) = \begin{cases} \Big(T(k,t)\phi_1(k,x,t)\mathrm{e}^{\mathrm{i}kx}, \phi(k,x,t)\mathrm{e}^{-\mathrm{i}kx}\Big), & k \in \mathbb{C}_c^U \\ \Big(\phi(-k,x,t)\mathrm{e}^{\mathrm{i}kx}, T(-k,t)\phi_1(-k,x,t)\mathrm{e}^{-\mathrm{i}kx}\Big), & k \in \mathbb{C}_c^L, \end{cases}$$

where $\mathbb{C}_c^U := \{k :\ \mathrm{Im}\, k > 0\} \setminus (0, \mathrm{i}c]$, $\mathbb{C}_c^L := \{k :\ \mathrm{Im}\, k < 0\} \setminus (0, -\mathrm{i}c]$. Then $m(k)$ is the unique solution to the following R-H problem:

Find a vector-valued function $m(k)$ which is holomorphic away from $\mathbb{R} \cup [-\mathrm{i}c, \mathrm{i}c]$, satisfying:

(i) The jump condition $m_+(k) = m_-(k)v(k)$

$$v(k) = \begin{cases} \begin{pmatrix} 1 - |R(k)|^2 & -\overline{R(k)}\mathrm{e}^{-t\Phi(k)} \\ R(k)\mathrm{e}^{t\Phi(k)} & 1 \end{pmatrix}, & k \in \mathbb{R}, \\ \begin{pmatrix} 1 & 0 \\ \chi(k)\mathrm{e}^{t\Phi(k)} & 1 \end{pmatrix}, & k \in (0, \mathrm{i}c], \\ \begin{pmatrix} 1 & \chi(k)\mathrm{e}^{-t\Phi(k)} \\ 0 & 1 \end{pmatrix}, & k \in [-\mathrm{i}c, 0), \end{cases}$$

(ii) the symmetry condition

$$m(-k) = m(k)\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \tag{2.2}$$

(iii) and the normalization condition

$$\lim_{k\to\infty} m(k) = \begin{pmatrix} 1 & 1 \end{pmatrix}. \tag{2.3}$$

Here the phase $\Phi(k) = \Phi(k,x,t)$ is given by

$$\Phi(k) = 8\mathrm{i}k^3 + 2\mathrm{i}k\frac{x}{t},$$

Remark 2.2. *We have excluded solitons in our analysis, as they do not contribute to the asymptotics in the region of interest. They can be included in the R-H formulation by requiring certain pole conditions on $m(k)$, which can then be transformed to jump conditions (see [8], [15]).*

Note that the jump matrix $v(k)$ also satisfies a symmetry condition

$$v(-k) = \sigma_1 v(k)^{-1}\sigma_1, \quad \sigma_1 := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}. \tag{2.4}$$

There are two methods to obtain $q(x,t)$ from $m(k,x,t) = (m_1(k,x,t) \;\; m_2(k,x,t))$ ([12]):

$$q(x,t) = \partial_x \lim_{k\to\infty} 2\mathrm{i}k(m_1(k,x,t)-1) = -\partial_x \lim_{k\to\infty} 2\mathrm{i}k(m_2(k,x,t)-1)$$

$$q(x,t) = \lim_{k\to\infty} 2k^2(m_1(k,x,t)m_2(k,x,t)-1). \tag{2.5}$$

The first formulas are more analytically demanding because of the differentiation, so we will use the second formula (2.5).

2.3. **Conjugation steps.** For further analysis we introduce the following function

$$g(k) = g(k,x,t) := 12\int_{\mathrm{i}c}^{k} (k^2+\mu^2)\sqrt{\frac{k^2+a^2}{k^2+c^2}}dk$$

which is holomorphic in $\mathbb{C}\setminus[-\mathrm{i}c,\mathrm{i}c]$ and approximates $\Phi(k)$ at infinity, while simplifying our R-H problem on $[-\mathrm{i}c,\mathrm{i}c]$. As has been shown in [16, Sect. 4], $a = a(\xi)$ and $\mu = \mu(\xi)$ can be chosen to depend continuously on the slowly varying parameter $\xi = \frac{x}{12t} \in (-\frac{c^2}{2}, \frac{c^2}{3})$ such that the following properties hold (see [8, Sect. 4]):

(i). The function g is odd, i.e. $g(-k) = -g(k)$, $k \in \mathbb{C}\setminus[-\mathrm{i}c,\mathrm{i}c]$;
(ii). $g_-(k) + g_+(k) = 0$ for $k \in [-\mathrm{i}c,\mathrm{i}c]\setminus(-\mathrm{i}a,\mathrm{i}a)$;
(iii). $g_-(k) - g_+(k) = B$ for $k \in [-\mathrm{i}a,\mathrm{i}a]$, with $B := -2g_+(\mathrm{i}a) > 0$;
(iv). for $k \to \infty$ we have

$$\frac{1}{2}\Phi(k,\xi) - \mathrm{i}g(k,\xi) = O(k^{-1}).$$

We modify the R-H problem from Theorem 2.1 by conjugating with the matrix $\mathrm{e}^{-(t\Phi(k)/2 - \mathrm{i}tg(k))\sigma_3}$, i.e.

$$\begin{aligned} m(k) &\longrightarrow \hat{m}(k) := m(k)\mathrm{e}^{-(t\Phi(k)/2-\mathrm{i}tg(k))\sigma_3}, \\ v(k) &\longrightarrow \hat{v}(k) := \mathrm{e}^{(t\Phi(k)/2-\mathrm{i}tg_-(k))\sigma_3} v(k) \mathrm{e}^{-(t\Phi(k)/2-\mathrm{i}tg_+(k))\sigma_3}, \\ \sigma_3 &:= \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \end{aligned}$$

The next step involves performing a standard factorization of the jump matrix on the real axis (see [8, Sect. 4]), and shall not be repeated here. To further simplify the jump matrices on $[-\mathrm{i}c,-\mathrm{i}a]\cup[\mathrm{i}a,\mathrm{i}c]$, we introduce another function $F(k)$ given by

$$F(k) := \exp\left\{\frac{\omega(k)}{2\pi\mathrm{i}}\left(\int_{\mathrm{i}c}^{\mathrm{i}a}\frac{f(s)}{s-k}ds + \int_{-\mathrm{i}c}^{-\mathrm{i}a}\frac{f(s)}{s-k}ds - \mathrm{i}\Delta\int_{-\mathrm{i}a}^{\mathrm{i}a}\frac{ds}{\omega(s)(s-k)}\right)\right\},$$

with

$$\omega(k) := \sqrt{(k^2+c^2)(k^2+a^2)}, \quad \omega(0) > 0,$$

$$f(k) := \frac{\log|\chi(k)|}{\omega_+(k)},$$

$$\Delta = \Delta(\xi) = 2\mathrm{i}\int_{\mathrm{i}a}^{\mathrm{i}c} \frac{\log|\chi(s)|}{\omega_+(s)}\, ds \Big(\int_{-\mathrm{i}a}^{\mathrm{i}a} \frac{ds}{\omega(s)}\Big)^{-1},$$

satisfying the following properties:

(i). $F_+(k)F_-(k) = |\chi(k)|$ for $k \in [\mathrm{i}c, \mathrm{i}a]$,
(ii). $F_+(k)F_-(k) = |\chi(k)|^{-1}$ for $k \in [-\mathrm{i}a, -\mathrm{i}c]$,
(iii). $F_+(k) = F_-(k)\mathrm{e}^{\mathrm{i}\Delta}$ for $k \in [-\mathrm{i}a, \mathrm{i}a]$,
(iv). $F(k) \to 1$ as $k \to \infty$ and $F(-k) = F^{-1}(k)$ for $k \in \mathbb{C} \setminus [-\mathrm{i}c, \mathrm{i}c]$.

That $F(k)$ has these properties on the open intervals follows from Plemelj formulas. The interval boundary points need a more careful analysis (see [20, Ch. 4]). Further comments regarding the possible singularities of $F(k)$ near $\mathrm{i}a$ and $\mathrm{i}c$ can be found in [12]. Again we conjugate our current R-H problem by the matrix $F(k)^{-\sigma_3}$.

Next, let us introduce the matrices

$$G^U(k) := \begin{pmatrix} 1 & -\frac{F^2}{\chi}\mathrm{e}^{-2\mathrm{i}tg} \\ 0 & 1 \end{pmatrix}, \quad G^L(k) := \begin{pmatrix} 1 & 0 \\ -\frac{1}{\chi F^2}\mathrm{e}^{2\mathrm{i}tg} & 1 \end{pmatrix}$$

where

$$\chi(k) := \frac{4k_1 k}{W(\widetilde{\phi}_1, \phi)W(\phi_1, \phi)}, \quad \chi(-k) = -\chi(k), \quad k \in \{k : 0 < \operatorname{Im} k < C_0\} \setminus [-\mathrm{i}c, \mathrm{i}c]$$

is the analytic continuation of $\chi(k)$ in the vicinity of $[-\mathrm{i}c, \mathrm{i}c]$, such that

$$\chi_+(k) = -\lim_{\varepsilon\to 0+} \overline{T(k+\varepsilon, 0)}T_1(k+\varepsilon, 0), \quad \chi_+(k) = -\chi_-(k), \quad k \in [-\mathrm{i}c, \mathrm{i}c].$$

The function $\widetilde{\phi}_1(k)$ is the analytic continuation of the function $\overline{\phi}_1(k)$ restricted to the imaginary segment $[0, \mathrm{i}c]$, and can be expressed via the transformation operator $K_1(x, y, t)$ (cf. [8, Sect. 2], [9]):

$$\widetilde{\phi}_1(k, x, t) = \mathrm{e}^{\mathrm{i}k_1 x} + \int_{-\infty}^{x} K_1(x, y, t)\mathrm{e}^{\mathrm{i}k_1 y} dy, \quad k \in \{k : 0 < \operatorname{Im} k < C_0\} \setminus [-\mathrm{i}c, \mathrm{i}c]$$

which converges because of the decay properties of $q(x, t)$. We conjugate with $G^U(k)$ and $G^L(k)$ in the domains Ω_1^U and Ω_1^L depicted in Figure 1, where all contours need to be confined to the strip $\{k : -C_0 < \operatorname{Im} k < C_0\}$. The resulting R-H problem takes the following form, where $b \in (0, a)$ can be chosen arbitrary (we shall only write down the resulting jump matrix and the contour, where the notation $v^{(2)}(k)$

is adopted from [12]):

$$v^{(2)}(k) = \begin{cases} \begin{pmatrix} 0 & \mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix}, & k \in [\mathrm{i}c, \mathrm{i}a], \\[2ex] \begin{pmatrix} \mathrm{e}^{-\mathrm{i}t\hat{B}} & 0 \\ \mathrm{i}|\chi|\mathrm{e}^{\mathrm{i}t(2g_- - \mathrm{i}t\hat{B})} & \mathrm{e}^{\mathrm{i}t\hat{B}} \end{pmatrix}, & k \in [\mathrm{i}a, \mathrm{i}b] \\[2ex] \begin{pmatrix} 1 & 0 \\ R(k)F^{-2}(k)\mathrm{e}^{2\mathrm{i}tg(k)} & 1 \end{pmatrix}, & k \in \Sigma^U \\[2ex] \begin{pmatrix} \mathrm{e}^{-\mathrm{i}t\hat{B}} & 0 \\ 0 & \mathrm{e}^{\mathrm{i}t\hat{B}} \end{pmatrix}, & k \in [-\mathrm{i}a, \mathrm{i}a], \\[2ex] \sigma_1 \left[v^{(2)}(-k)\right]^{-1}\sigma_1, & k \in [-\mathrm{i}a, -\mathrm{i}c] \cup [-\mathrm{i}b, -\mathrm{i}a] \cup \Sigma^L, \\[1ex] G^U(k)^{-1}, & k \in \Sigma_1^U, \\[1ex] G^L(k)^{-1}, & k \in \Sigma_1^L, \end{cases}$$

where $\hat{B} := B + \frac{\Delta}{t}$. We write Σ for the union of all contours listed above.

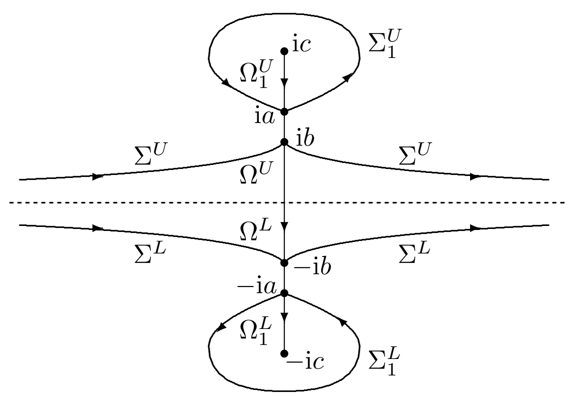

FIGURE 1. The contour of the model R-H problem with exponential correction

As all conjugation and deformation steps are invertible, we know from Theorem 2.1, that there exists a unique solution to the above R-H problem with the usual asymptotics at infinity (2.3) and symmetry condition (2.2), which shall be denoted by $m^{(2)}(k, x, t)$. Moreover, we assume that Σ^U and Σ^L remain a finite distance away from the real line, implying that for $t \to \infty$ the jump matrices on Σ^U, Σ^L, Σ_1^U and Σ_1^L converge exponentially fast to the identity matrix, and on $[\mathrm{i}a, \mathrm{i}b] \cup [-\mathrm{i}a, -\mathrm{i}b]$ to the diagonal matrix $\mathrm{e}^{-\mathrm{i}t\hat{B}\sigma_3}$. We shall refer to the above R-H problem as the *model R-H problem with exponential correction*, and the one where we ignore the jump matrices converging to the identity matrix as the *asymptotic model R-H problem* or just *model R-H problem*. The latter one is solved explicitly in [8] (see also [12], [16]) and its solution is unique:

Theorem 2.3. *The model R-H problem, given by the jump matrix*

$$v^{(3)} = \begin{cases} \begin{pmatrix} 0 & \mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix}, & k \in [\mathrm{i}a, \mathrm{i}c], \\ \\ \begin{pmatrix} 0 & -\mathrm{i} \\ -\mathrm{i} & 0 \end{pmatrix}, & k \in [-\mathrm{i}a, -\mathrm{i}c], \\ \\ \begin{pmatrix} \mathrm{e}^{-\mathrm{i}t\hat{B}} & 0 \\ 0 & \mathrm{e}^{\mathrm{i}t\hat{B}} \end{pmatrix}, & k \in [-\mathrm{i}a, \mathrm{i}a], \end{cases}$$

has a unique solution $m^{\mathrm{mod}}(k)$ satisfying the symmetry and normalization conditions (2.2), (2.3).

3. Parametrix problem

3.1. **Local change of variables.** We now turn to the jump condition near the points $\pm\mathrm{i}a$ (we will just consider $+\mathrm{i}a$, analogous results hold for $-\mathrm{i}a$). For k near $\mathrm{i}a$ we can write

$$\begin{aligned} g(k) &= 12 \int_{\mathrm{i}c}^{k} (s^2 + \mu^2) \sqrt{\frac{s^2 + a^2}{s^2 + c^2}} ds \\ &= g_{\pm}(\mathrm{i}a) + 12 \int_{\mathrm{i}a}^{k} (s^2 + \mu^2) \sqrt{\frac{s^2 + a^2}{s^2 + c^2}} ds \\ &= \mp \frac{B(\xi)}{2} - 8\mathrm{e}^{\mathrm{i}\pi/4} (a^2 - \mu^2) \sqrt{\frac{2a}{c^2 - a^2}} (k - \mathrm{i}a)^{3/2} + O((k - \mathrm{i}a)^{5/2}), \end{aligned}$$

where the roots in the last two lines have a branch cut on the positive imaginary axis (i.e. $\operatorname{Im} k > a$, $\operatorname{Re} k = 0$) and are chosen positive for $k - \mathrm{i}a > 0$. The upper (lower) sign is for the limit from the right (left), respectively.

Next, we have to define a local time-dependent holomorphic change of variables $k - \mathrm{i}a \to w$, such that

$$g(k) = \mp \frac{B(\xi)}{2} + \frac{\varsigma w(k)^{3/2}}{t}, \qquad k \in \mathbb{D}$$

where $\mathbb{D}$ is a disc around $\mathrm{i}a$ with a fixed radius smaller than $\min(c - a, a - b)$ such that $k - \mathrm{i}a \to w$ is bijective and $\varsigma = \mathrm{e}^{-3\pi\mathrm{i}/4}$. The branch cut is defined again on the positive imaginary axis. Furthermore, if convenient we will abuse notation by writing $f(w)$ for $f(k(w))$, if a f is a function of the variable k and vice versa.

We continue with the model R-H problem with exponential correction around the critical point $\mathrm{i}a$. To make it independent of our new variable w, we perform a conjugation by the matrix $\mathrm{e}^{-\mathrm{i}tg(w)\sigma_3}$. This results in a local R-H problem with the following jump matrices around $k = \mathrm{i}a$ ($w = 0$), where $\hat{h} := \mathrm{e}^{\mathrm{i}\Delta}$:

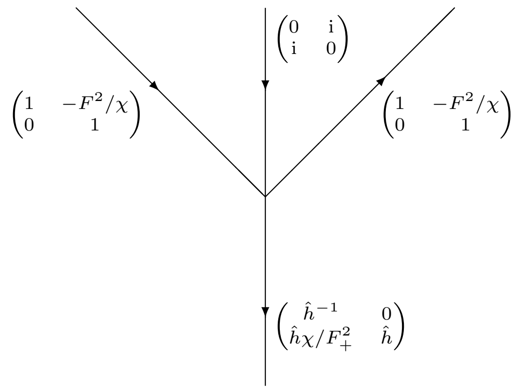

FIGURE 2. The local R-H problem

In the next step we conjugate around the origin in the w-domain by the matrix

$$\begin{pmatrix} p(w) & 0 \\ 0 & p(w)^{-1} \end{pmatrix}$$

where $p(w)$ should be a holomorphic function with a possible jump on the imaginary axis. The jump matrices transform as follows

$$\begin{pmatrix} 1 & -F^2/\chi \\ 0 & 1 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & -F^2/\chi p^{-2} \\ 0 & 1 \end{pmatrix}$$
$$\begin{pmatrix} 0 & \mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix} \longrightarrow \begin{pmatrix} 0 & \mathrm{i}p_+^{-1}p_-^{-1} \\ \mathrm{i}p_+p_- & 0 \end{pmatrix}$$
$$\begin{pmatrix} \hat{h}^{-1} & 0 \\ \hat{h}\chi/F_+^2 & \hat{h} \end{pmatrix} \longrightarrow \begin{pmatrix} \hat{h}^{-1}p_+p_-^{-1} & 0 \\ \hat{h}\chi/F_+^2 p_+p_- & \hat{h}p_+^{-1}p_- \end{pmatrix}.$$

To simplify the problem, we require that

$$\frac{F^2}{\chi}p^{-2} = 1$$

which implies

$$p := \frac{F}{\sqrt{\chi}}.$$

As the limit of F^2/χ from the right (left) to $\mathrm{i}a$ is equal to $-\mathrm{i}\hat{h}$ ($\mathrm{i}\hat{h}^{-1}$) which are of modulus 1, we see that we can find locally a square root. We choose the normalization

$$p_+(0) = \mathrm{e}^{-\pi\mathrm{i}/4}\hat{h}^{1/2}$$

and

$$p_-(0) = \mathrm{e}^{\pi\mathrm{i}/4}\hat{h}^{-1/2}.$$

The following boundary values can be computed explicitly:

$$p_+p_- = \frac{F_+F_-}{\sqrt{\chi_+}\sqrt{\chi_-}} = \frac{|\chi|}{\sqrt{\chi_+}\sqrt{-\chi_+}} = \frac{\mathrm{i}|\chi_+|}{\chi_+} = 1, \quad k(w) \in [\mathrm{i}a, \mathrm{i}c] \cap \mathbb{D},$$

$$p_+ p_-^{-1} = \frac{F_+ F_-^{-1}}{\sqrt{\chi_+}\sqrt{\chi_-}^{-1}} = \frac{\hat{h}}{\sqrt{\chi_+}\sqrt{-\chi_+}^{-1}} = -\mathrm{i}\hat{h}, \quad k(w) \in [0, \mathrm{i}a] \cap \mathbb{D}.$$

The conjugated R-H problem, referred to as the *Airy R-H problem*, takes on the form:

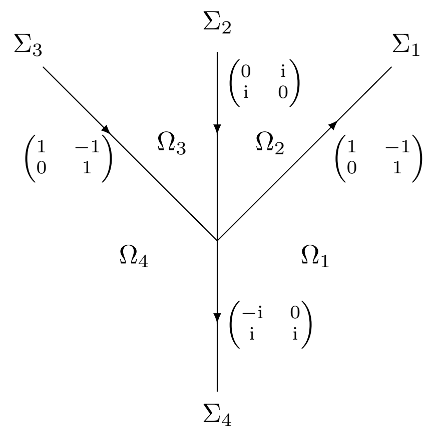

FIGURE 3. The Airy R-H problem

where we have used that

$$\frac{\hat{h}\chi p_+ p_-}{F_+^2} = \mathrm{i}$$

on the imaginary segment $k(w) \in [0, \mathrm{i}a] \cap \mathbb{D}$.

Note that matrix solutions without any normalization constraint at infinity correspond bijectively to matrices with entire entries. This follows from the cyclic condition satisfied by the jump matrices (cf. [24]):

$$\begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & \mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix}^{-1} \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}^{-1} \begin{pmatrix} -\mathrm{i} & 0 \\ \mathrm{i} & \mathrm{i} \end{pmatrix} = \mathbb{I}, \tag{3.1}$$

i.e. the product of the jump matrices when going around the point $w = 0$ and taking into account the orientation of Σ_i, $i = 1, \ldots, 4$ evaluates to the identity matrix. In particular, to construct the local parametrix solution we will need to use entire functions, such that later on the matching condition (3.7) can be satisfied.

To obtain the correct asymptotic for $w \to \infty$, we shall make an ansatz involving the Airy function (see [22, Ch. 9])

$$\mathrm{Ai}(z) := \int_{\infty \mathrm{e}^{-\mathrm{i}\pi/3}}^{\infty \mathrm{e}^{\mathrm{i}\pi/3}} \exp(s^3/3 - zs)\, ds,$$

$$\mathrm{Ai}(z) = \frac{1}{2\sqrt{\pi}}\Big(z^{-1/4} + O(z^{-7/4})\Big)\mathrm{e}^{-2/3 z^{3/2}}, \quad z \to \infty, \quad |\arg(z)| < \pi \tag{3.2}$$

which is a solution of the Airy differential equation. The notation $\infty\mathrm{e}^{\pm\mathrm{i}\pi/3}$ means that we integrate over a contour which asymptotically is a straight ray with angle $\pm\mathrm{i}\pi/3$. Here, the error term is uniform in closed sectors excluding the negative real axis. Two other solutions are given by $\mathrm{Ai}(\rho z)$ and $\mathrm{Ai}(\rho^2 z)$ with $\rho := \mathrm{e}^{2\mathrm{i}\pi/3}$. As the

Airy equation is linear and of second order, there must be a linear relation between these three solutions, which is given by

$$\mathrm{Ai}(z) + \rho\,\mathrm{Ai}(\rho z) + \rho^2\,\mathrm{Ai}(\rho^2 z) = 0.$$

As it will be needed later, let us also write down the asymptotics of the first derivative of the Airy function

$$\mathrm{Ai}'(z) = -\frac{1}{2\sqrt{\pi}}\Big(z^{1/4} + O(z^{-5/4})\Big)\mathrm{e}^{-2/3z^{3/2}}, \quad z \to \infty, \quad |\arg(z)| < \pi \tag{3.3}$$

where the error is again uniform in closed sectors excluding the negative real axis. Let us define

$$\begin{aligned} y_1(w) &:= 2\sqrt{\pi}\mathrm{e}^{\mathrm{i}\pi/8}(3/2)^{1/6}\,\mathrm{Ai}(\mathrm{i}w(3/2)^{2/3}) \\ y_2(w) &:= \rho y_1(\rho w) \\ y_3(w) &:= \rho^2 y_1(\rho^2 w) \end{aligned}$$

With our prior definition of $w^{3/2}$ (branch cut on the positive imaginary axis) we can deduce from (3.2) the following asymptotics

$$\begin{aligned} y_1(w) &= \Big(w^{-1/4} + O(w^{-7/4})\Big)\mathrm{e}^{\mathrm{i}\varsigma w^{3/2}}, \qquad \arg(w) \neq \pi/2 \\ y_2(w) &= \begin{cases} -\Big(w^{-1/4} + O(w^{-7/4})\Big)\mathrm{e}^{\mathrm{i}\varsigma w^{3/2}}, & \arg(w) \in (-\pi/6, \pi/2) \\ \mathrm{i}\Big(w^{-1/4} + O(w^{-7/4})\Big)\mathrm{e}^{-\mathrm{i}\varsigma w^{-3/2}}, & \arg(w) \in (\pi/2, 11\pi/6) \end{cases} \\ y_3(w) &= \begin{cases} -\Big(w^{-1/4} + O(w^{-7/4})\Big)\mathrm{e}^{\mathrm{i}\varsigma w^{3/2}}, & \arg(w) \in (\pi/2, 7\pi/6) \\ -\mathrm{i}\Big(w^{-1/4} + O(w^{-7/4})\Big)\mathrm{e}^{-\mathrm{i}\varsigma w^{3/2}}, & \arg(w) \in (-5\pi/6, \pi/2) \end{cases} \end{aligned}$$

with $\varsigma = \mathrm{e}^{-3\pi\mathrm{i}/4}$ and for the derivatives

$$\begin{aligned} y_1'(w) &= -\frac{3\mathrm{i}\varsigma}{2}\Big(-w^{1/4} + O(w^{-5/4})\Big)\mathrm{e}^{\mathrm{i}\varsigma w^{3/2}}, \qquad \arg(w) \neq \pi/2 \\ y_2'(w) &= \begin{cases} -\frac{3\mathrm{i}\varsigma}{2}\Big(w^{1/4} + O(w^{-5/4})\Big)\mathrm{e}^{\mathrm{i}\varsigma w^{3/2}}, & \arg(w) \in (-\pi/6, \pi/2) \\ -\frac{3\mathrm{i}\varsigma}{2}\Big(\mathrm{i}w^{1/4} + O(w^{-5/4})\Big)\mathrm{e}^{-\mathrm{i}\varsigma w^{-3/2}}, & \arg(w) \in (\pi/2, 11\pi/6) \end{cases} \\ y_3'(w) &= \begin{cases} -\frac{3\mathrm{i}\varsigma}{2}\Big(w^{1/4} + O(w^{-5/4})\Big)\mathrm{e}^{\mathrm{i}\varsigma w^{3/2}}, & \arg(w) \in (\pi/2, 7\pi/6) \\ -\frac{3\mathrm{i}\varsigma}{2}\Big(-\mathrm{i}w^{1/4} + O(w^{-5/4})\Big)\mathrm{e}^{-\mathrm{i}\varsigma w^{3/2}}, & \arg(w) \in (-5\pi/6, \pi/2) \end{cases} \end{aligned}$$

Next, we need to check whether we can indeed get the right boundary behaviour from an ansatz involving $y_i(w)$. Because of the cyclic relation (3.1) and the fact that the Airy function is entire, it is sufficient to specify the two vector components in one region, to automatically obtain a global vector solution to the Airy R-H problem. Assuming the form

$$\mathrm{ai}(w) = (y_3(w),\ -y_1(w))$$

in Ω_1 we get the following solution

$$\mathrm{ai}(w) = \begin{cases} \big(y_3(w), \ -y_1(w)\big), & w \in \Omega_1, \\ \big(y_3(w), \quad y_2(w)\big), & w \in \Omega_2, \\ \big(-\mathrm{i}y_2(w), \ -\mathrm{i}y_3(w)\big), & w \in \Omega_3, \\ \big(-\mathrm{i}y_2(w) \quad \mathrm{i}y_1(w)\big), & w \in \Omega_4. \end{cases}$$

As all jump matrices for the parametrix problem are constant, we know that $\mathrm{ai}'(w)$ is another solution to the Airy R-H problem. Conjugating back with the matrices $p(w)^{-\sigma_3}$ and $\mathrm{e}^{\mathrm{i}tg(w)\sigma_3}$ we can use these vector solutions to write down a local matrix solution $A(k)$ of the model R-H problem with exponential correction in the k-domain defined as follows:

(3.4)

$$A(k) = \begin{cases} \mathrm{e}^{\mathrm{i}\pi/4} t^{-\sigma_3/6} \begin{pmatrix} \frac{2\mathrm{i}}{3\varsigma} & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} y_3'(w) & -y_1'(w) \\ -y_3(w) & y_1(w) \end{pmatrix} p(k)^{-\sigma_3} \mathrm{e}^{\mathrm{i}tg(k)\sigma_3}, & w \in \Omega_1, \\ \mathrm{e}^{\mathrm{i}\pi/4} t^{-\sigma_3/6} \begin{pmatrix} \frac{2\mathrm{i}}{3\varsigma} & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} y_3'(w) & y_2'(w) \\ -y_3(w) & -y_2(w) \end{pmatrix} p(k)^{-\sigma_3} \mathrm{e}^{\mathrm{i}tg(k)\sigma_3}, & w \in \Omega_2, \\ \mathrm{e}^{\mathrm{i}\pi/4} t^{-\sigma_3/6} \begin{pmatrix} \frac{2\mathrm{i}}{3\varsigma} & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} -\mathrm{i}y_2'(w) & -\mathrm{i}y_3'(w) \\ \mathrm{i}y_2(w) & \mathrm{i}y_3(w) \end{pmatrix} p(k)^{-\sigma_3} \mathrm{e}^{\mathrm{i}tg(k)\sigma_3}, & w \in \Omega_3, \\ \mathrm{e}^{\mathrm{i}\pi/4} t^{-\sigma_3/6} \begin{pmatrix} \frac{2\mathrm{i}}{3\varsigma} & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} -\mathrm{i}y_2'(w) & \mathrm{i}y_1'(w) \\ \mathrm{i}y_2(w) & -\mathrm{i}y_1(w) \end{pmatrix} p(k)^{-\sigma_3} \mathrm{e}^{\mathrm{i}tg(k)\sigma_3}, & w \in \Omega_4, \end{cases}$$

in some fixed disc $\mathbb{D}$ around $\mathrm{i}a$ with radius smaller than $\min(a-b, c-a)$ in the k-domain, such that $k \to w$ is a change of variables for $k \in \mathbb{D}$.

Note that $m^{(2)}(k)A^{-1}(k)$ will have no jumps inside $\mathbb{D}$. For our subsequent analysis we need an analogous local solution for the model problem, i.e. we look for a matrix $N(k)$ defined on $\mathbb{D}$, such that $m^{\mathrm{mod}}(k)N(k)$ has no jumps inside $\mathbb{D}$. This translate to the following jump condition for $N(k)$

$$\begin{aligned} N_+(k) &= \begin{pmatrix} 0 & -\mathrm{i} \\ -\mathrm{i} & 0 \end{pmatrix} N_-(k), \quad k \in [\mathrm{i}a, \mathrm{i}c] \cap \mathbb{D} \\ N_+(k) &= \begin{pmatrix} \mathrm{e}^{\mathrm{i}t\hat{B}} & 0 \\ 0 & \mathrm{e}^{-\mathrm{i}t\hat{B}} \end{pmatrix} N_-(k), \quad k \in [0, \mathrm{i}a] \cap \mathbb{D}. \end{aligned} \tag{3.5}$$

where $\hat{B} = B + \frac{\Delta}{t}$. Furthermore, on the boundary $\partial\mathbb{D}$ we would like $N(k)$ to asymptotically cancel with $A^{-1}(k)$. The correct solution of (3.5) is given by

$$N(k) = \frac{1}{2} \mathrm{e}^{\pm \mathrm{i}(tB/2 + \pi/4)\sigma_3} p(k)^{\sigma_3} \begin{pmatrix} w^{-1/4} & -w^{1/4} \\ w^{-1/4} & w^{1/4} \end{pmatrix} t^{\sigma_3/6}, \tag{3.6}$$

which is obtained from $A^{-1}(k)$ by taking the first term in the expansion of the Airy function and their derivative (3.2), (3.3). Note that in $A(k)$ the $\exp(\mathrm{i}\varsigma w(k)^{3/2})$ factor cancels partially with $\exp(\pm \mathrm{i}tg(k)) = \exp(\mp \mathrm{i}B/2 + \mathrm{i}\varsigma w(k)^{3/2})$ leaving only $\exp(\pm \mathrm{i}tB/2)$, which is contained in formula (3.6). In fact, choosing $N(k)$ as above and using the normalization in (3.4) we obtain the estimate

$$A^{-1}(k) = N(k) + O(t^{-1}), \quad k \in \partial\mathbb{D}, \tag{3.7}$$

which follows from (3.2), (3.3) and the fact that $O(t^{1/6}w^{-7/4}) = O(t^{-1/6}w^{-5/4}) = O(t^{-1})$ on $\partial\mathbb{D}$. As both the determinants of $A(k)$ and $N(k)$ are constant equal to 1, we also have the estimate

$$N^{-1}(k) = A(k) + O(t^{-1}), \quad k \in \partial\mathbb{D}.$$

4. Singular integral equations

Our goal is to show that the contributions coming from the vicinities of $\pm\mathrm{i}a$ are small, such that they do not affect the leading asymptotics of the KdV equation. This can be achieved by reformulating our R-H problems as a singular integral equations and is a rigorous justification of Theorem 5.1 in [8], giving uniform error estimates of order $O(t^{-1})$. Again, the arguments follow a similar line to the ones given in [1], [12] and [21], except for the final analysis, which omits the construction of a model-matrix solution and the corresponding small norm R-H problem. Instead, we rely on Fredholm index theory for singular integral operators. Relevant literature on Cauchy operators and their connection with R-H problems can be found in [2], [17]. We shall review the essential results here.

We write $\mathfrak{C}^\Gamma$ for the Cauchy operator defined on $L^2(\Gamma)$,

$$\mathfrak{C}^\Gamma : L^2(\Gamma) \to \mathcal{O}(\mathbb{C}\setminus\Gamma), \quad f(k) \mapsto \mathfrak{C}^\Gamma(f)(k) := \frac{1}{2\pi\mathrm{i}}\int_\Gamma f(s)\frac{ds}{s-k},$$

where $\mathcal{O}(\mathbb{C}\setminus\Gamma)$ denotes the space of holomorphic functions on $\mathbb{C}\setminus\Gamma$. Here, we are assuming that Γ has an orientation and that the family of functions $(s-k)^{-1}$ are in $L^2(\Gamma)$ for $k \in \mathbb{C}\setminus\Gamma$. We define the operators $\mathfrak{C}^\Gamma_-$, $\mathfrak{C}^\Gamma_+$ by

$$\mathfrak{C}^\Gamma_\pm(f)(k) = \lim_{z\to k\pm}\mathfrak{C}^\Gamma(f)(z),$$

where the limit is assumed to be nontangential. Standard theory tells us that if Γ is a Carleson jump contour on the Riemann sphere $\hat{\mathbb{C}} = \mathbb{C}\cup\{\infty\}$, the nontangential limit exists a.e. and $\mathfrak{C}^\Gamma_\pm$ will be a bounded operator from $L^2(\Gamma)$ to itself (see [17, Prop. 3.11]). Note that the contour Σ of the model problem can be extended to a Carleson jump contour $\Sigma\cup\mathrm{i}\mathbb{R}$ (the same is true for $\tilde{\Sigma}$ defined in the next section). Hence, boundedness of $\mathfrak{C}^{\Sigma\cup\mathrm{i}\mathbb{R}}$ implies boundedness of $\mathfrak{C}^\Sigma$, which is all that is required in [25] for the underlying operators to be Fredholm.

For a general 2×2 matrix-valued function $u(k) \in L^\infty(\Gamma;\mathbb{C}^{2\times2})$ we can define with slight abuse of notation the following operator:

$$\mathfrak{C}^\Gamma_u : L^2(\Gamma;\mathbb{C}^2) \to L^2(\Gamma;\mathbb{C}^2), \quad f \mapsto \mathfrak{C}^\Gamma_-(f\cdot u),$$

where Γ is a Carleson jump contour. Here, $\mathfrak{C}^\Gamma_-$ acts componentwise on vector-valued entries. We assume that Γ is invariant with respect to the transformation $k \to -k$ and that sequences converging to the positive side still converge after this transformation to the positive side. This is a different convention then the one we used in [12] or the one from Theorem 2.1, in order to simplify computations. The jump matrix v (and also $u := v - \mathbb{I}$) must now satisfy

$$v(-k) = \sigma_1 v(k)\sigma_1$$

which is the same as (2.4) when taking into account the different orientation. By the previous arguments, we have that $\|\mathfrak{C}^\Gamma_u\|_{L^2(\Gamma;\mathbb{C}^2)} \le C\|u\|_{L^\infty(\Gamma;\mathbb{C}^{2\times2})}$, where C is the operator bound of $\mathfrak{C}^\Gamma_-$ in $L^2(\Gamma;\mathbb{C}^2)$.

Remark 4.1. *We emphasise that the orientation is essentially a free choice. While some theorems in the literature (c.f. [17]) require particular orientation conventions, one can always fulfill them by changing orientations of certain arcs and inverting the corresponding jump matrix.*

Next, let us consider the following integral equation

$$(\mathbb{I} - \mathfrak{C}_u^\Gamma)\phi(k) = \mathfrak{C}_-^\Gamma((1 \;\; 1)u),$$

where we also require that the matrix entries of u are in $L^2(\Gamma)$ in order for the right-hand side to be well-defined. It turns out that there is a bijective correspondence between solutions of the above equation, and vector solutions of the R-H problem on the contour Γ with jump matrices $v(k) = \mathbb{I} + u(k)$ (in the L^2-setting) given by

$$\begin{aligned}\phi(k) &\longrightarrow m(k) := (1 \;\; 1) + \frac{1}{2\pi \mathrm{i}}\int_\Gamma (\phi(s) + (1 \;\; 1))u(s)\frac{ds}{s-k}\\ m(k) &\longrightarrow \phi(k) := m_-(k) - (1 \;\; 1),\end{aligned}$$

see [25]. The following result is of central importance for this paper and holds also without our symmetry assumption on Γ:

Theorem 4.2. *For $u(k) \in L^\infty(\Gamma; \mathbb{C}^{2\times 2})$, the operator $\mathbb{I} - \mathfrak{C}_u^\Gamma$ is Fredholm of index 0.*

Proof. The fact that this operator is Fredholm is proven in [25, Prop. 4.1]. Note that $\tau \to \mathbb{I} - \mathfrak{C}_{\tau u}^\Gamma$ for $\tau \in [0,1]$ is a continuous deformation of $\mathbb{I} - \mathfrak{C}_u^\Gamma$ to $\mathbb{I}$ in the space of Fredholm operators. As the Fredholm index stays invariant with respect to such deformations, it follows that

$$\operatorname{ind}\,(\mathbb{I} - \mathfrak{C}_u^\Gamma) = \operatorname{ind}\,\mathbb{I} = 0.$$

As the uniqueness statements in Theorems 2.1 and 2.3 only hold for vector solutions satisfying the symmetry condition (2.2), we would need to restrict the operator $\mathfrak{C}_u^\Gamma$ to $L_s^2(\Gamma; \mathbb{C}^2)$, which is the Hilbert space of functions $\phi(k) \in L^2(\Gamma; \mathbb{C}^2)$, satisfying

$$\phi(-k) = \phi(k)\sigma_1.$$

We write $\mathfrak{S}_u^\Gamma$ for the restriction to the symmetric functions, and $\mathfrak{A}_u^\Gamma$ for the restriction to the antisymmetric function satisfying

$$\phi(-k) = -\phi(k)\sigma_1.$$

The space of antisymmetric functions is denoted by $L_a^2(\Gamma; \mathbb{C}^2)$. Let us define the operator

$$H\colon L^2(\Gamma; \mathbb{C}^2) \to L^2(\Gamma; \mathbb{C}^2), \qquad H\phi(k) := \phi(-k)\sigma_1.$$

Note that $L_s^2(\Gamma; \mathbb{C}^2)$ is the eigenspace of H with eigenvalue 1, while $L_a^2(\Gamma; \mathbb{C}^2)$ is the one with eigenvalue -1. Next, let us assume that $u(k)$ (or analogously $v(k)$) satisfies the symmetry condition

$$u(-k) = \sigma_1 u(k)\sigma_1.$$

We can then compute that H is a symmetry of $\mathfrak{C}_u^\Gamma$, i.e. commutes with it

$$\begin{aligned}\mathfrak{C}_u^\Gamma H\phi(k) &= \lim_{k'\to k-}\frac{1}{2\pi\mathrm{i}}\int_\Gamma H\phi(s)u(s)\frac{ds}{s-k'}\\ &= \lim_{k'\to k-}\frac{1}{2\pi\mathrm{i}}\int_\Gamma \phi(-s)\sigma_1 u(s)\frac{ds}{s-k'}\\ &= \lim_{-k'\to -k-}\frac{1}{2\pi\mathrm{i}}\int_\Gamma \phi(s)u(s)\frac{ds}{s-(-k')}\sigma_1\\ &= \mathfrak{C}_u^\Gamma\phi(-k)\sigma_1\\ &= H\mathfrak{C}_u^\Gamma\phi(k).\end{aligned}$$

Hence, we conclude

$$[\mathfrak{C}_u^\Gamma, H] = 0$$

which implies that the range of $\mathfrak{S}_u^\Gamma$ lies in the space of symmetric functions, while the range of $\mathfrak{A}_u^\Gamma$ lies in the space of antisymmetric functions. This implies that $\mathbb{I}-\mathfrak{S}_u^\Gamma$ and $\mathbb{I}-\mathfrak{A}_u^\Gamma$ can both be restricted to be Fredholm operators on $L^2_s(\Gamma;\mathbb{C}^2)$ and $L^2_a(\Gamma;\mathbb{C}^2)$ respectively. Using an argument as in the proof of Theorem 4.2, we conclude that both operators are of index 0. Hence, we see that injectivity of the operator $\mathbb{I}-\mathfrak{S}_u^\Gamma$ which is equivalent to the uniqueness of the corresponding R-H problem already implies invertibility. The same reasoning holds for $\mathbb{I}-\mathfrak{A}_u^\Gamma$.

5. Main result

Next, we define two new R-H problems for which we know their unique solutions. Similarly as in Section 3, denote by $\mathbb{D}^U$ a disc around $\mathrm{i}a$ with radius smaller than $\min(c-a, a-b)$ such that $k\to w$ is bijective for $k\in\mathbb{D}^U$. In particular, Σ^U is assumed to be disjoint from $\mathbb{D}^U$. Note that the radius can be chosen to be constant with respect to small variations of ξ (see Remark 5.1). Let $\mathbb{D}^L$ be the image of $\mathbb{D}^U$ under the transformation $k\to -k$. Furthermore, $\partial\mathbb{D}^U$ is oriented counterclockwise, $\partial\mathbb{D}^L$ clockwise and let $\mathcal{U} := \mathbb{D}^U\cup\mathbb{D}^L$. The two R-H problems satisfy by assumption the symmetry condition for the contour and for the jump matrices specified in the previous section. The same goes for the solutions, which are assumed to be symmetric.

Riemann-Hilbert problem I. Find a vector-valued function $\widetilde{m}^{\mathrm{mod}}(k)$, holomorphic in $\mathbb{C}\setminus([-\mathrm{i}c,\mathrm{i}c]\cup\partial\mathcal{U})$, satisfying:

(i) The jump condition $\widetilde{m}^{\mathrm{mod}}_+(k) = \widetilde{m}^{\mathrm{mod}}_-(k)v^I(k)$:

$$v^I(k) = \begin{cases} v^{(3)}(k), & k\in[-\mathrm{i}c,\mathrm{i}c]\setminus\mathcal{U}\\ N(k), & k\in\partial\mathbb{D}^U\\ \sigma_1 N(-k)\sigma_1, & k\in\partial\mathbb{D}^L\end{cases}$$

with $N(k)$ defined by (3.6),

(ii) the symmetry condition

$$\widetilde{m}^{\mathrm{mod}}(-k) = \widetilde{m}^{\mathrm{mod}}(k)\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix},$$

(iii) and the normalization condition

$$\lim_{k\to\infty}\widetilde{m}^{\mathrm{mod}}(k) = \begin{pmatrix}1 & 1\end{pmatrix}.$$

The solution has the form

$$
\begin{aligned}
\widetilde{m}^{\mathrm{mod}}(k) &= m^{\mathrm{mod}}(k), && k \in \mathbb{C}\setminus\mathcal{U}\\
\widetilde{m}^{\mathrm{mod}}(k) &= m^{\mathrm{mod}}(k)N(k), && k \in \mathcal{U}
\end{aligned}
\tag{5.1}
$$

where $m^{\mathrm{mod}}(k)$ satisfies the model R-H problem and is taken from [8] (for a proof of uniqueness see [12]), where it is given explicitly in terms of Jacobi theta functions:

$$
m_1^{\mathrm{mod}}(k) = \sqrt[4]{\frac{k^2+a^2}{k^2+c^2}}\,\frac{\theta\left(\mathbf{A}(k)-\mathrm{i}\pi-\frac{\mathrm{i}t\hat{B}}{2}\right)\theta\left(\mathbf{A}(k)-\frac{\mathrm{i}t\hat{B}}{2}\right)\theta^2\left(\frac{\pi\mathrm{i}}{2}\right)}{\theta\left(\mathbf{A}(k)-\mathrm{i}\pi\right)\theta\left(\mathbf{A}(k)\right)\theta\left(\frac{\pi\mathrm{i}}{2}-\frac{\mathrm{i}t\hat{B}}{2}\right)\theta\left(\frac{\pi\mathrm{i}}{2}+\frac{\mathrm{i}t\hat{B}}{2}\right)},
$$

$$
m_2^{\mathrm{mod}}(k) = \sqrt[4]{\frac{k^2+a^2}{k^2+c^2}}\,\frac{\theta\left(-\mathbf{A}(k)-\mathrm{i}\pi-\frac{\mathrm{i}t\hat{B}}{2}\right)\theta\left(-\mathbf{A}(k)-\frac{\mathrm{i}t\hat{B}}{2}\right)\theta^2\left(\frac{\pi\mathrm{i}}{2}\right)}{\theta\left(-\mathbf{A}(k)-\mathrm{i}\pi\right)\theta\left(-\mathbf{A}(k)\right)\theta\left(\frac{\pi\mathrm{i}}{2}-\frac{\mathrm{i}t\hat{B}}{2}\right)\theta\left(\frac{\pi\mathrm{i}}{2}+\frac{\mathrm{i}t\hat{B}}{2}\right)}.
$$

Here, $\mathbf{A}\colon \mathbb{C}\setminus[-\mathrm{i}c,\mathrm{i}c]\to\mathbb{C}$, is the Abel map restricted to the upper sheet (cf. [8, Sect. 5]). Furthermore, uniqueness of $m^{\mathrm{mod}}(k)$ implies uniqueness of $\tilde{m}^{\mathrm{mod}}(k)$, as any model vector solution would give rise to a solution to the R-H problem I via (5.1).

Riemann-Hilbert problem II. Find a vector-valued function $\widetilde{m}^{(2)}(k)$, holomorphic in $\mathbb{C}\setminus(\Sigma\cup\partial\mathcal{U})$, satisfying:

(i) The jump condition $\widetilde{m}_+^{(2)}(k)=\widetilde{m}_-^{(2)}(k)v^{II}(k)$:

$$
v^{II}(k)=\begin{cases} v^{(2)}(k), & k\in\Sigma\setminus\mathcal{U}\\ A^{-1}(k), & k\in\partial\mathbb{D}^U\\ \sigma_1A^{-1}(-k)\sigma_1, & k\in\partial\mathbb{D}^L\end{cases}
$$

with $A(k)$ defined by (3.4),

(ii) the symmetry condition

$$
\widetilde{m}^{(2)}(-k)=\widetilde{m}^{(2)}(k)\begin{pmatrix}0&1\\1&0\end{pmatrix},
$$

(iii) and the normalization condition

$$
\lim_{k\to\infty}\widetilde{m}^{(2)}(k)=\begin{pmatrix}1&1\end{pmatrix}.
$$

The solution has the form

$$
\begin{aligned}
\widetilde{m}^{(2)}(k) &= m^{(2)}(k), && k\in\mathbb{C}\setminus\mathcal{U}\\
\widetilde{m}^{(2)}(k) &= m^{(2)}(k)A^{-1}(k), && k\in\mathcal{U}
\end{aligned}
$$

where again uniqueness of $m^{(2)}(k)$ implies uniqueness of $\widetilde{m}^{(2)}(k)$.

Let us remark once again that $N(k)$ is chosen to cancel the jump matrix inside $\mathcal{U}$, and the same is true for $A^{-1}(k)$. Hence, while R-H problem I has only jumps on the imaginary segments $[-\mathrm{i}c,\mathrm{i}c]\setminus\mathcal{U}$ and $\partial\mathcal{U}$, R-H problem II additionally has jump exponentially converging to the identity matrix away from the discs.

Note that $v^I(k) = v^{II}(k)$ for $k \in [-\mathrm{i}c, \mathrm{i}c] \setminus \big(\mathcal{U} \cup [\mathrm{i}a, \mathrm{i}b] \cup [-\mathrm{i}a, -\mathrm{i}b]\big)$. On the boundary of the discs we have the estimate (3.7), hence we can conclude that

$$\|v^{II} - v^I\|_{L^\infty(\widetilde{\Sigma};\mathbb{C}^{2\times 2})} = O(t^{-1}),$$

where $\widetilde{\Sigma} := (\Sigma \setminus \mathcal{U}) \cup \partial\mathcal{U}$ and we set $v^I(k) \equiv \mathbb{I}$ for $k \in \Sigma \setminus [-\mathrm{i}c, \mathrm{i}c]$. We can now use the results from the previous sections, to prove that $m^{(2)}(k)$ and $m^{\mathrm{mod}}(k)$ are asymptotically close to one another as $t \to \infty$.

As we have an explicit solution of R-H problem I, we can use our uniqueness result and Fredholm theory to solve the equation

$$(\mathbb{I} - \mathfrak{S}^{\widetilde{\Sigma}}_{u^I})\phi^I(k) = \mathfrak{S}^{\widetilde{\Sigma}}_{-}((1\ \ 1)u^I).$$

with $u^I := v^I - \mathbb{I}$, by inverting $\mathbb{I} - \mathfrak{S}^{\widetilde{\Sigma}}_{u^I}$. As u^I is periodic in time, we can conclude that the continuous family of operators $\mathbb{I} - \mathfrak{S}^{\widetilde{\Sigma}}_{u^I}$ is uniformly invertible

$$\|(\mathbb{I} - \mathfrak{S}^{\widetilde{\Sigma}}_{u^I})^{-1}\|_{L^2(\widetilde{\Sigma};\mathbb{C}^2)} \leq C.$$

Here, C can be chosen locally uniformly in the parameter $\xi = \frac{x}{12t}$ (see remark below and Appendix A).

Remark 5.1. *It should be emphasized that $u^{I,II}$ are not only time dependent but also dependent on $\xi = \frac{x}{12t}$. As shown in appendix A, we can choose the contour $\widetilde{\Sigma}$ such that it does not depend on the parameter ξ, as long as ξ stays in some compact subinterval of $(-\frac{c^2}{2}, \frac{c^2}{3})$. As the jump matrices are continuous functions of ξ, we can vary ξ when letting $t \to \infty$ and all estimates would still hold. In our subsequent computations we suppress the ξ-dependence as it does not change the asymptotics as long it stays in $(-c^2/2 + \varepsilon, c^2/3 - \varepsilon)$, for some $\varepsilon > 0$.*

From now on we abbreviate the norms $\|.\|_{L^p(\widetilde{\Sigma};\mathbb{C}^2)}$ and $\|.\|_{L^p(\widetilde{\Sigma};\mathbb{C}^{2\times 2})}$ by $\|.\|_p$, for $p \in [1, \infty]$. We note that

$$\|v^I - v^{II}\|_\infty = \|u^I - u^{II}\|_\infty = O(t^{-1})$$
$$\|v^I - v^{II}\|_2 = \|u^I - u^{II}\|_2 = O(t^{-1})$$

and

$$\|\mathfrak{S}^{\widetilde{\Sigma}}_{u^I} - \mathfrak{S}^{\widetilde{\Sigma}}_{u^{II}}\|_2 = \|\mathfrak{S}^{\widetilde{\Sigma}}_{u^I - u^{II}}\|_2 = O(t^{-1}).$$

As the set of bounded invertible operators is open in the operator norm topology, we conclude that for t large enough the operator $\mathbb{I} - \mathfrak{S}^{\widetilde{\Sigma}}_{u^{II}}$ must also be uniformly invertible. Denote by $\phi^{II}(k)$ the unique solution of

$$(\mathbb{I} - \mathfrak{S}^{\widetilde{\Sigma}}_{u^{II}})\phi^{II}(k) = \mathfrak{S}^{\widetilde{\Sigma}}_{-}((1\ \ 1)u^{II}). \tag{5.2}$$

The following computations shows that $\phi^I(k)$ and $\phi^{II}(k)$ are in fact also close to one another in $L^2(\widetilde{\Sigma}; \mathbb{C}^2)$:

$$\|\phi^I(k) - \phi^{II}(k)\|_2 = \|(\mathbb{I} - \mathfrak{S}^{\widetilde{\Sigma}}_{u^I})^{-1}\mathfrak{S}^{\widetilde{\Sigma}}_{-}((1\ \ 1)u^I) - (\mathbb{I} - \mathfrak{S}^{\widetilde{\Sigma}}_{u^{II}})^{-1}\mathfrak{S}^{\widetilde{\Sigma}}_{-}((1\ \ 1)u^{II})\|_2$$
$$\|[(\mathbb{I} - \mathfrak{S}^{\widetilde{\Sigma}}_{u^I})^{-1} - (\mathbb{I} - \mathfrak{S}^{\widetilde{\Sigma}}_{u^{II}})^{-1}]\mathfrak{S}^{\widetilde{\Sigma}}_{-}((1\ \ 1)u^I)\|_2 + \|(\mathbb{I} - \mathfrak{S}^{\widetilde{\Sigma}}_{u^{II}})^{-1}\mathfrak{S}^{\widetilde{\Sigma}}_{-}((1\ \ 1)(u^I - u^{II}))\|_2$$
$$\leq C_1\|(\mathbb{I} - \mathfrak{S}^{\widetilde{\Sigma}}_{u^I})^{-1}\mathfrak{S}^{\widetilde{\Sigma}}_{u^I - u^{II}}(\mathbb{I} - \mathfrak{S}^{\widetilde{\Sigma}}_{u^{II}})^{-1}\|_2 + C_2\|u^I - u^{II}\|_2 = O(t^{-1})$$

where we use the second resolvent formula. Furthermore, $\|\phi^{I,II}(k)\|_2$ are uniformly bounded by the uniform invertibility of the corresponding singular integral operators, as well as the uniform boundedness of $\|u^{I,II}\|_2$. Now, developing $1/(s-k)$ into a truncated Neumann series

$$\frac{1}{s-k} = -\frac{1}{k} - \frac{s}{k^2} - \frac{s^2}{k^3}\frac{1}{1-s/k}$$

and taking into account the exponential decay of the matrices u^I, u^{II} at infinity, one obtains for $k \to \infty$ such that $|1 - s/k| \geq \varepsilon > 0$ for all $s \in \tilde{\Sigma}$

$$\begin{aligned} m^{\mathrm{mod}}(k) = (1 \;\; 1) - \frac{1}{k}\int_{\widetilde{\Sigma}}(\phi^I(s) + (1 \;\; 1))u^I(s)\; ds \\ -\frac{1}{k^2}\int_{\widetilde{\Sigma}}(\phi^I(s) + (1 \;\; 1))u^I(s)s\; ds + O(k^{-3}) \end{aligned}$$

and

$$\begin{aligned} m^{(2)}(k) = (1 \;\; 1) - \frac{1}{k}\int_{\widetilde{\Sigma}}(\phi^{II}(s) + (1 \;\; 1))u^{II}(s)\; ds \\ -\frac{1}{k^2}\int_{\widetilde{\Sigma}}(\phi^{II}(s) + (1 \;\; 1))u^{II}(s)s\; ds + O(k^{-3}). \end{aligned}$$

Next we compute

$$\begin{aligned} &\Big|\int_{\widetilde{\Sigma}}(\phi^I(s) + (1 \;\; 1))u^I(s)\; ds - \int_{\widetilde{\Sigma}}(\phi^{II}(s) + (1 \;\; 1))u^{II}(s)\; ds\Big| \\ &\leq \Big|\int_{\widetilde{\Sigma}}(\phi^I(s) - \phi^{II}(s))u^I(s)\; ds\Big| + \Big|\int_{\widetilde{\Sigma}}\phi^{II}(s)(u^I(s) - u^{II}(s))\; ds\Big| \\ &\quad + \Big|\int_{\widetilde{\Sigma}}(1 \;\; 1)(u^I(s) - u^{II}(s))\; ds\Big| = O(t^{-1}) \end{aligned}$$

and analogously

$$\Big|\int_{\widetilde{\Sigma}}(\phi^I(s) + (1 \;\; 1))u^I(s)s\; ds - \int_{\widetilde{\Sigma}}(\phi^{II}(s) + (1 \;\; 1))u^{II}(s)s\; ds\Big| = O(t^{-1}).$$

Hence, we can conclude

$$m^{(2)}(k) = m^{\mathrm{mod}}(k) + \frac{1}{k}O(t^{-1}) + \frac{1}{k^2}O(t^{-1}) + O(k^{-3}).$$

We will now make use of the formula (2.5)

$$m_1(k,x,t)m_2(k,x,t) = 1 + \frac{q(x,t)}{2k^2} + O(k^{-4}).$$

There is no need to trace back our deformation and conjugation steps, as we are only interested in the asymptotics of $m(k)$ at infinity, and conjugation by $\mathrm{e}^{(t\Phi(k)/2 - \mathrm{i}tg(k))\sigma_3}F(k)^{\sigma_3}$ will not change the product $m_1^{(2)}(k)m_2^{(2)}(k)$. Hence we can conclude that

$$m_1(k)m_2(k) = m_1^{(2)}(k)m_2^{(2)}(k) = m_1^{\mathrm{mod}}(k)m_2^{\mathrm{mod}}(k) + \frac{1}{k^2}O(t^{-1}) + O(k^{-4}).$$

Note that the odd terms $k^{-1}, k^{-3}\ldots$ drop out because of the symmetry condition (2.2) satisfied by $m(k)$, $m^{(2)}(k)$ and $m^{\mathrm{mod}}(k)$, implying that all of the above products are even functions. For the solution of $q(x,t)$ of the KdV equation we obtain

$$q(x,t) = q^{\mathrm{mod}}(x,t) + O(t^{-1})$$

where

$$m_1^{\mathrm{mod}}(k) m_2^{\mathrm{mod}}(k) = 1 + \frac{q^{\mathrm{mod}}(x,t)}{2k^2} + O(k^{-4}).$$

In [10] it has been shown that $q^{\mathrm{mod}}(x,t)$ has the form of a periodic Its–Matveev solution modulated by the parameter ξ. A general theorem summarizing the above argumentation in the abstract setting is given in Appendix B.

6. DISCUSSION

The main difference in our nonlinear steepest descent analysis compared to the usual one (see [2], [5], [6]), has been the avoidance of a small norm R-H problem. Instead, to obtain invertibility of the associated singular integral operators, we relied on Fredholm index theory (c.f. [25]). This Fredholm R-H approach could be used in other scenarios where an invertible model matrix solution fails to exist. For this purpose we have also included a general theorem in Appendix B.

An issue not covered here in detail, is the computation of a full asymptotic expansion of the R-H solution, as it is done in [4] for the case of orthogonal polynomials. This works analogously in our case, as we have a shifted Neumann series given by:

$$(\mathbb{I} - \mathfrak{S}_{u^{II}}^{\tilde{\Sigma}})^{-1} = \sum_{n=0}^{\infty} \Big[(\mathbb{I} - \mathfrak{S}_{u^{I}}^{\tilde{\Sigma}})^{-1} (\mathfrak{S}_{u^{II}}^{\tilde{\Sigma}} - \mathfrak{S}_{u^{I}}^{\tilde{\Sigma}}) \Big]^n (\mathbb{I} - \mathfrak{S}_{u^{I}}^{\tilde{\Sigma}})^{-1}.$$

This in itself is not enough to write down an expansion of the solution to the singular integral equation (5.2) in powers of t^{-1}. We also need an expansion of

$$\mathfrak{S}_{u^{II}-u^{I}}^{\tilde{\Sigma}} = \mathfrak{S}_{u^{II}}^{\tilde{\Sigma}} - \mathfrak{S}_{u^{I}}^{\tilde{\Sigma}}$$

which is equivalent to an expansion of

$$u^{II}(k) - u^{I}(k) = A^{-1}(k) - N(k)$$

on $\partial\mathcal{U}$, as $u^{II}(k) - u^{I}(k) = o(t^{-\ell})$ for $\ell \in \mathbb{N}$ on the rest of the contour. To this end we make use of the full expansion of the Airy functions [22, Ch. 9] in powers of $w^{-3/2}$ which translates to an expansion in t^{-1}. This results in

$$\phi^{II}(k,x,t) = \sum_{j=0}^{\ell} \frac{\phi_j^{II}(k,x,t)}{t^j} + Er(k,x,t)$$

for $\ell \in \mathbb{N}$ and ξ fixed, with $\phi_j^{II}(k,x,t)$ being a periodic function in t, $\phi_0^{II} = \phi^I$ and $\|Er(.,x,t)\|_2 = O(t^{-\ell-1})$. Consequently, one obtains a similar expansion of $m(k,x,t)$ in terms of t^{-1} and k^{-1}

$$m^{(2)}(k,x,t) = m^{\mathrm{mod}}(k,x,t) + \sum_{i=1}^{\ell} \sum_{j=1}^{n} \frac{c_{ij}(x,t)}{k^i t^j} + O(k^{-\ell-1}t^{-1}) + O(k^{-1}t^{-n-1}).$$

where $c_{ij}(x,t)$ are periodic in t for fixed ξ. Hence, we see that the existence of an asymptotic expansion of the R-H solution follows from an asymptotic expansion of the jump matrices, just as with the traditional small norm approach.

Another future challenge would be characterizing those R-H problems which admit a vector solution but in general no matrix solution. Note that precisely for $n\pi = t\hat{B}(\xi)$, $n \in \mathbb{Z}$, the model R-H problem has an additional symmetry:

$$v^{(3)}(k) = \sigma_1 v^{(3)}(k) \sigma_1$$

where σ_1 is the first Pauli matrix. In particular, one can check that for those values of t, $m^{\text{mod}}(k)\sigma_1$ is also a solution. From uniqueness it follows that $m^{\text{mod}}(k)\sigma_1 = m^{\text{mod}}(k)$, or equivalently $m_1^{\text{mod}}(k) = m_2^{\text{mod}}(k)$. This makes it easier to satisfy the equation $m^{\text{mod}}(k) = (0\ \ 0)$ for some k, which is related to the nonexistence of an invertible matrix model solution (see [12, Sect. 3]). Indeed for odd n, $m^{\text{mod}}(0) = (0\ \ 0)$ holds and no holomorphic invertible matrix solution exists. It would be interesting to explore the question whether such symmetric problems have a distinguished role in the R-H analysis of integrable equations.

Appendix A. Uniformity of operator bounds

Observe that the contour Σ of the model RH problem with the exponentially converging matrices depends on the parameter $\xi = \frac{x}{12t}$, via the point $a = a(\xi) \in (0, c)$. However, it is possible to make the contour $\widetilde{\Sigma} = (\Sigma \setminus \mathcal{U}) \cup \partial\mathcal{U}$ at least locally in ξ, independent of ξ. To see this, note that while we for simplicity always assumed $\mathrm{i}a$ to be the center of $\mathbb{D}^U$, this is not essential. Furthermore, we can always choose the rays emanating from $\mathrm{i}a$ to hit the boundary of the disc at the same points. This then allows us to choose the rest of the contour $\widetilde{\Sigma}$ independent of ξ as long as $\mathrm{i}a$ stays in the interior of $\mathbb{D}^U$.

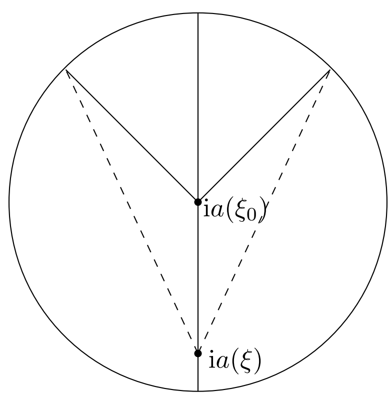

Figure 4. The inner rays in dependence of ξ.

This greatly simplifies the analysis, as we now have to deal with only one ξ-independent Hilbert space $L^2_s(\widetilde{\Sigma}; \mathbb{C}^2)$. While we might not be able to choose $\mathcal{U}$ as large as possible because of the constraint that $k \to w$ should be bijective, we certainly can cover any compact interval contained in $(0, \mathrm{i}c)$ with finitely many discs. Hence all our estimates will be uniform, as long as ξ stays in some compact subinterval of $(-c^2/2, c^2/3)$.

Another issue neglected in the main text is the uniform boundedness of

$$(\mathbb{I} - \mathfrak{S}^{\tilde{\Sigma}}_{u^{I,II}})^{-1}.$$

Note that for two operators O and P where O is invertible and P is some perturbation of O we have the formulas

$$\|(O + P)^{-1}\| \le \|O^{-1}\| \frac{1}{1 - \|O^{-1}\| \|P\|} \tag{A.1}$$

and similarly

$$\|(O+P)^{-1}\| \geq \|O^{-1}\| \frac{1}{1+\|O^{-1}\|\|P\|}.$$

whenever $\|O^{-1}\|\|P\| < 1$. This implies continuity of the norm of the inverse. In particular, we can conclude that for $(\xi, t) \in K \times [T_1, T_2]$, where $K \subset (-c^2/2, c^2/3)$ is compact and $T_1 < T_2$, we have the estimate

$$\|(\mathbb{I} - \mathfrak{S}_{u^I}^{\widetilde{\Sigma}})^{-1}\|_2 \leq C < \infty.$$

By periodicity of u^I in time this inequality can be extended to $t \in \mathbb{R}$. Analogously, because of $\|u^I - u^{II}\|_\infty = O(t^{-1})$ we get

$$\|(\mathbb{I} - \mathfrak{S}_{u^{II}}^{\widetilde{\Sigma}})^{-1}\|_2 \leq C' < \infty.$$

for t large enough.

Appendix B. A General theorem

We now mention a theorem generalizing the argumentation given in the proof of the main result (c.f [2, Cor. 7.108], [14, Ch. 3], [25, Prop. 4.4]). Let Γ be an oriented contour, such that the associated Cauchy operators $\mathfrak{C}_\pm^\Gamma$ are bounded operators from $L^2(\Gamma)$ to itself. Explicit conditions on the contour for the above statement to hold can be found in [17]. Furthermore, let an $n \times n$ matrix-valued function $v \in \mathbb{I} + L^2(\Gamma; \mathbb{C}^{n\times n})$ be given, such that $v^{-1} \in \mathbb{I} + L^2(\Gamma; \mathbb{C}^{n\times n})$. We associate to v a factorization data $u = (u^+, u^-) \in L^2(\Gamma; \mathbb{C}^{n\times n}) \cap L^\infty(\Gamma; \mathbb{C}^{n\times n})$, such that $v = (\mathbb{I} - u^-)^{-1}(\mathbb{I} + u^+)$ on the contour Γ. Note that the factorization data is nonunique, but always exists, as one can choose $u^- = 0$ and $u^+ = v - \mathbb{I}$, as it is done in the main text. For any factorization data we define a singular integral operator

$$\mathfrak{C}_u^\Gamma : L^2(\Gamma; \mathbb{C}^n) \to L^2(\Gamma; \mathbb{C}^n), \quad \phi \mapsto \mathfrak{C}_+^\Gamma(\phi u^-) + \mathfrak{C}_-^\Gamma(\phi u^+).$$

Again, we are interested in solutions of the R-H problem on the contour Γ with jump matrix v. The normalization for the vector-valued solution $m(k)$ is assumed to take the simple form

$$\lim_{k\to\infty} m(k) = m_\infty \in \mathbb{C}^n$$

where the limit is taken such that $|1 - s/k| \geq \varepsilon > 0$ for all $s \in \Gamma$ and some positive constant ε. As before the above R-H problem is equivalent to the following singular integral equation

$$\big(\mathbb{I} - \mathfrak{C}_u^\Gamma\big)\phi = \mathfrak{C}_u^\Gamma(m_\infty) \tag{B.1}$$

where $m(k)$ can be obtained by the formula

$$\begin{aligned} m(k) &= m_\infty + \frac{1}{2\pi \mathrm{i}} \int_\Gamma (\phi(s) + m_\infty)(u^+(s) + u^-(s)) \frac{ds}{s-k} \\ &= m_\infty + \mathfrak{C}^\Gamma((\phi + m_\infty)(u^+ + u^-)). \end{aligned} \tag{B.2}$$

Indeed, assume ϕ satisfies (B.1) and define $m(k)$ as above. Then

$$\begin{aligned}
m_+ &= m_\infty + \mathfrak{C}_+^\Gamma((\phi + m_\infty)(u^+ + u^-)) \\
&= m_\infty + \mathfrak{C}_-^\Gamma(\phi u^+) + \phi u^+ + \mathfrak{C}_+^\Gamma(\phi u^-) + \mathfrak{C}_-^\Gamma(m_\infty u^+) + m_\infty u^+ + \mathfrak{C}_+^\Gamma(m_\infty u^-) \\
&= m_\infty(\mathbb{I} + u^+) + \underbrace{\mathfrak{C}_u^\Gamma(\phi) + \mathfrak{C}_u^\Gamma(m_\infty)}_{\phi} + \phi u^+ \\
&= (m_\infty + \phi)(\mathbb{I} + u^+)
\end{aligned}$$

where we used $\mathfrak{C}_+^\Gamma - \mathfrak{C}_-^\Gamma = \mathbb{I}$. Analogously one computes

$$m_- = (m_\infty + \phi)(\mathbb{I} - u^-),$$

which then implies

$$m_+ = m_-(\mathbb{I} - u^-)^{-1}(\mathbb{I} + u^+) = m_- v.$$

Hence, $m(k)$ is a solution of the R-H problem with $\lim_{k\to\infty} m(k) = m_\infty$. Conversely, assume $m(k)$ be a solution to the R-H problem. Then by the Sokhotski–Plemelj formula for additive scalar R-H problems, $m(k)$ can be written as

$$m(k) = m_\infty + \mathfrak{C}^\Gamma(m_+(\mathbb{I} - v^{-1})) = m_\infty + \mathfrak{C}^\Gamma(m_-(v - \mathbb{I})).$$

Define

$$\phi := m_-(\mathbb{I} - u^-)^{-1} - m_\infty = m_+(\mathbb{I} + u^+)^{-1} - m_\infty.$$

Then relation (B.2) is fulfilled. From the definition of ϕ it follows that

$$\begin{aligned}
m_+ &= (\phi + m_\infty)(\mathbb{I} + u^+), \\
m_- &= (\phi + m_\infty)(\mathbb{I} - u^-).
\end{aligned}$$

Meanwhile, (B.2) together with $\mathfrak{C}_+^\Gamma - \mathfrak{C}_-^\Gamma = \mathbb{I}$, implies as before

$$\begin{aligned}
m_+ &= m_\infty(\mathbb{I} + u^+) + \mathfrak{C}_u^\Gamma(\phi) + \mathfrak{C}_u^\Gamma(m_\infty) + \phi u^+, \\
m_- &= m_\infty(\mathbb{I} - u^-) + \mathfrak{C}_u^\Gamma(\phi) + \mathfrak{C}_u^\Gamma(m_\infty) - \phi u^-.
\end{aligned}$$

Comparing the two expressions of either m_+ or m_- results in (B.1). We are now in a position to state a theorem generalising the arguments given in Section 5.

Theorem B.1. *Let Γ be a contour such that the Cauchy operators $\mathfrak{C}_\pm^\Gamma$ are bounded operators from $L^2(\Gamma)$ to itself with operator bounds less than $C > 0$, and let for $i = 1, 2$*

$$v_i : \mathbb{R}_+ \to \mathbb{I} + L^2(\Gamma; \mathbb{C}^{n\times n}), \quad t \mapsto v_i(t) = v_i(t, k)$$

together with a factorization

$$v_i(t) = (\mathbb{I} - u_i^-(t))^{-1}(\mathbb{I} + u_i^+(t)), \quad t > 0$$

be given, such that $v_i^{-1}(t) \in \mathbb{I} + L^2(\Gamma; \mathbb{C}^{n\times n})$ and $u_i^\pm(t) \in L^2(\Gamma; \mathbb{C}^{n\times n}) \cap L^\infty(\Gamma; \mathbb{C}^{n\times n})$. Furthermore assume that the operator $\mathbb{I} - \mathfrak{C}_{u_1}^\Gamma$ is invertible for all $t > 0$ with

$$\|(\mathbb{I} - \mathfrak{C}_{u_1}^\Gamma)^{-1}\|_2 \le \rho(t)$$

and

$$\|u_1^\pm - u_2^\pm\|_2 \le \epsilon(t), \quad \|u_1^\pm - u_2^\pm\|_\infty \le \delta(t)$$

where $\rho(t), \epsilon(t)$ and $\delta(t)$ are given positive functions and $\|.\|_p := \|.\|_{L^p(\Gamma)}$, $p \in [1, \infty]$, where the norm is naturally generalized to matrix and vector functions. Then $\mathbb{I} - \mathfrak{C}^{\Gamma}_{u_2}$ is also invertible as long as $C\rho(t)\delta(t) < 1$ with

$$\|(\mathbb{I} - \mathfrak{C}^{\Gamma}_{u_2})^{-1}\|_2 \le \frac{\rho(t)}{1 - C\rho(t)\delta(t)}.$$

For $t > 0$ such that $C\rho(t)\delta(t) < 1$, denote by $\phi_{1,2}$ the unique solutions of

$$(\mathbb{I} - \mathfrak{C}^{\Gamma}_{u_{1,2}})\phi = \mathfrak{C}^{\Gamma}_{u_{1,2}}(m_\infty)$$

for some fixed $m_\infty \in \mathbb{C}^n$. Then

$$\|\phi_1 - \phi_2\|_2 \le \frac{2C\rho(t)\epsilon(t)}{1 - C\rho(t)\delta(t)}\|m_\infty\|_\infty + \frac{2C^2\rho^2(t)\delta(t)}{1 - C\rho(t)\delta(t)}\|m_\infty\|_\infty \|u_1\|_2.$$

Now, assume that the i-th moments of $u_1^\pm$ and $u_2^\pm$ exist, in the sense that

$$\|u_j^\pm(k)k^i\|_p \le \infty$$

for $j = 1, 2$, $p = 1, 2$ and $i = 0, \dots, \ell$. Then for the vector solutions of the R-H problem m_1 and m_2 associated to ϕ_1 and ϕ_2 respectively, we have the formula

$$m_j(k) = m_\infty - \sum_{i=1}^{\ell} \frac{1}{k^i}\int_\Gamma (\phi_j(s) + m_\infty)(u_j^+(s) + u_j^-(s))s^{i-1}ds + O(k^{-\ell-1})$$

for $j = 1, 2$ where $k \to \infty$ such that $|1 - s/k| \ge c > 0$ for $s \in \Gamma$. Furthermore, if

$$\|u_1^\pm(k)k^i - u_2^\pm(k)k^i\|_2 \le \rho_i(t)$$
$$\|u_1^\pm(k)k^i - u_2^\pm(k)k^i\|_1 \le \sigma_i(t)$$

for $i = 0, \dots, \ell - 1$, then

$$m_1(k) - m_2(k) = \sum_{i=1}^{\ell} \frac{c_i}{k^i} + O(k^{-\ell-1})$$

with

$$|c_i| \le \|\phi_1 - \phi_2\|_2 \|(u_1^+(k) + u_1^-(k))k^{i-1}\|_2 + 2\|\phi_2\|_2 \rho_{i-1}(t) + 2\|m_\infty\|_\infty \sigma_{i-1}(t).$$

Moreover,

$$|m_1(k) - m_2(k)| \le \operatorname{dist}(k, \Gamma)^{-1}\Big[\|\phi_1 - \phi_2\|_2 \|(u_1^+(k) + u_1^-(k))\|_2 + 2\|\phi_2\|_2 \rho_0(t) + 2\|m_\infty\|_\infty \sigma_0(t)\Big].$$

Proof. The statement concerning the existence and bound of $(\mathbb{I} - \mathfrak{C}^{\Gamma}_{u_2})^{-1}$ follows directly from formula (A.1) in Appendix A. The estimates for $\|\phi_1 - \phi_2\|_2$ and c_i can be computed as is done in the proof of our main result concerning the KdV equation, where we identify u_1 with u^I and u_2 with u^{II}. The last estimate is obtained similarly by bounding $(k - s)^{-1}$ by $\operatorname{dist}(k, \Gamma)$ instead of writing down the Neumann series. □

Remark B.2. *With the identification of u_1 with u^{II} and u_2 with u^I one obtains analogously the estimates*

$$\|\phi_1 - \phi_2\|_2 \le 2C\rho(t)\epsilon(t)\|m_\infty\|_\infty + \frac{2C^2\rho^2(t)\delta(t)}{1 - C\rho(t)\delta(t)}\|m_\infty\|_\infty \|u_2\|_2,$$

$$|c_i| \leq \|\phi_1 - \phi_2\|_2 \|(u_2^+(k) + u_2^-(k))k^{i-1}\|_2 + 2\|\phi_1\|_2 \rho_{i-1}(t) + 2\|m_\infty\|_\infty \sigma_{i-1}(t),$$

$$|m_1(k) - m_2(k)| \leq \operatorname{dist}(k, \Gamma)^{-1} \Big[\|\phi_1 - \phi_2\|_2 \|(u_2^+(k) + u_2^-(k))\|_2 + 2\|\phi_1\|_2 \rho_0(t) + 2\|m_\infty\|_\infty \sigma_0(t) \Big].$$

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Faculty of Mathematics, University of Vienna, Oskar-Morgenstern-Platz 1, 1090 Wien
E-mail address: Mateusz.Piorkowski@univie.ac.at

A SCALAR RIEMANN–HILBERT PROBLEM ON THE TORUS: APPLICATIONS TO THE KDV EQUATION

MATEUSZ PIORKOWSKI AND GERALD TESCHL

Abstract. We take a closer look at the Riemann–Hilbert problem associated to one-gap solutions of the Korteweg–de Vries equation. To gain more insight, we reformulate it as a scalar Riemann–Hilbert problem on the torus. This enables us to derive deductively the model vector-valued and singular matrix-valued solutions in terms of Jacobi theta functions. We compare our results with those obtained in recent literature.

1. Introduction

1.1. **Background.** The main goal of this short note is to present an alternative approach to the existence/uniqueness results for the model Riemann–Hilbert (R-H) problem presented in [6] and the construction of a singular matrix-valued solution found in [8, Sect. 6] (see also [10, Sect. 3]). Recall, that the objective of [6] and [8] was to apply rigorously the nonlinear steepest descent method to the initial value problem for the Korteweg–de Vries (KdV) equation,

$$q_t(x,t) = 6q(x,t)q_x(x,t) - q_{xxx}(x,t), \quad (x,t) \in \mathbb{R} \times \mathbb{R}_+,$$

with steplike initial data $q(x,0) = q(x)$:

$$\lim_{x\to\infty} q(x) = 0, \qquad \lim_{x\to-\infty} q(x) = -c^2, \quad c > 0.$$

For large t, solutions to this problem display different behaviours in three regions of the (x,t)-plane characterized by the ratio x/t (see [6, Sect. 1]). Of particular interest to us is the transition region given by $-6c^2t < x < 4c^2t$, where solutions asymptotically converge to a modulated elliptic wave. This result was proven in [8], where an ill-posedness of the corresponding holomorphic matrix model R-H problem was found.

The ill-posedness is closely related to the fact that the R-H problem for the KdV equation is formulated as a vector-valued problem. Note that, the standard Liouville-type argument relating existence to uniqueness for matrix-valued R-H problems having jump matrices with unit determinant (see for example [13, Thm. 5.6]), cannot be generalized to the vector case in a straightforward manner. In fact, uniqueness can fail despite existence, as demonstrated in [11, Sect. 2] for the simple case of a one soliton solution. Uniqueness was restored by assuming an additional symmetry condition.

Another feature of the KdV equation playing an important role in the present note is the relationship between finite-gap solutions and elliptic Riemann surfaces

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(see [1, Ch. 3]). Algebro-geometric finite-gap solutions to the KdV equation can be given explicitly in terms of Jacobi theta functions via the Its–Matveev formula [12] (see also [7]). Unsurprisingly, the solution of the corresponding model R-H problem is also expressed in terms of Jacobi theta functions. Given that these functions can be regarded as multivalued functions on an underlying Riemann surface, the natural question arises whether the model R-H problem in the plane found in [6] can be viewed as a R-H problem on a Riemann surface instead. In our simple one-gap case, that would correspond to a R-H problem on a torus.

1.2. **Outline of this work.** In the next section we will show that the one-gap KdV model R-H problem can in fact be formulated as a *scalar-valued* R-H problem on the torus. Equivalently, solutions to this problem can be characterized by quasiperiodic meromorphic functions in the complex plane (see Eq. (2.12)), leading to the explicit R-H model solution found in [6] and singular solutions similar to the one described in [8] (see also [10]) in a straightforward manner. Moreover, we show that the symmetry condition from [11, Sect. 2] translates to halving the period (see Eq. (2.13)), while uniqueness follows from Liouville's Theorem.

Section 3 compares different regular and singular matrix-valued model solutions. As shown in [8], there is no regular matrix-valued model solution satisfying all the standard assumptions, hence it is necessary to drop some of them. We also comment on the regularity of the determinant of the solution in each case.

In the final section we compare our singular matrix-valued model solution to the ones found previously (see [8], [10]). We point out that the corresponding vanishing problem has a nontrivial solution, meaning that there is no uniqueness for the associated singular model problem. In particular, the solutions described in [8] and [10] differ from the one we presented in Section 3.

2. The model Riemann–Hilbert problem

In the following we recall the model vector-valued R-H problem for one-gap solutions of the KdV equation. For the underlying scattering theory and nonlinear steepest descent analysis leading to this problem in the transition region, we refer to [6, Sect. 4].

Find a vector-valued function $m^{\mathrm{mod}}(k) = (m_1^{\mathrm{mod}}(k),\ m_2^{\mathrm{mod}}(k))$ holomorphic in the domain $\mathbb{C} \setminus [\mathrm{i}c, -\mathrm{i}c]$, continuous up to the boundary except at points $\mathcal{G} := \{\mathrm{i}c, \mathrm{i}a, -\mathrm{i}a, -\mathrm{i}c\}$ and satisfying the jump condition (with $\tilde{\Lambda} = \frac{1}{2\pi}(\Lambda + tB)$, cf. [8, Sect. 3]):

$$m_+^{\mathrm{mod}}(k) = m_-^{\mathrm{mod}}(k) v^{\mathrm{mod}}(k), \tag{2.1}$$

where

$$v^{\mathrm{mod}}(k) = \begin{cases} \begin{pmatrix} 0 & \mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix}, & k \in [\mathrm{i}c, \mathrm{i}a], \\ \begin{pmatrix} 0 & -\mathrm{i} \\ -\mathrm{i} & 0 \end{pmatrix}, & k \in [-\mathrm{i}a, -\mathrm{i}c], \\ \begin{pmatrix} \mathrm{e}^{-2\pi\mathrm{i}\tilde{\Lambda}} & 0 \\ 0 & \mathrm{e}^{2\pi\mathrm{i}\tilde{\Lambda}} \end{pmatrix}, & k \in [\mathrm{i}a, -\mathrm{i}a], \end{cases} \tag{2.2}$$

the symmetry condition

$$m^{\mathrm{mod}}(-k) = m^{\mathrm{mod}}(k) \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \tag{2.3}$$

and the normalization condition

$$\lim_{k \to \mathrm{i}\infty} m^{\mathrm{mod}}(k) = (1 \;\; 1). \tag{2.4}$$

At any point $\kappa \in \mathcal{G}$ the vector function $m^{\mathrm{mod}}(k)$ can have at most a fourth root singularity: $m^{\mathrm{mod}}(k) = O((k-\kappa)^{-1/4})$, $k \to \kappa$.

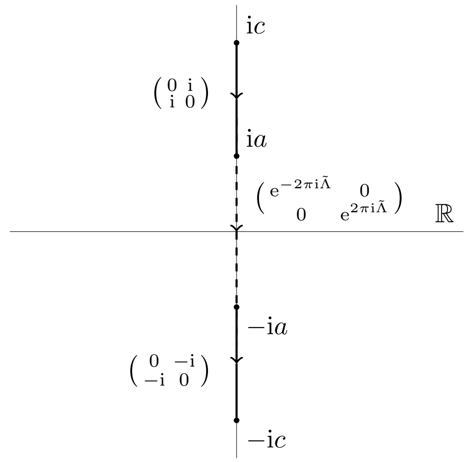

FIGURE 1. Jump contour for the model R-H problem

The solution to this problem was given in [6]. As mentioned in the introduction, we want to solve this problem in a slightly different way, which should shed some further light on the model problem. For this, it will be convenient to denote by $m(k)$ a generic vector-valued meromorphic function satisfying the jump condition (2.2).

For our first transformation we define

$$\tilde{\gamma}(k) = \sqrt[4]{\frac{k^2 + a^2}{k^2 + c^2}}$$

with the branch cuts along $[\pm \mathrm{i}a, \pm \mathrm{i}c]$ and the branch chosen such that $\tilde{\gamma}(k) > 0$ for $k \in [\mathrm{i}c, \infty)$. Note that we have $\tilde{\gamma}(-k) = \tilde{\gamma}(k)$ and $\tilde{\gamma}(k) > 0$ for $k \in \mathbb{R}$. Then $\tilde{\gamma}(k)$ solves the scalar R-H problem

$$\tilde{\gamma}_+(k) = \pm \mathrm{i} \tilde{\gamma}_-(k), \qquad k \in [\pm \mathrm{i}a, \pm \mathrm{i}c],$$

and we set

$$m(k) = \tilde{\gamma}(k) n(k) \tag{2.5}$$

such that $n(k)$ satisfies the jump condition (2.2), except that the jumps on $[-\mathrm{i}c, -\mathrm{i}a]$ and $[\mathrm{i}c, \mathrm{i}a]$ are replaced by

$$n_+(k) = n_-(k) \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}. \tag{2.6}$$

The reason for this change is that it will be convenient to look at this problem on the elliptic Riemann surface X associated with the function

$$w(k) = \sqrt{(k^2 + c^2)(k^2 + a^2)},$$

defined on $\mathbb{C}\backslash([-\mathrm{i}c,-\mathrm{i}a]\cup[\mathrm{i}a,\mathrm{i}c])$ with $w(0)>0$. The two sheets of X are glued along the cuts $[\mathrm{i}c,\mathrm{i}a]$ and $[-\mathrm{i}a,-\mathrm{i}c]$. Points on this surface are denoted by $p=(k,\pm)$. To simplify formulas we keep the notation $k=(k,+)$ for points on the upper sheet of X.

In this setup, the two components n_1, n_2 of the vector $n\colon\mathbb{C}\setminus[-\mathrm{i}c,\mathrm{i}c]\to\mathbb{C}^2$ can be regarded as the values of a single function $N\colon X\to\mathbb{C}$ on the upper, lower sheet, respectively. Explicitly,

$$n(k)=(N((k,+)),N((k,-))). \tag{2.7}$$

In this case the jump condition (2.6) implies that N will have no jump along the cuts, where the two sheets are glued together. However, the other jump will remain. In fact, the jump contour on X is a circle through the two branch points $-\mathrm{i}a$ and $\mathrm{i}a$, on which we have the jump condition

$$N_+(p)=N_-(p)\mathrm{e}^{-2\pi\mathrm{i}\tilde{\Lambda}}. \tag{2.8}$$

Note that the symmetry condition (2.3) translates to

$$N(p^*)=N(-p), \tag{2.9}$$

where $(k,\pm)^*=(k,\mp)$ denotes the sheet exchange map and $-(k,\pm)=(-k,\pm)$.

Next we choose a canonical homology basis of cycles $\{\mathbf{a},\mathbf{b}\}$ as follows: The $\mathbf{a}$-cycle surrounds the points $-\mathrm{i}a,\mathrm{i}a$ starting on the upper sheet from the left side of the cut $[\mathrm{i}c,\mathrm{i}a]$ and continues on the upper sheet to the left part of $[-\mathrm{i}a,-\mathrm{i}c]$ and returns after changing sheets. The cycle $\mathbf{b}$ surrounds the points $\mathrm{i}a,\mathrm{i}c$ counterclockwise on the upper sheet.

Then the normalized holomorphic differential is given by

$$d\omega=\Gamma\frac{d\zeta}{w(\zeta)},\quad\text{where }\Gamma:=\left(\int_{\mathbf{a}}\frac{d\zeta}{w(\zeta)}\right)^{-1}\in\mathrm{i}\mathbb{R}_-,$$

such that $\int_{\mathbf{a}}d\omega=1$ and

$$\tau=\int_{\mathbf{b}}d\omega\in\mathrm{i}\mathbb{R}_+.$$

Let

$$\theta_3(z\,|\,\tau)=\sum_{n\in\mathbb{Z}}\exp\big((n^2\tau+2nz)\pi\mathrm{i}\big),\quad z\in\mathbb{C},$$

be the associated Jacobi theta function (see for example [2]). Recall that θ_3 is even, $\theta_3(-z\,|\,\tau)=\theta_3(z\,|\,\tau)$, and satisfies

$$\theta_3(z+n+\tau\ell\,|\,\tau)=\theta_3(z\,|\,\tau)\mathrm{e}^{-\pi\mathrm{i}\tau\ell^2-2\pi\mathrm{i}\ell z}\quad\text{for}\quad\ell,n\in\mathbb{Z}. \tag{2.10}$$

Furthermore, let $A(p)=\int_{\mathrm{i}c}^{p}d\omega$ be the Abel map on X. We identify the upper sheet of X with the complex plane $\mathbb{C}\setminus([\mathrm{i}c,\mathrm{i}a]\cup[-\mathrm{i}a,-\mathrm{i}c])$. Restricting the path of integration to $\mathbb{C}\setminus[\mathrm{i}c,-\mathrm{i}c]$ we observe that $A(k)$ is a holomorphic function in that given domain with the following properties:

- $A_+(k)=-A_-(k)\pmod 1$, for $k\in[\mathrm{i}c,\mathrm{i}a]\cup[-\mathrm{i}a,-\mathrm{i}c]$;
- $A_+(k)-A_-(k)=-\tau$, for $k\in[\mathrm{i}a,-\mathrm{i}a]$;
- $A(-k)=-A(k)+\frac{1}{2}$, for $k\in\mathbb{C}\setminus[\mathrm{i}c,-\mathrm{i}c]$,
- $A_+(\mathrm{i}a)=-\frac{\tau}{2}=-A_-(\mathrm{i}a)$, $A_+(-\mathrm{i}a)=-\frac{\tau}{2}+\frac{1}{2}$, $A_-(-\mathrm{i}a)=\frac{\tau}{2}+\frac{1}{2}$.
- $A(\infty)=\frac{1}{4}$, $A(k)=\frac{1}{4}-\Gamma k^{-1}+O(k^{-3})$, as $k\to\infty$.

For points on the lower sheet we set $A(p^*) = -A(p)$. Finally, denote by $K = \frac{1+\tau}{2}$ the Riemann constant associated with X and abbreviate $\infty_\pm = (\infty, \pm)$, $0_\pm = (0, \pm)$. Note that $A(0_+) = \frac{1}{4} + \frac{\tau}{2}$. By Riemann's vanishing theorem [9] the zeros of θ_3 are simple and given by $z = K + \mathbb{Z} + \tau\mathbb{Z}$.

According to the Jacobi inversion theorem [9], the Abel map A maps our Riemann surface X bijectively to its associated Jacobi variety $\mathbb{C}/(\mathbb{Z} + \tau\mathbb{Z})$ depicted in Figure 2. The jump contour is indicated by the dashed line, while the dark/light

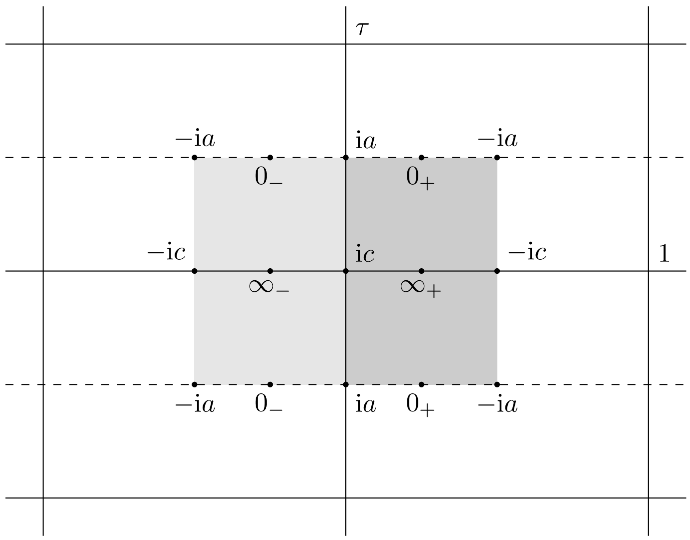

FIGURE 2. Jacobi variety (dark/light gray denotes the upper/lower sheet)

shaded region correspond to the upper/lower sheet. Moreover, a meromorphic function $E(z)$ given by

$$E(A(p)) = N(p) \tag{2.11}$$

will satisfy our original jump condition if and only if

$$E(z+1) = E(z), \qquad E(z+\tau) = E(z)\mathrm{e}^{2\pi\mathrm{i}\tilde{\Lambda}}, \tag{2.12}$$

and it will satisfy the symmetry condition if and only if

$$E(z + \tfrac{1}{2}) = E(z). \tag{2.13}$$

If this latter condition holds we will call E symmetric. If we have $E(z+\frac{1}{2}) = -E(z)$, we will call E anti-symmetric.

At this stage we remind the reader that we have four equivalent ways of describing vector-valued functions satisfying the jump condition (2.2):

$$m(k) \quad \longleftrightarrow \quad n(k) \quad \longleftrightarrow \quad N(p) \quad \longleftrightarrow \quad E(z)$$

related via (2.5), (2.7) and (2.11) respectively. The most convenient framework will be given through the quasiperiodic meromorphic functions $E(z)$. Let us consider the space $\mathcal{F}(\tilde{\Lambda})$ of all quasiperiodic meromorphic functions on $\mathbb{C}/(\mathbb{Z} + \tau\mathbb{Z})$ satisfying (2.12), without imposing any symmetry requirements. Note that $E \in \mathcal{F}(\tilde{\Lambda})$ is uniquely determined up to a constant by its divisor $(E) = \sum_{j=1}^n \mathcal{D}_{z_j} - \sum_{j=1}^n \mathcal{D}_{p_j}$, since the quotient of two such functions with the same divisor is elliptic without poles, hence a constant. Moreover, since $\frac{E'}{E}$ is elliptic, integrating this function along a fundamental polygon shows that the number of zeros and poles must be equal. Note also that there must be at least one pole (unless $\tilde{\Lambda} = 0$). Integrating $z\frac{E'(z)}{E(z)}$ along a fundamental polygon gives

$$\sum_{j=1}^{n} z_j - \sum_{j=1}^{n} p_j = \tilde{\Lambda} \pmod{\mathbb{Z} + \tau\mathbb{Z}}. \tag{2.14}$$

Choosing representatives z_j, $p_j \in \mathbb{C}$ such that

$$\sum_{j=1}^{n} z_j - \sum_{j=1}^{n} p_j = \tilde{\Lambda} \pmod{\mathbb{Z}}, \tag{2.15}$$

we can represent E as

$$E(z) = E_0 \prod_{j=1}^{n} \frac{\theta_3(z - z_j - K|\tau)}{\theta_3(z - p_j - K|\tau)}. \tag{2.16}$$

Indeed the right-hand side has the required zeros and poles while (2.10) and (2.15) ensure that it is elliptic.

Lemma 2.1. *The divisor of $E \in \mathcal{F}(\tilde{\Lambda})$ is invariant with respect to translations of $\frac{1}{2}$ if and only if E is either symmetric or anti-symmetric*

Proof. Observe that $C = \frac{E(z+\frac{1}{2})}{E(z)}$ is elliptic without poles and hence constant. Moreover, $C^2 = \frac{E(z+\frac{1}{2})}{E(z)} \cdot \frac{E(z+1)}{E(z+\frac{1}{2})} = 1$ shows $C = \pm 1$. The converse is trivial. □

Lemma 2.2. *If $E \in \mathcal{F}(\tilde{\Lambda})$ is (anti-)symmetric and has at most two poles, it is already uniquely determined up to a constant by its poles. Conversely, for each choice of two poles p_1, $p_2 = p_1 + \frac{1}{2}$ there is a unique (up to constants) symmetric and a unique anti-symmetric function $E \in \mathcal{F}(\tilde{\Lambda})$, with at most simple poles at p_1 and p_2. In fact p_1, p_2 are simple poles, unless $\tilde{\Lambda} \in \mathbb{Z}$, in which case the symmetric solution is constant.*

Proof. Let E be (anti-)symmetric and nonconstant. Denote its poles by p_1, $p_2 = p_1 + \frac{1}{2} \pmod{\mathbb{Z} + \tau\mathbb{Z}}$ and its zeros by z_1, $z_2 = z_1 + \frac{1}{2} \pmod{\mathbb{Z} + \tau\mathbb{Z}}$. Choosing representatives in $\mathbb{C}$, (2.14) implies $2(z_1 - p_1) = \tilde{\Lambda} + m + n\tau$ for some $m, n \in \mathbb{Z}$. In particular, since adding a period to z_1 is irrelevant, we can assume $m, n \in \{0, 1\}$. If $m = 1$ this just amounts to exchanging z_1 and z_2 and hence we can assume $m = 0$ without loss of generality. Now using $z_1 = p_1 + \frac{\tilde{\Lambda}}{2} + n\frac{\tau}{2}$, we can set $z_2 = z_1 + \frac{1}{2} - n\tau$ and $p_2 = p_1 + \frac{1}{2}$ such that (2.15) holds. Now one computes using (2.10) that (2.16) fulfills $E(z+\frac{1}{2}) = (-1)^n E(z)$. In other words, $p_1 \in \mathbb{C}/(\mathbb{Z}+\tau\mathbb{Z})$ and $n \in \mathbb{Z}_2$ uniquely determine E up to a constant. One can check, that the zeros and poles cancel if and only if $n = 0$ and $\tilde{\Lambda} \in \mathbb{Z}$, corresponding to a constant symmetric solution. □

Corollary 2.3. *If $E \in \mathcal{F}(\tilde{\Lambda})$ has at most two poles p_1, $p_2 = p_1 + \frac{1}{2}$ then there exist unique $c_s, c_a \in \mathbb{C}$ such that $E = c_s E_s + c_a E_a$, where E_s, E_a are the symmetric, anti-symmetric solutions constructed in the previous lemma, respectively.*

Returning to our original model problem, we want the poles of E to lie at the images of ia and $-$ia under the Abel map A, that is $p_1 = \frac{\tau}{2}$ and $p_2 = \frac{1+\tau}{2} = K$. The reason is that we require $m^{\mathrm{mod}}(k)$ to be holomorphic, with at most fourth root singularities at points of $\mathcal{G}$. As $\tilde{\gamma}(k)$ has fourth root zeros at $\pm$ia and the Abel map A has square root singularities at the points of $\mathcal{G}$, simple poles at $\frac{\tau}{2}$, $\frac{1+\tau}{2}$ in $\mathbb{C}/(\mathbb{Z}+\tau\mathbb{Z})$ translate to fourth root singularities at $\pm$ia of $m^{\mathrm{mod}}(k)$ under the inverse of the Abel map. In fact, this is the only choice of the pole structure leading to a holomorphic $m^{\mathrm{mod}}(k)$ with at most fourth root singularities.

For the zeros of the symmetric ($n=0$) and the anti-symmetric ($n=1$) solution we use $z_1 = \frac{\tilde{\Lambda}}{2} + \frac{(n+1)\tau}{2}$, $z_2 = \frac{\tilde{\Lambda}}{2} + \frac{1-(n-1)\tau}{2}$. Denote by

$$E_s(z) = \frac{\theta_3(z - \frac{\tilde{\Lambda}}{2} + \frac{1}{2}|\tau)\theta_3(z - \frac{\tilde{\Lambda}}{2}|\tau)}{\theta_3(z + \frac{1}{2}|\tau)\theta_3(z|\tau)} \tag{2.17}$$

the corresponding symmetric and by

$$E_a(z) = \frac{\theta_3(z - \frac{\tilde{\Lambda}}{2} - \frac{1+\tau}{2}|\tau)\theta_3(z - \frac{\tilde{\Lambda}}{2} + \frac{\tau}{2}|\tau)}{\theta_3(z + \frac{1}{2}|\tau)\theta_3(z|\tau)} \tag{2.18}$$

the corresponding anti-symmetric solution. Using the identity (cf. [5] formula (1.4.3))

$$\theta_3(z|\tau)\theta_3(z + \tfrac{1}{2}|\tau) = \theta_3(2z + \tfrac{1}{2}|2\tau)\,\theta_3(\tfrac{1}{2}|2\tau),$$

(note that the quotient of both sides is a holomorphic elliptic function which equals 1 at $z = \frac{1}{2}$) we can write the formula for E_s somewhat more compactly as

$$E_s(z) = \frac{\theta_3(2z - \tilde{\Lambda} + \frac{1}{2}|2\tau)}{\theta_3(2z + \frac{1}{2}|2\tau)}.$$

So for the symmetric case $n=0$, we have $\mathrm{i}\,\mathrm{Im}(z_j) = \frac{\tau}{2} \pmod{\mathbb{Z}+\tau\mathbb{Z}}$ and both zeros will be on $[-\mathrm{i}a, \mathrm{i}a]$ (see Figure 2). In the anti-symmetric case $n=1$ we have $\mathrm{i}\,\mathrm{Im}(z_j) = 0 \pmod{\mathbb{Z}+\tau\mathbb{Z}}$ and both zeros will be on $(\infty, -\mathrm{i}c] \cup [\mathrm{i}c, \infty]$. In particular, if $\tilde{\Lambda} = \frac{1}{2} \pmod 1$ the two zeros of the anti-symmetric solution will be at $\infty_\pm$ and we cannot normalize at this point. Moreover, if $\tilde{\Lambda} = 0 \pmod 1$ such that we are looking for elliptic functions, we have $E_s(z) = 1$ (i.e. zeros and poles coincide) and the zeros of $E_a(z)$ will be at $z_1 = 0$ and $z_2 = \frac{1}{2} + \tau$.

Hence all solutions of (2.8) with poles at most at $\pm$ia are given by

$$N(p) = c_s N_s(p) + c_a N_a(p), \quad N_s(p) = E_s(A(p)),\ N_a(p) = E_a(A(p)), \quad c_s, c_a \in \mathbb{C},$$

and we have

$$N(\infty_\pm) = c_s E_s(\tfrac{1}{4}) \pm c_a E_a(\tfrac{1}{4}), \qquad N(0_\pm) = c_s E_s(\tfrac{1}{4} + \tfrac{\tau}{2}) \pm c_a E_a(\tfrac{1}{4} + \tfrac{\tau}{2}),$$

with $E_s(\frac{1}{4}) \neq 0$ for all $\tilde{\Lambda} \in \mathbb{R}$ and $E_a(\frac{1}{4}) \neq 0$ for all $\tilde{\Lambda} \neq \frac{1}{2} \pmod 1$. Moreover, N will satisfy (2.9) if and only if $c_a = 0$.

Note that in the special case $\tilde{\Lambda} = 0 \pmod 1$ we have (up to constants):

$$N_s((k,\pm)) = 1 \qquad N_a((k,\pm)) = \frac{k^2 + c^2}{\pm w(k)}$$

In the case $\tilde{\Lambda} = \frac{1}{2} \pmod 1$ we have (again up to constants):

$$N_s((k,\pm)) = \frac{k}{\sqrt{k^2 + a^2}} \qquad N_a((k,\pm)) = \frac{1}{\sqrt{k^2 + a^2}},$$

where the root has the branch cut along $[-\mathrm{i}a, \mathrm{i}a]$.

Returning to our original problem we have shown:

Lemma 2.4. *The function*

$$m^{\mathrm{mod}}(k) = \frac{\tilde{\gamma}(k)}{N_s(\infty_+)}\big(N_s((k,+)), N_s((k,-))\big)$$

is the unique vector-valued function which is holomorphic in the domain $\mathbb{C}\setminus[\mathrm{i}c, -\mathrm{i}c]$*, has square integrable boundary values, and satisfies the jump condition* (2.1)*, the symmetry condition* (2.3) *and the normalization condition* (2.4)*.*

Specifically, $m^{mod}(k)$ *is continuous up to the boundary except at points of the set* $\mathcal{G} := \{\mathrm{i}c, \mathrm{i}a, -\mathrm{i}a, -\mathrm{i}c\}$ *where it has at most a fourth root singularity:* $m^{mod}(k) = O((k-\kappa)^{-1/4}))$*,* $k \to \kappa$*.*

3. Matrix-valued solutions

In the framework of the nonlinear steepest descent analysis one usually needs to construct a matrix-valued R-H solutions which is invertible. This is a necessary step to arrive at a small-norm R-H problem which can be solved via a Neumann series. However, while many integrable wave equations like the modified KdV equation [3] or the nonlinear Schrödinger equation [4] have a matrix-valued R-H formulation, this is not the case for the KdV equation.

Recall that the model matrix-valued R-H problem related to the KdV equation has a jump matrix satisfying (see [6], [11])

$$v^{\mathrm{mod}}(-k) = \sigma_1 (v^{\mathrm{mod}}(k))^{-1}\sigma_1, \qquad \det v(k) = 1, \quad k \in \Sigma. \tag{3.1}$$

For the corresponding holomorphic matrix-valued solution $M^{\mathrm{mod}}(k)$, one would then require

$$\lim_{k\to\infty} M^{\mathrm{mod}}(k) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}. \tag{3.2}$$

Moreover, given holomorphicity of $M^{\mathrm{mod}}(k)$, we can derive from (3.1) and (3.2):

$$M^{\mathrm{mod}}(-k) = \sigma_1 M^{\mathrm{mod}}(k)\sigma_1, \qquad \det M^{\mathrm{mod}}(k) \equiv 1. \tag{3.3}$$

Note that (3.3) implies that $M^{\mathrm{mod}}(k)$ must have the form

$$M^{\mathrm{mod}}(k) = \begin{pmatrix} \tilde{\alpha}(k) & \tilde{\beta}(-k) \\ \tilde{\beta}(k) & \tilde{\alpha}(-k) \end{pmatrix}$$

and that $(1,1)M^{\mathrm{mod}}(k)$ will satisfy the symmetry condition (2.3).

Let us now present three different ways of writing down matrix-valued solutions of the model R-H problem corresponding to the KdV equation with steplike initial data in the transition region. Each one will violate some standard assumption described above, as satisfying all of them is in general impossible, see [8, Rem. 4.1].

3.1. **Partial normalization at infinity.** We start with the function

$$M_1^{\mathrm{mod}}(k) = \tilde{\gamma}(k)\begin{pmatrix} \alpha_1(k) & \beta_1(-k) \\ \beta_1(k) & \alpha_1(-k) \end{pmatrix}$$

where

$$\alpha_1(k) = \frac{1}{2}\big(N_a(\infty_+)N_s(k) + N_s(\infty_+)N_a(k)\big),$$
$$\beta_1(k) = \frac{1}{2}\big(N_a(\infty_+)N_s(k) - N_s(\infty_+)N_a(k)\big).$$

Note that $M_1^{\text{mod}}(k)$ satisfies the symmetry condition in (3.3) and satisfies *partially* the normalization (3.2). In fact we have

$$\lim_{k\to\infty} M_1^{\text{mod}}(k) = N_s(\infty_+)N_a(\infty_+)\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix},$$

with

$$\det M_1^{\text{mod}}(k) = N_s(\infty_+)^2 N_a(\infty_+)^2.$$

The problem here is that the prefactor vanishes for $\tilde{\Lambda} = \frac{1}{2} \pmod 1$ as $N_a(\infty_+) = 0$, hence we cannot enforce the normalization (3.2) for all $\tilde{\Lambda}$. In particular, $M_1^{\text{mod}}(k)$ is not invertible for these values of $\tilde{\Lambda}$. The relation to $m^{\text{mod}}(k)$ is given through

$$m^{\text{mod}}(k) = \frac{1}{N_s(\infty_+)N_a(\infty_+)}(1,1)M^{\text{mod}}(k).$$

where one needs to use the rule of l'Hôspital for $\tilde{\Lambda} = \frac{1}{2} \pmod 1$.

3.2. **Different symmetry condition.** The function

$$M_2^{\text{mod}}(k) = \tilde{\gamma}(k)\begin{pmatrix}N_s(k) & N_s(-k)\\ N_a(k) & N_a(-k)\end{pmatrix}$$

is a matrix-valued solution which satisfies the new symmetry condition $M(-k) = \sigma_3 M(k)\sigma_1$ and is nondiagonal at infinity. Moreover,

$$\det M_2^{\text{mod}}(k) = -2\tilde{\gamma}(0)N_s(0_+)N_a(0_+) = -2N_s(\infty_+)N_a(\infty_+),$$

and the relation to the vector-valued model solution is given by

$$m^{\text{mod}}(k) = \frac{1}{N_s(\infty_+)}(1,0)M_2^{\text{mod}}(k),$$

which does not require the rule of l'Hôsptial as $N_s(\infty_+) \neq 0$ for all $\tilde{\Lambda}$. The advantage of this matrix-valued solution is that its determinant has only first order zeros.

3.3. **Singularity at the origin.** Finally, we write down a matrix-valued solution $M_3^{\text{mod}}(k)$ with determinant constant equal to 1. The price we have to pay is a singularity at the origin, making $M_3^{\text{mod}}(k)$ a meromorphic solution. To be precise, we will move the poles of the anti-symmetric solution to $\hat{p}_{1,2} = \pm\frac{1}{4} + \frac{\tau}{2}$, which corresponds to a pole at $(0,\pm)$ on X. Following Section 2, this gives rise to an anti-symmetric solution of the form

$$\hat{E}_a(z) = \frac{\theta_3(z + \frac{1}{4} - \frac{\tilde{\Lambda}}{2} - \frac{\tau}{2}|\tau)\theta_3(z - \frac{1}{4} - \frac{\tilde{\Lambda}}{2} + \frac{\tau}{2}|\tau)}{\theta_3(z+\frac{1}{4}|\tau)\theta_3(z-\frac{1}{4}|\tau)}, \qquad \hat{N}_a(p) = \hat{E}_a(A(p)). \tag{3.4}$$

We can now define $M_3^{\text{mod}}(k)$ analogously to $M_1^{\text{mod}}(k)$, but substituting $\hat{N}_a(k)$ for $N_a(k)$, and including the correct normalization at infinity for $\tilde{\Lambda} \neq 0 \pmod 1$:

$$M_3^{\text{mod}}(k) = \frac{\tilde{\gamma}(k)}{N_s(\infty_+)\hat{N}_a(\infty_+)}\begin{pmatrix}\alpha_3(k) & \beta_3(-k)\\ \beta_3(k) & \alpha_3(-k)\end{pmatrix}, \qquad \tilde{\Lambda} \neq 0 \pmod 1,$$

where

$$\alpha_3(k) = \frac{1}{2}\big(\hat{N}_a(\infty_+)N_s(k) + N_s(\infty_+)\hat{N}_a(k)\big),$$
$$\beta_3(k) = \frac{1}{2}\big(\hat{N}_a(\infty_+)N_s(k) - N_s(\infty_+)\hat{N}_a(k)\big),$$

Note that $N_s(\infty_+)\hat{N}_a(\infty_+) \neq 0$ for $\tilde{\Lambda} \neq 0 \pmod 1$ and thus we have:

$$\lim_{k\to\infty} M_3^{\mathrm{mod}}(k) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \qquad \tilde{\Lambda} \neq 0 \pmod 1.$$

Moreover, $\det M_3^{\mathrm{mod}}(k)$ is an even meromorphic function with at most a simple pole at the origin, hence $\det M_3^{\mathrm{mod}}(k) \equiv \det M_3^{\mathrm{mod}}(\infty) = 1$. We do not define $M_3^{\mathrm{mod}}(k)$ for $\tilde{\Lambda} = 0 \pmod 1$, which should not pose a problem in applications, as explained in the next section.

4. Comparison to previous work

It turns out that sacrificing holomorphicity, while retaining (3.2) and (3.3), is the most convenient way to deal with the ill-posedness of the holomorphic matrix-valued model problem for the KdV equation. Indeed, this was the strategy in [8]. Note however, that the anti-symmetric meromorphic vector solutions found [8] and [10] are not the same as given by (3.4). The reason is, that while we assumed that $\hat{N}_a(p) = \hat{E}_a(A(p))$ has only poles at $0_\pm$, the pole condition was not necessary, as we could still allow for singularities at $\pm \mathrm{i}a$, as is the case for $m^{\mathrm{mod}}(k)$. Indeed, $m^{\mathrm{mod}}(k)/k$ is an anti-symmetric solution to the vanishing problem where solutions are required to vanish at infinity. Hence, there is no chance for uniqueness if we allow for poles at 0 and fourth root singularities at $\pm \mathrm{i}a$. Moreover for $\tilde{\Lambda} = 0 \pmod 1$, $m^{\mathrm{mod}}(k)$ has no singularities at $\pm \mathrm{i}a$, and hence by our uniqueness Lemma 2.2 must coincide with the solution generated by $\hat{N}_a(p)$.

Interestingly, any anti-symmetric solution with a simple pole at the origin and fourth root singularities at $\pm \mathrm{i}a$, which is normalized to $(-1, 1)$ at infinity, is adequate for the analysis performed in [8]. The reason is that the pole cancellation in the final step of the nonlinear steepest descent analysis is due to the underlying symmetry class, rather than the exact form of the second vector-valued solution (see Lemma 6.4 in [8]). While the solution generated by (3.4) is not normalizable for $\tilde{\Lambda} = 0 \pmod 1$, this is not an issue, as for these values of $\tilde{\Lambda}$ there is a regular matrix-valued model solution given in terms of (2.17), (2.18) anyways.

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Faculty of Mathematics, University of Vienna, Oskar-Morgenstern-Platz 1, 1090 Wien

E-mail address: Mateusz.Piorkowski@univie.ac.at

Faculty of Mathematics, University of Vienna, Oskar-Morgenstern-Platz 1, 1090 Wien

E-mail address: Gerald.Teschl@univie.ac.at

URL: http://www.mat.univie.ac.at/~gerald/

RIEMANN–HILBERT THEORY WITHOUT LOCAL PARAMETRIX PROBLEMS: APPLICATIONS TO ORTHOGONAL POLYNOMIALS

MATEUSZ PIORKOWSKI

Abstract. We study whether in the setting of the Deift–Zhou nonlinear steepest descent method one can avoid solving local parametrix problems, while still obtaining asymptotic results. We show that this can be done, provided an a priori estimate for the exact solution of the Riemann–Hilbert problem is known. This enables us to derive asymptotic results for orthogonal polynomials on $[-1, 1]$ with a new class of weight functions. These weight functions are in general too badly behaved to allow a reformulation of the local parametrix problem to a global one with constant jump matrices. Possible implications for edge universality in random matrix theory are also discussed.

1. Introduction

1.1. **Background.** Local parametrix problems appear frequently in the context of the nonlinear steepest descent method for Riemann–Hilbert (R-H) problems, formulated by Deift and Zhou ([12, Sect. 4], [14, Sect. 7], [16], [18, Sect. 4], for details see [17]). Throughout this paper, *local parametrix problem* will refer, as the name suggests, to the restriction of a R-H problem to a bounded set together with a matching condition on the boundary[1] (for an example see Section 4.3). This should be contrasted with the term *limiting parametrix problem*, which corresponds to the R-H problem one gets after taking a limit of the local parametrix problems as the asymptotic parameter tends to infinity. Most common examples of such R-H problems are the Airy R-H problem ([6], [9, Ch. 5], [13], [14], [53], see Appendix A), the Bessel R-H problem [33] and the cylindrical parabolic R-H problem ([16], [17], [29], [32], [35]). We introduce this distinction, as we do not assume that the local parametrix problems in this paper converge to a limiting parametrix problem. Note that in most applications the limiting parametrix problem exists, and the term *local parametrix problem* or more generally *model problem*[2] is also used for it.

Solutions to limiting parametrix problems can determine either the leading asymptotics ([17], [27], [32]), or contribute to higher-order corrections ([9, Ch. 7], [12], [14], [33]). Interestingly, even in the former case one has to study the limiting parametrix problem to obtain rigorous leading asymptotics. Usually this is done

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[1]Some results on the matching condition for higher-dimensional R-H problems have been obtained in [47].

[2]In this work however, we will reserve the term *model problem* for the global parametrix problem.

by explicitly constructing the solution. There are however a few exception which serve as the motivation for this paper. In [13, Sect. 5] (see also [7, Sect. 2.2]) a solution to the limiting parametrix problem was shown to exist without an explicit construction. The approach was based on Fredholm theory for singular operators and a vanishing lemma.[3] A similar technique was used in [31], where polynomials orthogonal with respect to the Freud weight $e^{-\beta|x|}$, $\beta > 0$ were studied. Note that the weight function is not analytic locally around 0, giving rise to a new kind of limiting parametrix problem. For $\beta \in (0, 1)$ a vanishing lemma was used to show existence of the solution, while for $\beta \geq 1$ the authors relied on a small-norm argument in the Wiener space to show invertibility of the corresponding singular integral operator via a Neumann series.

Note that in these cases the formulation of a limiting local parametrix problem was still necessary. However, the existence of such a problem puts additional constraints on the regularity of the weight function. Hence, the natural question arises, whether in general the construction of explicit solutions, or even the computation/existence of a limiting parametrix problem is necessary. A negative answer would be useful in applications with no known limiting parametrix problems, see for example [8] which we discuss towards the end of this subsection.

In this paper we show how the computation of a limiting parametrix problem can be avoided and use our method to obtain new error estimates for Plancherel–Rotach asymptotics of orthogonal polynomials [54]. Asymptotics of more general orthogonal polynomials were studied thoroughly by Bernstein and Szegő on the unit interval and unit circle (see [56, Ch. 12] and references in therein). There has been renewed interested in these asymptotics motivated by the Wigner–Dyson–Mehta universality conjecture in random matrix theory ([21], [22], [45], [46]). In this setting orthogonal polynomials can be applied most naturally to unitary ensembles ([6], [9], [13], [52]), but also to orthogonal and symplectic ensembles ([10], [11], [49], [50]). The R-H formulation, first introduced by Fokas, Its and Kitaev ([24], [25]), in conjunction with the nonlinear steepest descent method is particularly useful in this context. The nonlinear steepest descent method was applied to orthogonal polynomials on the real line by Bleher and Its in [6], (see also [12]) and by Deift et al. in [13] and [14] (for an introduction see the book by Deift [9]). Based on this work Kuijlaars et al. computed in [33] the asymptotics of polynomials orthogonal on $[-1, 1]$ and related quantities. The leading asymptotic terms were already known [56, Thm. 12.1.1–4]. However, the R-H analysis leads to more explicit error terms and in the case of [33] even an asymptotic expansion of the orthogonal polynomials was obtained. The follow-up paper [34] relates these results to bulk and edge universality in random matrix theory (see also [49]). We comment more on this topic in relation to our results in the discussion section.

As part of the R-H analysis performed in [33] one is confronted with local parametrix problems at $x = \pm 1$. The convergence to a limiting Bessel parametrix problem puts constraints on the local behaviour of the weight function at the corresponding endpoints. In particular, the authors considered the modified Jacobi weight function $\rho^{\alpha,\beta}_{Jac}$:

$$\rho^{\alpha,\beta}_{Jac}(x) := (1-x)^{\alpha}(1+x)^{\beta}h(x), \quad x \in (-1, 1),\ \alpha, \beta > -1, \tag{1.1}$$

[3] See [53] for a similar application of Fredholm index theory to the R-H problem for the KdV equation.

where h is strictly positive on $[-1,1]$ and assumed to have an analytic continuation to a neighbourhood of $[-1,1]$ (note the inclusion of the endpoints ± 1). In our work we do not consider the prefactor $(1-x)^{\alpha}(1+x)^{\beta}$, but rather assume that our weight function ρ has an analytic continuation only in a lens-shaped neighbourhood of $(-1,1)$, together with some growth conditions near $x = \pm 1$. The difference might seem to be minor, but the possibility that the weight function does not have an analytic continuation in a neighbourhood of $x = \pm 1$ makes the R-H analysis performed in [33, Sect. 6] impossible. In particular, one cannot write down the limiting parametrix problem with constant jump matrices, as this step relies on the local analyticity of the function h in (1.1) around $x = \pm 1$ (see Eq. (6.7) in [33]). For an alternative approach to this problem see [44], where the $\overline{\partial}$ steepest descent method is used instead, which can also deal with nonanalytic varying weight functions $e^{-NV(x)}$ on the real line via an analytic approximation, leading to known local parametrix problems. The external field $V(x)$ is assumed to have only two Lipschitz continuous derivatives. Subsequently, the $\overline{\partial}$ method was used in [3] for the case of Hölder perturbations of the Jacobi weight. For more recent developments in Plancherel–Rotach asymptotics see the work by Lubinsky [42], [43].

As already mentioned, the methods described in this papers can be used for problems that do not have known limiting parametrix solutions. However, they can be also used in cases where the limiting parametrix problem cannot be derived. A recent example can be found in [8], where the weight function $\rho_{\log}$ on $[-1,1]$ with a logarithmic singularity at $x = 1$ was considered:

$$\rho_{\log}(x) := \log \frac{2k}{1-x}, \quad x \in (-1,1),\ k > 1.$$

To the best of the authors knowledge the corresponding R-H problem has no known limiting parametrix problem around $x = 1$. This issue was circumvented through a comparison argument with the Legendre problem ($\rho_{Leg}(x) \equiv 1$). However, the analytic continuation of $\rho_{\log}$ around the point $x = 1$, which introduces an explicit jump condition on $(1, 1+\delta)$, $\delta > 0$, has been crucial. The weights considered in this paper, are not required to have such analytic continuation around the endpoints.

1.2. **Outline of this paper.** In the next section we set the stage by discussing the notion of approximating solutions to R-H problems. We then show that the construction of a limiting parametrix solution can be avoided, provided a certain a priori L^p-estimate of the exact solution to the global R-H problem and a regular enough global parametrix solution is known. Our method uses the connection between R-H problems and singular integral equations, which will be briefly summarized.

In Section 3 we describe local parametrix problems as they appear in practice, and analyze them using our method. We summarize our findings in Theorem 3.1 and also state Lemma 3.3 which will be crucial for obtaining the a priori L^p-estimate.

Following this, an application of our approach is presented for the case of orthogonal polynomials on the interval $[-1,1]$. We consider a new class of weight functions and obtain a bound of the error term in the pointwise asymptotics of the orthogonal polynomials, as the degree goes to infinity. The main result is summarized in Theorem 4.4. We then elaborate on why the local parametrix problem does not converge in our setting to a limiting parametrix problem with constant jump matrices (cf. [33, Sect. 6]). Moreover, we illustrate how our method uses the exact

R-H solution consisting of orthogonal polynomials and their Cauchy transforms, as a local parametrix solution.

In the discussion section we elaborate on connections with random matrix theory, in particular eigenvalue universality near the edge of the spectrum. Moreover, we describe the advantages and limitations of our approach, mention possible applications and point towards future challenges related to obtaining the a priori L^p-estimate.

The first appendix contains a description of the Airy R-H problem, including the heuristics by which the explicit parametrix solution can be found. In the second appendix it is shown that a local a priori L^p-estimate, instead of the one described in Section 3, is also sufficient. This is crucial in applications different than the one considered in this paper.

2. Approximating solutions of Riemann–Hilbert problems

2.1. **Two Riemann–Hilbert problems.** Consider a R-H problem with data $(v_{\mathcal{S}}, \Sigma)$, meaning with jump matrix $v_{\mathcal{S}}$ and jump contour Σ. We are looking for a matrix-valued function S, normalized at infinity, such that:

(i) $S(z)$ is analytic for $z \in \mathbb{C} \setminus \Sigma$,

(ii) $S_+(k) = S_-(k) v_{\mathcal{S}}(k)$, for $k \in \Sigma$,

(iii) $S(z) = \mathbb{I} + O(z^{-1})$, as $z \to \infty$.

Note that condition (iii) is not specified by the data $(v_{\mathcal{S}}, \Sigma)$ and has to be stated separately. Also, to guarantee uniqueness, one usually has to assume additional regularity requirements for $S_\pm$, see for example [33, Eqs. (4.6)–(4.8)]. Here Σ is a 'sufficiently smooth' oriented contour and the $+(-)$ sign corresponds to taking the left (right) limit to Σ. The precise conditions on Σ and the sense in which the limits are taken can be found [36].

Next, consider a global parametrix R-H problem with data $(v_{\mathcal{N}}, \Sigma^{mod})$, which is suppose to approximate, or *model*, the R-H problem for S. We assume $\Sigma^{mod} \subseteq \Sigma$, and the same normalization at infinity as for S. Hence we are looking for a matrix valued-function N, such that:

(i) $N(z)$ is analytic for $z \in \mathbb{C} \setminus \Sigma^{mod}$,

(ii) $N_+(k) = N_-(k) v_{\mathcal{N}}(k)$, for $k \in \Sigma^{mod}$,

(iii) $N(z) = \mathbb{I} + O(z^{-1})$, as $z \to \infty$.

We assume that both R-H problems are solvable. Despite the aforementioned uniqueness issue, denote by S and N two special solutions (fixed by e.g. regularity assumptions for $S_\pm$ and $N_\pm$ on Σ). We refer to S as the *exact solution* and to N as the *model solution*.[4] The general question we try to answer in this section is the following:

[4] We refrain from using the term *global parametrix solution* for N to simplify nomenclature.

(2.1) Under what conditions is the model solution N a good approximation to the exact solution S?

Before tackling (2.1), we need to answer the more basic question:

(2.2) What does it mean for a model solution N to be a good approximation to the exact solution to S?

The answer to question (2.2) depends on the problem at hand. In the case of orthogonal polynomials pointwise estimates are of interest, meaning we would like $S(z)$ to be close to $N(z)$ for $z \in \mathbb{C} \setminus \Sigma$. In problems involving Toeplitz and Hankel determinants, or scattering theory, we are interested in the first Laurent term of S at infinity, meaning the complex number $\theta_{\mathcal{S}}$ given by

$$S(z) = \mathbb{I} + \frac{\theta_{\mathcal{S}}}{z} + O(z^{-2}), \quad z \to \infty.$$

Hence, we would like $\theta_{\mathcal{N}}$ with

$$N(z) = \mathbb{I} + \frac{\theta_{\mathcal{N}}}{z} + O(z^{-2}), \quad z \to \infty,$$

to be close to $\theta_{\mathcal{S}}$.

Usually, the jump matrix $v_{\mathcal{S}}$ (and sometimes $v_{\mathcal{N}}$)[5] depends on some auxiliary continuous or discrete parameter. In the case of orthogonal polynomials this parameter is the polynomial degree n and we demand for the n-dependent solution S:

$$S(z, n) = (\mathbb{I} + o(1))N(z), \quad z \in \mathbb{C} \setminus \Sigma, \quad n \to \infty.$$

In the case of scattering theory this parameter, denoted by t, is time and we demand:

$$\theta_{\mathcal{S}}(t) = \theta_{\mathcal{N}}(t) + o(1), \quad t \to \infty,$$

where the error term is a measure of the accuracy of the approximation. Having answered question (2.2), we now move to question (2.1). For this we need to reformulate a R-H problem as a singular integral equation.

2.2. **Singular integral formulation of Riemann–Hilbert problems.** To find approximations to solutions of R-H problems as described in the last section, we need to reformulate a R-H problem as an equivalent singular integral equation. The underlying theory can be found in [5], [48], [60], for more recent developments see [36], [53]. Let us define the Cauchy operator $\mathcal{C}^{\Sigma}$ associated to an oriented contour Σ:

$$\mathcal{C}^{\Sigma} : L^p(\Sigma) \to \mathcal{O}(\mathbb{C} \setminus \Sigma), \quad f \to \mathcal{C}^{\Sigma}(f)(z) := \frac{1}{2\pi \mathrm{i}} \int_{\Sigma} \frac{f(k)}{k - z}\, dk,$$

with $p \in (1, \infty)$, which shall be assumed throughout this section. The only further requirement needed for $\mathcal{C}^{\Sigma}$ to be well-defined is that $(k - z)^{-1}$ is in $L^q(\Sigma)$ with $p^{-1} + q^{-1} = 1$ for some, and hence for every $z \in \mathbb{C} \setminus \Sigma$. Given some further

[5]In the case of orthogonal polynomials $v_{\mathcal{N}}$ is independent of the degree n. For integrable wave equations in the elliptic wave region, $v_{\mathcal{N}}$ is periodic in the time parameter (see [23]).

regularity assumptions on Σ which are fulfilled in most applications including ours [36], we can define two bounded operators given by:

$$\mathcal{C}^{\Sigma}_{\pm} : L^p(\Sigma) \to L^p(\Sigma), \quad f \mapsto \mathcal{C}^{\Sigma}_{\pm}(f)(k) := \lim_{z \to k\pm} \mathcal{C}^{\Sigma}(f)(z),$$

where the limits are assumed to be nontangential, in which case they exist a.e. on Σ.

We now turn to a bijection between solutions of R-H problems and solutions of a certain singular integral equation. These results can be found in [60]. We assume that $w_{\mathcal{R}} := v_{\mathcal{R}} - \mathbb{I} \in L^p(\Sigma)$[6], where we abuse notation by denoting as $L^p(\Sigma)$ the space of matrix functions with entries in $L^p(\Sigma)$ (the subscript $\mathcal{R}$ is chosen for later convenience). Associated to $w_{\mathcal{R}}$ and Σ, let $M^{\Sigma}_{w_{\mathcal{R}}}$ be the maximal domain of the multiplication operator defined by $w_{\mathcal{R}}$, meaning

$$M^{\Sigma}_{w_{\mathcal{R}}} := \{f \in L^p(\Sigma) : f w_{\mathcal{R}} \in L^p(\Sigma)\}.$$

With this we can define the operator $\mathcal{C}^{\Sigma}_{w_{\mathcal{R}}}$ associated to a R-H problem with data $(v_{\mathcal{R}}, \Sigma)$:

$$\mathcal{C}^{\Sigma}_{w_{\mathcal{R}}} : M^{\Sigma}_{w_{\mathcal{R}}} \to L^p(\Sigma), \quad f \mapsto \mathcal{C}^{\Sigma}_{-}(f w_{\mathcal{R}}).$$

The operator $\mathcal{C}^{\Sigma}_{-}$ is evaluated componentwise for matrix inputs. To next proposition found in [60, Prop. 3.3] describes the correspondence between R-H problems and certain singular integral equation and is central for our approach.

Proposition 2.1. *Let $(v_{\mathcal{R}}, \Sigma)$ be the data of a R-H problem, and assume that $w_{\mathcal{R}} := v_{\mathcal{R}} - \mathbb{I} \in L^p(\Sigma)$. Then there is a bijection between R-H solutions R, satisfying*

$$\lim_{z \to \infty} R(z) \to \mathbb{I}, \quad R_{\pm} - \mathbb{I} \in L^p(\Sigma)$$

and solutions $\Phi \in M^{\Sigma}_{w_{\mathcal{R}}}$ of

$$(\mathbb{I} - \mathcal{C}^{\Sigma}_{w_{\mathcal{R}}})\Phi = \mathcal{C}^{\Sigma}_{-}(w_{\mathcal{R}}). \tag{2.3}$$

Moreover the relation between R and Φ is given by

$$R = \mathbb{I} + \mathcal{C}^{\Sigma}((\Phi + \mathbb{I})w_{\mathcal{R}}), \tag{2.4}$$

$$\Phi = R_{-} - \mathbb{I}. \tag{2.5}$$

Proof. Let R be a solution of the R-H problem and define $\Phi := R_{-} - \mathbb{I}$. Then by assumption $\Phi \in L^p(\Sigma)$ and as

$$\Phi w_{\mathcal{R}} = (R_{-} - \mathbb{I})(v_{\mathcal{R}} - \mathbb{I}) = R_{+} - R_{-} - w_{\mathcal{R}} \in L^p(\Sigma)$$

we indeed see that $\Phi \in M^{\Sigma}_{\mathcal{R}}$. Next, using the Sokhotski–Plemelj formula for additive R-H problems, we obtain from

$$R_{+} - R_{-} = R_{-} w_{\mathcal{R}}$$

the equality

$$R = \mathbb{I} + \mathcal{C}^{\Sigma}(R_{-} w_{\mathcal{R}}). \tag{2.6}$$

Taking the limit to the contour Σ from the right, we get

$$R_{-} = \mathbb{I} + \mathcal{C}^{\Sigma}_{w_{\mathcal{R}}}(R_{-}).$$

[6] Usually it is assumed that $w_{\mathcal{R}} \in L^{\infty}(\Sigma)$, which would imply $M^{\Sigma}_{w_{\mathcal{R}}} = L^p(\Sigma)$ and the boundedness of $\mathcal{C}^{\Sigma}_{w_{\mathcal{R}}}$. This assumption will not be needed and even violated in our application to orthogonal polynomials

which after substituting $R_- = \Phi + \mathbb{I}$ is equivalent to (2.3).

Next, let $\Phi \in M^{\Sigma}_{w_{\mathcal{R}}}$ satisfy (2.3) and define $R := \mathbb{I} + \mathcal{C}^{\Sigma}((\Phi + \mathbb{I})w_{\mathcal{R}})$. Note that from the assumptions on Φ and $w_{\mathcal{R}}$ it follows that $R(z) \to \mathbb{I}$, as $z \to \infty$. By the Sokhotski–Plemelj formula we have

$$R_+ - R_- = (\Phi + \mathbb{I})w_{\mathcal{R}}. \tag{2.7}$$

On the other hand we compute

$$\begin{aligned} R_- - \mathbb{I} &= \mathcal{C}^{\Sigma}_-((\Phi + \mathbb{I})w_{\mathcal{R}}) \\ &= \mathcal{C}^{\Sigma}_{w_{\mathcal{R}}}(\Phi) + C^{\Sigma}_-(w_{\mathcal{R}}) \\ &= \Phi, \end{aligned}$$

as Φ satisfies (2.3). Substituting this into (2.7) results in

$$R_+ = R_- v_{\mathcal{R}}.$$

Furthermore, we have

$$\begin{aligned} R_- - \mathbb{I} &= \Phi \in L^p(\Sigma) \\ R_+ - \mathbb{I} &= \Phi w_{\mathcal{R}} + \Phi + w_{\mathcal{R}} \in L^p(\Sigma). \end{aligned}$$

Hence we see that R is a solution of the R-H problem with the required properties and the proof is finished. □

For analogous results in the case of inhomogeneous R-H problems consult [20, Prop. 2.5].

2.3. **Residual Riemann-Hilbert problem.** Let us now return to the solutions S and N from the beginning of the section, under the assumption that $v_{\mathcal{S}}$ and hence S depends on a discrete parameter $n \in \mathbb{N}_0$. Moreover let N and its limits to the contour Σ^{mod} be invertible. As $v_{\mathcal{N}} = N_+ N_-^{-1}$, this implies that $v_{\mathcal{N}}$ is also invertible.

We can now define a new matrix-valued function $R := SN^{-1}$. Assuming $\Sigma^{mod} \subseteq \Sigma$, we see that R will have jumps only on Σ. We call R the *residual solution* and it satisfies the residual R-H problem with data $(v_{\mathcal{R}}, \Sigma)$, where

$$v_{\mathcal{R}} := N_- v_{\mathcal{S}} N_+^{-1}.$$

Assuming that $R_- w_{\mathcal{R}}$ is integrable on Σ, where $w_{\mathcal{R}} = v_{\mathcal{R}} - \mathbb{I}$, we know from the Sokhotski–Plemelj formula that R can be also written in integral form:

$$\begin{aligned} R(z,n) &= \mathbb{I} + \mathcal{C}^{\Sigma}((R_- w_{\mathcal{R}})(z,n) \\ &= \mathbb{I} + \frac{1}{2\pi \mathrm{i}} \int_{\Sigma} \frac{R_-(k,n) w_{\mathcal{R}}(k,n)}{k-z}\, dk \\ &= \mathbb{I} + \frac{1}{2\pi \mathrm{i}} \int_{\Sigma} \frac{S_-(k,n)(v_{\mathcal{S}}(k,n) - v_{\mathcal{N}}(k)) N_+^{-1}(k)}{k-z}\, dk. \end{aligned} \tag{2.8}$$

The quantity of interest is the n-dependent $L^1(\Sigma)$-norm of the integrand without $(k-z)^{-1}$:

$$\left\| S_-(v_{\mathcal{S}} - v_{\mathcal{N}}) N_+^{-1} \right\|_{L^1(\Sigma)}. \tag{2.9}$$

Let us assume that (2.9) is of order $\varepsilon(n)$ where $\varepsilon : \mathbb{N}_0 \to \mathbb{R}_+$. Observe that in this case we have

$$R(z,n) = \mathbb{I} + O(\varepsilon(n) \operatorname{dist}(z,\Sigma)^{-1}),$$

which implies

$$S(z,n) = (\mathbb{I} + O(\varepsilon(n)\operatorname{dist}(z,\Sigma)^{-1})N(z).$$

Here $\operatorname{dist}(z,\Sigma)$ denotes the distance between z and Σ. Hence, we see that in order to show convergence of N to S away from the contour Σ, we need to control (2.9).

3. Riemann–Hilbert problems in applications

We now compare the setting described in the last section with R-H problems appearing in practise. In the case of orthogonal polynomials ([14], [33]) and scattering theory ([1], [53]), the contour Σ is a disjoint union of the contours Σ^{mod} and Σ^{exp}, except for finitely many points of intersections. The n-independent matrix $v_{\mathcal{N}}$ is given by

$$v_{\mathcal{N}}(k) = \begin{cases} v_{\mathcal{S}}(k,n), & k \in \Sigma^{mod}, \\ \mathbb{I}, & k \in \Sigma^{exp} \setminus \Sigma^{mod}. \end{cases} \tag{3.1}$$

In particular, $v_{\mathcal{S}}(k,n)$ is assumed to be n-independent for $k \in \Sigma^{mod}$. Moreover, assume that $\det v_{\mathcal{S}} \equiv \det v_{\mathcal{N}} \equiv 1$ and also that $\det S \equiv \det N \equiv 1$. Note that the latter assumption usually follows from the former one under some additional regularity requirements [9].

On Σ^{exp} the jump matrix $v_{\mathcal{S}}$ converges uniformly exponentially fast as to the identity matrix as $n \to \infty$, except in the vicinity of a finite number of points $\kappa \in \mathcal{K} \subset \Sigma^{exp}$. For each $\kappa \in \mathcal{K}$ the local behaviour of $w_{\mathcal{S}} = v_{\mathcal{S}} - \mathbb{I}$ is given by:

$$|w_{\mathcal{S}}(k,n)| = O(\mathrm{e}^{-cn|k-\kappa|^{\chi}}), \tag{3.2}$$

for two positive constants c and χ. Moreover, $v_{\mathcal{N}}$ is uniformly bounded for $z \in \mathbb{C} \setminus \Sigma^{mod}$. The same holds true for the n-independent model solution N, except in the vicinities of the points $\kappa \in \mathcal{K}$, where N it can have fourth root singularities[7]

$$|N(z)| = O(|z-\kappa|^{-1/4}). \tag{3.3}$$

The same is true for N^{-1} as $\det N \equiv 1$.

In applications, a local parametrix problem has to be solved in vicinities of the points $\kappa \in \mathcal{K}$. The value of χ is critical, as it determines the class of special functions from which the explicit limiting parametrix solution can be constructed. For $\chi = 2$, these are the parabolic cylindrical functions, and this case occurs in the study of nonlinear integrable systems in the dispersive region ([16], [17], [29], [32], [35]). The cases $\chi = 1/2$ and $\chi = 3/2$ are common in the R-H analysis of orthogonal polynomials. Here, the exponent χ depends on the behaviour of the associated equilibrium measure at the endpoints of its support. An example for $\chi = 1/2$ leading to a Bessel parametrix problem can be found in [33], while cases of $\chi = 3/2$ leading to an Airy parametrix problem are covered in [6], [9, Ch. 5], [13], [14], [53]. Moreover, $\chi = 3/2$ is also related to the Painléve II equation ([2], [18]) and appears in the analysis of integrable systems with rarefaction/steplike initial data ([1], [30], [53]). For higher-dimensional R-H problems, the Meijer-G function has been used to construct explicit parametrix solutions ([4], [47]).

[7] In our application to orthogonal polynomials in the next section, this condition can be violated for certain singular weight functions.

Let us now apply Hölder's inequality to (2.9) assuming condition (3.1):

$$\begin{aligned}\left\|S_-(v_{\mathcal{S}}-v_{\mathcal{N}})N_+^{-1}\right\|_{L^1(\Sigma)} &= \left\|S_- w_{\mathcal{S}} N_+^{-1}\right\|_{L^1(\Sigma^{exp})}\\ &\leq \|S_-\|_{L^p(\Sigma^{exp})}\left\|w_{\mathcal{S}}N_+^{-1}\right\|_{L^q(\Sigma^{exp})}.\end{aligned}\tag{3.4}$$

Note that we made use of $v_{\mathcal{N}}(k)=\mathbb{I}$ for $k\in\Sigma^{exp}$, and as $v_{\mathcal{N}}(k)=v_{\mathcal{S}}(k,n)$ for $k\in\Sigma^{mod}$, we only need to integrate over Σ^{exp}. Now, from the assumptions (3.2) and (3.3), it follows that

$$\|k^i w_{\mathcal{S}}(k,n)N_+^{-1}(k)\|_{L^q(\Sigma^{exp})} = O(n^{\frac{1}{4\chi}-\frac{1}{q\chi}}),\quad q\in[1,4),\ i\in\mathbb{N}_0,\tag{3.5}$$

where the main contributions come from the points $\kappa\in\mathcal{K}$. The motivation for including k^i will become clear in the next theorem. Note that the condition on q implies $p\in(4/3,\infty]$.

We see that in order to guarantee that (3.4) goes to 0, it is sufficient to show that

$$\|S_-\|_{L^p(\Sigma^{exp})} = O(n^r)\tag{3.6}$$

with

$$r<\frac{1}{q\chi}-\frac{1}{4\chi}.$$

We call estimates of the form (3.6) a priori L^p-estimates, as they have to be established before an approximation for S is known.[8] We show in the next section that such estimates can be computed in the case of the R-H problem associated with orthogonal polynomials on the interval $[-1,1]$.

We can now state the following theorem:

Theorem 3.1. *Suppose $v_{\mathcal{S}},v_{\mathcal{N}},S,N$ satisfy (3.1), (3.2), (3.3), (3.6) and $\Sigma=\Sigma^{mod}\cup\Sigma^{exp}$. Let*

$$s:=\frac{1}{q\chi}-\frac{1}{4\chi}-r>0,\quad l\in\mathbb{N}_0.$$

Then

$$S(z,n)=(\mathbb{I}+O(n^{-s}\operatorname{dist}(z,\Sigma^{exp})^{-1}))N(z).\tag{3.7}$$

for $z\in\mathbb{C}\setminus\Sigma^{exp}$. Moreover,

$$S(z,n)=\Big(\mathbb{I}+\sum_{i=1}^{\ell}\frac{\theta_i(n)}{z^i}+O(z^{-\ell-1})\Big)N(z),\tag{3.8}$$

for $z\to\infty$ such that $|1-k/z|\geq\varepsilon>0$ for all $k\in\Sigma^{exp}$, with the matrices θ_i satisfying

$$|\theta_i(n)|_\infty=O(n^{-s}).$$

Proof. Equation (3.7) follows from writing $S=RN$ and using the expression (2.8) for R, together with Hölder's inequality (3.4) and the estimates (3.5) and (3.6). The asymptotic expansion follows analogously after substituting the partial Neumann series

$$\frac{1}{k-z}=-\sum_{i=1}^{\ell}\frac{k^{i-1}}{z^i}-\frac{k^\ell}{z^{\ell+1}}\frac{1}{1-k/z}.$$

[8] A priori L^p-estimates of solutions have been considered in R-H theory previously by Deift and Zhou [19] in their study of long-time asymptotics of solutions to the perturbed nonlinear Schrödinger equation on the real line.

into the integrand in (2.8). $\square$

Remark 3.2. *Instead of the a priori L^p-estimate (3.6), local L^p-estimates around the points $\kappa \in \mathcal{K}$ together with some auxiliary assumptions are sufficient for the conclusion of Theorem 3.1 to hold. For a proof and additional references, see the appendix.*

The next lemma tells us that a priori L^p-estimates of solutions to R-H problems can be uniformly extended to larger contours which are often introduced in the nonlinear steepest descent method (cf. [20, Eq. 2.4]).

Lemma 3.3. *Let Σ and Γ be oriented contours in $\hat{\mathbb{C}} := \mathbb{C} \cup \{\infty\}$. Assume that the Cauchy boundary operators $\mathcal{C}_{\pm}^{\Sigma\cup\Gamma}$ are well-defined and bounded on $L^p(\Sigma \cup \Gamma)$. Let $f \in \mathcal{O}(\hat{\mathbb{C}} \setminus \Sigma)$ be given, such that f is continuous at infinity in the case that $\infty \in \Sigma$ and the $\pm$-limits on Σ exist in the usual sense satisfying $f_+ - f_- \in L^p(\Sigma)$. Then*

$$\|f - f(\infty)\|_{L^p(\Gamma)} \leq C(\Gamma \cup \Sigma)\|f_+ - f_-\|_{L^p(\Sigma)}$$

for some positive constant $C(\Gamma \cup \Sigma)$ independent of f.

Proof. Note that because of the conditions on f it follows from the properties of the Cauchy integral operator that

$$f - f(\infty) = \mathcal{C}^{\Sigma}(f_+ - f_-) = \mathcal{C}^{\Sigma\cup\Gamma}(f_+ - f_-).$$

where the last equality is true because $f_+ = f_-$ on $\Gamma \setminus \Sigma$, as $f \in \mathcal{O}(\hat{\mathbb{C}} \setminus \Sigma)$. Hence we conclude

$$\|f - f(\infty)\|_{L^p(\Gamma)} \leq \|\mathcal{C}_{\pm}^{\Sigma\cup\Gamma}\|_{L^p(\Sigma\cup\Gamma)\to L^p(\Sigma\cup\Gamma)} \underbrace{\|f_+ - f_-\|_{L^p(\Sigma\cup\Gamma)}}_{=\|f_+ - f_-\|_{L^p(\Sigma)}}$$

which shows that we can choose $C(\Sigma \cup \Gamma) = \|\mathcal{C}_{\pm}^{\Sigma\cup\Gamma}\|_{L^p(\Sigma\cup\Gamma)\to L^p(\Sigma\cup\Gamma)}$. $\square$

Remark 3.4. *Similar arguments work in the case that the R-H problem is not stated for a matrix S, but rather for a vector s which is normalized to be s_∞ at infinity. However, we still need a matrix-valued model solution N, which is normalized to the identity matrix at infinity. As before we define a vector-valued function $r := sN^{-1}$ which can be written in integral form*

$$r(z,n) = s_\infty + \frac{1}{2\pi \mathrm{i}} \int_{\Sigma} \frac{s_-(k,n)(v_{\mathcal{S}}(k,t) - v_{\mathcal{N}}(k))N_+^{-1}(k,n)}{k - z}\, dk.$$

The rest of the analysis is analogous. An example of a vector-valued R-H problem comes from the inverse scattering transform of the KdV equation ([23], [27] [53]).

4. Application to orthogonal polynomials on $[-1,1]$

4.1. **Riemann–Hilbert formulation of orthogonal polynomials.** Let us consider an example for which our method can provide new results, namely the R-H problem for orthogonal polynomials on $[-1,1]$. We assume that the corresponding measure $d\mu$ on $[-1,1]$ is absolutely continuous and thus can be written as

$$d\mu(x) = \rho(x)\, dx, \quad x \in (-1,1),$$

for some real-valued function $\rho \geq 0$. Following [25], [24] the R-H problem characterizing the n-th orthogonal polynomial is stated as follows:

For any $n \in \mathbb{N}_0$ find a 2×2 matrix-valued function X on $\mathbb{C} \setminus [-1,1]$, such that:

(i) $X(z,n)$ is analytic in for $z \in \mathbb{C} \setminus [-1,1]$,

(ii) $X_+(x,n) = X_-(x,n) \begin{pmatrix} 1 & \rho(x) \\ 0 & 1 \end{pmatrix}$, for $x \in (-1,1)$,

(iii) $X(z,n) = (\mathbb{I} + O(z^{-1})) \begin{pmatrix} (2z)^n & 0 \\ 0 & (2z)^{-n} \end{pmatrix}$, as $z \to \infty$.

The following theorem explains how the R-H solution X is related to orthogonal polynomials:

Theorem 4.1. *(Fokas, Its, Kitaev ([24], [25])) The R-H problem for X is solved by*

$$X(z,n) = \begin{pmatrix} p_n(z) & \mathcal{C}^{(-1,1)}(p_n\rho)(z) \\ \eta_{n-1}p_{n-1}(z) & \eta_{n-1}\mathcal{C}^{(-1,1)}(p_{n-1}\rho)(z) \end{pmatrix} \tag{4.1}$$

where $p_n(z)$ is the n-th orthogonal polynomial with leading coefficient 2^n and

$$\eta_n := -\pi \mathrm{i} \|p_n\|^{-2}_{L^2((-1,1),\rho(x)dx)}.$$

Note that given some additional regularity assumptions on solutions, one also obtains uniqueness (c.f. [33]). Usually the R-H problem for orthogonal polynomials is normalized without the factor $2^{\pm n}$ at infinity, in which case the $p_n(z)$ would be the monic orthogonal polynomials. The next theorem found in [56, Thm. 12.7.1] explains this discrepancy:

Theorem 4.2. *Assume the weight $\rho(x)$ on $(-1,1)$ satisfies the Szegő condition, given by*

$$\int_{-1}^{1} \frac{\log \rho(x)}{\sqrt{1-x^2}}\, dx > -\infty.$$

Then

$$\lim_{n\to\infty} \|p_n\|_{L^2([-1,1],\rho(x)dx)} = \sqrt{\pi} \exp\Big(\frac{1}{2\pi} \int_{-1}^{1} \frac{\log \rho(x)}{\sqrt{1-x^2}}\, dx\Big).$$

Hence, we see that with our normalization, the $L^2((-1,1),\rho(x)dx)$-norm of p_n converges as n goes to infinity. This uniform boundedness of the norm is similar, though not identical, to the a priori L^p-estimate needed for our approach to local parametrix problems. The difference is the measure $\rho(x)dx$, as we need a priori L^p-estimates in the space $L^p((-1,1),dx)$.

4.2. **Large n limit without a parametrix solution.** In [14] the authors were considering orthogonal polynomials with an exponential weight of the form $\mathrm{e}^{-Q(x)}$ on the real line, where $Q(x)$ was a polynomial of even degree and positive leading coefficient. To extract asymptotic results, they introduced a series of transformations of R-H problems and their solutions, starting with the solution

$$Y := 2^{-n\sigma_3}X, \quad \sigma_3 := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

for monic polynomials:

$$Y \longrightarrow U \longrightarrow T \longrightarrow S \longrightarrow R \tag{4.2}$$

In each step a conjugation or deformation step has been performed to obtain a new R-H problem. An analogous procedure has been done by Kuijlaars et al. in [33] for the modified Jacobi weight function $\rho_{Jac}^{\alpha,\beta}$ on $[-1,1]$ (see (1.1)). In this case the R-H problem for U was not needed. We will not repeat the steps (4.2) here, but rather simply define the R-H problem for S and N, for details see [33].

In our setting, we will assume that ρ has an analytic continuation to a lens-shaped neighbourhood $\mathcal{L}$ of $(-1,1)$ as shown in Figure 1:

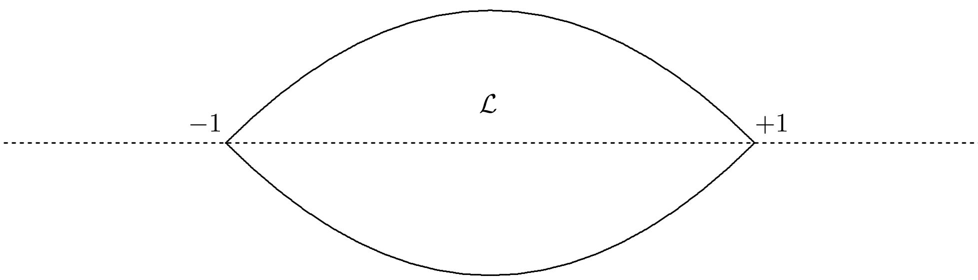

FIGURE 1. Neighbourhood $\mathcal{L}$.

Moreover, $|\rho^{\pm 1}|$ should remain bounded in $\mathcal{L}$, except possibly near the points $z = \pm 1$, where it can have the behaviour

$$|\rho(z)| = O(|z \pm 1|^{-1/\nu_+ + \varepsilon}), \quad |\rho(z)^{-1}| = O(|z \pm 1|^{-1/\nu_- + \varepsilon}), \tag{4.3}$$

for two constants $\nu_\pm > 1$ and some $\varepsilon > 0$. This implies that for any smooth curve γ going through $\mathcal{L}$ from -1 to 1, we have

$$\|\rho\|_{L^{\nu_+}(\gamma,dz)} < \infty, \quad \text{and} \quad \|\rho^{-1}\|_{L^{\nu_-}(\gamma,dz)} < \infty.$$

The R-H problem for S is defined on the contour $\Sigma := \Sigma_1 \cup \Sigma_2 \cup \Sigma_3$ as shown in Figure 2.

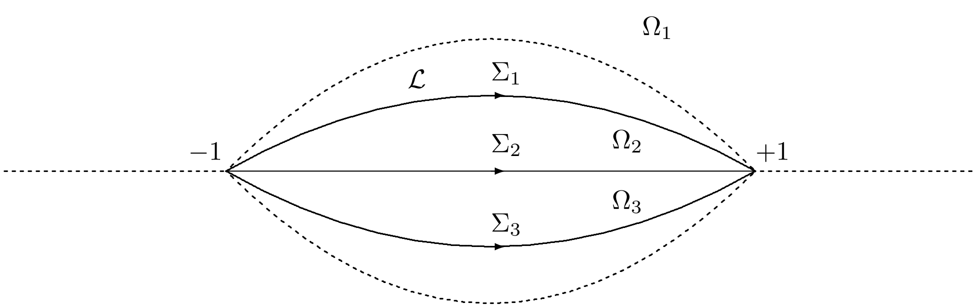

FIGURE 2. Jump contour for S.

Here the region enclosed by Σ_1 and Σ_2 is denoted by Ω_2 and assumed to be a subset of $\mathcal{L}$, analogously for Ω_3. Moreover, we set $\Omega_1 := \mathbb{C} \setminus \Omega_2 \cup \Omega_3$. The corresponding R-H problem for S is now stated as follows:

For any $n \in \mathbb{N}$ find a 2×2 matrix-valued function S such that:

(i) $S(z,n)$ is analytic for $z \in \mathbb{C} \setminus \Sigma$,

(ii) $S_+(k,n) = S_-(k,n) v_{\mathcal{S}}(k,n)$, for $k \in \Sigma$ with:

$$v_{\mathcal{S}}(k,n) = \begin{cases} \begin{pmatrix} 1 & 0 \\ \rho(k)^{-1}\varphi(k)^{-2n} & 1 \end{pmatrix}, & \text{for } k \in \Sigma_1 \cup \Sigma_3, \\ \begin{pmatrix} 0 & \rho(k) \\ -\rho(k)^{-1} & 0 \end{pmatrix}, & \text{for } k \in \Sigma_2 = (-1,1), \end{cases} \tag{4.4}$$

(iii) $S(z,n) = \mathbb{I} + O(z^{-1})$, as $z \to \infty$.

Here
$$\varphi(z) := z + \sqrt{z^2 - 1}$$
maps $\mathbb{C} \setminus [-1,1]$ biholomorphically to the exterior of the unit disc. In particular $|\varphi(z)| > 1$ for $z \in \mathbb{C} \setminus [-1,1]$ and $|\varphi(z)| \to 1$ as z approaches $[-1,1]$. Looking at the jump matrix (4.4) and observing that
$$\log \varphi(z) = O(\sqrt{|z \mp 1|}), \quad \text{as } z \to \pm 1,$$
we identify $z = \pm 1$ as the oscillatory points with $\chi = 1/2$ (cf. (3.2)).

A solution S can be expressed in terms of the solution X from Theorem 4.1 via
$$S(z,n) = \begin{cases} X(z,n)\varphi(z)^{-n\sigma_3}, & z \in \Omega_1, \\ X(z,n)\varphi(z)^{-n\sigma_3} \begin{pmatrix} 1 & 0 \\ -\rho(z)^{-1}\varphi(z)^{-2n} & 1 \end{pmatrix}, & z \in \Omega_2, \\ X(z,n)\varphi(z)^{-n\sigma_3} \begin{pmatrix} 1 & 0 \\ \rho(z)^{-1}\varphi(z)^{-2n} & 1 \end{pmatrix}, & z \in \Omega_3. \end{cases}$$
and satisfies
$$\lim_{z\to\infty} S(z,n) = \lim_{z\to\infty} X(z,n)\varphi^{-n\sigma_3} = \mathbb{I}. \tag{4.5}$$
The corresponding model problem has the same normalization at infinity, but only a jump condition on $(-1,1)$,
$$N_+(x) = N_-(x) \underbrace{\begin{pmatrix} 0 & \rho(x) \\ -\rho(x)^{-1} & 0 \end{pmatrix}}_{=: v_{\mathcal{N}}(x)}, \quad x \in (-1,1),$$
and is independent of n. The explicit solution given in [33, Eq. 5.5] has the form
$$N(z) = D_\infty^{\sigma_3} \begin{pmatrix} \dfrac{a(z) + a(z)^{-1}}{2} & \dfrac{a(z) - a(z)^{-1}}{2\mathrm{i}} \\ \dfrac{a(z) - a(z)^{-1}}{-2\mathrm{i}} & \dfrac{a(z) + a(z)^{-1}}{2} \end{pmatrix} D(z)^{-\sigma_3}$$

where $D(z)$ is the Szegő function associated to ρ [56]:

$$D(z) := \exp\left(\frac{\sqrt{z^2-1}}{2\pi}\int_{-1}^{1}\frac{\log\rho(x)}{\sqrt{1-x^2}}\frac{dx}{z-x}\right), \quad z\in\mathbb{C}\setminus[-1,1],$$

$$D_\infty := \lim_{z\to\infty} D(z) = \exp\left(\frac{1}{2\pi}\int_{-1}^{1}\frac{\log\rho(x)}{\sqrt{1-x^2}}\,dx\right)$$

and

$$a(z) := \left(\frac{z-1}{z+1}\right)^{1/4}$$

with a branch cut on $[-1,1]$ and $a(\infty)=1$. We note that

$$\Sigma^{\exp} := \Sigma_1\cup\Sigma_3, \qquad \Sigma^{mod} := \Sigma_2 = (-1,1).$$

Remark 4.3. *It is important to note that $v_{\mathcal{N}}$ might not be uniformly bounded and N might not satisfy condition (3.3). This is because of the singular behaviour of the weight function ρ, and hence D, near the endpoints (4.3). However, we will still use Hölder's inequality to get similar results as in Theorem 3.1.*

Next we want to arrive at estimates for $\|S_-w_{\mathcal{S}}N\|_{L^1(\Sigma^{exp},dz)}$. As N has no jumps on Σ^{exp}, the $\pm$ subscripts can be left out:

$$\begin{aligned}\|S_-w_{\mathcal{S}}N^{-1}\|_{L^1(\Sigma^{exp},dz)} &= \left\| X\varphi^{-n\sigma_3}\begin{pmatrix}0 & 0\\ \rho^{-1}\varphi^{-2n} & 0\end{pmatrix}N^{-1}\right\|_{L^1(\Sigma^{exp},dz)}\\ &\lesssim \|X\varphi^{-n\sigma_3}\|_{L^p(\Sigma^{exp},dz)}\,\|\rho^{-1}\|_{L^\vartheta(\Sigma^{exp},dz)}\\ &\quad\times\|\varphi^{-2n}(z\pm1)^{-1/4}\|_{L^\tau(\Sigma^{exp},dz)}\,\|D^{\sigma_3}\|_{L^\omega(\Sigma^{exp},dz)},\end{aligned} \tag{4.6}$$

with

$$\frac{1}{p}+\frac{1}{\vartheta}+\frac{1}{\tau}+\frac{1}{\omega}=1. \tag{4.7}$$

We have used for simplicity $\varphi^{-2n}(z\pm1)^{-1/4}$, which has the same growth behaviour as the more complicated expression

$$\left\|\begin{pmatrix}0 & 0\\ \varphi^{-2n} & 0\end{pmatrix}\begin{pmatrix}\frac{a(z)+a(z)^{-1}}{2} & \frac{a(z)-a(z)^{-1}}{-2\mathrm{i}}\\ \frac{a(z)-a(z)^{-1}}{2\mathrm{i}} & \frac{a(z)+a(z)^{-1}}{2}\end{pmatrix}D_\infty^{-\sigma_3}\right\|_{L^\tau(\Sigma^{exp},dz)}.$$

Using (3.5) with $\chi=1/2$ and $q=\tau$, we see that

$$\|\varphi^{-2n}(z\pm1)^{-1/4}\|_{L^\tau(\Sigma^{exp},dz)} = O(n^{-2/\tau+1/2}).$$

Hence, we should try to choose τ as small as possible, which translates into maximizing p, ϑ and ω under the constraint that all the corresponding terms in (4.6) remain bounded.

Next we want to show that $\|X\varphi^{-n\sigma_3}\|_{L^p(\Sigma^{exp},dz)}$ remains bounded for an appropriate $p\geq1$. Using Lemma 3.3 together with (4.5), it is enough to establish the $L^p((-1,1),dx)$-boundedness of $X\varphi_\pm^{-n\sigma_3}$, which, as $|\varphi_\pm(x)|=1$ for $x\in(-1,1)$, is equivalent to showing that

$$\|X_\pm\|_{L^p((-1,1),dx)} \leq C_1 < \infty.$$

Looking at the components of X in (4.1) and using the L^p-boundedness of the Cauchy boundary operators $\mathcal{C}_\pm^{(-1,1)}$ on $L^p((-1,1),dx)$ for $p \in (1,\infty)$, we need to show that

$$\|p_n\|_{L^p((-1,1),dx)} \le C_2, \quad \|p_n\rho\|_{L^p((-1,1),dx)} \le C_3, \quad C_2, C_3 < \infty.$$

For both cases we can use Hölder's inequality and theorem 4.2 which tells us that

$$\|p_n\sqrt{\rho}\|_{L^a((-1,1),dx)} \le C_4 < \infty$$

for $a = 2$ and thus automatically for $a \le 2$, as we integrate over a finite interval. Using this we can write

$$\|p_n\|_{L^p((-1,1),dx)} \le \|p_n\sqrt{\rho}\|_{L^a((-1,1),dx)}\|\sqrt{\rho}^{-1}\|_{L^b((-1,1),dx)}$$
$$\|p_n\rho\|_{L^p((-1,1),dx)} \le \|p_n\sqrt{\rho}\|_{L^{a'}((-1,1),dx)}\|\sqrt{\rho}\|_{L^{b'}((-1,1),dx)}$$

with

$$\frac{1}{p} = \frac{1}{a} + \frac{1}{b} = \frac{1}{a'} + \frac{1}{b'}.$$

As we want to maximize p, we choose $a = a' = 2$. The condition (4.3) tells us that we can take $b \le 2\nu_-$ and $b' \le 2\nu_+$. Again, maximizing p, we choose

$$b = b' = 2\nu_0 := 2\min\{\nu_+, \nu_-\}$$

which gives us

$$p := \frac{2\nu_0}{1+\nu_0}.$$

The condition $p > 1$ translates to $\nu_0 > 1$, which was assumed right after (4.3).

Next, let us consider the term $\|\rho^{-1}\|_{L^\vartheta(\Sigma^{exp},dz)}$. This is the simplest case, as (4.3) implies that we can choose $\vartheta := \nu_-$.

The term $\|D^{\sigma_3}\|_{L^\omega(\Sigma^{exp},dz)}$ is more challenging. Recall that the Szegő function satisfies [8, Eq. 2.14–15]

$$D_+(x)D_-(x) = \rho(x), \quad x \in (-1,1)$$

and

$$\overline{D(z)} = D(\overline{z}), \quad \mathbb{C} \setminus [-1,1].$$

These two identities imply

$$|D_\pm(x)| = \sqrt{\rho(x)} \quad \text{and} \quad D_+(x) - D_-(x) = 2\,\mathrm{Im}(D_+(x)), \quad x \in (-1,1).$$

Hence, D satisfies an additive R-H problem, with

$$\|D_+ - D_-\|_{L^\omega((-1,1),dx)} \le \|\sqrt{\rho}\|_{L^\omega((-1,1),dx)}.$$

As D has a limit at infinity, namely D_∞, we can apply lemma 3.3 to conclude that

$$\|D\|_{L^\omega(\Sigma^{exp},dz)} = O(\|\sqrt{\rho}\|_{L^\omega((-1,1),dx)}).$$

From (4.3) it follows that we must have $\omega \le 2\nu_+$. The same argument works with D^{-1}, and in the end we can choose $\omega := 2\nu_0$.

We have computed the optimal values for p, ϑ and ω:

$$p = \frac{2\nu_0}{1+\nu_0}, \qquad \vartheta = \nu_-, \qquad \omega = 2\nu_0.$$

A quick calculation using the relation (4.7) shows that

$$\tau = \frac{2\nu_0\nu_-}{\nu_0\nu_- - 2(\nu_0+\nu_-)},$$

which implies

$$\|(z \pm 1)^{-1/4} \varphi^{-2n}\|_{L^r(\Sigma^{exp}, dz)} = O(n^{-\lambda}),$$

with

$$\lambda := \frac{1}{2} - \frac{2(\nu_0 + \nu_-)}{\nu_0 \nu_-}. \tag{4.8}$$

As all the other terms on the right-hand side of (4.6) remain bounded as $n \to \infty$, we conclude:

$$\|S_- w_{\mathcal{S}} N^{-1}\|_{L^1(\Sigma^{exp}, dz)} = O(n^{-\lambda}).$$

To ensure that λ is positive it is enough to assume that

$$\nu_0 > 8, \tag{4.9}$$

or in the case $\nu_0 = \nu_+ \in (4, 8)$, one would need

$$\nu_- > \frac{4\nu_+}{\nu_+ - 4}. \tag{4.10}$$

For positive λ we can conclude that

$$S(z, n) = (\mathbb{I} + O(n^{-\lambda} \operatorname{dist}(z, \Sigma^{exp})^{-1})) N(z). \tag{4.11}$$

In particular

$$X(z, n) = (\mathbb{I} + O(n^{-\lambda} \operatorname{dist}(z, \Sigma^{exp})^{-1})) N(z) \varphi(z)^{n\sigma_3}, \qquad z \in \Omega_1, \tag{4.12}$$

$$X(z, n) = (\mathbb{I} + O(n^{-\lambda} \operatorname{dist}(z, \Sigma^{exp})^{-1})) N(z) \times \begin{pmatrix} 1 & 0 \\ \rho^{-1}(z) \varphi^{-2n}(z) & 1 \end{pmatrix} \varphi(z)^{n\sigma_3}, \qquad z \in \Omega_2, \tag{4.13}$$

$$X(z, n) = (\mathbb{I} + O(n^{-\lambda} \operatorname{dist}(z, \Sigma^{exp})^{-1}) N(z) \times \begin{pmatrix} 1 & 0 \\ -\rho^{-1}(z) \varphi^{-2n}(z) & 1 \end{pmatrix} \varphi(z)^{n\sigma_3}, \qquad z \in \Omega_3. \tag{4.14}$$

We can state our main result concerning orthogonal polynomials:

Theorem 4.4. *Let the weight function ρ have an analytic continuation to a lens-shaped neighbourhood $\mathcal{L}$ of $(-1, 1)$ and satisfy condition (4.3), with $\nu_\pm$ fulfilling either (4.9) or (4.10). Moreover, let U be a compact subset of the Riemann sphere contained in $\mathbb{C} \setminus [-1, 1] \cup \{\infty\}$ and V be a compact set contained in $\mathcal{L}$. Then, we have for the n-th orthogonal polynomial with leading coefficient 2^n:*

$$\begin{aligned} p_n(z) &= \frac{D_\infty \varphi(z)^{n+1/2}}{\sqrt{2} D(z)(z^2 - 1)^{1/4}} + O(n^{-\lambda} z^{-1} \varphi(z)^n) \\ &= \Big(\frac{D_\infty \varphi(z)^{n+1/2}}{\sqrt{2} D(z)(z^2 - 1)^{1/4}} \Big) \big(1 + O(n^{-\lambda} z^{-1})\big), \quad z \in U, \end{aligned} \tag{4.15}$$

and

$$p_n(z) = \frac{1}{\sqrt{2}(z^2 - 1)^{1/4}} \Big(\frac{D_\infty}{D(z)} \varphi(z)^{n+1/2} \pm \mathrm{i} \frac{D_\infty D(z)}{\rho(z)} \varphi(z)^{-n-1/2} \Big) + O(n^{-\lambda} \varphi(z)^n), \qquad z \in V, \tag{4.16}$$

with the $+(-)$ sign in the case of z in the upper(lower)-half plan, and both signs giving the same result for $z \in (-1, 1)$. The constant λ is given by (4.8).

Proof. First observe the two identities

$$\frac{a(z)+a(z)^{-1}}{2}=\frac{\varphi(z)^{1/2}}{\sqrt{2}(z^2-1)^{1/4}} \tag{4.17}$$

and

$$\frac{a(z)-a(z)^{-1}}{2\mathrm{i}}=\mathrm{i}\frac{\varphi(z)^{-1/2}}{\sqrt{2}(z^2-1)^{1/4}}. \tag{4.18}$$

Next, let us consider the asymptotics in U. As U is assumed to be compact on the Riemann sphere, it must be a finite distance away from $[-1,1]$. In particular, we can choose the contour $\Sigma^{\mathrm{exp}}\subset\mathcal{L}$ such that

$$\operatorname{dist}(U,\Sigma^{exp})>0.$$

It follows that for $z\in U$, we have that asymptotic terms of order $O(z^{-1})$ and $O(\operatorname{dist}(z,\Sigma^{exp})^{-1})$ become equivalent. Again by compactness of U, we see that the functions $D^{\pm1}$ and $a^{\pm1}$ are bounded in U. With this information, (4.15) is obtained by multiplying out (4.12) and using $p_n(z)=X_{11}(z,n)$ from Theorem 4.1.

For the set V we can again choose $\Sigma^{exp}\subset\mathcal{L}$ such that

$$\operatorname{dist}(V,\Sigma^{exp})>0.$$

As V is bounded, asymptotic terms of order $O(\operatorname{dist}(z,\Sigma^{exp})^{-1})$ and $O(1)$ become equivalent. Similar to before the functions $D^{\pm1}$ and $a^{\pm1}$ are uniformly bounded, as they are continuous in $V\setminus(-1,1)$ and take continuous limits on $V\cap(-1,1)$. Multiplying out (4.13) and (4.14) gives then (4.16). Moreover, using for $x\in(-1,1)$,

$$\begin{aligned}(x^2-1)_+^{1/4}&=\mathrm{i}(x^2-1)_-^{1/4}\\ D_+(x)D_-(x)&=\rho(x)\\ \varphi_+(x)\varphi_-(x)&=1\end{aligned}$$

one can verify that both signs in (4.16) give the same result on $V\cap(-1,1)$. ☐

Remark 4.5. *The leading terms in* (4.15) *and* (4.16) *have been obtained by Bernstein and Szegő in* [56, Thm. 12.1.1–4]. *However, the R-H method allows for more explicit bounds on the error terms.*

As a corollary, we obtain bulk universality for the unitary matrix ensemble associated to ρ.

Corollary 4.6. *Let the weight function ρ satisfy the assumption* (4.3) *and let $\{\pi_n\}_{n\in\mathbb{N}_0}$ be the associated monic polynomials. Then the corresponding kernel K_n given by*

$$K_n(x,\,y)=\sqrt{\rho(x)}\sqrt{\rho(y)}\sum_{j=0}^{n-1}\pi_j(x)\pi_j(y),$$

satisfies

$$\frac{1}{n\xi(x)}K_n\Big(x+\frac{u}{n\xi(x)},\,x+\frac{v}{n\xi(x)}\Big)=\frac{\sin\pi(u-v)}{\pi(u-v)}+O(n^{-1}),\qquad x\in(-1,1),\ \ u,v\in\mathbb{R}, \tag{4.19}$$

where $\xi(x)=(\pi\sqrt{1-x^2})^{-1}$. The estimate holds uniformly for x in compact subsets of $(-1,1)$ and u,v in compact subsets of $\mathbb{R}$.

Proof. The proof is analogous to the one given in [34, Sect. 3.2].[9] Only the local boundedness of $\frac{d}{dx}S_\pm(x)$ for $x \in (-1,1)$, uniformly as $n \to \infty$ requires additional comments. It follows from the local boundedness of $N_\pm(x)$, $\frac{d}{dx}N_\pm(x)$, $R(x,n)$ and $\frac{d}{dx}R(x,n)$ for $x \in (-1,1)$, uniformly as $n \to \infty$ (recall $S = RN$). The statement for $N_\pm(x)$ and $\frac{d}{dx}N_\pm(x)$ follows directly from smoothness of the limits of the Szegő function to $(-1,1)$ and n-independence. The statement for $R(x,n)$ and $\frac{d}{dx}R(x,n)$ follows from (2.8), and the fact that the integral representation of R contains only integrals over Σ^{exp}, where $\Sigma^{exp} \cap [-1,1] = \{-1,1\}$. □

Remark 4.7. *Interestingly, the error term in* (4.19) *is bounded by* $O(n^{-1})$ *instead of* $O(n^{-\lambda})$*, where* λ *is defined in* (4.8)*. The reason is that convergence to the* sine-*kernel follows from the relations* (4.13) *and* (4.14) *between* X *and* S*, rather than the exact form of* S*.*

4.3. **Riemann–Hilbert problem with constant jump matrices.** Finally, let us explain why our choice of weight functions makes a reformulation of the local parametrix problem to a limiting parametrix problem with constant jump matrices, as found in [33, Sect. 6], impossible. In that paper the authors considered the following local parametrix problem for P in a small but fixed disc U_δ of radius δ around $z = 1$ (and later analogously around $z = -1$):

Find a 2×2 matrix-valued function P, such that

(i) $P(z,n)$ is analytic for $z \in U_\delta \setminus \Sigma$,

(ii) $P_+(k,n) = P_-(k,n)v_{\mathcal{S}}(k,n)$, for $k \in \Sigma \cap U_\delta$,

(iii) $P(k,n)N^{-1}(k) = \mathbb{I} + o(1)$, uniformly for $k \in \partial U_\delta$ as $n \to \infty$.[10]

Hence, P should satisfy locally around $z = 1$ the same R-H problem as S, but the normalization at infinity is changed to a matching condition with N. To transform this R-H problem to an explicitly solvable one, a further conjugation step is needed to make the jump matrices independent of $k \in \Sigma$. This is a crucial step that cannot be performed in our case. Assume for a moment that ρ has an analytic nowhere vanishing continuation in U_δ. In [33, Eq. 6.7] the authors define the function W by[11]

$$W(z) := \sqrt{\rho(z)}, \quad z \in U_\delta. \tag{4.20}$$

This function can be used to construct $P^{(1)}$:

$$P^{(1)}(z,n) := P(z,n)\varphi^{n\sigma_3}(z)W(z)^{\sigma_3}, \quad U_\delta.$$

Then $P^{(1)}$ satisfies the following R-H problem:

Find a 2×2 matrix-valued function $P^{(1)}$, such that

[9] The authors denote our matrix S by L.

[10] In [33] an error term of order $O(n^{-1})$ is used instead of $o(1)$.

[11] In [33] a Jacobi-weight prefactor $(1-x)^\alpha(1+x)^\beta$ was considered. In our case any singular behaviour around ± 1 is assumed to be absorbed into the weight function ρ (compare Eq. (1.1) with (4.3)).

(i) $P^{(1)}(z,n)$ is analytic in $z \in U_\delta \setminus \Sigma$,

(ii) $P_+^{(1)}(k,n) = P_-^{(1)}(k,n) v_{\mathcal{P}}(k,n)$, for $k \in \Sigma \cap U_\delta$, with

$$v_{\mathcal{P}}(k) = \begin{cases} \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, & \text{for } k \in \Sigma^{exp} \cap U_\delta, \\ \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, & \text{for } k \in \Sigma^{mod} \cap U_\delta. \end{cases}$$

(iii) $P^{(1)}(k,n)\big(N(k)\varphi(k)^{n\sigma_3} W(k)^{\sigma_3}\big)^{-1} = \mathbb{I} + o(1)$,
uniformly for $k \in \partial U_\delta$ as $n \to \infty$.

After an n-dependent change of variables $z \to \zeta$, the matching condition (iii) for $P^{(1)}$ can be transformed into a normalization at infinity for the matrix-valued function

$$\Psi(\zeta) := P^{(1)}(z(\zeta)).$$

The corresponding jump contour $\Sigma^{\mathcal{B}} := \Sigma_1^{\mathcal{B}} \cup \Sigma_2^{\mathcal{B}} \cup \Sigma_3^{\mathcal{B}}$ consists of three rays emanating from the origin as in Figure 3:

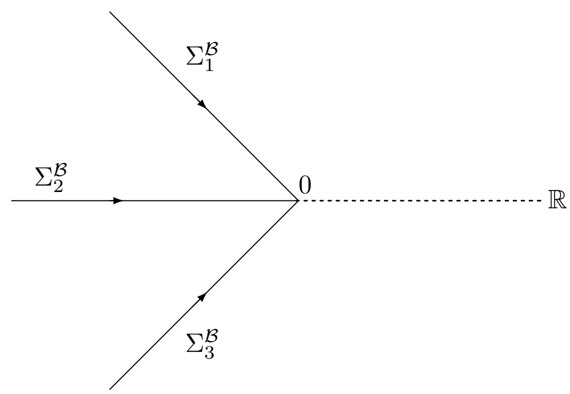

FIGURE 3. Contour for the Bessel R-H problem

and the jump matrix $v_{\mathcal{B}}$ has the form

$$v_{\mathcal{B}}(\zeta) = \begin{cases} \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, & \text{for } \zeta \in \Sigma_1^{\mathcal{B}} \cup \Sigma_3^{\mathcal{B}}, \\ \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, & \text{for } \zeta \in \Sigma_2^{\mathcal{B}}. \end{cases}$$

The R-H problem for Ψ is stated as follows:

Find a 2×2 matrix-valued function Ψ, such that

(i) $\Psi(\zeta)$ is analytic for $\zeta \in \mathbb{C} \setminus \Sigma^{\mathcal{B}}$,

(ii) $\Psi_+(\zeta) = \Psi_-(\zeta) v_{\mathcal{B}}(\zeta)$, for $\zeta \in \Sigma^{\mathcal{B}}$,

(iii) $\Psi(\zeta) \to (2\pi\zeta^{1/2})^{-\sigma_3/2} \frac{1}{\sqrt{2}} \left(\begin{pmatrix} 1 & i \\ i & 1 \end{pmatrix} + o(1) \right) \mathrm{e}^{2\zeta^{1/2}\sigma_3}$, uniformly as $\zeta \to \infty$.

This R-H problem is the corresponding limiting parametrix problem as it is obtained by taking the limit of the local parametrix problems as $n \to \infty$. It is solved explicitly using the Bessel functions [33, Eq. 6.23–25]. A deductive argument leading to the explicit solution for a similar limiting parametrix problem, namely the Airy parametrix problem, can be found [28, Sect. 3.2].

As ρ is not assumed to have an analytic continuation to a disc U_δ around $z = 1$ ($z = -1$), the function W in (4.20) cannot be defined in general. Hence, the convergence of the parametrix problem to a limiting parametrix problem with constant jump matrices, at least with the usual approach, is not possible. However, we point out that under the assumptions of Theorem 4.4 the local parametrix problem for P indeed has a solution.

Theorem 4.8. *The exact solution S, satisfies the local parametrix R-H problem for P.*

Proof. The matrix-valued function S satisfies trivially condition (i) and (ii) for P. The only remaining condition (iii) is the matching condition:

$$S(k,n)N(k,n)^{-1} = R(k,n) = \mathbb{I} + o(1), \quad \text{for } k \in \partial U_\delta.$$

uniformly as $n \to \infty$. We can substitute $O(n^{-\lambda}\,\mathrm{dist}(k, \Sigma^{exp})$ for the error term $o(1)$, because of (4.11). However, by deforming the contour Σ^{exp} we see that the points where Σ^{exp} and U_δ meet are movable, and that the error term is in fact uniform on ∂U_δ, meaning

$$S(k,n)N(k,n)^{-1} = R(k,n) = \mathbb{I} + O(n^{-\lambda}), \quad \text{for } k \in \partial U_\delta,$$

uniformly as $n \to \infty$. Hence, S is a solution to the local parametrix R-H problem. □

The solution S is unique in the following sense: For any other solution $\widetilde{P}$ having sufficient regularity, for example at most fourth root singularities at $z = \pm 1$ (c.f. [33]), the matrix-valued function $H := \widetilde{P} S^{-1}$ will be analytic in U_δ, and satisfies

$$H(k,n) = \widetilde{P}(k,n)S(k,n)^{-1} = \mathbb{I} + o(1) \quad \text{for } k \in \Sigma \cap U_\delta,$$

uniformly as $n \to \infty$. By the maximum principle for analytic function, we therefore know that

$$H(z,n) = \widetilde{P}(z,n)S(z,n)^{-1} = \mathbb{I} + o(1) \quad \text{for } z \in U_\delta,$$

uniformly as $n \to \infty$. Hence we see that $\widetilde{P}$ has the form

$$\widetilde{P}(z,n) = H(z,n)S(z,n), \quad z \in U_\delta,$$

where $H(z,n)$ is a sequence of matrix-valued function with analytic entries, such that it converges uniformly to the identity matrix as $n \to \infty$. Moreover, every solution of the local parametrix problem can be obtained in this way.

5. Discussion

We have shown that the formulation of a limiting parametrix problem is not necessary for a rigorous R-H analysis, provided an a priori L^p-estimate of the exact solution S is known. Our method has been illustrated on the example of orthogonal polynomials on $[-1,1]$ with a new class of weight functions ρ. We impose analytic continuation of ρ to a lens-shaped neighbourhood of $(-1,1)$ and a growth condition at the endpoints (cf. Figure 1 and Eq. (4.3)). In particular, we do not require any sort of analytic continuation around $x=\pm 1$.

However, the error bound obtained that way is in general worse than the actual error term. In [33] the authors show

$$S(z,n)=\Big(\mathbb{I}+\sum_{k=1}^{\ell}\frac{R_k(z)}{n^k}+O(n^{-\ell-1})\Big)N(z),\quad \text{as } n\to\infty,\ \ell\in\mathbb{N}_+, \tag{5.1}$$

uniformly away from $x=\pm 1$. They considered modified Jacobi weight functions of the form (1.1). The series expansion stems from the series expansion of the Bessel functions, which are contained in the solution of the limiting parametrix problem. Our approach would only result in $O(n^{-1/2})$ without a series expansion, for weight functions ρ satisfying that $|\rho^{\pm 1}|$ is uniformly bounded. This case corresponds to $\nu_\pm\to\infty$ in (4.3). This is unsurprising as the a priori L^p-estimate contains very limited information on the local structure of the exact solution S around the oscillatory points.

Whether an expansion of the form (5.1) holds in our case, seems to be an open problem. Also the optimal error bound in (4.15) and (4.16) is, to the best of our knowledge, unknown. These questions relate to one of the main motivations for obtaining large n asymptotics of orthogonal polynomials, namely the study of eigenvalue statistics for ensembles of random matrices. This field of study has been initiated by Wigner [59]. The statistics in the bulk of the spectrum have been further studied for special cases by Dyson in [21], [22] and Mehta in [45] and confirmed instances of the Wigner–Dyson–Mehta universality conjecture. This universality conjecture states that the local statistics in the bulk of the spectrum depend only on the type of the ensemble, which is either unitary, orthogonal or symplectic.

More general instances of the universality conjecture were obtained by Lubinsky in [38], [41] (see also [37], [39], [40], [55]), showing that bulk and edge universality depend essentially only on the local behaviour of the weight function and do not require smoothness. This generalizes the universality results in [34], by allowing for continuity and positivity instead of analyticity of the function h in (1.1). The proof is not based on the R-H method, but rather works directly with the corresponding Christoffel–Darboux kernel and uses localization and smoothing techniques. In particular, Plancherel–Rotach asymptotics are not needed. In fact, recently the implication from asymptotics of orthogonal polynomials to universality has been inverted, meaning that pointwise asymptotics for orthogonal polynomials were obtained from the corresponding universality results (see [42] for the bulk and [43] for the edge case).

Our Theorem 4.4 implies the known universality in the bulk, but has more interesting connections with universality near the edge. Just as for the bulk, most of the previous work has been focused on proving edge universality for various ensembles. Here one has to distinguish between two classes, the soft edge ([6], [9],

[14], [26, Sect. 3]) and the hard edge ([26, Sect. 2], [33], [34], [51]). While the former leads to local statistics described by the Airy kernel, the latter leads to the Bessel kernel. Both kernels have been further studied by Tracy and Widom ([57], [58]).

Using the R-H method, the cited results on edge universality can be obtained through the explicit solution of a limiting parametrix problem with constant jump matrices. However, as shown in Section 4.3, for weight functions that do not have an analytic continuation outside the lens $\mathcal{L}$ (cf. Figure 1) one cannot formulate the corresponding limiting parametrix problem with the usual R-H approach (see [3], [44] for an alternative $\overline{\partial}$ R-H approach). The natural question arises, whether the local parametrix problems converge in any other sense as the degree n goes to infinity to some other limiting parametrix problem. A hypothetical limiting R-H problem would likely determine the behaviour of the eigenvalues near the edge of the spectrum. Our work suggests that such a limiting R-H problem might not exist, as the nonlinear steepest descent method can be rigorously applied without computing any limit of the local parametrix problems. More research is needed to formalize and prove or disprove this statement.

Regarding future work, it would be interesting to see how the methods presented in this paper can be used in other settings. For orthogonal polynomials, one can consider different kinds of nonanalytic singularities in the interior of the support, generalizing the work in [31]. In particular, one can apply our results to show that certain families of local parametrix problems are solvable and use this fact for other R-H applications with similar local parametrix problems, but without a priori estimates. On the other hand one could try to derive more powerful a priori information than the one given in Theorem 4.2. This could lead to qualitative Plancherel–Rotach asymptotics, even in the vicinity of nonanalytic singularities like the one considered in [8]. Another future challenge would be proving the a priori L^p-estimate for R-H problems related to other areas. In particular, the inverse scattering transform for integrable PDEs can be framed as a R-H problem, and here it would be interesting to show and interpret the associated a priori estimates.

Appendix: Local a priori L^p-estimate

Let us consider the general setting of Section 3, with the additional assumption that Σ^{exp} is unbounded. This was not the case in our application to orthogonal polynomials on $[-1, 1]$, for further examples of R-H problems having a bounded Σ^{exp} see ([2], [29], [32]). However, there are numerous examples of R-H problems having an unbounded Σ^{exp} ([1], [14], [15], [16], [17], [18], [19], [23], [27], [30], [35], [53]). Figure 4 displays the contour in the case the KdV equation with steplike initial data [23], where the solid part is Σ^{mod} and the dashed Σ^{exp}. We see that Σ^{exp} extends to $\pm\infty$.

The unboundedness assumption on Σ^{exp} poses a hurdle for obtaining the a priori L^p-estimate. Namely, given that $S(z) \to \mathbb{I}$, as $z \to \infty$, we see that

$$\|S_-\|_{L^p(\Sigma^{exp})} = \infty, \quad \text{for } p \in (1, \infty). \tag{A.1}$$

Luckily, it turns out that only local L^p-estimates around the points $\kappa \in \mathcal{K}$ are needed. To show this, we choose a bounded domain $\Delta \subset \mathbb{C}$, which contains all the oscillatory points $\kappa \in \mathcal{K}$. Next, write Σ^{exp} as a disjoint union of an unbounded part Σ^{exp}_∞ and a bounded part $\Sigma^{exp}_\mathcal{K}$ with $\mathcal{K} \subset \Sigma^{exp}_\mathcal{K}$:

$$\Sigma^{exp}_\infty := \Sigma^{exp} \setminus \Delta, \qquad \Sigma^{exp}_\mathcal{K} := \Sigma^{exp} \cap \Delta.$$

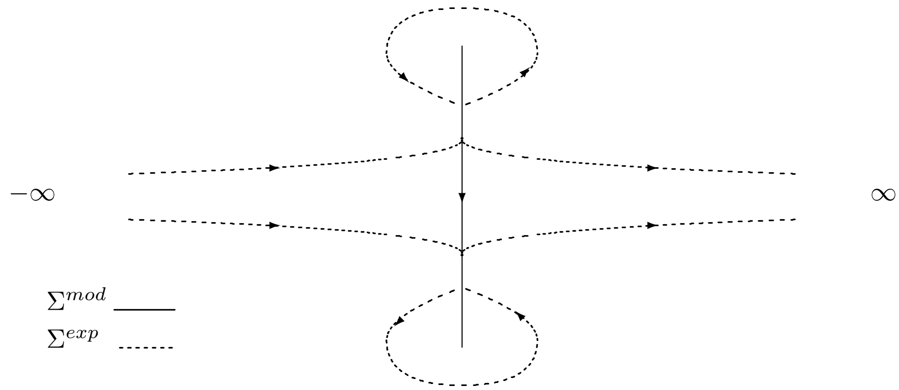

FIGURE 4. Contour for the KdV equation with steplike initial data

We make the additional assumption

$$\|k^i w_{\mathcal{S}}(k,n)\|_{L^1(\Sigma_\infty^{exp})} = O(\mathrm{e}^{-cn}) \tag{A.2}$$

for some positive c and $i = 0, \dots, \ell - 1$, where ℓ will play the same role as in Theorem 3.1. The condition (A.2) is satisfied in most applications.

We can now introduce the following R-H problem with data $(v_{\mathcal{G}}, \Sigma^{\mathcal{G}})$, where $\Sigma^{\mathcal{G}} := \Sigma^{mod} \cup \Sigma_\infty^{exp}$ and the jump matrix is given by $v_{\mathcal{G}}(k) = v_{\mathcal{S}}(k)$ for $k \in \Sigma^{\mathcal{G}}$.

Find a 2×2 matrix-valued function G such that

(i) $G(z)$ is analytic for $z \in \mathbb{C} \setminus \Sigma^{\mathcal{G}}$,

(ii) $G_+(k) = G_-(k) v_{\mathcal{G}}(k)$, for $k \in \Sigma^{\mathcal{G}}$

(iii) $G(z) = \mathbb{I} + O(z^{-1})$, as $z \to \infty$.

The solution G will be later used as a substitute for the model solution N. From (A.2) it follows that

$$\|v_{\mathcal{G}} - v_{\mathcal{N}}\|_{L^1(\Sigma)} = \|w_{\mathcal{S}}\|_{L^1(\Sigma_\infty^{exp})} = O(\mathrm{e}^{-cn}) \tag{A.3}$$

Moreover, in (3.2) it was assumed that $|w_{\mathcal{S}}(k,n)| = O(\mathrm{e}^{-cn})$, without loss of generality with the same c, uniformly away from the points $\kappa \in \mathcal{K}$. Hence, we also have

$$\|v_{\mathcal{G}} - v_{\mathcal{N}}\|_{L^\infty(\Sigma)} = \|w_{\mathcal{S}}\|_{L^\infty(\Sigma_\infty^{exp})} = O(\mathrm{e}^{-cn}). \tag{A.4}$$

Let us now take a look at the corresponding singular integral equations for both R-H problems (cf. Section 2.1)

$$(\mathbb{I} - \mathcal{C}^{\Sigma}_{w_{\mathcal{G}}})\Phi_{\mathcal{G}} = \mathcal{C}^{\Sigma}_{-}(w_{\mathcal{G}}), \tag{A.5}$$

$$(\mathbb{I} - \mathcal{C}^{\Sigma}_{w_{\mathcal{N}}})\Phi_{\mathcal{N}} = \mathcal{C}^{\Sigma}_{-}(w_{\mathcal{N}}), \tag{A.6}$$

where $w_{\mathcal{G}} := v_{\mathcal{G}} - \mathbb{I}$, and for notational convenience we regard everything defined on the larger contour Σ. Note that

$$\|(\mathbb{I} - \mathcal{C}^{\Sigma}_{w_{\mathcal{G}}}) - (\mathbb{I} - \mathcal{C}^{\Sigma}_{w_{\mathcal{N}}})\|_{L^q(\Sigma)\to L^q(\Sigma)} = \|\mathcal{C}^{\Sigma}_{w_{\mathcal{N}} - w_{\mathcal{G}}}\|_{L^q(\Sigma)\to L^q(\Sigma)} \leq C(q)\|w_{\mathcal{S}}\|_{L^\infty(\Sigma^{exp}_\infty)} = O(\mathrm{e}^{-cn}). \tag{A.7}$$

Moreover, as for $q \in (1,4)$ a unique solution $\Phi_{\mathcal{N}} := N_- - \mathbb{I}$ of (A.6) exists, we know that $\mathbb{I} - \mathcal{C}^{\Sigma}_{w_{\mathcal{N}}}$ must be invertible as an operator on $L^q(\Sigma)$. As the set of invertible operators is open in the operator norm topology, it follows from (A.7) that for n large enough $\mathbb{I} - \mathcal{C}^{\Sigma}_{w_{\mathcal{G}}}$ is uniformly invertible on $L^q(\Sigma)$. Moreover, using (A.3) and (A.4) we see that

$$\|w_{\mathcal{G}} - w_{\mathcal{N}}\|_{L^q(\Sigma)} = \|w_{\mathcal{S}}\|_{L^q(\Sigma^{exp}_\infty)} = O(\mathrm{e}^{-cn}).$$

Altogether this implies that for n large enough the unique solution $\Phi_{\mathcal{G}}$ of (A.5) satisfies

$$\|\Phi_{\mathcal{G}} - \Phi_{\mathcal{N}}\|_{L^q(\Sigma)} = O(\mathrm{e}^{-cn}).$$

In particular, $\|\Phi_{\mathcal{G}}\|_{L^q(\Sigma)}$ is uniformly bounded as $\Phi_{\mathcal{N}}$ is n-independent.

As the goal will be to use G as a substitute of the model solution N, we need to understand the behaviour of G in the vicinities of the oscillatory points $\kappa \in \mathcal{K}$. As described in Section 2.2, we can relate G and N via:

$$G(z,n) = \left(\mathbb{I} + \frac{1}{2\pi\mathrm{i}}\int_{\Sigma^{exp}_\infty} \frac{G_-(k,n)w_{\mathcal{S}}(k,n)N^{-1}(k)}{k-z}\, dk\right)N(z).$$

Note that

$$\|G_- w_{\mathcal{S}} N^{-1}\|_{L^1(\Sigma^{exp}_\infty)} = \|(\Phi_{\mathcal{G}} + \mathbb{I})w_{\mathcal{S}}N^{-1}\|_{L^1(\Sigma^{exp}_\infty)}$$
$$\leq \Big(\|\Phi_{\mathcal{G}}\|_{L^2(\Sigma^{exp}_\infty)}\|w_{\mathcal{S}}\|_{L^2(\Sigma^{exp}_\infty)} + \|w_{\mathcal{S}}\|_{L^1(\Sigma^{exp}_\infty)}\Big)\|N^{-1}\|_{L^\infty(\Sigma^{exp}_\infty)} = O(\mathrm{e}^{-cn}),$$

which follows from the uniform boundedness of $\|\Phi_{\mathcal{G}}\|_{L^2(\Sigma)}$, and $|N^{-1}(k)|$ away from the points $\kappa \in \mathcal{K}$. Hence, we conclude that

$$G(z,n) = (\mathbb{I} + O(\mathrm{e}^{-cn}\operatorname{dist}(z,\Sigma^{exp}_\infty)^{-1}))N(z). \tag{A.8}$$

In particular, we see that G has locally around $\kappa \in \mathcal{K}$ the same behaviour as N,

$$|G(z,n)| = O(|z-\kappa|^{-1/4}), \tag{A.9}$$

uniformly for $n \to \infty$. However, $G(z,n)$ might not be uniformly bounded away from the points $\kappa \in \mathcal{K}$, as it can blow up near the contour Σ^{exp}_∞.

Let us now reconsider (3.4), but with G instead of N:

$$\big\|S_-(v_{\mathcal{S}} - v_{\mathcal{G}})G_+^{-1}\big\|_{L^1(\Sigma)} = \big\|S_- w_{\mathcal{S}} G^{-1}\big\|_{L^1(\Sigma^{exp}_{\mathcal{K}})} \leq \|S_-\|_{L^p(\Sigma^{exp}_{\mathcal{K}})}\big\|w_{\mathcal{S}}G^{-1}\big\|_{L^q(\Sigma^{exp}_{\mathcal{K}})}.$$

Because of $\det G \equiv 1$ and the boundedness of $\Sigma^{exp}_{\mathcal{K}}$, we have

$$\|G^{-1}\|_{L^q(\Sigma^{exp}_{\mathcal{K}})} = \|G\|_{L^q(\Sigma^{exp}_{\mathcal{K}})} = \|\Phi_{\mathcal{G}} + \mathbb{I}\|_{L^q(\Sigma^{exp}_{\mathcal{K}})} \leq C < \infty$$

Together with the local assumption (3.2) on $w_{\mathcal{S}}$ and (A.9), it follows that

$$\|k^i w_{\mathcal{S}}(k,n)G^{-1}(k)\|_{L^q(\Sigma^{exp}_{\mathcal{K}})} = O(n^{\frac{1}{4\chi} - \frac{1}{q\chi}}), \quad q \in [1,4),\ i \in \mathbb{N}_0,$$

which is in line with (3.5). Moreover, we see that we are now required to find only a local a priori L^p-estimate of the form

$$\|S_-\|_{L^p(\Sigma^{exp}_{\mathcal{K}})} = O(n^r). \tag{A.10}$$

Thus, Theorem 3.1 remains valid with only the local a priori L^p-estimate for S_- (A.10) instead of (3.6) and G substituted for N. In fact, because of (A.8) the two relations (3.7) and (3.8) between S and N continue to be true and Theorem 3.1 holds in its original form with condition (A.10) instead of (3.6).

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Faculty of Mathematics, University of Vienna, Oskar-Morgenstern-Platz 1, 1090 Wien

E-mail address: Mateusz.Piorkowski@univie.ac.at