DISSERTATION

Titel der Dissertation

On the Inverse Scattering Transform for the Korteweg–de Vries Equation with Steplike Initial Profile

Verfasser

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Abstract

The Korteweg–de Vries (KdV) equation is an integrable wave equation modeling shallow water waves and is one of the most prominent soliton equations. The corresponding Cauchy problem was solved by Gardner, Green, Kurskal, and Miura by the inverse scattering transform. In the classical case the initial data will vanish asymptotically and this case is well understood. Another case, modeling shock and rarefaction waves, is when the initial conditions asymptotically tend to different constants, known as steplike initial conditions.

In the first part of this thesis we study the underlying direct and inverse scattering problem for the one-dimensional Schrödinger equation with steplike potentials. We give necessary and sufficient conditions for the scattering data to correspond to a potential with prescribed smoothness and prescribed spatial decay. This problem has been considered before but our results generalize all previous known results.

In the second part these results are then applied to the Cauchy problem of the KdV equation with steplike initial data. More specifically, we look at the case corresponding to rarefaction waves. For this case we formulate the inverse scattering problem as an oscillatory Riemann–Hilbert factorization problem and apply the nonlinear steepest descent method to determine the long-time behaviour of solutions. To analyse the problem one needs to change to a new phase function, the so-called g function, which will depend on a slow variable $\xi = \frac{x}{12t}$. After this change the problem can be deformed to an explicitly solvable model problem. Depending on the value of ξ there are three main regions as $t \to \infty$: For $\xi < -\xi_0$ the solution is close to the left constant. For $-\xi_0 < \xi < 0$ there is a rarefaction region where the solution behaves like $\frac{x}{6t}$. For $0 < \xi$ there is a soliton region where the solution is given by a sum of solitons.

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Contents

Chapter 1

Introduction

1.1 Objective

The aim of the dissertation is to provide a rigorous treatment of the inverse scattering transform (IST) for solving the Cauchy problem for the Korteweg–de Vries equation

$$q_t(x,t) = -q_{xxx}(x,t) + 6q(x,t)q_x(x,t), \tag{1.1.1}$$

with steplike initial data $q(x) := q(x,0)$, such that

$$q(x) \to c_\pm, \quad \text{as} \quad x \to \pm\infty, \tag{1.1.2}$$

where c_+, c_- are different real valued constants. We assume that $q \in L^1_{\mathrm{loc}}(\mathbb{R})$ and tends to its background asymptotics c_+ and c_- with m "moments" finite:

$$\int_0^{+\infty} (1+|x|^m)(|q(x)-c_+| + |q(-x)-c_-|)dx < \infty, \tag{1.1.3}$$

where $m \geq 1$ is a fixed integer. We need the following definitions.

Definition 1.1.1. *Let $m \geq 0$ and $n \geq 0$ be integers and $f : \mathbb{R} \to \mathbb{R}$ be an n times differentiable function. We say that $f \in \mathcal{L}_m^n(\mathbb{R}_\pm)$ if $f^{(n)}(x)(1+|x|^m) \in L^1(\mathbb{R}_\pm)$ for $j = 0, 1, \ldots, n$.*

Definition 1.1.2. *Let $c_\pm$ be given real values and let $m \geq 1$, $n \geq 0$ be given integers. We say that $q \in \mathcal{L}_m^n(c_+, c_-)$ if $q_\pm(\cdot) := q(\cdot) - c_\pm \in \mathcal{L}_m^n(\mathbb{R}_\pm)$.*

The main concern is to investigate in the direct and inverse scattering for the one–dimensional Schrödinger operator $-\frac{d^2}{dx^2} + q(x)$, which has left and right Jost solutions ϕ_+, ϕ_- . We start with this transformation operators for the Schrödinger operator with steplike background and are giving estimates for them and their kernels and its derivatives. Furthermore we describe the analytical properties of underlying scattering data, e.g. we prove that the Wronskian $W(\lambda) := W(\phi_-(\lambda,\cdot), \phi_+(\lambda,\cdot))$ $(W(f,g) = fg' - gf')$ of the Jost solutions has a zero of the first order at the edge of continuous spectrum in the resonance case $W = 0$ for $q \in \mathcal{L}_1^0(c_+, c_-)$. This problem was solved for the second moment only [17].

Furthermore we establish the Gelfand-Levitan-Marchenko equation for potential of form (1.1.2) and get estimates for the kernel of the GLM equation and its derivatives. This was done before for decaying initial data $c_+ = c_- = 0$ of first and second moment in [59] and [20], and steplike data of second moment [17].

We finish our consideration of the scattering by proofing that the described analytic properties of the scattering are necessary and sufficient for the solution of the inverse problem.

Finally we use the nonlinear steepest descent method for oscillatory Riemann–Hilbert problems from [22] and apply it to rigorously establish long-time asymptotics in all principal regions for the rarefaction problem $c_+ = 0$, $c_- = c^2$. We assume here for some $\varepsilon > 0$

$$\int_0^{+\infty} \mathrm{e}^{\varepsilon x}(|q(x)| + |q(-x) - c^2|dx < \infty,$$

$$\int_0^{\infty} x^4 \left(|q(x)| + |q(-x) - c^2| + |q^{(i)}(x)|\right) dx < \infty, \quad i = 1, \dots, 8.$$

It is known (cf. [25], [30]), that this Cauchy problem has a unique solution satisfying $q(\cdot, t) \in C^3(\mathbb{R})$ and

$$\int_0^{+\infty} |x|(|q(x,t) - c_+| + |q(-x,t) - c_-|)dx < \infty, \qquad t \in \mathbb{R}.$$

If $q(x, 0)$ is a Schwartz type perturbation, that is $q \in \mathcal{L}_m^n(c_+, c_-)$ for all $m \geq 1$ and $n \geq 0$, then the solution $q(x,t)$ of the KdV equation behaves asymptotically as $t \to \infty$ as follows:

(i) In the region $x > \varepsilon$ the solution $q(x,t)$ splits into classical solitons;

(ii) In the region $(-6c^2 + \varepsilon)t < x < -\varepsilon$ we have $q(x,t) \sim \dfrac{x}{t}$;

(iii) In the region $x < (-6c^2 - \varepsilon)t$ we obtain asymptotics $q(x,t) = c^2$ plus oscillatory term.

1.2 Literature

First we will describe the literature for the scattering problem. Among various direct/inverse spectral problems the scattering problem on the whole axis for one-dimensional Schrödinger operators with decaying potentials takes a particular place as being one of the most rigorously investigated spectral problems. Being considered first by Kay and Moses [47] on a physical level of rigor, it was rigorously studied by Faddeev [31], and then revisited independently by Marchenko [59] and by Deift and Trubowitz [20]. In particular, Faddeev [31] considered the inverse problem in the class of potentials which have a finite first moment (i.e., (1.1.3) with $c_- = c_+ = 0$ and $m = 1$) but the importance of the behaviour of the scattering coefficients at the bottom of the continuous spectrum was missed. A complete solution was independently given by Marchenko [59] (see also Levitan [58]) for the first moment ($m = 1$) and by Deift and Trubowitz [20] for the second moment (2) who also gave an example showing that some condition on the aforementioned behaviour is necessary to solve the inverse problem.

The next simplest case is the the so-called steplike case where the potential tends to different constants one the left and right half axis. The corresponding scattering problem was first considered on an informal level by Buslaev and Fomin in [15] who studied mostly the direct scattering problem and derived the main equation of the inverse problem — the Gelfand–Levitan–Marchenko (GLM) equation. A complete solution of the direct and inverse scattering problem for steplike potentials with a finite second moment (i.e., (1.1.3) with $m = 2$) was solved rigorously by Cohen and Kappeler [17] (see also [18] and [37]). While several aspects in the steplike case are similar to the decaying case, there are also some distinctive differences due to the presence of spectrum of multiplicity one. Moreover, there have also been further generalizations to the case of periodic backgrounds by Firsova [32, 34, 33] and to steplike finite-gap backgrounds by Ira Egorova, Boutet de Monvel and Gerald Teschl [11] (see also [60]) and to steplike almost periodic backgrounds by Grunert [38, 39]. We refer to these publications for further information.

Our first aim is to use the Marchenko approach to generalize the results of [17] for the case of steplike potentials with finite first moment which in fact turns out to be much more delicate than the second moment and has been done in [24]. In fact, we will also give a complete solution of the inverse problem for potentials with any given number of moments $m \geq 1$ and any given number of derivatives $n \geq 0$ which has important applications for the solution of the Korteweg–de Vries (KdV) equation.

In fact, as is well known, the inverse scattering transform (IST) is the main ingredient for solving and understanding the solutions of the KdV (and the associated modified KdV) equation. In fact, applications to KdV were already considered in the work of the aforementioned authors [59], [20], [46], [35], [28, 29, 30], [25] and we refer to these papers for further details. Concerning the long-time asymptotics of solutions we refer to the review [40] and to [49], [61], [53], [62],[23] for more recent developments.

For the asymptotic behaviour of $q(x,t)$ as $t \to \infty$ it is known from several results ([6]–[10], [41], [42], [36], [64]), obtained on a physical level of rigor, that the solution can be split into three main regions: In fact, the long-time asymptotics for this problem were first studied by Gurevich and Pitaevskii [41], [42]. These authors have used the Whitham multi-phase averaging method and obtained the main term of the asymptotics of the solution in terms the Jacobi elliptic function. Moreover, they gave a qualitative picture of the splitting of an initial step into solitons. Since the Schrödinger operator with the Heaviside step function as potential has no discrete spectrum, this picture refuted the general idea that solitons arise only from the discrete spectrum. This phenomenon was explained by Khruslov [48], [50] with the help of the inverse scattering transform (IST) in the form of the Marchenko equation. The IST not only made it possible to obtain an explicit form of these asymptotic solitons but also to give a rigorous proof that the solitons are generated by a small vicinity of the edge of the continuous spectrum. Further developments of this method can be found in [67] and [51], while the Cauchy problem for the Heaviside functions and generalizations has been studied in [16] as well. The first finite-gap description of the asymptotics for the steplike initial problem of the KdV equation was given by Bikbaev and Novokshenov [8] only in 1987 (see also [6]–[7], [10], [9], and the review [64]). The results are based on an analysis of the Whitham equations and the theory of analytic functions on a hyperelliptic surface.

The aim here is to confirm the known leading asymptotics of the solution in these principal regions and rigorously justify them using the nonlinear steepest descent method for oscillatory Riemann–Hilbert (RH) problems. This approach is based on the inverse scattering transform for steplike initial data. Scattering theory for the Schrödinger operator with step-like potential was originally developed by Buslaev and Fomin [15] with later contributions by Cohen and Kappeler [17]. The rarefaction problem has been solved by us in [4]. It has been studied before in [56] using matched–asymptotic coordinate expansion analysis. For the corresponding shock problem we refer to [1, 6, 23, 41, 42, 51, 57, 67].

The IST with the steepest descent method is also used in several other papers for different equations like Toda in [45] or [27] together with a g-function ansatz, or for Camassa–Holm [13] or very recently for the Ostrovsky–Vakhnenko equation [14].

1.3 Structure

The thesis is based on two papers [24, 4], written during the Ph.D.. We will investigate the direct and inverse scattering problem in chapter 2. We start with transformation operators for the Schrödinger operator with step-like background and are giving estimates for them and their kernels. Furthermore we discuss analytical properties of the scattering data and establish the Gelfand-Levitan-Marchenko equation for potential of form (1.1.2). We finish our consideration of the scattering by proving uniqueness of the inverse problem. In chapter 3 we formulate the inverse scattering problem in terms of a Riemann–Hilbert problem and analyse it, using nonlinear steepest descent to obtain the solution $q(x,t)$. Thus we formulate the inverse step as a Riemann–Hilbert problem, solving it by reduction to a model problem with conjugation and deformation of the Phase Φ. The model problem gives the solution and by transforming back we obtain the solution of our initial RHP. Considering the long-time asymptotics we achieve the KdV solution.

Chapter 2

Scattering problem for Schrödinger operator with steplike background and uniqueness

For the following chapter we will need results for the direct and inverse scattering. Therefore we will start to consider the Sturm–Liouville spectral problem

$$(Lf)(x) := -\frac{d^2}{dx^2}f(x) + q(x)f(x) = \lambda f(x), \qquad x \in \mathbb{R}, \tag{2.0.1}$$

with a steplike potential $q(x)$ satisfying (1.1.2) and (1.1.3). We will investigate in the transformation operator and give estimates for them. The Jost solutions are giving rise to scattering data. We repeat known analytical properties and additionally give the behaviour of the scattering data in the resonance case. Using properties and estimates for the transformation operators we obtain the GLM equations and estimates for their kernels. We proof that our properties of the scattering data is necessary and sufficient to solve the inverse problem.

We will introduce some notations, before we start with the direct scattering. Note, that $f \in \mathcal{L}_m^0(\mathbb{R}_\pm)$ means that $\int_{\mathbb{R}_\pm} |f(x)|(1+|x|^m)dx < \infty$. By this definition $\mathcal{L}_0^0(\mathbb{R}_\pm) = L^1(\mathbb{R}_\pm) \cap L^1_{\text{loc}}(\mathbb{R})$ and $\mathcal{L}_0^j(\mathbb{R}_\pm) = \{f : \ f^{(i)} \in \mathcal{L}_0^0(\mathbb{R}_\pm), \ 0 \leq i \leq j\}$.

Note that $q \in \mathcal{L}_m^0(c_+, c_-)$ if condition (1.1.3) holds. If $q \in \mathcal{L}_m^n(c_+, c_-)$

with $n \geq 1$ then in addition

$$\int_{\mathbb{R}} (1 + |x|^m)|q^{(i)}(x)|dx < \infty, \quad i = 1, \dots, n. \tag{2.0.2}$$

We will denote $\underline{c} := \min\{c_+, c_-\}$, $\overline{c} := \max\{c_+, c_-\}$ to consider both steplike initial data cases, namely shock with $c_- < c_+$ and rarefaction with $c_- > c_+$, and abbreviate $\mathcal{D} := \mathbb{C} \setminus [\underline{c}, \infty)$. We consider equation (2.0.1) with the spectral parameter $\lambda \in \operatorname{clos}(\mathcal{D})$, where $\operatorname{clos}(A)$ denotes the closure of a set A. Along with λ we use two more spectral parameters

$$k_\pm := \sqrt{\lambda - c_\pm},$$

which map the domains $\mathbb{C} \setminus [c_\pm, \infty)$ conformally onto $\mathbb{C}^+$. Thus there is a one to one correspondence between the parameters $k_\pm$ and λ.

2.1 The Direct scattering problem

2.1.1 Properties of the Jost solutions

In this subsection we collect some well-known properties of the Jost solutions for (2.0.1) with $q \in \mathcal{L}_1^0(c_+, c_-)$ and establish additional properties of these solutions for a potential from the class $\mathcal{L}_m^n(c_+, c_-)$ with $m \geq 2$ or $n \geq 1$. All the estimates below are one-sided and hence are generated by the behaviour of the potential on one half axis. For $q_\pm(\cdot) = q(\cdot) - c_\pm \in \mathcal{L}_m^n(\mathbb{R}_\pm)$, $m \geq 1$, $n \geq 0$, introduce nonnegative, as $x \to \pm\infty$ nonincreasing functions

$$\sigma_{\pm,i}(x) := \pm \int_x^{\pm\infty} |q_\pm^{(i)}(\xi)|d\xi, \quad \hat{\sigma}_{\pm,i}(x) := \pm \int_x^{\pm\infty} \sigma_{\pm,i}(\xi)d\xi, \quad i = 0, 1, \dots, n. \tag{2.1.1}$$

Evidently,

$$\sigma_{\pm,i}(\cdot) \in \mathcal{L}_{m-1}^1(\mathbb{R}_\pm),\ m \geq 1,\ \hat{\sigma}_{\pm,i}(\cdot) \in \mathcal{L}_{m-2}^2(\mathbb{R}_\pm),\ m \geq 2, \tag{2.1.2}$$

$$\hat{\sigma}_{\pm,i}(x) \downarrow 0 \ \text{ as } \ x \to \pm\infty, \ \text{ for } q_\pm \in \mathcal{L}_1^n(\mathbb{R}_\pm), \ i = 0, 1, \dots, n. \tag{2.1.3}$$

Lemma 2.1.1. *([59, Lemmas 3.1.1–3.1.3]). Let $q_\pm(\cdot) = q(\cdot) - c_\pm \in \mathcal{L}_1^0(\mathbb{R}_\pm)$. Then for all $\lambda \in \operatorname{clos}(\mathcal{D})$ equation (2.0.1) has a solution $\phi_\pm(\lambda, x)$ which can be represented as*

$$\phi_\pm(\lambda, x) = \mathrm{e}^{\pm\mathrm{i}k_\pm x} \pm \int_x^{\pm\infty} K_\pm(x, y)\mathrm{e}^{\pm\mathrm{i}k_\pm y}dy, \tag{2.1.4}$$

where the kernel $K_\pm(x,y)$ is real-valued and satisfies the inequality

$$|K_\pm(x,y)| \le \frac{1}{2}\sigma_{\pm,0}\left(\frac{x+y}{2}\right)\exp\left\{\hat\sigma_{\pm,0}(x) - \hat\sigma_{\pm,0}\left(\frac{x+y}{2}\right)\right\}. \qquad (2.1.5)$$

Moreover,

$$K_\pm(x,x) = \pm\frac{1}{2}\int_x^{\pm\infty} q_\pm(\xi)d\xi.$$

The function $K_\pm(x,y)$ has first order partial derivatives which satisfy the inequality

$$\left|\frac{\partial K_\pm(x_1,x_2)}{\partial x_j} \pm \frac{1}{4}q_\pm\left(\frac{x_1+x_2}{2}\right)\right| \le \qquad (2.1.6)$$

$$\le \frac{1}{2}\sigma_{\pm,0}\left(x\right)\sigma_{\pm,0}\left(\frac{x_1+x_2}{2}\right)\exp\left\{\hat\sigma_{\pm,0}(x_1) - \hat\sigma_{\pm,0}\left(\frac{x_1+x_2}{2}\right)\right\}.$$

The solution $\phi_\pm(\lambda,x)$ is an analytic function of $k_\pm$ in $\mathbb{C}^+$ and is continuous up to $\mathbb{R}$. For all $\lambda\in\operatorname{clos}(\mathcal{D})$ the following estimate is valid

$$\left|\phi_\pm(\lambda,x) - \mathrm{e}^{\pm \mathrm{i}k_\pm x}\right| \le \left(\hat\sigma_{\pm,0}(x) - \hat\sigma_{\pm,0}\left(x\pm\frac{1}{|k_\pm|}\right)\right)\mathrm{e}^{-\operatorname{Im}(k_\pm)x + \hat\sigma_{\pm,0}(x)}. \qquad (2.1.7)$$

For $k_\pm\in\mathbb{R}\setminus\{0\}$ the functions $\phi_\pm(\lambda,x)$ and $\overline{\phi(\lambda,x)}$ are linearly independent with

$$W(\phi_\pm(\lambda,\cdot),\overline{\phi_\pm(\lambda,\cdot)}) = \mp 2\mathrm{i}k_\pm, \qquad (2.1.8)$$

where $W(f,g) = fg' - gf'$ denotes the usual Wronski determinant.

Formulas (2.1.5) and (2.1.6) together with (2.1.4) and (2.1.2) imply

Corollary 2.1.2. *Let $q_\pm\in\mathcal{L}_m^0(\mathbb{R}_\pm)$, $m\ge 1$. Then*

$$K_\pm(x,\cdot),\quad \frac{\partial K_\pm(x,\cdot)}{\partial x}\in\mathcal{L}^0_{m-1}(\mathbb{R}_\pm),\quad m\ge 1, \qquad (2.1.9)$$

and the function $\phi_\pm(\lambda,x)$ is $m-1$ times differentiable with respect to $k_\pm\in\mathbb{R}$.

Note also that for $m\ge 2$ Lemma 2.1.1 implies $xK_\pm(x,x)\to 0$ and

$$\frac{\partial^l K_\pm(x,\cdot)}{\partial x^l}\in L^0_{m-1}(\pm\infty),\ l=0,1.$$

It allows us to compute the following Wronskian $W(\phi_\pm,\frac{\partial}{\partial k}\phi)(0)$. Namely, if $\frac{\partial}{\partial k_\pm}\phi_\pm(k_\pm,x) = \dot\phi_\pm(k_\pm,x)$ exists then it solves the Schrödinger equation for $k\pm = 0$, and, therefore, the Wronskian $W\Big(\phi_\pm(0,\cdot),\dot\phi_\pm(0,\cdot)\Big)$ does not depend on spatial variable and can be estimated for large values of x. Formula (2.1.4) implies

$$\phi_\pm(0,x) = 1 \pm \int_x^{\pm\infty} K_\pm(x,y)dy, \quad \dot{\phi}_\pm(0,x) = \pm \mathrm{i}x + \mathrm{i}\int_x^{\pm\infty} yK_\pm(x,y)dy,$$

$$\begin{aligned}\dot{\phi}'_\pm(0,x) &= \mathrm{i}(1 \mp xK_\pm(x,x) + \int_x^{\pm\infty} \frac{\partial K_\pm(x,y)}{\partial x} ydy), \ \phi'_\pm(0,x) \\ &= \pm \int_x^{\pm\infty} \frac{\partial}{\partial x} K_\pm(x,y)dy\end{aligned}$$

Thus, $\phi_\pm(0,x)\dot{\phi}'_\pm(0,x) = \mathrm{i} + o(1)$ and $|\phi'_\pm(0,x)\dot{\phi}_\pm(0,x)| \le Cx\sigma_\pm^{(0)}(x)\hat{\sigma}_\pm(2x) = o(1)$. We proved

Corollary 2.1.3. *Let* $q_\pm \in L_2^0(\pm\infty)$. *Then*

$$W(\phi_\pm, \dot{\phi}_\pm)(0) = \pm \mathrm{i}.$$

Note, that the key ingredient for proving the estimates (2.1.5) and (2.1.6) is a rigorous investigation of the following integral equation (formula (3.1.12) of [59])

$$K_\pm(x,y) = \pm\frac{1}{2}\int_{\frac{x+y}{2}}^{\pm\infty} q_\pm(\xi)d\xi + \int_{\frac{x+y}{2}}^{\pm\infty} d\alpha \int_0^{\frac{y-x}{2}} q_\pm(\alpha-\beta)K_\pm(\alpha-\beta,\alpha+\beta)d\beta. \tag{2.1.10}$$

To further study the properties of the Jost solution we represent (2.1.4) in the form proposed in [20]:

$$\phi_\pm(\lambda,x) = \mathrm{e}^{\mathrm{i}k_\pm x}\left(1 \pm \int_0^{\pm\infty} B_\pm(x,y)\mathrm{e}^{\pm 2\mathrm{i}k_\pm y}dy\right), \tag{2.1.11}$$

where

$$B_\pm(x,y) = 2K_\pm(x,x+2y), \quad B_\pm(x,0) = \pm\int_x^{\pm\infty} q_\pm(\xi)d\xi, \tag{2.1.12}$$

and equation (2.1.10) transforms into the following integral equation with respect to $\pm y \ge 0$

$$B_\pm(x,y) = \pm\int_{x+y}^{\pm\infty} q_\pm(s)ds + \int_{x+y}^{\pm\infty} d\alpha \int_0^y d\beta q_\pm(\alpha-\beta)B_\pm(\alpha-\beta,\beta). \tag{2.1.13}$$

This equation is the basis for proving the following

Lemma 2.1.4. *Let $n \geq 1$ and $m \geq 1$ be fixed natural numbers and let $q_\pm \in \mathcal{L}_m^n(\mathbb{R}_\pm)$. Then the functions $B_\pm(x,y)$ have $n+1$ partial derivatives and the following estimates are valid*

$$\left|\frac{\partial^s}{\partial x^l\,\partial y^{s-l}}B_\pm(x,y) \pm q_\pm^{(s-1)}(x+y)\right| \leq C_\pm(x)\nu_{\pm,s}(x)\nu_{\pm,s}(x+y),\ l \leq s \leq n+1, \tag{2.1.14}$$

where

$$\nu_{\pm,l}(x) = \sum_{i=0}^{l-2}\left(\sigma_{\pm,i}(x) + |q_\pm^{(i)}(x)|\right), \quad l \geq 2, \quad \nu_{\pm,1}(x) := \sigma_{\pm,0}(x), \tag{2.1.15}$$

and $C_\pm(x) = C_\pm(x,n) \in \mathcal{C}(\mathbb{R})$ are positive functions which are nonincreasing as $x \to \pm\infty$.

Proof. Differentiating equation (2.1.13) with respect to each variable we get

$$\frac{\partial B_\pm(x,y)}{\partial x} = \mp q_\pm(x+y) - \int_x^{x+y} q_\pm(s)B_\pm(s,x+y-s)ds; \tag{2.1.16}$$

$$\frac{\partial B_\pm(x,y)}{\partial y} = \mp q_\pm(x+y) - \int_x^{x+y} q_\pm(s)B_\pm(s,x+y-s)ds + \int_x^{\pm\infty} q_\pm(\alpha)B_\pm(\alpha,y)d\alpha. \tag{2.1.17}$$

From these formulas and (2.1.12) we obtain

$$\frac{\partial B_\pm(x,0)}{\partial x} = \mp q_\pm(x); \quad \frac{\partial B_\pm(x,y)}{\partial y}|_{y=0} = \mp q_\pm(x) \pm \frac{1}{2}\left(\int_x^{\pm\infty} q_\pm(\alpha)d\alpha\right)^2,$$

$$\frac{\partial B_\pm(x,y)}{\partial y} = \frac{\partial B_\pm(x,y)}{\partial x} + \int_x^{\pm\infty} q_\pm(\alpha)B_\pm(\alpha,y)d\alpha. \tag{2.1.18}$$

We observe that the partial derivatives of $B_\pm$ which contain at least one differentiation with respect to x have the structure

$$\begin{aligned}\frac{\partial^p}{\partial x^k\partial y^{p-k}}B_\pm(x,y) &= \mp q_\pm^{(p-1)}(x+y) + D_{\pm,p,k}(x,y) + \\ &+ \int_{x+y}^{x} q_\pm(\xi)\frac{\partial^{p-1}}{\partial y^{p-1}}B_\pm(\xi,x+y-\xi)d\xi, \quad p > k \geq 1,\end{aligned} \tag{2.1.19}$$

where $D_{\pm,p,k}(x,y)$ is the sum of all derivatives of all integrated terms which appeared after $p-1$ differentiation of the upper and lower limits of the integral on the right hand side of (2.1.16). Since the integrand in (2.1.19) at the lower limit of integration has value

$$q_\pm(\xi)\frac{\partial^{p-1}}{\partial y^{p-1}}B_\pm(\xi, x+y-\xi)|_{\xi=x+y} = q_\pm(x+y)B_{\pm,p-1}(x+y),$$

where

$$B_{\pm,r}(\xi) = \frac{\partial^r}{\partial t^r}B_\pm(\xi,t)|_{t=0}. \tag{2.1.20}$$

Thus, further derivatives of such a term do not depend on whether we differentiate it with respect to x or y. The same integrand at the upper limit has the value $q_\pm(x)\frac{\partial^{r-1}}{\partial y^{r-1}}B_\pm(x,y)$, and it will appear only after a differentiation with respect to x. Taking all this into account we conclude that $D_{\pm,p,k}(x,y)$ in (2.1.19) can be represented as

$$D_{\pm,p,k}(x,y) = (1-\delta(k,1))\frac{\partial^{p-k}}{\partial y^{p-k}}\sum_{s=2}^{k}\frac{\partial^{k-s}}{\partial x^{k-s}}\left(q_\pm(x)\frac{\partial^{s-2}}{\partial y^{s-2}}B_\pm(x,y)\right)-D_{\pm,p}(x+y),$$

where $\delta(r,s)$ is the Kronecker delta (i.e. the first summand is absent for $k=1$) and

$$D_{\pm,p}(\xi) := \sum_{s=0}^{p-2}\frac{d^{p-s}}{d\xi^{p-s}}\left(q_\pm(\xi)B_{\pm,s}(\xi)\right), \tag{2.1.21}$$

see (2.1.20). If we differentiate (2.1.17) with respect to y, we get for $p \geq 2$

$$\frac{\partial^p}{\partial y^p}B(x,y) = \mp q_\pm^{(p-1)}(x+y) + D_{\pm,p}(x+y)+$$

$$+\int_{x+y}^{x} q_\pm(\xi)\frac{\partial^{p-1}}{\partial y^{p-1}}B_\pm(\xi,x+y-\xi)d\xi + \int_x^{\pm\infty} q_\pm(\xi)\frac{\partial^{p-1}}{\partial y^{p-1}}B_\pm(\xi,y)d\xi,$$

where $D_{\pm,p}(\xi)$ is defined by (2.1.21). We complete the proof by induction taking into account (2.1.12) and the estimates (2.1.5), (2.1.6) in which the exponent factors are replaced by the more crude estimate of type $C_\pm(x)$. □

2.1.2 Analytical properties of the scattering data

The spectrum of L with steplike potential consists of an absolutely continuous and a discrete part. Introduce the sets

$$\Sigma^{(2)} := [\overline{c}, +\infty), \quad \Sigma^{(1)} = [\underline{c},\ \overline{c}], \quad \Sigma := \Sigma^{(2)} \cup \Sigma^{(1)}.$$

The set Σ is the (absolutely) continuous spectrum of operator L, and $\Sigma^{(1)}$, respectively $\Sigma^{(2)}$, are the parts which are of multiplicity one, respectively two.

We will distinguish two sides of the cuts along the spectrum, namely $\Sigma = \Sigma^u \cup \Sigma^l$ with $\Sigma^u = \{\lambda^u = \lambda + \mathrm{i}0, \lambda \in [\underline{c}, \infty)\}$ and $\Sigma^l = \{\lambda^l = \lambda - \mathrm{i}0, \lambda \in [\underline{c}, \infty)\}$. Note that the set Σ is the preimage of the real axis $\mathbb{R}$ under the conformal map $k_\pm(\lambda) : \mathrm{clos}(\mathcal{D}) \to \overline{\mathbb{C}^+}$ when $c_\pm < c_\mp$. For $q \in \mathcal{L}_m^n(c_+, c_-)$ with $m \geq 1$ and $n \geq 0$ the operator L has a finite discrete spectrum (see [2]), which we denote as $\Sigma_d = \{\lambda_1, \dots, \lambda_p\}$, where $\lambda_1 < \cdots < \lambda_p < \underline{c}$. Our next step is to briefly describe some well-known analytical properties of the scattering data ([15], [17]). Most of these properties follow from analytical properties of the Wronskian of the Jost solutions $W(\lambda) := W(\phi_-(\lambda, \cdot), \phi_+(\lambda, \cdot))$. The representations (2.1.4) imply that the Jost solutions, together with their derivatives, decays exponentially fast as $x \to \pm\infty$ for $\mathrm{Im}(k_\pm) > 0$. Evidently, the discrete spectrum Σ_d of L coincides with the set of points, where ϕ_+ is proportional to ϕ_- and, correspondingly, their Wronskian vanishes. The Jost solutions at these points are called the left and the right eigenfunctions. They are real-valued and we denote the corresponding norming constants by

$$\gamma_j^\pm := \left(\int_{\mathbb{R}} \phi_\pm^2(\lambda_j, x) dx \right)^{-1}.$$

Lemma 2.1.5. *Let $q \in \mathcal{L}_m^n(c_+, c_-)$ with $m \geq 1$, $n \geq 0$. Then the function $W(\lambda)$ possesses the following properties*

(i) It is holomorphic in the domain $\mathcal{D}$ and continuous up to the boundary Σ of this domain.

(ii) $W(\lambda) \neq 0$ as $\lambda \in \Sigma \setminus \underline{c}$.

(iii) On the set Σ it satisfies the symmetry property, namely $W(\lambda^u) = \overline{W(\lambda^l)}$. Moreover, $W(\lambda) \in \mathbb{R}$ as $\lambda \in (-\infty, c_-)$.

(iv) It has simple zeros in the domain $\mathcal{D}$ only at the points $\lambda_1, \dots, \lambda_p$, where

$$\left(\frac{dW}{d\lambda}(\lambda_j) \right)^{-2} = \gamma_j^+ \gamma_j^-. \tag{2.1.22}$$

Proof. Properties (i) and (iii) follows immediately from Lemma 2.1.1. To prove (ii) consider first $\lambda = \lambda_0 \in \Sigma^{(2)}$. If $W(\lambda_0) = 0$ for such a λ_0 then $\phi_+(k_+(\lambda_0), x) = C\phi_-(k_-(\lambda_0), x)$ with $C = C(k_0)$. But then $\overline{\phi_+(k_+(\lambda_0), x)} = \overline{C}\,\overline{\phi_-(k_-(\lambda_0), x)}$ since $W(\lambda_0) = 0$. From (2.1.8) then follows

$$-2\mathrm{i}k_+(\lambda_0) = W\Big(\phi_+, \overline{\phi}_+\Big)(\lambda_0) = |C|^2 W\Big(\phi_-, \overline{\phi}_-\Big)(\lambda_0) = 2\mathrm{i}k_-(\lambda_0)|C|^2$$

which contradicts to the definition of k_+ and k_-. If $W(\lambda_0) = 0$ for $\lambda_0 \in \Sigma^{(1)}$ then $C\phi_-(\lambda_0, x) = \overline{C}\,\overline{\phi_-(\lambda_0, x)}$ since $\phi(\lambda_0, x) \in \mathbb{R}$. It means that $\phi_-(\lambda_0, x)$

and $\overline{\phi_-(\lambda_0, x)}$ are dependent solutions, which contradicts to (2.1.8). Thus the only possible zero of function $W(\lambda)$ on Σ is the point $\underline{c}$. For proving (iv) we use [11]. $\square$

If $W(\underline{c}) = 0$ we will refer to this as the resonant case.

Remark 2.1.6. *Using iv and the inner derivation we have the important identity*

$$\dot{W}(\mathrm{i}\kappa_j) = 2\mathrm{i}\kappa_j \int \phi_- \phi_+.$$

To study further spectral properties of L we consider the usual scattering relations

$$T_\mp(\lambda)\phi_\pm(\lambda, x) = \overline{\phi_\mp(\lambda, x)} + R_\mp(\lambda)\phi_\mp(\lambda, x), \quad k_\pm \in \mathbb{R}, \tag{2.1.23}$$

where the transmission and reflection coefficients are defined as usual,

$$T_\pm(\lambda) := \frac{W(\overline{\phi_\pm(\lambda)}, \phi_\pm(\lambda))}{W(\phi_\mp(\lambda), \phi_\pm(\lambda))}, \qquad R_\pm(\lambda) := -\frac{W(\phi_\mp(\lambda), \overline{\phi_\pm(\lambda)})}{W(\phi_\mp(\lambda),\, \phi_\pm(\lambda))}, \quad k_\pm \in \mathbb{R}. \tag{2.1.24}$$

Their properties are given in the following

Lemma 2.1.7. *Let $q \in \mathcal{L}_m^n(c_+, c_-)$ with $m \geq 1$, $n \geq 0$. Then the entries of the scattering matrix possess the following properties:*

I. **(a)** $T_\pm(\lambda + \mathrm{i}0) = \overline{T_\pm(\lambda - \mathrm{i}0)}$ *and* $R_\pm(\lambda + \mathrm{i}0) = \overline{R_\pm(\lambda - \mathrm{i}0)}$ *for* $k_\pm(\lambda) \in \mathbb{R}$.

(b) $\dfrac{T_\pm(\lambda)}{\overline{T_\pm(\lambda)}} = R_\pm(\lambda)$ *for* $\lambda \in \Sigma^{(1)}$ *when* $c_\pm = \underline{c}$.

(c) $1 - |R_\pm(\lambda)|^2 = \dfrac{k_\mp}{k_\pm}\, |T_\pm(\lambda)|^2$ *for* $\lambda \in \Sigma^{(2)}$.

(d) $\overline{R_\pm(\lambda)} T_\pm(\lambda) + R_\mp(\lambda) \overline{T_\pm(\lambda)} = 0$ *for* $\lambda \in \Sigma^{(2)}$.

(e) $T_\pm(\lambda) = 1 + O(\lambda^{-1/2})$ *and* $R_\pm(\lambda) = O(\lambda^{-1/2})$ *for* $\lambda \to \infty$.

II. **(a)** *The functions $T_\pm(\lambda)$ can be analytically continued to the domain $\mathcal{D}$ satisfying*

$$2\mathrm{i}k_+(\lambda)T_+^{-1}(\lambda) = 2\mathrm{i}k_-(\lambda)T_-^{-1}(\lambda) =: W(\lambda), \tag{2.1.25}$$

where $W(\lambda)$ possesses the properties (i)–(ii) from Lemma 2.1.5.

(b) *If $W(\underline{c}) = 0$ then $W(\lambda) = \mathrm{i}\gamma\sqrt{\lambda - \underline{c}}\,(1 + o(1))$, where $\gamma \in \mathbb{R} \setminus \{0\}$.*

III. *$R_\pm(\lambda)$ is continuous for $k_\pm(\lambda) \in \mathbb{R}$.*

Proof. Properties **I. (a)–(e)**, **II. (a)** are proved in [11] for $m = 2$, and the proof remains valid for $m = 1$. Property **III** is evidently valid for $k_\pm \neq 0$ by (2.1.24), continuity of the Jost solutions, and absence of resonances. Since $W(\overline{c}) \neq 0$ by Lemma 2.1.5 it remains to establish that in the case $\underline{c} = c_\pm$ the function $R_\pm$ is continuous as $k_\pm \to 0$. Since $\overline{\phi_\pm(c_\pm, x)} = \phi_\pm(c_\pm, x)$, the property

$$R_\pm(c_\pm) = -1 \quad \text{if} \;\; W(c_\pm) \neq 0, \tag{2.1.26}$$

follows immediately from (2.1.24). In the resonant case the proof of **II. (b)** will be deferred to Subsection 2.1.4. □

Since we have deferred the proof of **II. (b)** we will not use it until then. However, we will need the following weakened version of property **II. (b)**.

Lemma 2.1.8. *If $W(\underline{c}) = 0$ then, in a vicinity of point $\underline{c}$, the Wronskian admits the estimates*

$$W^{-1}(\lambda) = \begin{cases} O\left((\lambda - \underline{c})^{-1/2}\right) & \textit{for} \;\; \lambda \in \Sigma, \\ O\left((\lambda - \underline{c})^{-1/2-\delta}\right) & \textit{for} \; \lambda \in \mathbb{C} \setminus \Sigma, \end{cases} \tag{2.1.27}$$

where $\delta > 0$ is an arbitrary small number.

Proof. We give the proof for the case $c_- = \underline{c}$, $c_+ = \overline{c}$. The other case is analogous. In this case the point $k_- = 0$ corresponds to the point $\lambda = \underline{c}$. To study the Wronskian we use (2.1.25) for $T_-(\lambda)$. First we prove that T_- is bounded on the set $V_\varepsilon : \{\lambda(k_-) : -\varepsilon < k_- < \varepsilon\}$, for some $\varepsilon > 0$. In fact, due to the continuity of $\phi_-(\lambda, x)$ with respect to both variables we can choose a point x_0 such that $\phi_-(\underline{c}, x_0) \neq 0$ and $|\phi(\lambda, x_0)| > C > 0$ in V_ε for sufficiently small ε. Then by (2.1.23)

$$|T_-(\lambda)| = \frac{|R_-(\lambda)\phi_-(\lambda, x_0) + \overline{\phi_-(\lambda, x_0)}|}{|\phi_+(\lambda, x_0)|} \leq C, \quad \lambda \in V_\varepsilon.$$

Thus, for real λ near $\underline{c}$ we have $W^{-1}(\lambda) = O((\lambda - \underline{c})^{-1/2})$. For nonreal λ we use that the diagonal of the kernel of the resolvent $(L - \lambda I)^{-1}$

$$G(\lambda, x, x) = \frac{\phi_+(\lambda, x)\phi_-(\lambda, x)}{W(\lambda)}, \;\; \lambda \in \mathcal{D} \setminus \Sigma_d,$$

is a Herglotz–Nevanlinna function. Hence it can be represented as

$$G(\lambda, x_0, x_0) = \int_{\underline{c}}^{\underline{c}+\varepsilon^2} \frac{\operatorname{Im} G(\xi + \mathrm{i}0, x_0, x_0)}{\xi - \lambda} d\xi + G_1(\lambda),$$

where $G_1(\lambda)$ is a bounded in a vicinity of $\underline{c}$. But $G(\xi + \mathrm{i}0, x_0, x_0) = O((\xi - \underline{c})^{-1/2})$ and by [63, Chap. 22] we get (2.1.27). □

In what follows we set $\kappa_j^\pm := \sqrt{c_\pm - \lambda_j}$, such that $\mathrm{i}\kappa_j^\pm$ is the image of the eigenvalue λ_j under the map $k_\pm$. Then we have the following

Remark 2.1.9. *For the function $T_\pm(\lambda)$, regarded as a function of variable $k_\pm$,*

$$\mathrm{Res}_{\mathrm{i}\kappa_j^\pm} T_\pm(\lambda) = \mathrm{i}(\mu_j)^{\pm 1}\gamma_j^\pm, \quad \textit{where} \quad \phi_+(\lambda_j, x) = \mu_j \phi_-(\lambda_j, x). \tag{2.1.28}$$

Proof. Since $T_\pm = \frac{2\mathrm{i}k_\pm}{W}$ has a zero of order one in $\mathrm{i}\kappa_j$ we have

$$\begin{aligned}
\operatorname*{Res}_{\mathrm{i}\kappa_j} T_\pm = \operatorname*{Res}_{\mathrm{i}\kappa_j} \frac{1}{\frac{1}{T_\pm}} &= \frac{1}{\dot{T}_\pm^{-1}}(\mathrm{i}\kappa_j) = \frac{1}{-T_\pm^{-2}\dot{T}_\pm}(\mathrm{i}\kappa_j) = -\frac{T_\pm^2}{\dot{T}_\pm}(\mathrm{i}\kappa_j) \\
&= -\frac{\left(\frac{2\mathrm{i}k_\pm}{W}\right)^2}{\frac{-2\mathrm{i}k_\pm \dot{W}}{W^2}}(\mathrm{i}\kappa_j) = \frac{(2\mathrm{i}k_\pm)^2}{2\mathrm{i}k_\pm \dot{W}}(\mathrm{i}\kappa_j) = 2\mathrm{i}\frac{k_\pm}{\dot{W}}(\mathrm{i}\kappa_j) \\
&= \frac{2\mathrm{i}\mathrm{i}\kappa_j^\pm}{2\mathrm{i}\kappa_j^\pm \int \phi_-\phi_+} = \mathrm{i}\left(\int \phi_-\phi_+\right)^{-1} = \mathrm{i}\gamma_\pm \mu_j^{\pm 1}.
\end{aligned}$$

□

Scattering data example

Before we will continue with the GLM equation, we give simple examples for the scatterings data.
EXAMPLE 1
We start with considering rarefaction initial data of the form

$$q(x) = \begin{cases} 1, & \text{as } \; x < 0, \\ 0, & \text{as } \; x > 0. \end{cases}$$

In the end of the first chapter we will have proven, that it is necessary and sufficient for the scattering data to have the form

$$\begin{aligned}
\mathcal{S}_m^n(c_+, c_-) := \Big\{ & R_+(\lambda),\, T_+(\lambda),\, \sqrt{\lambda - c_+} \in \mathbb{R};\; R_-(\lambda),\, T_-(\lambda),\, \sqrt{\lambda - c_-} \in \mathbb{R}; \\
& \lambda_1, \dots, \lambda_p \in \mathbb{R} \setminus \sigma,\, \gamma_1^\pm, \dots, \gamma_p^\pm \in \mathbb{R}_+ \Big\}
\end{aligned}$$

and that it uniquely determines our solution. We will compute all of the terms here.

On the right half-axis we have $\phi_+(\lambda, x) = e^{\mathrm{i}k_+ x}$, on the left ϕ is given by $\phi_-(\lambda, x) = e^{-\mathrm{i}k_- x}$. Thus in zero both functions equals one. Computing the Wronskian for $x = 0$ we have

$$W(\lambda, 0) = \mathrm{i}k_+ + \mathrm{i}k_- = \mathrm{i}(k_+ + k_-).$$

Since the eigenvalues are given by the zeros of the Wronskian, which means $k_+ + \sqrt{k_+^2 - c^2} = 0$ there are no eigenvalues. Thus $\gamma_j^\pm$ is not defined as well. It remains to consider the reflection coefficients $R_\pm$ and transmission coefficients $T_\pm$. By 2.1.24

$$R_\pm(\lambda) := -\frac{W(\phi_\mp(\lambda), \overline{\phi_\pm(\lambda)})}{W(\phi_\mp(\lambda), \phi_\pm(\lambda))} = -\frac{\overline{\phi_\pm}'\phi_\mp - \overline{\phi_\pm}\phi_\mp'}{\phi_\pm'\phi_\mp - \phi_\pm\phi_\mp'}, \quad k_\pm \in \mathbb{R}.$$

This gives

$$R(\lambda) = -\frac{\overline{\mathrm{i}k_+} + \mathrm{i}k_-}{\mathrm{i}k_+ + \mathrm{i}k_-} = \frac{k_+ - k_-}{k_+ + k_-} = \frac{k_+ - \sqrt{k_+^2 - 1}}{k_+ + \sqrt{k_+^2 - 1}}, \quad \lambda \in \Sigma$$

$$R_-(\lambda) = -\frac{\overline{-\mathrm{i}k_-} - \mathrm{i}k_+}{-\mathrm{i}k_- - \mathrm{i}k_+} = -\frac{\mathrm{i}k_- - \mathrm{i}k_+}{-\mathrm{i}k_- - \mathrm{i}k_+} = \frac{\sqrt{k_+^2 - 1} - k_+}{\sqrt{k_+^2 - 1} + k_+}, \quad \lambda \in \Sigma^{(2)}.$$

On the other hand the transmission coefficients are given by 2.1.25

$$T_\pm(\lambda) = \frac{2\mathrm{i}k_\pm}{W(\lambda)} = \frac{2\mathrm{i}k_\pm}{\mathrm{i}k_+ + \mathrm{i}k_-} = \frac{2k_\pm}{k_+ + \sqrt{k_+^2 - 1}}.$$

This finishes the computation of the scattering data.

Now we will change the perspective to find out, if we can find simple examples producing the resonance case.

EXAMPLE 2

As a second example we have a look at a potential being zero everywhere, except from some gap from zero to $a \in \mathbb{R}_+$ with depth $-b^2$. If we would have $+b^2$ there don't exist discrete spectrum, with $-b^2$ the discrete spectrum only exists if $ba = n\pi$.

In the gap we consider the spectral problem $-y'' + qy = (b^2 + k^2)y$ which gives us the spectral variable $k_1 = \sqrt{k^2 + b^2}$. For the left half-axis we have $\phi_-(\lambda, x) = \mathrm{e}^{-\mathrm{i}kx}$. We expand it to the gap by $\phi_-(\lambda, x) = A\mathrm{e}^{\mathrm{i}k_1 x} + B\mathrm{e}^{-\mathrm{i}k_1 x}$ on $0 < x < a$. At point zero we have $1 = A + B$, where the derivative gives $-\mathrm{i}k = \mathrm{i}k_1(A - B)$ that means $B = \frac{1}{2}\left(1 + \frac{k}{k_1}\right)$, $A = \frac{1}{2}\left(1 - \frac{k}{k_1}\right)$. This leads to

$$\phi_-(\lambda, x) = \frac{1}{2}\left(1 - \frac{k}{k_1}\right)\mathrm{e}^{\mathrm{i}k_1 x} + \frac{1}{2}\left(1 + \frac{k}{k_1}\right)\mathrm{e}^{-\mathrm{i}k_1 x}$$
$$\phi_-'(\lambda, x) = \frac{\mathrm{i}k_1}{2}\left(1 - \frac{k}{k_1}\right)\mathrm{e}^{\mathrm{i}k_1 x} - \frac{\mathrm{i}k_1}{2}\left(1 + \frac{k}{k_1}\right)\mathrm{e}^{-\mathrm{i}k_1 x}.$$

Evaluating it in a it is convenient to reformulate it as

$$\phi_-(\lambda, a) = \cos k_1 a - \mathrm{i}k \frac{\sin k_1 a}{k_1}$$
$$\phi'_-(\lambda, a) = -k_1 \sin k_1 a - \mathrm{i}k \cos k_1 a.$$

Now we compute the Wronskian in point a using the function $\phi_+(\lambda, a) = \mathrm{e}^{\mathrm{i}ka}$ from the right, so $\phi'_+(\lambda, a) = \mathrm{i}k\mathrm{e}^{\mathrm{i}ka}$. The Wronskian in point $x = a$ has the form

$$W(\lambda) = \mathrm{e}^{\mathrm{i}ka}\{2\mathrm{i}k \cos k_1 a + \frac{k^2}{k_1} \sin k_1 a + k_1 \sin k_1 a\}.$$

The only possible zero of the real axis can occur on the edge of the spectrum so we consider this point. $W(0) = 0$ gives us $k = 0$, $k_1 = b$ and thus $\sin k_1 a = \sin ba = 0$. So we have the resonant case if $ba = n\pi$.

EXAMPLE 3
If the potential is steplike with $-c^2$ on $x < 0$, $-b^2$ up to a and zero for $x > a$ we have discrete spectrum for $c < b$ if $\cot(\sqrt{b^2 - c^2}a) = \dfrac{\sqrt{b^2 - c^2}}{c}$ and for $b < c$ if $\coth(\sqrt{c^2 - b^2}a) = \dfrac{\sqrt{c^2 - b^2}}{c}$.

Compared to the former example ϕ_+ has not changed. On the left axis we have now $k_1 = \sqrt{k^2 + c^2}$ instead of k and the previous k_1 we call now $k_2 = \sqrt{k^2 + b^2}$. Again we build the linear combination in $x = 0$ and compute the Wronskian in a. The linear combination is exactly as before with the only difference, that we have on the left side k_1. For $c < b$ this leads to

$$\phi_-(\lambda, x) = \frac{1}{2}\left(1 - \frac{k_1}{k_2}\right)\mathrm{e}^{\mathrm{i}k_2 x} + \frac{1}{2}\left(1 + \frac{k_1}{k_2}\right)\mathrm{e}^{-\mathrm{i}k_2 x}$$
$$\phi'_-(\lambda, x) = \frac{\mathrm{i}k_2}{2}\left(1 - \frac{k_1}{k_2}\right)\mathrm{e}^{\mathrm{i}k_2 x} - \frac{\mathrm{i}k_2}{2}\left(1 + \frac{k_1}{k_2}\right)\mathrm{e}^{-\mathrm{i}k_2 x}$$
$$\phi_-(\lambda, a) = \cos k_2 a - \mathrm{i}k_1 \frac{\sin k_2 a}{k_2}$$
$$\phi'_-(\lambda, a) = -k_2 \sin k_2 a - \mathrm{i}k_1 \cos k_2 a.$$

Now we compute the Wronskian in point a, using the function $\phi_+(\lambda, a) = \mathrm{e}^{\mathrm{i}ka}, \phi'_+(\lambda, a) = \mathrm{i}k\mathrm{e}^{\mathrm{i}ka}$ and $\phi_-(-c^2, a) = \cos k_2 a, \phi'_-(-c^2, a) = -k_2 \sin k_2 a$. The Wronskian in point $x = a$ on the edge of the spectrum has the form $W(-c^2) = \mathrm{e}^{\mathrm{i}ka}\{k_2 \sin k_2 a + \mathrm{i}k \cos k_2 a\}$. Here $k = \mathrm{i}c, k_2 = \sqrt{b^2 - c^2}$ and the requirement to be zero leads to $k_2 \sin k_2 a - c \cos k_2 a = 0$ e.g $\cot \sqrt{b^2 - c^2}a =$

$\frac{\sqrt{b^2-c^2}}{c}$.

For $b < c$ we proceed analogous by writing $k_2 = \mathrm{i}k_3$

$$\begin{aligned}
\phi_-(\lambda, x) &= \frac{1}{2}(1 - \frac{k_1}{\mathrm{i}k_3})\mathrm{e}^{-k_3 x} + \frac{1}{2}(1 + \frac{k_1}{\mathrm{i}k_3})\mathrm{e}^{k_3 x} \\
\phi'_-(\lambda, x) &= \frac{-k_3}{2}(1 - \frac{k_1}{\mathrm{i}k_3})\mathrm{e}^{-k_3 x} + \frac{k_3}{2}(1 + \frac{k_1}{\mathrm{i}k_3})\mathrm{e}^{k_3 x} \\
\phi_-(\lambda, a) &= \cosh k_3 a + \frac{\mathrm{i}k_1}{k_3}\sinh k_3 a \\
\phi'_-(\lambda, a) &= -\mathrm{i}k_1 \cosh k_3 a + k_3 \sinh k_3 a.
\end{aligned}$$

Now we compute the Wronskian in point a using again the function $\phi_+(\lambda, a) = \mathrm{e}^{\mathrm{i}ka}, \phi'_+(\lambda, a) = \mathrm{i}k\mathrm{e}^{\mathrm{i}ka}$ and $\phi_-(-c^2, a) = \cosh k_3 a, \phi'_-(-c^2, a) = k_3 \sinh k_3 a$. The Wronskian in point $x = a$ on the edge of the spectrum has the form

$$W(-c^2) = \mathrm{e}^{\mathrm{i}ka}\{-k_3 \sinh k_3 a + \mathrm{i}k \cosh k_3 a\}$$

Considering $k(-c^2), k_3(-c^2)$ and the requirement to be zero we have
$-k_3 \sinh k_3 a - c \cosh k_3 a = 0$ e.g $\coth \sqrt{c^2 - b^2} a = -\frac{\sqrt{c^2-b^2}}{c}$.

2.1.3 The Gelfand–Levitan–Marchenko equations

Our next aim is to derive the Gelfand–Levitan–Marchenko equations, which is crucial for the inverse scattering. In addition to to **I. (e)** we will need another property of the reflection coefficients.

Lemma 2.1.10. *Let $q \in \mathcal{L}_1^0(c_+, c_-)$. Then the reflection coefficient $R_\pm(\lambda)$ regarded as a function of $k_\pm \in \mathbb{R}$ belongs to the space $L_1(\mathbb{R}) = L_{1,\{k_\pm\}}(\mathbb{R})$.*

Proof. Throughout this proof we will denote by $f_{s,\pm} := f_{s,\pm}(k_\pm)$, $s = 1, 2, \ldots,$ functions whose Fourier transforms are in $L_1(\mathbb{R}) \cap L^2(\mathbb{R})$ (with respect to $k_\pm$). Note that $f_{s,\pm}$ are continuous. Moreover, a function $f_{s,\pm}$ is continuous with respect to $k_\mp$ for $k_\mp = k_\mp(\lambda)$ with $\lambda \in \Sigma^{(2)}$ and $f_{s,\pm} \in L^2_{\{k_\mp\}}(\mathbb{R} \setminus (-a, a))$ where the set $\mathbb{R} \setminus (-a, a)$ is the image of the spectrum $\Sigma^{(2)}$ under the map $k_\mp(\lambda)$.

Denote by a prime the derivative with respect to x. Then (2.1.4)–(2.1.6) and (2.1.1) imply

$$\overline{\phi_\pm(\lambda, 0)} = 1 + f_{1,\pm}, \quad \overline{\phi'_\pm(\lambda, 0)} = \mp\mathrm{i}k_\pm\, \overline{\phi_\pm(\lambda, 0)} + f_{2,\pm},$$

$$\phi_\pm(\lambda, 0) = 1 + f_{3,\pm}, \quad \phi'_\pm(\lambda, 0) = \pm\mathrm{i}k_\pm\, \phi_\pm(\lambda, 0) + f_{4,\pm}.$$

Since

$$k_\pm - k_\mp = \frac{c_\mp - c_\pm}{2k_\pm}(1+o(1)) \quad \text{as} \quad |k_\pm| \to \infty, \tag{2.1.29}$$

then $W(\phi_\mp(\lambda), \overline{\phi_\pm(\lambda)}) = f_{5,\pm}$ for large $k_\pm$. By the same reason

$$W(\lambda) = 2\mathrm{i}\sqrt{\lambda}(1+o(1)) \quad \text{as} \quad \lambda \to \infty.$$

Remembering that the reflection coefficient is a bounded function with respect to $k_\pm \in \mathbb{R}$ by **I. (b), (c)** and that for $|k_\pm| \gg 1$ it admits the representation $R_\pm(\lambda) = f_{6,\pm}k_\pm^{-1}$ finishes the proof. $\square$

Lemma 2.1.11. *Let $q \in \mathcal{L}_1^0(c_+, c_-)$. Then the kernels of the transformation operators $K_\pm(x,y)$ satisfy the integral equations*

$$K_\pm(x,y) + F_\pm(x+y) \pm \int_x^{\pm\infty} K_\pm(x,s)F_\pm(s+y)ds = 0, \quad \pm y > \pm x, \tag{2.1.30}$$

where, if $c_+ > c_-$

$$\begin{aligned} F_+(x) =& \frac{1}{2\pi}\int_{\mathbb{R}} R_+(\lambda)e^{\mathrm{i}k_+x}dk_+ + \sum_{j=1}^p \gamma_j^+ \mathrm{e}^{-\kappa_j^+ x} \qquad (2.1.31)\\ &+ \frac{1}{4\pi}\int_{c_-}^{c_+} \frac{|T_-(\lambda)|^2}{|k_-|}\mathrm{e}^{\mathrm{i}k_+x}d\lambda, \\ F_-(x) =& \frac{1}{2\pi}\int_{\mathbb{R}} R_-(\lambda)e^{-\mathrm{i}k_-x}dk_- + \sum_{j=1}^p \gamma_j^- \mathrm{e}^{-\kappa_j^- x} \qquad (2.1.32) \end{aligned}$$

and if $c_- > c_+$

$$\begin{aligned} F_+(x) =& \frac{1}{2\pi}\int_{\mathbb{R}} R_+(\lambda)e^{\mathrm{i}k_+x}dk_+ + \sum_{j=1}^p \gamma_j^+ \mathrm{e}^{-\kappa_j^+ x} \qquad (2.1.33)\\ F_-(x) =& \frac{1}{2\pi}\int_{\mathbb{R}} R_-(\lambda)e^{-\mathrm{i}k_-x}dk_- + \sum_{j=1}^p \gamma_j^- \mathrm{e}^{-\kappa_j^- x} \qquad (2.1.34)\\ &+ \frac{1}{4\pi}\int_{c_+}^{c_-} \frac{|T_+(\lambda)|^2}{|k_+|}\mathrm{e}^{-\mathrm{i}k_-x}d\lambda, \end{aligned}$$

Proof. We consider $c_+ > c_-$ and mention the differences to $c_- > c_+$ when they occur. To derive the GLM equations introduce two functions

$$G_\pm(\lambda, x, y) = \left(T_\pm(\lambda)\phi_\mp(\lambda, x) - \mathrm{e}^{\mp\mathrm{i}k_\pm x}\right)\mathrm{e}^{\pm\mathrm{i}k_\pm y}, \quad \pm y > \pm x,$$

where x, y are considered as parameters. As a function of λ both functions are meromorphic in the domain $\mathcal{D}$, with simple poles at the points λ_j of the discrete spectrum. By property **II** they are continuous up to the boundary $\Sigma^{\mathrm{u}} \cup \Sigma^{\mathrm{l}}$, except at the point c_-, where $G_+(\lambda, x, y)$ can have a singularity of order $O((\lambda - c_-)^{-1/2-\delta})$ in the resonant case by Lemma 2.1.8.

By the scattering relations

$$\begin{aligned} T_\pm(\lambda)\phi_\mp(\lambda, x) - \mathrm{e}^{\mp \mathrm{i}k_\pm x} &= R_\pm(\lambda)\phi_\pm(\lambda, x) + (\overline{\phi_\pm(\lambda, x)} - \mathrm{e}^{\mp \mathrm{i}k_\pm x}) \\ &= S_{\pm,1}(\lambda, x) + S_{\pm,2}(\lambda, x) \end{aligned}$$

it follows from (2.1.4) that

$$\frac{1}{2\pi}\int_{\mathbb{R}} S_{\pm,2}(\lambda, x)\mathrm{e}^{\pm \mathrm{i}k_\pm y} dk_\pm = K_\pm(x, y).$$

Next, according to Lemma 2.1.10 and (2.1.9), we get

$$R_\pm(\lambda)K_\pm(x, s)\mathrm{e}^{\mathrm{i}k_\pm(y+s)} \in L_{1,\{k_\pm\}}(\mathbb{R}) \times L_{1,\{s\}}([x, \pm\infty)) \quad \text{for } x, y \text{ fixed.}$$

Using again (2.1.4) and Fubini's theorem we get

$$\begin{aligned} &\frac{1}{2\pi}\int_{\mathbb{R}} S_{\pm,1}(\lambda)\mathrm{e}^{\pm \mathrm{i}k_\pm y} dk_\pm = \\ &\quad = F_{r,\pm}(x + y) \pm \frac{1}{2\pi}\int_{\mathbb{R}}\int_x^{\pm\infty} K_\pm(x, s)R_\pm(\lambda)\mathrm{e}^{\pm \mathrm{i}k_\pm(y+s)} ds\, dk_\pm \\ &\quad = F_{r,\pm}(x + y) \pm \int_x^{\pm\infty} K_\pm(x, s)F_{r,\pm}(y + s)ds, \end{aligned}$$

where we have set (r for "reflection")

$$F_{r,\pm}(x) := \frac{1}{2\pi}\int_{\mathbb{R}} R_\pm(\lambda)\mathrm{e}^{\pm \mathrm{i}k_\pm x} dk_\pm. \tag{2.1.35}$$

Thus, for $\pm y > \pm x$

$$\frac{1}{2\pi}\int_{\mathbb{R}} G_\pm(\lambda, x, y)dk_\pm = K_\pm(x, y) + F_{r,\pm}(x + y) \pm \int_x^{\pm\infty} K_\pm(x, s)F_{r,\pm}(y + s)ds. \tag{2.1.36}$$

Now let $\mathcal{C}_\rho$ be a closed semicircle of radius ρ lying in the upper half plane with the centre at the origin and set $\Gamma_\rho = \mathcal{C}_\rho \cup [-\rho, \rho]$. Estimates (2.1.3), (2.1.7), (2.1.29), and **I. (e)** imply that the Jordan lemma is applicable to the function $G_\pm(\lambda, x, y)$ as a function of $k_\pm$ when $\pm y \geq \pm x$. Moreover, formula (2.1.28) implies

$$\phi_\mp(\lambda_j, x)\,\mathrm{Res}_{\mathrm{i}\kappa_j^\pm} T_\pm(\lambda) = \mathrm{i}\gamma_j^\pm \phi_\pm(\lambda_j, x),$$

and thus

$$\sum_{j=1}^{p} \operatorname{Res}_{\mathrm{i}\kappa_j^\pm} G_\pm(\lambda,x,y) = \mathrm{i}\sum_{j=1}^{p} \gamma_j^\pm \phi_\pm(\lambda_j,x)\mathrm{e}^{\mp\kappa_j^\pm y}$$
$$= \mathrm{i}\left(F_{d,\pm}(x+y) \pm \int_x^{\pm\infty} K_\pm(x,s)F_{d,\pm}(s+y)ds\right), \tag{2.1.37}$$

where we denote (d for discrete spectrum)

$$F_{d,\pm}(x) := \sum_{j=1}^{p} \gamma_j^\pm \mathrm{e}^{\mp\kappa_j^\pm x}.$$

Consider first the left scattering data of L. In this case $k_- \in \mathbb{R}$ covers the whole continuous spectrum of L. The function $G_-(\lambda,x,y)$ as a function of k_- has a meromorphic continuation to the domain $\mathbb{C}^+$ with poles at the points $\mathrm{i}\kappa_j^-$. By use of the Cauchy theorem, of the Jordan lemma and (2.1.36) we get for $-x < -y$

$$\lim_{\rho\to\infty} \frac{1}{2\pi}\oint_{\Gamma_\rho} G_-(\lambda,x,y)dk_- = \mathrm{i}\sum_{j=1}^{p} \operatorname{Res}_{\mathrm{i}\kappa_j^-} G_-(\lambda,x,y) = K_-(x,y)$$
$$+ F_{r,-}(x+y) - \int_x^{-\infty} K_-(x,s)F_{r,-}(y+s)ds.$$

Joining this with (2.1.37) we get equation (2.1.32). Unlike to this, the real values of variable k_+ corresponds to the spectrum of multiplicity two only. In this case the function $G_+(\lambda,x,y)$ considered as a function of k_+ in $\mathbb{C}^+$ has a jump along the interval $[0,\mathrm{i}b_+]$ with $b_+ = \sqrt{c_+ - c_-} > 0$. It does not have a pole in b_+ because by Lemma 2.1.8 the estimate is valid $G_+(\lambda,x,y) = O((k_+ - b_+)^\alpha)$ with $-1 < \alpha \le -1/2$.

For large $\rho > 0$ put $b_\rho = b_+ + \rho^{-1}$, introduce a union of three intervals

$$\mathcal{C}'_\rho = [-\rho^{-1}, \mathrm{i}b_\rho - \rho^{-1}] \cup [\rho^{-1}, \mathrm{i}b_\rho + \rho^{-1}] \cup [\mathrm{i}b_\rho - \rho^{-1}, \mathrm{i}b_\rho + \rho^{-1}],$$

and consider a closed contour $\Gamma'_\rho = \mathcal{C}_\rho \cup \mathcal{C}'_\rho \cup [-\rho,-\rho^{-1}] \cup [\rho^{-1},\rho]$ oriented counterclockwise. The function $G_+(\lambda,x,y)$ is meromorphic inside the domain bounded by Γ'_ρ (we suppose that ρ is sufficiently large such that all poles are inside this domain). Thus,

$$\lim_{\rho\to\infty}\frac{1}{2\pi}\oint_{\Gamma'_\rho} G_+(\lambda,x,y)dk_+ = \mathrm{i}\sum_{j=1}^{p}\mathrm{Res}_{\mathrm{i}\kappa_j^+} G_+(\lambda,x,y) = K_+(x,y) \tag{2.1.38}$$
$$+ F_{r,+}(x+y) + \int_x^\infty K_+(x,s)F_{r,+}(y+s)ds$$
$$+ \frac{1}{2\pi}\int_{\mathrm{i}b_+}^0 \left(G_+(\lambda+\mathrm{i}0,x,y) - G_+(\lambda-\mathrm{i}0,x,y)\right)dk_+.$$

In the case under consideration, the variable $k_+ = \mathrm{i}\kappa$, $\kappa > 0$, does not have a jump along the spectrum of multiplicity one, and the same is true for the solution $\phi_+(\lambda,x)$. Thus the jump $[G_+] := G_+(\lambda+\mathrm{i}0,x,y) - G_+(\lambda-\mathrm{i}0,x,y)$ stems from the function $T_+(\lambda)\phi_-(\lambda,x)$. By (2.1.25) and **I. (b)** we have $T_+\overline{T_+^{-1}} = -T_-\overline{T_-^{-1}} = -R_-$ on $\Sigma^{(1)}$. To simplify notations we omit the dependence on λ and x. The scattering relations (2.1.24) then imply

$$T_+\phi_- - \overline{T_+\phi_-} = -\overline{T_+}\left(\overline{\phi_-} + R_-\phi_-\right) = -\overline{T_+}T_-\phi_+,$$

and therefore $[G_+] = -\mathrm{e}^{k_+y}\overline{T_+(\lambda+\mathrm{i}0)}T_-(\lambda+\mathrm{i}0)\phi_+(\lambda,x)$. Set

$$\chi(\lambda) := -\overline{T_+(\lambda+\mathrm{i}0)}T_-(\lambda+\mathrm{i}0), \quad \lambda\in[c_-,c_+].$$

By use of (2.1.4) we get

$$\frac{1}{2\pi}\int_{\mathrm{i}b_+}^0 \left(G_+(\lambda+\mathrm{i}0,x,y) - G_+(\lambda-\mathrm{i}0,x,y)\right)dk_+$$
$$= F_{\chi,+}(x+y) + \int_x^\infty K_+(x,s)F_{\chi,+}(s+y)ds,$$

where

$$F_{\chi,+}(x) = \frac{1}{2\pi}\int_{\mathrm{i}b_+}^0 \chi(\lambda)\mathrm{e}^{+\mathrm{i}k_+x}dk_+ = \frac{1}{4\pi}\int_{c_-}^{c_+}\chi(\lambda)\mathrm{e}^{+\mathrm{i}k_+x}\frac{d\lambda}{\sqrt{\lambda-c_+}}.$$

Combining this with (2.1.38), (2.1.37), and (2.1.35) and taking into account that by (2.1.25)

$$\frac{\chi(\lambda)}{\sqrt{\lambda-c_+}} = |T_-(\lambda)|^2|k_-|^{-1} > 0, \quad \lambda\in(c_-,c_+),$$

gives (2.1.31).

For the rarefaction case $c_- > c_+$ the proof is analogue. Now G_- might have a singularity of order $O((\lambda - c_+)^{-1/2-\delta})$ in c_+ in the resonant case. Again we use scattering relation and Jordan lemma together with residuum theorem. Here $k_+ \in \mathbb{R}$ covers the whole continuous spectrum and k_- belongs to the spectrum of multiplicity two, where $G_-(k_-, x, y)$ has a jump on $[0, \mathrm{i}b_-]$, $b_- = \sqrt{c_- - c_+}$. Suitable contours with residuum theorem together with scatterings relations are giving the claim. □

Corollary 2.1.12. *Put $\hat{F}_\pm(x) := 2F_\pm(2x)$. Then equation* (2.1.30) *reads*

$$\hat{F}_\pm(x+y) + B_\pm(x,y) \pm \int_0^{\pm\infty} B_\pm(x,s)\hat{F}_\pm(x+y+s)ds = 0, \qquad (2.1.39)$$

where $B_\pm(x,y)$ is the transformation operator from (2.1.11).

This equation and Lemma 2.1.4 allows us to establish the decay properties of $F_\pm(x)$.

Lemma 2.1.13. *Let $q \in \mathcal{L}_m^n(c_+, c_-)$, $m \geq 1$, $n \geq 0$. Then the kernels of the GLM equations* (2.1.30) *possess the property:*

IV. *The function $F_\pm(x)$ is $n+1$ times differentiable with $F'_\pm \in \mathcal{L}_m^n(\mathbb{R}_\pm)$.*

Proof. Differentiation of (2.1.39) j times with respect to y gives

$$\hat{F}_\pm^{(j)}(x+y) + B_{\pm,y}^{(j)}(x,y) \pm \int_0^{\pm\infty} B_\pm(x,s)\hat{F}_\pm^{(j)}(x+y+s)ds = 0.$$

Set here $y = 0$ and abbreviate $H_{\pm,j}(x) = B_{\pm,y}^{(j)}(x,0)$. Recall that the estimates (2.1.14) and (2.1.15) imply $H_{\pm,j} \in \mathcal{L}_m^{n+1-j}(\mathbb{R}_\pm)$, $j = 1, \dots, n+1$. Changing variables $x + s = \xi$ we get

$$\hat{F}_\pm^{(j)}(x) + H_{\pm,j}(x) \pm \int_x^{\pm\infty} B_\pm(x, \xi - x)\hat{F}_\pm^{(j)}(\xi)d\xi = 0. \qquad (2.1.40)$$

Formula (2.1.12) and the estimate (2.1.5) imply

$$|B_\pm(x, \xi - x)| \leq \sigma_{\pm,0}(\xi)\mathrm{e}^{\hat{\sigma}_{\pm,0}(x) - \hat{\sigma}_{\pm,0}(\xi)}$$

and from (2.1.40) it follows

$$\begin{aligned} |\hat{F}_\pm^{(j)}(x)|\mathrm{e}^{-\hat{\sigma}_{\pm,0}(x)} \leq & |H_{\pm,j}(x)|\mathrm{e}^{-\hat{\sigma}_{\pm,0}(x)} \\ & \pm \int_x^{\pm\infty} \sigma_{\pm,0}(s)\mathrm{e}^{-\hat{\sigma}_{\pm,0}(s)}|\hat{F}_\pm^{(j)}(s)|ds \\ = & |H_{\pm,j}(x)|\mathrm{e}^{-\hat{\sigma}_{\pm,0}(x)} + \Phi_{\pm,j}(x), \end{aligned} \qquad (2.1.41)$$

where $\Phi_{\pm,j}(x) := \pm\int_x^{\pm\infty}|F_\pm^{(j)}(s)|\mathrm{e}^{-\hat{\sigma}_{\pm,0}(s)}\sigma_{\pm,0}(s)ds$. Multiplying the last inequality by $\sigma_{\pm,0}(x)$ and using (2.1.1) we get

$$\mp\frac{d}{dx}(\Phi_{\pm,j}(x)\mathrm{e}^{-\hat{\sigma}_{\pm,0}(x)}) \le |H_{\pm,j}(x)|\sigma_{\pm,0}(x)\mathrm{e}^{-2\hat{\sigma}_{\pm,0}(x)}.$$

By integration we have

$$\Phi_{\pm,j}(x) \le \pm C\mathrm{e}^{\hat{\sigma}_{\pm,j}(x)}\int_x^{\pm\infty} H_{\pm,j}(s)\sigma_{\pm,0}(s)ds.$$

This inequality implies $\Phi_\pm(\cdot) \in \mathcal{L}_m^1(\mathbb{R}_\pm)$ because $H_{\pm,j} \in \mathcal{L}_m^{n+1-j}(\mathbb{R}_\pm)$, $j \ge 1$, $\sigma_{\pm,0} \in \mathcal{L}_{m-1}^1(\mathbb{R}_\pm)$. Property **IV** now follows from (2.1.41). □

2.1.4 The Marchenko and Deift–Trubowitz conditions

In this subsection we give the proof of property **II. (b)** and also prove the continuity of the reflection coefficient $R_\pm$ at the edge of the spectrum $\underline{c}$ when $c_\pm = \underline{c}$ in the resonant case. As is known, these properties are crucial for solving the inverse problem but were originally missed in the seminal work of Faddeev [31] as pointed out by Deift and Trubowitz [20] who also gave a counterexample which showed that some restrictions on the scattering coefficients at the bottom of the continuous spectrum is necessary for solvability of the inverse problem. The behaviour of the scattering coefficients at the bottom of the continuous spectrum is easy to understand for $m = 2$, both for decaying and steplike cases, because the Jost solutions are differentiable with respect to the local parameters $k_\pm$ in this case. For $m = 1$ the situation is more complicated. For the case $q \in \mathcal{L}_1^0(0,0)$ continuity of the scattering coefficients was established independently by Guseinov [43] and Klaus [52] (see also [3]). For the case $q \in \mathcal{L}_1^0(c_+, c_-)$ property **II. (b)** is proved in [2]. We propose here another proof following the approach of Guseinov which will give as some additional formulas which are of independent interest (e.g. when tying to understand the dispersive decay of solutions to the time-dependent Schrödinger equation, see e.g. [26]). Nevertheless, one has to emphasize that the Marchenko approach does not require these properties of the scattering data: In [59] the direct/inverse scattering problem for $q \in \mathcal{L}_1^0(0,0)$ was solved under the following less restrictive conditions:

1) The transmission coefficient $T(k)$, where $k^2 = \lambda$, is bounded for $k \in \mathbb{C}^+$ in a vicinity of $k = 0$ (at the edge of the continuous spectrum);

2) $\lim_{k\to 0} kT^{-1}(k)(R_\pm(k) + 1) = 0$.

Our conditions **I. (b)** and **II** imply the Marchenko condition at point $\underline{c}$. Namely, if $W(\underline{c}) \neq 0$ then property (i) of Lemma 2.1.5 implies $W(\underline{c}) \in \mathbb{R}$ and from **I. (b)** it follows that $R_\pm(c_\pm) = -1$ for $\underline{c} = c_\pm$. The other reflection coefficient $R_\mp(c_-)$ is simply not defined at this point. Of course, it has the property $R_\mp(\overline{c}) = -1$ (cf. (2.1.26)), because $W(\overline{c}) \neq 0$, but we do not use this fact when solving the inverse problem. Our choice to give conditions **I–III** as a part of necessary and sufficient ones is stipulated by the following. First of all, getting an analogue of the Marchenko condition 1) directly, without **II. (b)**, requires additional efforts. The second reason is that in fact we additionally justify here that the conditions proposed for $m = 2$ in [17] are valid for the first finite moment of perturbation too. The proof is given for the shock case $\underline{c} = c_-$, the rarefaction case $\underline{c} = c_+$ is analogous.

Denote by $h_\pm(\lambda, x) = \phi_\pm(\lambda, x)\mathrm{e}^{\mp \mathrm{i} k_\pm x}$ for $k_\pm \in \mathbb{R}$, then (2.1.11) implies

$$
\begin{aligned}
h_\pm(\lambda) = h_\pm(\lambda, 0) &= 1 \pm \int_0^{\pm\infty} B_\pm(0, y)\mathrm{e}^{\pm 2\mathrm{i} y k_\pm} dy,\\
h'_\pm(\lambda) = h'_\pm(\lambda, 0) &= \pm \int_0^{\pm\infty} \frac{\partial}{\partial x} B_\pm(0, y)\mathrm{e}^{\pm 2\mathrm{i} y k_\pm} dy.
\end{aligned}
$$

We observe that for $\underline{c} = c_-$ we have $2\mathrm{i}k_+(\underline{c}) = -b = -2\sqrt{c_+ - c_-} < 0$, and therefore, in a vicinity of $\underline{c}$

$$
h_+(\lambda) = 1 + \int_0^\infty B_+(0, y)\mathrm{e}^{-by}\mathrm{e}^{\mathrm{i}\tau(\lambda) y} dy, \quad \tau(\lambda) = 2\frac{\lambda - \underline{c}}{k_+ - \mathrm{i}b/2},
$$

where $\tau(\lambda)$ is differentiable in a vicinity of $\underline{c}$ and $\tau(\underline{c}) = 0$. Since $B_+(0, y)\mathrm{e}^{-by} \in L^1_s(\mathbb{R}_+)$ and $B_{+,x}(0, y)\mathrm{e}^{-by} \in L^0_s(\mathbb{R}_+)$, $s = 1, 2, \dots$, then

$$
\begin{aligned}
-\phi_+(\underline{c}, 0)\phi'_+(\lambda, 0) + \phi_+(\lambda, 0)\phi'_+(\underline{c}, 0) &= h_+(\lambda)h'_+(\underline{c}) - h_+(\underline{c})h'_+(\lambda)\\
&\quad + (2\mathrm{i}k_+ + b)h_+(\underline{c})h_+(\lambda) = C(\lambda - \underline{c})(1 + o(1)), \quad \lambda \to \underline{c}.
\end{aligned}
\tag{2.1.42}
$$

Now consider the function $\Phi(\lambda) = h_-(\lambda)h'_-(\underline{c}) - h_-(\underline{c})h'_-(\lambda)$, where $k_- \in \mathbb{R}$. One can show following (cf. [26]) that it has a representation

$$\begin{aligned}\Phi(\lambda) =& h_-(\lambda)h'_-(\underline{c}) - h'_-(\lambda)h_-(\underline{c}) = \\ & (h_-(0) + 2ik_- \int_0^{-\infty} K(y)e^{-2ik_- y}dy)(D(0) + 2ik_- \int_0^{-\infty} D(y)e^{-by}dy) \\ & - (D(0) + 2ik_- \int_0^{-\infty} De^{-2ik_- y}dy)(h_-(0) + 2ik_- \int_0^{-\infty} K(y)e^{-by}dy) \\ =& 2ik_- \int_{-\infty}^{0} \left\{D(y)h_-(\underline{c}) - K(y)h'_-(\underline{c})\right\} e^{-2iyk_-}dy \\ =& 2ik_-\Psi(k_-), \quad \text{where} \quad \Psi(k_-) = \int_{\mathbb{R}_-} H(y)\mathrm{e}^{-2iyk_-}dy,\end{aligned} \tag{2.1.43}$$

with $H(x) := D(x)h_-(\underline{c}) - K(x)h'_-(\underline{c})$,

$$K(x) = \int_{-\infty}^{x} B_-(0,y)dy, \quad D(x) = \int_{-\infty}^{x} \frac{\partial}{\partial x} B_-(0,y)dy.$$

Note that the integral in (2.1.43) has to be understood as an improper integral.

Lemma 2.1.14 (Sublemma)**.** *The function $H(x)$ satisfies the following integral equation*

$$H(x) - \int_{\mathbb{R}_-} H(y)\hat{F}_-(x+y)dy = h_-(\underline{c}) \left(\int_{\mathbb{R}_-} B_-(0,y)\hat{F}_-(x+y)dy - F_-(x) \right).$$

Proof. We repeat the proof of [43]. Starting with the integral equation (2.1.39)

$$\hat{F}_-(x+y) + B_-(x,y) - \int_0^{-\infty} B_-(x,s)\hat{F}_-(x+y+s)ds = 0,$$

we take the derivative in x and set x to zero to obtain

$$\hat{F}'_-(y) + B'_-(0,y) - \int_0^{-\infty} B'_-(0,z)\hat{F}_-(y+z) + B_-(0,z)\hat{F}'_-(y+z)dz = 0. \tag{2.1.44}$$

Integrating in y from x to $-\infty$ using partial integration we have

$$-\int_x^{-\infty} \hat{F}'_-(y) + B'_-(0,y) - \int_0^{-\infty} B'_-(0,z)\hat{F}_-(y+z) + B_-(0,z)\hat{F}'_-(y+z)dzdy \tag{2.1.45}$$

$$\begin{aligned}
=& -\hat{F}_-(x) + D(x) + \int_0^{-\infty} B'_-(0,z) \int_x^{-\infty} \hat{F}_-(y+z)dydz- \\
& - \int_0^{-\infty} B_-(0,z)\hat{F}_-(x+z)dz = 0.
\end{aligned}$$

On the other hand, we set for (2.1.39) first $x = 0$ and then integrate in y from x to $-\infty$

$$-\int_x^{-\infty} \hat{F}_-(y) + B_-(0,y) - \int_0^{-\infty} B_-(0,z)\hat{F}_-(y+z)dzdy = 0.$$

We get rid of the double integration by our previous notations and further more use $\frac{\partial}{\partial z}\int_x^{-\infty} \hat{F}_-(y+z)dy = \hat{F}_-(x+z)$ and partial integration. Thus (2.1.44) becomes

$$\hat{F}_-(y) + D(0,y) + \left\{ -\int_z^{-\infty} \frac{\partial}{\partial x} B_-(0,s)ds \int_x^{-\infty} \hat{F}_-(y+z)dy \right\} |_0^{-\infty} \tag{2.1.46}$$

$$\begin{aligned}
&-\int_0^{-\infty} -\int_z^{-\infty} \frac{\partial}{\partial x} B_-(0,s)ds(-\hat{F}_-(x+z))dz - \int_0^{-\infty} B_-(0,z)\hat{F}_-(z+x)dz \\
=&\hat{F}_-(y) + D(0,y) - h'_-(0)\int_x^{-\infty} \hat{F}_-(y)dy + \int_0^{-\infty} D(z)\hat{F}_-(x+z)dz \\
&- \int_0^{-\infty} B_-(0,z)\hat{F}_-(z+x)dz
\end{aligned}$$

and for the second equation (2.1.45) we have

$$
\begin{aligned}
&-\int_0^{-\infty} \hat{F}_-(y)dy + K_-(x) + \left\{K_-(z)\int_x^{-\infty} \hat{F}_-(y+z)dy\right\}\Big|_0^{-\infty} \\
&-\int_0^{-\infty} K_-(z)(-\hat{F}_-(x+z))dz \\
&= -(1+K(0))\int_x^{-\infty} \hat{F}_-(y)dy + K(x) + \int_0^{-\infty} K(z)\hat{F}_-(x+z)dz \\
&= K(x) - h_-(0)\int_x^{-\infty} \hat{F}_-(y)dy + \int_0^{-\infty} K(z)\hat{F}_-(x+z)dz = 0 \qquad (2.1.47)
\end{aligned}
$$

Combining the results by multiplying (2.1.47) by $h'_-(0)$ and subtracting from it (2.1.46) times $h_-(0)$

$$
\begin{aligned}
0 = &\left\{K(x) - h_-(0)\int_x^{-\infty} \hat{F}_-(y)dy + \int_0^{-\infty} K(z)\hat{F}_-(x+z)dz\right\} h'_-(0) - \\
&\left\{\hat{F}_-(y) + D(0,y) - h'_-(0)\int_x^{-\infty} \hat{F}_-(y)dy + \int_0^{-\infty} D(z)\hat{F}_-(x+z)dz\right. \\
&\left.-\int_0^{-\infty} B_-(0,z)\hat{F}_-(z+x)dz\right\} h_-(0) \\
\iff &D(x)h_-(0) - K_-(x)h'_-(0) + \int \left\{D(y)h_-(0) - K_-(y)h'_-(0)\right\}\hat{F}_-(x+y)dy \\
&= h_-(0)\left\{\int_0^{-\infty} B_-(0,y)\hat{F}_-(x+y)dy - \hat{F}_-(x)\right\} \\
\iff &H_- + \int_0^{-\infty} H_-(y)\hat{F}_-(x+y) = h_-(0)\left\{\int_0^{-\infty} B_-(0,y)\hat{F}_-(x+y)dy - \hat{F}_-(x)\right\}.
\end{aligned}
$$

And this was the claim. □

By property **IV** we have $\hat{F}'_- \in \mathcal{L}^0_m(\mathbb{R}_-)$. Using this and (2.1.5) one can prove that $H \in L_1(\mathbb{R}_-)$ and therefore $\Phi(\lambda) = 2\mathrm{i}k_-\Psi(0)(1+o(1))$, with $\Psi(0) \in \mathbb{R}$. Moreover,

$$
\begin{aligned}
\phi_-(\lambda,0)\phi'_-(\underline{c},0) - \phi_-(\underline{c},0)\phi'_-(\lambda,0) &= -2\mathrm{i}k_-h_-(\lambda)h_-(\underline{c}) + \Phi(\lambda) \\
&= 2\mathrm{i}k_-(h_-(\underline{c})^2 + \Psi(0))(1+O(1)), \quad \lambda \to \underline{c},
\end{aligned}
$$

where $h_-(\underline{c}) \in \mathbb{R}$. Combining this with (2.1.42) we get the following

Lemma 2.1.15 ([2]). *Let $\underline{c} = c_-$. Then in a vicinity of $\underline{c}$ the following asymptotics are valid:*

(a) If $\phi_-(\underline{c},0)\phi_+(\underline{c},0) \neq 0$ then

$$\frac{\phi_+'(\lambda,0)}{\phi_+(\lambda,0)} - \frac{\phi_+'(\underline{c},0)}{\phi_+(\underline{c},0)} = O(\lambda-\underline{c}), \qquad \frac{\phi_-'(\lambda,0)}{\phi_-(\lambda,0)} - \frac{\phi_-'(\underline{c},0)}{\phi_-(\underline{c},0)} = \mathrm{i}\alpha\sqrt{\lambda-\underline{c}}(1+o(1));$$

(b) If $\phi_-'(\underline{c},0)\phi_+'(\underline{c},0) \neq 0$ then

$$\frac{\phi_+(\lambda,0)}{\phi_+'(\lambda,0)} - \frac{\phi_+(\underline{c},0)}{\phi_+'(\underline{c},0)} = O(\lambda-\underline{c}), \qquad \frac{\phi_-(\lambda,0)}{\phi_-'(\lambda,0)} - \frac{\phi_-(\underline{c},0)}{\phi_-'(\underline{c},0)} = \mathrm{i}\hat{\alpha}\sqrt{\lambda-\underline{c}}(1+o(1)),$$

where $\alpha, \hat{\alpha} \in \mathbb{R}$.

Now suppose that $W(\underline{c}) = 0$, that is, $\phi_-(\underline{c},x) = C\phi_+(\underline{c},x)$ with $C \in \mathbb{R}\setminus\{0\}$ some constant. Therefore at least one of two cases described in Lemma 2.1.15 holds true. Since the functions ϕ_+ and ϕ_- are continuous in a vicinity of $\underline{c}$, then in the case (a) we have $\phi_-(\lambda,0)\phi_+(\lambda,0) = \beta(1+o(1))$ with $\beta \in \mathbb{R}\setminus\{0\}$. Thus

$$\begin{aligned} W(\lambda) =& \phi_-(\lambda,0)\phi_+(\lambda,0)\left(\frac{\phi_-'(\lambda,0)}{\phi_-(\lambda,0)} - \frac{\phi_-'(\underline{c},0)}{\phi_-(\underline{c},0)}\right.\\ &\left.- \frac{\phi_+'(\lambda,0)}{\phi_+(\lambda,0)} + \frac{\phi_+'(\underline{c},0)}{\phi_+(\underline{c},0)}\right) = \mathrm{i}\alpha\beta\sqrt{\lambda-\underline{c}}(1+o(1)), \end{aligned}$$

where $\alpha\beta \in \mathbb{R}$. In fact $\gamma = \alpha\beta \neq 0$ because of property (2.1.27). The case (b) is analogous and **II. (b)** is proved. To prove the continuity of the reflection coefficient R_- at $\underline{c}$ when $\underline{c} = c_+$ it is sufficient to apply a "conjugated" version of Lemma 2.1.15, which is valid if we consider the asymptotics as $\lambda \to \underline{c}$, $\lambda \in \Sigma^{(1)}$, to formula (2.1.24).

We summarize our findings by listing those conditions of the scattering data which haven shown to be necessary in the present section and will be shown to be sufficient for solving the inverse problem in the next section:

Theorem 2.1.16 (necessary conditions for the scattering data). *The scattering data of a potential $q \in \mathcal{L}_m^n(c_+,c_-)$*

$$\begin{aligned} \mathcal{S}_m^n(c_+,c_-) := \Big\{ & R_+(\lambda),\, T_+(\lambda),\, \sqrt{\lambda-c_+} \in \mathbb{R};\, R_-(\lambda),\, T_-(\lambda),\, \sqrt{\lambda-c_-} \in \mathbb{R};\\ & \lambda_1,\dots,\lambda_p \in \mathbb{R}\setminus\sigma,\, \gamma_1^\pm,\dots,\gamma_p^\pm \in \mathbb{R}_+ \Big\} \end{aligned} \tag{2.1.48}$$

*possess the properties **I–III** listed in Lemma 2.1.7. The functions $F_\pm(x,y)$, defined in* (2.1.30)*, possess property **IV** from Lemma 2.1.13.*

2.2 The inverse scattering problem

Let $\mathcal{S}_m^n(c_+, c_-)$ be a given set of data as in (2.1.48) satisfying the properties listed in Theorem 2.1.16.

We begin by showing that, given $F_\pm(x,y)$ (constructed from our data via (2.1.31),(2.1.32) or (2.1.33),(2.1.34)) , the GLM equations (2.1.30) can be solved for $K_\pm(x,y)$ uniquely. First of all we observe that condition **IV** implies $F_\pm \in \mathcal{L}_{m-1}^{n+1}(\mathbb{R}_\pm)$ (and therefore $F_\pm \in L^1(\mathbb{R}_\pm) \cap L^1_{\mathrm{loc}}(\mathbb{R})$) as well as $F_\pm$ absolutely continuous on $\mathbb{R}$ for $m = 1$. Introduce the operator

$$(\mathcal{F}_{\pm,x} f)(y) = \pm \int_0^{\pm\infty} F_\pm(t + y + 2x) f(t) dt.$$

This operator is compact by [59, Lem. 3.3.1]. To prove that $I + \mathcal{F}_{\pm,x}$ is invertible for every $x \in \mathbb{R}$ it is hence sufficient to prove that the respective homogeneous equation $f(y) + \int_{\mathbb{R}_\pm} F_\pm(y + t + 2x) f(t) dt = 0$ has only the trivial solution in the space $L^1(\mathbb{R}_\pm)$. Consider first the case $\underline{c} = c_-$ and the equation

$$f(y) + \int_0^\infty F_+(y + t + 2x) f(t) dt = 0, \quad f \in L_1(\mathbb{R}_+). \tag{2.2.1}$$

Suppose that $f(y)$ is a nontrivial solution of (2.2.1). Since $F_+(x)$ is real-valued we can assume $f(y)$ being real-valued too. By property **IV** the function $F_+(t)$ is bounded as $t \geq x$ and hence the solution $f(y)$ is bounded too. Thus $f \in L^2(\mathbb{R}_\pm)$ and

$$\begin{aligned} 0 =& 2\pi \left(\int_{\mathbb{R}_+} f(y)\overline{f(y)} dy + \iint_{\mathbb{R}_+^2} F_+(y + t + 2x) f(t) \overline{f(y)} dy dt \right) \\ =& \sum_{j=1}^p \gamma_j^+ (\tilde{f}(\lambda_j, x))^2 + \int_{c_-}^{c_+} \frac{|T_-(\lambda)|^2}{|\lambda - c_-|^{1/2}} (\tilde{f}(\lambda, x))^2 d\lambda \\ &+ \int_{\mathbb{R}} R_+(\lambda) \mathrm{e}^{2\mathrm{i}kx} \widehat{f}(-k) \widehat{f}(k) dk + \int_{\mathbb{R}} |\widehat{f}(k)|^2 dk, \end{aligned}$$

where $k := k_+ = \sqrt{\lambda - c_+}$,

$$\tilde{f}(\lambda, x) = \int_{\mathbb{R}_+} \mathrm{e}^{-\sqrt{c_+ - \lambda}(y+x)} f(y) dy, \quad \text{and} \quad \widehat{f}(k) = \int_x^\infty \mathrm{e}^{\mathrm{i}ky} f(y) dy.$$

Since $\tilde{f}(\lambda, x)$ is real-valued for $\lambda < c_+$ the corresponding summands are non-negative. Omitting them and taking into account that (cf. [59, Lem. 3.5.3])

$$\int_{\mathbb{R}} R_+(\lambda) \mathrm{e}^{2\mathrm{i}kx} \widehat{f}(-k) \widehat{f}(k) dk \leq \int_{\mathbb{R}} |R_+(\lambda)| |\widehat{f}(k)|^2 dk,$$

we come to the inequality $\int_{\mathbb{R}}(1-|R_+(\lambda)|)|\widehat{f}(k)|^2dk \le 0$. By property **I. (c)** $|R_+(\lambda)| < 1$ for $\lambda \neq c_+$, therefore, $\widehat{f}(k) = 0$, i.e. f is the trivial solution of (2.2.1).

For the solution f of the homogeneous equation $(I + \mathcal{F}_{-,x})f = 0$ we proceed in the same way and come to the inequality $\int_{\mathbb{R}}(1-|R_-(\lambda)|)|\widehat{f}(k_-)|^2dk_- \le 0$, where $|R_-(\lambda)| < 1$ for $\lambda > c_+$. Thus $\widehat{f}(k)$ is a holomorphic function for $k \in \mathbb{C}^+$, continuous up to the boundary, and $\widehat{f}(k) = 0$ on the rays $k^2 > c_+ - c_-$. Continuing $\widehat{f}(k)$ analytically in the symmetric domain $\mathbb{C}^+$ via these rays we come to the equality $\widehat{f}(k) = 0$ for $k \in \mathbb{R}$. The case $\underline{c} = c_+$ can be studied similarly. These considerations show that condition **IV** can in fact be weakened:

Theorem 2.2.1. *Given $\mathcal{S}_m^n(c_+, c_-)$ satisfying conditions* **I–III**, *let the function $F_\pm(x)$ be defined by* (2.1.31),(2.1.32) *or* (2.1.33),(2.1.34). *Suppose it satisfies the condition*

IV$^{\text{weak}}$. *The function $F_\pm(x)$ is absolutely continuous with $F_\pm' \in L^1(\mathbb{R}_\pm) \cap L^1_{\text{loc}}(\mathbb{R})$. For any $x_0 \in \mathbb{R}$ there exists a positive continuous function $\tau_\pm(x, x_0)$, decreasing as $x \to \pm\infty$, with $\tau_\pm(\cdot, x_0) \in L_1(\mathbb{R}_\pm)$ and such that $|F_\pm(x)| \le \tau_\pm(x, x_0)$ for $\pm x \ge x_0$.*

Then

(i) For each x equation (2.1.30) *has a unique solution $K_\pm(x, \cdot) \in L_1[x, \pm\infty)$.*

(ii) This solution has first order partial derivatives satisfying

$$\frac{d}{dx}K_\pm(x,x) \in L^1(\mathbb{R}_\pm) \cap L^1_{\text{loc}}(\mathbb{R}).$$

(iii) The function

$$\phi_\pm(\lambda, x) = \mathrm{e}^{\pm \mathrm{i}k_\pm x} \pm \int_x^{\pm\infty} K_\pm(x,y)\mathrm{e}^{\pm\mathrm{i}k_\pm y}dy \qquad (2.2.2)$$

solves the equation

$$-y''(x) \mp 2y(x)\frac{d}{dx}K_\pm(x,x) = (k_\pm)^2 y(x), \quad x \in \mathbb{R}.$$

(iv) If $F_\pm$ satisfies condition **IV** *then $q_\pm(x) := \mp 2\frac{d}{dx}K_\pm(x,x) \in \mathcal{L}_m^n(\mathbb{R}_\pm)$.*

Proof. If $F_\pm$ satisfies condition **IV** for any $m \ge 1$ and $n \ge 0$, then at least $F_\pm' \in L_1^0(R_\pm)$ and we can choose $\tau_\pm(x, x_0) = \tau_\pm(x) = \int_{\mathbb{R}_\pm} |F'(x+t)|dt$. Since

$|F_\pm(x)| \le \tau_\pm(x)$, $\tau_\pm(\cdot) \in L_1(\mathbb{R}_\pm)$, and since it is decreasing as $x \to \pm\infty$, condition $\mathbf{IV}^{\mathrm{weak}}$ is fulfilled.

Item (i) is already proved under the condition $F_\pm \in L^1(\mathbb{R}_\pm) \cap L^1_{\mathrm{loc}}(\mathbb{R})$ and $F' \in L^1_{\mathrm{loc}}(\mathbb{R})$, which is weaker than $\mathbf{IV}^{\mathrm{weak}}$. Therefore, we have a solution $K_\pm(x,y)$. To prove (ii) it is sufficient to prove $B'_{\pm,x} = \frac{\partial}{\partial x} B_\pm(x,0) \in L_1[x_0, \pm\infty)$ for any x_0 fixed, where $B_\pm(x,y) = 2K_\pm(x, x+2y)$.

Let $\pm x \ge \pm x_0$. Consider the GLM equation in the form (2.1.39). By (i) the operator $I + \hat{\mathcal{F}}_{\pm,x}$ generated by the kernel $\hat{F}_\pm$ is also invertible and admits estimate $\|\{I+\hat{\mathcal{F}}_{x,\pm}\}^{-1}\| \le C_\pm(x)$, where $C_\pm(x)$, $x \in \mathbb{R}$ is a continuous function with $C_\pm(x) \to 1$ as $x \to \pm\infty$. Introduce notations

$$\tau_{\pm,1}(x) = \int_{\mathbb{R}_\pm} |\hat{F}'_\pm(t+x)|dt, \quad \tau_{\pm,0}(x) = \int_{\mathbb{R}_\pm} |\hat{F}_\pm(t+x)|dt.$$

Note that $|\hat{F}_\pm(x)| \le \tau_{\pm,1}(x)$. From the other side, $|\hat{F}_\pm(x)| \le 2\tau_\pm(2x, 2x_0)$, where $\tau_\pm(x,x_0)$ is the function from condition $\mathbf{IV}^{\mathrm{weak}}$. From (2.1.39) we have

$$\int_{\mathbb{R}_\pm} |B_\pm(x,y)|dy \le \|\{I+\hat{\mathcal{F}}_{\pm,x}\}^{-1}\| \int_{\mathbb{R}_\pm} |\hat{F}_\pm(y+x)|dy \le C_\pm(x)\tau_{\pm,0}(x), \tag{2.2.3}$$

and, therefore

$$\begin{aligned} |B_\pm(x,y)| \le& |\hat{F}(x+y)| + \int_{\mathbb{R}_\pm} |B_\pm(x,s)\hat{F}(x+y+s)|ds \\ \le& \tau_\pm(2x+2y, 2x_0)(1 + C_\pm(x)\tau_{\pm,0}(x)) \le C(x_0)\tau_\pm(2x+2y, 2x_0). \end{aligned} \tag{2.2.4}$$

Being the solution of (2.1.39) with absolutely continuous kernel $\hat{F}_\pm$, the function $B_\pm(x,y)$ is also absolutely continuous with respect to x for every y. Differentiate (2.1.39) with respect to x gives

$$\begin{aligned} \hat{F}'_\pm(y+x) + B'_\pm(x,y) \pm \int_0^{\pm\infty} B'_\pm(x,t)\hat{F}_\pm(t+y+x)dt \\ \pm \int_0^{\pm\infty} B_\pm(x,t)\hat{F}'_\pm(t+y+x)dt = 0. \end{aligned}$$

Proceeding as in (2.2.3) we get then

$$\begin{aligned} \int_{\mathbb{R}_\pm} |B'_{\pm,x}(x,y)|dy \le& \|\{I + \hat{\mathcal{F}}_{\pm,x}\}^{-1}\| \left(\int_{\mathbb{R}_\pm} \int_{\mathbb{R}_\pm} |B_\pm(x,t)\hat{F}'(t+y+x)|dtdy \right. \\ & \left. + \int_{\mathbb{R}_\pm} |\hat{F}'_\pm(y+x)|dy \right) \\ \le& C_\pm(x)\left(\tau_{\pm,1}(x) + C_\pm(x)\tau_{\pm,1}(x)\tau_{\pm,0}(x)\right). \end{aligned} \tag{2.2.5}$$

Now set $y = 0$ in the derivative of (2.1.39) with respect to x. By use of (2.2.3), (2.2.5) and $\mathbf{IV}^{\text{weak}}$ we have then

$$\begin{aligned}|\hat{F}'_\pm(x) + B'_{\pm,x}(x,0)| \le &\int_{\mathbb{R}_\pm} |B'_{\pm,x}(x,t)\hat{F}_\pm(t+x)|dt + \int_{\mathbb{R}_\pm} |B_\pm(x,t)\hat{F}'_\pm(t+x)|dt \\ \le &C_\pm(x)(1 + C_\pm(x)\tau_{\pm,0}(x))\tau_{\pm,1}(x)\tau_\pm(2x, 2x_0) + H_\pm(x),\end{aligned}$$

where $H_\pm(x) = \int_{\mathbb{R}_\pm} |B_\pm(x,t)\hat{F}'_\pm(x+t)|dt$. By (2.2.4)

$$H_\pm(x) \le C(x_0)\int_{\mathbb{R}_\pm} \tau_\pm(2x + 2t, 2x_0)|\hat{F}'_\pm(x+t)|dt \le C(x_0)\tau_\pm(2x, 2x_0)\tau_{\pm,1}(x),$$

which implies

$$|B'_{\pm,x}(x,0)| \le |\hat{F}'(x)| + C(x_0)\tau_{\pm,1}(x)\tau_\pm(2x, 2x_0). \tag{2.2.6}$$

Therefore, under condition $\mathbf{IV}^{\text{weak}}$ we get $q_\pm(x) := B_{\pm,x}(x,0) \in L^1(\mathbb{R}^\pm) \cap L^1_{\text{loc}}(\mathbb{R})$, which proves (ii).

Repeating the corresponding part of the proof for Theorem 3.3.1 from [59] we start with the observation that $K_\pm$ is $n-$ times differentiable since $F_\pm$ is and by using the fundamental equation. Differentiating the GLM equation in the form 2.1.30 two times in y we obtain

$$F''_\pm(x+y) + K_{\pm,yy}(x,y) \pm \int_x^{\pm\infty} K_\pm(x,t)F''_\pm(t+y)dt = 0$$

and by using partial integration two times

$$\begin{aligned}F''_\pm(x+y) + K_{\pm,yy}(x,y) \mp K_\pm(x,x)F'_\pm(x+y) \pm K_{\pm,t}(x,t)F_\pm(t+y)|_{t=x} \pm \\ \pm \int_0^{\pm\infty} K_{\pm,tt}(x,t)F_\pm(t+y)dt = 0.\end{aligned}$$

On the other hand, if we differentiate the GLM equation in x direction two times

$$\begin{aligned}F''_\pm(x+y) + K_{\pm,xx}(x,y) \mp \frac{d}{dx}\{K_\pm(x,x)F_\pm(x+y)\} \mp K_{\pm,x}(x,t)F_\pm(t+y)|_{t=x} \\ \pm \int_x^{\pm\infty} K_{\pm,xx}(x,t)F_\pm(t+y)dt = 0.\end{aligned}$$

Subtracting this one from the former one we have

$$\begin{aligned}&K_{\pm,xx}(x,y) - K_{\pm,yy}(x,y) + q_\pm(x)F_\pm(x+y) \\ &\qquad \pm \int_x^{\pm\infty} \{K_{\pm,xx}(x,t) - K_{\pm,tt}(x,t)\}F_\pm(t+y)dt = 0.\end{aligned}$$

where $q_\pm = \mp 2\frac{d}{dx}K_\pm(x,x)$. From the GLM equation we have

$$q_\pm(x)F_\pm(x+y) = -q_\pm(x)K_\pm(x,y) \mp \int_x^{\pm\infty} q_\pm(x)K_\pm(x,t)F_\pm(t+y)dt.$$

Using this, the former equation becomes

$$\begin{aligned}&K_{\pm,xx}(x,y) - K_{\pm,yy}(x,y) - q_\pm(x)K_\pm(x,y)\\ \pm&\int_x^{\pm\infty}\{K_{\pm,xx}(x,t) - K_{\pm,tt}(x,t) - q_\pm(x)K(x,t)\}F_\pm(t+y)dt = 0.\end{aligned}$$

This means the function $f(y) = K_{\pm,xx}(x,y) - K_{\pm,yy}(x,y) - q_\pm(x)K_\pm(x,y)$ is a solution of the homogeneous equation $f(y) \pm \int_x^{\pm\infty} f(t)F_\pm(t+y)dt = 0, (\pm x \leq \pm y < \infty)$. Using our previous considerations the equation has only the zero solution for $\pm x > 0$ and so $f \equiv 0$. This means $K_\pm(x,y)$ is a solution of the equation $K_{\pm,xx}(x,y) - K_{\pm,yy}(x,y) - q_\pm(x)K_\pm(x,y) = 0$ and $q_\pm(x) = \mp 2\frac{d}{dx}K_\pm(x,x)$ with $\lim_{x+y\to\pm\infty} K_{\pm,x}(x,y) =$ $= \lim_{x+y\to\pm\infty} K_{\pm,y}(x,y) = 0$. By this properties [59] remark to Lemma 3.1.2 gives us, that $K_\pm(x,y)$ is the kernel of the transformation operator. In the end we have achieved, that $\phi_\pm(\lambda,x) = \mathrm{e}^{\pm\mathrm{i}\lambda x} \pm \int_x^{\pm\infty} K(x,t)\mathrm{e}^{\pm\mathrm{i}\lambda t}dt$ solves the Sturm Liouville equation $-y'' + q_\pm(x)y = \lambda^2 y, 0 < \pm x < \infty$. Since this condition is too strong for $\mathbf{IV}^{\text{weak}}$, we have to weaken it and consider a sequence $F_{n,\pm}(x)$ of twice continuously differentiable functions such that, for every $\varepsilon > 0$ on the one hand

$$\lim_{n\to\infty}\int_\varepsilon^{\pm\infty}|F_{n,\pm}(x) - F_\pm(x)|dx = 0$$

and on the other hand

$$\lim_{n\to\infty}\int_\varepsilon^{\pm\infty}x|F'_{n,\pm}(x) - F'_\pm(x)|dx = 0.$$

For big enough n, all the equations

$$F_{n,\pm}(x+y) + K_{n,\pm}(x,y) \pm \int_x^{\pm\infty} K_{n,\pm}(x,t)F_{n,\pm}(t+y) = 0$$

have a unique solution and furthermore $\lim\limits_{n\to\infty}\sup\limits_{\pm\varepsilon\leq x\leq\pm\infty}\int\limits_x^{\pm\infty}|K_{n,\pm}(x,t) - K_\pm(x,y)|$ and $\lim\limits_{n\to\infty}\int\limits_\varepsilon^{\infty}|K'_{n,\pm}(x,x) - K'_\pm(x,x)|dx = 0$ for all $\varepsilon > 0$. As we have proven before, all of the functions $\phi_{n,\pm}(\lambda,x) = \mathrm{e}^{\pm\mathrm{i}k_\pm x} \pm \int_x^{\pm\infty} K_{n,\pm}(x,y)\mathrm{e}^{\pm\mathrm{i}k_\pm y}dy$, satisfying

$$-y'' + q_{n,\pm}y = \lambda^2 y.$$

Taking the limit we have proven the claim.
Now let $\hat{F}_\pm$ satisfy condition **IV** for some $m \ge 1$ and $n \ge 0$. As we already discussed, in this case one can replace $\tau_\pm(x,x_0)$ by $\tau_{\pm,1}(x)$, and then formulas (2.2.6) and (2.2.4) read

$$|B_\pm(x,y)| \le C(x_0)\tau_{\pm,1}(x+y), \quad |B'_{\pm,x}(x,0)| \le C(x_0)\tau^2_{\pm,1}(x).$$

Since $\tau_{\pm,1}(x) \in \mathcal{L}^1_{m-1}(\mathbb{R}_\pm)$ and $\tau^2_{\pm,1}(x) \in \mathcal{L}^0_m(\mathbb{R}_\pm)$ for $m \ge 1$ then $q_\pm(x) \in \mathcal{L}^0_m(\mathbb{R}_\pm)$. To prove the claim for higher derivatives, we proceed similarly. Namely, in agreement with previous notations denote

$$\tau_{\pm,i}(x) := \int_{\mathbb{R}_\pm} \hat{F}^{(i)}_\pm(t+x)dt, \quad i = 0,\dots,n+1,$$

and also denote $D^{(i)}_\pm(x,y) := \frac{\partial^i}{\partial x^i}B_\pm(x,y)$. Let $\binom{i}{j}$ be the binomial coefficients. Differentiating (2.1.39) i times with respect to x implies

$$\hat{F}^{(i)}_\pm(x+y) + D^{(i)}_\pm(x,y) = -\sum_{j=0}^{i}\binom{i}{j}\int_{\mathbb{R}_\pm}\hat{F}^{(j)}_\pm(x+y+t)D^{(i-j)}_\pm(x,t)dt,$$

and therefore

$$\begin{aligned}\int_{\mathbb{R}_\pm}|D^{(i)}_\pm(x,y)|dy \le & \|\{I+\hat{F}_{\pm,x}\}^{-1}\|\left\{\int_{\mathbb{R}_\pm}|\hat{F}^{(i)}_\pm(x+y)|dy\right. \\ & \left.\sum_{j=1}^{i}\binom{i}{j}\int_{\mathbb{R}_\pm}\int_{\mathbb{R}_\pm}|\hat{F}^{(j)}_\pm(x+y+t)D^{(i-j)}_\pm(x,t)|dtdy\right\} \\ \le & C_{\pm,i}(x)[\tau_{\pm,i-1}(x) + \sum_{j=1}^{i}\tau_{\pm,j}(x)\rho_{\pm,i-j}(x)],\end{aligned}$$

where $C_{\pm,i}(x) := K_i\|\{I+F_{\pm,x}\}^{-1}\| = K_iC_\pm(x)$ with $K_i = \max_{j\le i}\binom{i}{j}$, and $\rho_{\pm,j}(x)$ is defined by the recurrence formula

$$\rho_{\pm,0}(x) := C_\pm(x)\tau_{\pm,0}(x), \quad \rho_{\pm,s} := C_{\pm,s}(x)[\tau_{\pm,s-1}(x) + \sum_{j=1}^{s}\tau_{\pm,j}(x)\rho_{\pm,s-j}(x)].$$

Thus for every $i = 1,\dots,n+1$

$$\int_{\mathbb{R}_\pm}|D^{(i)}_\pm(x,y)|dy \le \rho_{\pm,i}(x) \in \mathcal{L}^0_{m-1}(\mathbb{R}_\pm).$$

Respectively

$$|q_\pm^{(i)}(x)| = |D_\pm^{(i)}(x,0)| \le |F^{(i)}(x)| + \sum_{j=1}^{i} \binom{i}{j} \tau_{\pm,j}(x)\rho_{\pm,i-j}(x) \in \mathcal{L}_m^0(\mathbb{R}_\pm),$$

which finishes the proof. □

Our next aim is to prove that the two functions $q_+(x)$ and $q_-(x)$ from the previous theorem do in fact coincide.

Theorem 2.2.2. *Let the set $\mathcal{S}_m^n(c_+,c_-)$ defined by* (2.1.48) *satisfy conditions* **I–III** *and* $\mathbf{IV}^{\text{weak}}$*. Then $q_-(x) \equiv q_+(x) =: q(x) \in \mathcal{L}_m^n(c_+,c_-)$. Moreover, $\mathcal{S}_m^n(c_+,c_-)$ are the scattering data for the Schrödinger operator with potential $q(x)$.*

Proof. This proof is a slightly modified version of the proof proposed in [59]. We give it for the case $\underline{c} = c_-$. We continue to use the notation $\Sigma^{(2)}$ for the two sides of the cut along the interval $[\overline{c},\infty) = [c_+,\infty)$ and $\mathcal{D} = \mathbb{C} \setminus \Sigma$, and the notation Σ for the two sides of the cut along the interval $[\underline{c},\infty) = [c_-,\infty)$.

The main differences between the present proof and the one from [59] concern the presence of the spectrum of multiplicity one and the use of condition $\mathbf{IV}^{\text{weak}}$. Namely, recall that the kernels of the GLM equations can be split naturally into the following summands (2.1.30) according to $F_+ = F_{\chi,+} + F_{d,+} + F_{r,+}$ and $F_- = F_{r,-} + F_{d,-}$ (cf. (2.1.31), (2.1.32)).

We begin by considering the following part of the GLM equations

$$G_\pm(x,y) := F_{r,\pm}(x+y) \pm \int_x^{\pm\infty} K_\pm(x,t)F_{r,\pm}(t+y)dt,$$

where $K_\pm(x,y)$ are the solutions of GLM equations obtained in Theorem 2.2.1. By condition $\mathbf{IV}^{\text{weak}}$ we have $F_{r,\pm} \in L^2(\mathbb{R})$, therefore for any fixed x

$$\int_{\mathbb{R}} F_{r,\pm}(x+y)\mathrm{e}^{\mp \mathrm{i} y k_\pm} dy = R_\pm(\lambda)\mathrm{e}^{\pm \mathrm{i} x k_\pm},$$

and consequently

$$\int_{\mathbb{R}} G_\pm(x+y)\mathrm{e}^{\mp \mathrm{i} k_\pm y} dy = R_\pm(\lambda)\phi_\pm(\lambda,x), \quad k_\pm \in \mathbb{R}, \tag{2.2.7}$$

where $\phi_\pm$ are the functions obtained in Theorem 2.2.1 and the integral is considered as a principal value. On the other hand, invoking the GLM equations and the same functions $\phi_\pm$ we have

$$G_+(x,y) = -K_+(x,y) - \sum_{j=1}^{p} \gamma_j^+ \mathrm{e}^{-\kappa_j y} \phi_+(\lambda_j, x) \\ - \frac{1}{4\pi} \int_{\underline{c}}^{\overline{c}} \frac{|T_-(\xi)|^2}{k_-(\xi)} \mathrm{e}^{\mathrm{i}k_+(\xi)y} \phi_+(\xi, x) d\xi, \quad y > x,$$

and

$$G_-(x,y) = -K_-(x,y) + \sum_{j=1}^{p} \gamma_j^- \mathrm{e}^{\kappa_j y} \phi_-(\lambda_j, x), \quad y < x.$$

Since for two points $k' \neq k''$

$$\int_x^{\pm\infty} \mathrm{e}^{\pm\mathrm{i}(k'-k'')y} dy = \mathrm{i} \frac{\mathrm{e}^{\pm\mathrm{i}(k'-k'')x}}{k'-k''},$$

then

$$\int_{\mathbb{R}} G_+(x,y) \mathrm{e}^{-\mathrm{i}k_+ y} dy = \int_{-\infty}^{x} G_+(x,y) \mathrm{e}^{-\mathrm{i}k_+ y} dy - \int_x^{+\infty} K_+(x,y) \mathrm{e}^{-\mathrm{i}k_+ y} dy \tag{2.2.8}$$

$$+ \frac{1}{4\pi\mathrm{i}} \int_{\underline{c}}^{\overline{c}} \frac{|T_-(\xi)|^2 \phi_+(\xi,x) \mathrm{e}^{\mathrm{i}(k_+(\xi) - k_+(\lambda))x}}{(k_+(\xi) - k_+(\lambda))\sqrt{\xi - c_-}} d\xi - \sum_{j=1}^{p} \gamma_j^+ \phi_+(\lambda_j, x) \frac{\mathrm{e}^{(-\mathrm{i}k_+ - \kappa_j^+)x}}{\kappa_j^+ + \mathrm{i}k_+},$$

and

$$\int_{\mathbb{R}} G_-(x,y) \mathrm{e}^{\mathrm{i}k_- y} dy = \int_x^{+\infty} G_-(x,y) \mathrm{e}^{\mathrm{i}k_- y} dy - \int_{-\infty}^{x} K_-(x,y) \mathrm{e}^{\mathrm{i}k_- y} dy \\ - \sum_{j=1}^{p} \gamma_j^- \phi_-(\lambda_j, x) \frac{\mathrm{e}^{(\mathrm{i}k_- + \kappa_j^-)x}}{\kappa_j^- + \mathrm{i}k_-}. \tag{2.2.9}$$

Since for $k_\pm \in \mathbb{R}$

$$\pm \int_x^{\pm\infty} K_\pm(x,y) \mathrm{e}^{\mp\mathrm{i}k_\pm y} dy = \overline{\phi_\pm(\lambda, x)} - \mathrm{e}^{\mp\mathrm{i}k_\pm x},$$

then combining (2.2.8) and (2.2.9) with (2.2.7) we infer the relations

$$R_\pm(\lambda)\, \phi_\pm(\lambda, x) + \overline{\phi_\pm(\lambda, x)} = T_\pm(\lambda) \theta_\mp(\lambda, x), \quad k_\pm \in \mathbb{R}, \tag{2.2.10}$$

where

$$\theta_-(\lambda, x) := \frac{1}{T_+(\lambda)} \Bigg(e^{-ik_+ x} + \int_{-\infty}^{x} G_+(x,y) e^{-ik_+ y} dy \\ - \int_{c_-}^{c_+} \frac{|T_-(\xi)|^2 W_+(\xi, \lambda, x)}{4\pi(\xi - \lambda)\sqrt{\xi - c_-}} d\xi + \sum_{j=1}^{p} \gamma_j^+ \frac{W_+(\lambda_j, \lambda, x)}{\lambda - \lambda_j} \Bigg), \tag{2.2.11}$$

$$\theta_+(\lambda, x) := \frac{1}{T_-(\lambda)} \Bigg(e^{ik_- x} + \int_{x}^{+\infty} G_-(x,y) e^{ik_- y} dy + \sum_{j=1}^{p} \gamma_j^- \frac{W_-(\lambda_j, \lambda, x)}{\lambda - \lambda_j} \Bigg),$$

and

$$W_\pm(\xi, \lambda, x) := i\phi_\pm(\xi, x) e^{\pm i(k_\pm(\xi) - k_\pm(\lambda))x} (k_\pm(\xi) + k_\pm(\lambda)). \tag{2.2.12}$$

It turns out that, in spite of the fact that $\theta_\pm(\lambda, x)$ is defined via the background solutions corresponding to the opposite half-axis $\mathbb{R}_\mp$, it shares a series of properties with $\phi_\pm(\lambda, x)$.

Lemma 2.2.3. *The function $\theta_\pm(\lambda, x)$ possesses the following properties:*

(i) It admits an analytic continuation to the set $\mathcal{D} \setminus \{c_+, c_-\}$ and is continuous up to its boundary Σ.

(ii) It has no jump along the interval $(-\infty, c_\pm]$, and it takes complex conjugated values on the two sides of the cut along $[c_\pm, \infty)$.

(iii) For large $\lambda \in \operatorname{clos}(\mathcal{D})$ it has the asymptotic behaviour $\theta_\pm(\lambda, x) = e^{\pm ik_\pm x}(1+o(1))$.

(iv) The formula $W(\theta_\pm(\lambda, x), \phi_\mp(\lambda, x)) = \mp W(\lambda)$ is valid for $\lambda \in \operatorname{clos}(\mathcal{D})$, where $W(\lambda)$ is defined by formula (2.1.25).

Proof. The function $T_\mp^{-1}(\lambda)$ admits an analytic continuation to $\mathcal{D}$ by property **II. (a)**. Moreover, we have $G_\mp(x, \cdot) \in L_1([x, \pm\infty))$. Since $e^{\pm ik_\mp y}$ does not grow as $\pm y \geq 0$ then the respective integral (the second summand in the representation for $\theta_\pm$) admits analytical continuation also. Function $\theta_\pm$ has not singularities at points $\{\lambda_1, \dots, \lambda_p\}$, since $T_\mp^{-1}(\lambda)$ has simple zeros at λ_j. The function $W_\mp(\xi, \lambda, x)$ can be continued analytically with respect to λ for ξ and x fixed. Next, consider the Cauchy type integral term in (2.2.11). The only singularity of the integrand can appear at point $\underline{c} = c_-$, because in in the resonance case $T_-(c_-) \neq 0$. Thus if $W(c_-) = 0$ then the integrand in (2.2.11) behaves as $O(\xi - c_-)^{-1/2}$. By [63] the integral is of order $O(\xi - c_-)^{-1/2-\delta}$

for arbitrary small positive delta, moreover, $T_+^{-1}(\lambda) = C\sqrt{\lambda - c_-}(1 + o(1))$. Therefore for $\lambda \to c_-$

$$\theta_-(\lambda, x) = \begin{cases} O((\lambda - c_-)^{-\delta}), & \text{if } W(c_-) = 0, \\ O(1), & \text{if } W(c_-) \neq 0. \end{cases} \tag{2.2.13}$$

Since $W(c_+) \neq 0$ by **II. (a)** then $T_+^{-1}(\lambda) = O(\lambda - c_+)^{-1/2}$, respectively

$$\theta_-(\lambda, x) = O\left((\lambda - c_+)^{-1/2}\right), \quad \theta_+(\lambda, x) = O(1), \quad \lambda \to c_+. \tag{2.2.14}$$

Properties (i) of Lemma 2.1.5, and **II. (a)** together with (2.2.11) and (2.2.12) imply that θ_+ and θ_- take complex conjugated values on the sides of cut along $[\underline{c}, \infty)$. Since $W_\pm(\xi, \lambda, x) \in \mathbb{R}$ when $\lambda, \xi \leq c_\pm$, then $\theta_\pm(\lambda, x) \in \mathbb{R}$ as $\lambda \leq c_-$. Due to property **I. (b)** we have $\overline{T_-^{-1}}T_- = R_-$ on both sides of cut along $[\underline{c}, \overline{c}]$, and from (2.2.10) it follows that

$$\theta_+ = \phi_- \overline{T_-^{-1}} + \overline{\phi_-}\, T_-^{-1} \in \mathbb{R}.$$

Therefore θ_+ has no jump along the interval $[\underline{c}, \overline{c}]$. At the point $\underline{c} = c_-$ function $\theta_+(x, \lambda)$ has an isolated nonessential singularity, i.e. a pole at most. But at the vicinity of point c_- $\theta_+(\lambda, x) = O(T_-^{-1}(\lambda)) = O(\lambda - c_-)^{-1/2}$. Thus this singularity is removable,

$$\theta_+(\lambda, x) = O(1), \quad \lambda \to c_-. \tag{2.2.15}$$

Items (i) and (ii) are proved.

The main term of asymptotical behaviour for $\theta_\pm(\lambda, x)$ as $\lambda \to \infty$ is the first summand in (2.2.11). Thus by **I. (e)** and (2.1.29)

$$\theta_\pm(\lambda, x) = T_\mp^{-1}(\lambda) \mathrm{e}^{\pm \mathrm{i} k_\mp x} + o(1) = \mathrm{e}^{\pm \mathrm{i} k_\pm x}(1 + o(1)),$$

which proves (iii). Property (iv) follows from (2.2.10), (2.2.2), and (2.1.25) by analytic continuation

$$\begin{aligned} W(\theta_\pm, \phi_\mp) &= \phi'_\mp \theta_\pm - \theta'_\pm \phi_\mp = \phi'_\mp \frac{R_\mp \phi_\mp + \overline{\phi}_\mp}{T_\mp} - \frac{R_\mp \phi'_\mp + \overline{\phi}'_\mp}{T_\mp} \phi_\mp \\ &= \frac{R_\mp}{T_\mp}(\phi'_\mp \phi_\mp - \phi'_\mp \phi_\mp) + \frac{1}{T_\mp}(\phi'_\mp \overline{\phi}_\mp - \overline{\phi}'_\mp \phi_\mp) \\ &= \frac{1}{T_\mp}(\phi'_\mp \overline{\phi}_\mp - \overline{\phi}'_\mp \phi_\mp) = \frac{1}{T_\mp} W(\overline{\phi}_\mp, \phi_\mp) \\ &= \frac{\mp 2\mathrm{i} k_\mp}{T_\mp} = \mp W(\lambda). \end{aligned}$$

. $\square$

Now conjugate equality (2.2.10) and eliminate $\overline{\phi_\pm}$ from the system

$$\begin{cases} \overline{R_\pm\phi_\pm} + \phi_\pm & = \overline{\theta_\mp T_\pm}, \\ R_\pm\phi_\pm + \overline{\phi}_\pm & = \theta_\mp T_\pm, \end{cases} \qquad k_\pm \in \mathbb{R},$$

to obtain

$$\phi_\pm(1 - |R_\pm|^2) = \overline{\theta_\mp T_\pm} - \overline{R_\pm}\theta_\mp T_\pm.$$

Using **I. (c), (d)** and **II** shows for $\lambda \in \Sigma^{(2)}$, that is for $k_+ \in \mathbb{R}$, that

$$T_\mp\phi_\pm = \overline{\theta_\mp} + R_\mp\theta_\mp \quad \lambda \in \Sigma^{(2)}.$$

This equation together with (2.2.10) gives us a system from which we can eliminate the reflection coefficients $R_\pm$. We get

$$T_\pm(\phi_\pm\phi_\mp - \theta_\pm\theta_\mp) = \phi_\pm\overline{\theta}_\pm - \overline{\phi}_\pm\theta_\pm, \quad \lambda \in \Sigma^{(2)}. \tag{2.2.16}$$

Next introduce a function

$$\Phi(\lambda) := \Phi(\lambda, x) = \frac{\phi_+(\lambda, x)\phi_-(\lambda, x) - \theta_+(\lambda, x)\theta_-(\lambda, x)}{W(\lambda)},$$

which is analytic in the domain $\operatorname{clos}(D) \setminus \{\lambda_1, \ldots, \lambda_p, \underline{c}, \overline{c}\}$. Our aim is to prove that this function has no jump along the real axis and has removable singularities at the points $\{\lambda_1, \ldots, \lambda_p, \underline{c}, \overline{c}\}$. Indeed, from (2.2.16) and (2.1.25) we see that

$$\Phi(\lambda) = \pm\frac{\phi_\pm(\lambda, x)\overline{\theta_\pm(\lambda, x)} - \overline{\phi_\pm(\lambda, x)}\theta_\pm(\lambda, x)}{2\mathrm{i}k_\pm}, \quad \lambda \in \Sigma^{(2)}.$$

By the symmetry property (cf. **II. (a)**, (iii), Theorem 2.2.1 and (ii), Lemma 2.2.3) we observe that both the nominator and denominator are odd functions of k_+, therefore $\Phi(\lambda + \mathrm{i}0) = \Phi(\lambda - \mathrm{i}0)$, as $\lambda \geq \overline{c}$, i.e., the function $\Phi(\lambda)$ has no jump along this interval. By the same properties **II. (a)**, (iii) of Theorem 2.2.1 and (ii) of Lemma 2.2.3 the function $\Phi(\lambda)$ has no jump on the interval $\lambda \leq \underline{c}$ as well. Let us check that it has no jump along the interval $(\underline{c}, \overline{c})$ also. Lemma 2.2.3, (ii) shows that the function $\theta_+(\lambda, x)$ has no jump here. Abbreviate

$$[\Phi] = \Phi(\lambda + \mathrm{i}0) - \Phi(\lambda - \mathrm{i}0) = \phi_+\left[\frac{\phi_-}{W}\right] - \theta_+\left[\frac{\theta_-}{W}\right], \quad \lambda \in (\underline{c}, \overline{c}),$$

and drop some dependencies for notational simplicity. Using property **I , (b)** and formula (2.2.10) we get

$$\left[\frac{\phi_-}{W}\right] = \frac{\phi_- T_- + \overline{\phi_- T_-}}{2\mathrm{i}k_-} = \frac{(\phi_- R_- + \overline{\phi_-})\overline{T}_-}{2\mathrm{i}k_-} = \frac{\theta_+ T_-\overline{T}_-}{2\mathrm{i}k_-},$$

that is

$$\phi_+ \left[\frac{\phi_-}{W}\right] = \frac{\theta_+\phi_+|T_-|^2}{2\mathrm{i}k_-}. \tag{2.2.17}$$

On the other hand, since $\mathrm{i}k_+ \in \mathbb{R}$ as $\lambda < \overline{c}$, we have

$$\left[\frac{\theta_-}{W}\right] = \left[\frac{\theta_- T_+}{2\mathrm{i}k_+}\right] = \frac{1}{2\mathrm{i}k_+}\left[\theta_- T_+\right]. \tag{2.2.18}$$

By formula (2.2.11) the jump of this function appears from the Cauchy type integral only. Represent this integral as

$$-\frac{1}{2\pi\mathrm{i}}\int_{\underline{c}}^{\overline{c}} \frac{\phi_+(x,\xi)(-\mathrm{i})(k_+(\lambda)+k_+(\xi))\mathrm{e}^{\mathrm{i}x(k_+(\xi)-k_+(\lambda))}|T_-(\xi)|^2}{2\mathrm{i}k_-(\xi)}\frac{d\xi}{\xi-\lambda},$$

and apply the Sokhotski–Plemejl formula. Then (2.2.18) implies

$$\theta_+ \left[\frac{\theta_-}{W}\right] = \frac{\theta_+\phi_+|T_-|^2}{2\mathrm{i}k_-}.$$

Comparing this with (2.2.17) we conclude that the function $\Phi(\lambda)$ has no jumps on $\mathbb{C}$, but may have isolated singularities at the points $E = \lambda_1, \dots, \lambda_p, c_-, c_+$ and ∞. Since all these singularities are at most isolated poles it is sufficient to check that $\Phi(\lambda) = o((\lambda - E)^{-1})$, from some direction in the complex plane, to show that they are removable. First of all properties **I. (e)** and (iii), Lemma 2.2.3 together with (2.1.25) and (2.2.2) imply $\Phi(\lambda) \to 0$ as $\lambda \to \infty$. The desired behaviour $\Phi(\lambda) = o((\lambda - c_\pm)^{-1})$ for $\lambda \to c_\pm$ is due to property **II** and estimates (2.2.13), (2.2.14), (2.2.15). Next, to prove that there is no singularities at the points of the discrete spectrum, we have to check that

$$\phi_+(x,\lambda_j)\phi_-(x,\lambda_j) = \theta_+(x,\lambda_j)\theta_-(x,\lambda_j). \tag{2.2.19}$$

Passing to the limit in both formulas (2.2.11) and taking into account (2.1.25) and (2.2.12) gives

$$\theta_\mp(\lambda_k, x) = \frac{dW}{d\lambda}(\lambda_k)\,\phi_\pm(\lambda_k, x)\,\gamma_j^\pm,$$

which together with (2.1.22) implies (2.2.19). Since $\Phi(\lambda)$ is analytic in $\mathbb{C}$ and $\Phi(\lambda) \to 0$ as $\lambda \to \infty$, Liouville's theorem shows

$$\Phi(x,\lambda) \equiv 0 \quad \text{for} \quad \lambda \in \mathbb{C}, \quad x \in \mathbb{R}. \tag{2.2.20}$$

Corollary 2.2.4. *$R_\pm(c_\pm) = -1$ if $W(c_\pm) \neq 0$.*

Proof. In the case $\underline{c} = c_-$ discussed above we have $W(c_+) \neq 0$. Formula (2.2.20) implies that instead of (2.2.14) we have in fact $\theta_-(x, \lambda) = O(1)$ as $\lambda \to c_+$. Since $T_+(c_+) = 0$ and $\phi(x, c_+) = \overline{\phi(x, c_+)}$ then by (2.2.10) we conclude $R_+(c_+) = -1$. Property $R_-(c_-) = -1$ in the nonresonant case is due to **I. (b)**, (2.1.25), and property $W(c_-) \in \mathbb{R} \setminus \{0\}$, which follows in turn from the symmetry property (iii) of Lemma 2.1.5. □

Formula (2.2.20) implies

$$\phi_+(\lambda, x)\phi_-(\lambda, x) = \theta_+(\lambda, x)\theta_-(\lambda, x), \quad \lambda \in \mathbb{C}, \quad x \in \mathbb{R}. \tag{2.2.21}$$

Moreover,

$$\phi_\pm(\lambda, x)\overline{\theta_\pm(\lambda, x)} = \overline{\phi_\pm(\lambda, x)}\theta_\pm(\lambda, x), \quad \lambda \in \Sigma^{(2)}. \tag{2.2.22}$$

It remains to show that $\phi_\pm(\lambda, x) = \theta_\pm(\lambda, x)$, or equivalently, that for all $\lambda \in \mathbb{C}$ and $x \in \mathbb{R}$

$$p(\lambda, x) := \frac{\phi_-(\lambda, x)}{\theta_-(\lambda, x)} = \frac{\theta_+(\lambda, x)}{\phi_+(\lambda, x)} \equiv 1.$$

We proceed as in [59], Section 3.5, or as in in [11], Section 5. We first exclude from our consideration the discrete set $\mathcal{O}$ of parameters $x \in \mathbb{R}$ for which at least one of the following equalities is fulfilled: $\phi(E, x) = 0$ for $E \in \{\lambda_1, \dots, \lambda_p, c_-, c_+\}$. We begin by showing that for each $x \notin \mathcal{O}$ the equality $\phi_+(\hat{\lambda}, x) = 0$ implies the equality $\theta_+(\hat{\lambda}, x) = 0$. Indeed, since $\hat{\lambda} \notin \{\lambda_1, \dots, \lambda_p, c_-, c_+\}$ we have $W(\hat{\lambda}) \neq 0$ and therefore by (iv) of Lemma 2.2.3 that $\theta_-(\hat{\lambda}, x) \neq 0$. But then from (2.2.21) the equality $\theta_+(\hat{\lambda}, x) = 0$ follows. Thus the function $p(\lambda, x)$ is holomorphic in $\mathcal{D}$. By (ii) of Lemma 2.2.3 it has no jump along the set (c_-, c_+), and by (2.2.22) it has no jump along $\lambda \geq c_+$. Since $\phi_+(c_\pm, x) \neq 0$ then (2.2.14) and (2.2.15) imply that $p(\lambda, x)$ has removable singularities at c_+ and c_-. By (iii) of Lemma 2.2.3 $p(\lambda) \to 1$ as $\lambda \to \infty$, and by Liouville's theorem $p(\lambda, x) \equiv 1$ for $x \notin \mathcal{O}$. But the set $\mathcal{O}$ is discrete, therefore by continuity $\phi_\pm(\lambda, x) = \theta_\pm(\lambda, x)$ for all $\lambda \in \mathbb{C}$ and $x \in \mathbb{R}$. In turn this implies that $q_-(x) = q_+(x)$ and completes the proof of Theorem 2.2.2. □

2.3 Additional properties of the scattering data

In this section we study the behaviour of the reflection coefficients as $\lambda \to \infty$ and its connection to the smoothness of the potential. One should emphasize that the rough estimate **I. (e)** is sufficient for solving the inverse scattering problem (independent of the number of derivatives n), because this information is contained in property **IV** of the Fourier transforms of the reflection coefficients. That is why we did not include the estimate from Theorem 2.3.1 proved below in the list of necessary and sufficient conditions. On the other hand, this estimate plays an important role in application of the IST for solving the Cauchy problem for KdV equation with steplike initial profile. Lemma 2.3.3 and Theorem 2.3.1 clarify and improve corresponding results of [11] and are of independent interest for the spectral analysis of L.

We introduce the following notation: We will say that a function $g(\lambda)$, defined on the set $\mathcal{A} := \Sigma \cap \{\lambda \geq a \gg \overline{c}\}$, belongs to the space $L^2(\infty)$ if it satisfies the symmetry property $g(\lambda + \mathrm{i}0) = \overline{g(\lambda - \mathrm{i}0)}$ on $\mathcal{A}$ and

$$\int_a^{+\infty} |g(\lambda)|^2 \frac{d\lambda}{|\sqrt{\lambda}|} < \infty.$$

Note that this definition implies $g(\lambda) \in L^2_{\{k_\pm\}}(\mathbb{R} \setminus (-a, a))$ for sufficiently large a.

Theorem 2.3.1. *Let $q \in \mathcal{L}_m^n(c_+, c_-)$, $m, n \geq 1$. Then for $\lambda \to \infty$*

$$\frac{d^s}{dk_\pm^s} R_\pm(\lambda) = g_{\pm,s}(\lambda)\, \lambda^{-\frac{n+1}{2}}, \quad s = 0, 1, \ldots, m-1,$$

where $g_{\pm,s}(\lambda) \in L^2(\infty)$.

Note that the case $n = 0$ and $m = 1$ already follows Lemma 2.1.10 since (using the notation of its proof) $R_\pm(\lambda) = f_{6,\pm} k_\pm^{-1}$ admits $m - 1$ derivatives with respect to $k_\pm$ for $m > 1$, and $f_{6,\pm}^{(s)} \in L^2_{\{k_\pm\}}(\mathbb{R} \setminus (-a, a))$. The general case will be shown at the end of this section. Using Lemma 2.1.4 and formula (2.1.18) we can specify an asymptotical expansion for the Jost solution of equation (2.0.1) with a smooth potential.

Lemma 2.3.2. *Let $q \in \mathcal{L}_m^n(c_+, c_-)$ and $q_\pm(x) = q(x) - c_\pm$. Then for large $k_\pm \in \mathbb{R}$ the Jost solution $\phi_\pm(\lambda, x)$ of the equation $L\phi_\pm = \lambda\phi_\pm$ admits an asymptotical expansion*

$$\phi_\pm(\lambda, x) = \mathrm{e}^{\pm \mathrm{i} k_\pm x} \left(u_{\pm,0}(x) \pm \frac{u_{\pm,1}(x)}{2\mathrm{i}k_\pm} + \cdots + \frac{u_{\pm,n}(x)}{(\pm 2\mathrm{i}k_\pm)^n} + \frac{U_{\pm,n}(\lambda, x)}{(\pm 2\mathrm{i}k_\pm)^{n+1}} \right), \tag{2.3.1}$$

where

$$u_0(x) = 1, \quad u_{\pm,l+1}(x) = \int_x^{\pm\infty} (u''_{\pm,l}(\xi) - q_\pm(\xi) u_{\pm,l}(\xi)) d\xi, \quad l = 1, \ldots, n. \tag{2.3.2}$$

Moreover, the functions $U_{\pm,n}(\lambda,x)$ *and* $\frac{\partial}{\partial x}U_{\pm,n}(\lambda,x)$ *are* $m-1$ *times differentiable with respect to* $k_\pm$ *with the following behaviour as* $\lambda\to\infty$ *and* $0\le s\le m-1$*:*

$$\frac{\partial^s}{\partial k_\pm^s}U_{\pm,n}(\lambda,x)\in L^2(\infty),\quad \frac{\partial^s}{\partial k_\pm^s}\left(\frac{1}{k_\pm}\frac{\partial}{\partial x}U_{\pm,n}(\lambda,x)\right)\in L^2(\infty). \tag{2.3.3}$$

Proof. Formula (2.1.18) implies

$$\frac{\partial^s B_\pm(x,y)}{\partial y^s}=\frac{\partial^s B_\pm(x,y)}{\partial x\partial y^{s-1}}+\int_x^{\pm\infty}q_\pm(\alpha)\frac{\partial^{s-1}B_\pm(\alpha,y)}{\partial y^{s-1}}d\alpha,\quad s\ge 1. \tag{2.3.4}$$

Integrating (2.1.11) by parts and taking into account (2.3.4) with $s=n+1$ and Lemma 2.1.4 we get

$$\begin{aligned}\phi_\pm(k_\pm,x)\mathrm{e}^{\mp\mathrm{i}k_\pm x}=&1\mp\frac{1}{2\mathrm{i}k_\pm}B_\pm(x,0)\pm\dots+\frac{(-1)^n}{(\pm 2\mathrm{i}k_\pm)^n}\frac{\partial^{n-1}B_\pm(x,0)}{\partial y^{n-1}}\\ &+\frac{(-1)^{n+1}}{(\pm 2\mathrm{i}k)^{n+1}}\left\{\frac{\partial^n B_\pm(x,0)}{\partial y^n}\pm\int_0^{\pm\infty}\left(\frac{\partial}{\partial x}\frac{\partial^n}{\partial y^n}B_\pm(x,y)\right.\right.\\ &\left.\left.+\int_x^{\pm\infty}q_\pm(\alpha)\frac{\partial^n}{\partial y^n}B_\pm(\alpha,y)d\alpha\right)\mathrm{e}^{\pm 2\mathrm{i}k_\pm y}dy\right\}.\end{aligned} \tag{2.3.5}$$

Set

$$u_{\pm,l}(x):=(-1)^l\frac{\partial^{l-1}B_\pm(x,0)}{\partial y^{l-1}},\quad l\le n+1.$$

Then (2.3.4) implies (2.3.2). Put

$$u_{\pm,l+1}(x,y)=(-1)^{l+1}\frac{\partial^l B_\pm(x,y)}{\partial y^l},\quad l\le n. \tag{2.3.6}$$

By (1.1.3), (2.0.2), (2.1.1), (2.1.15), and (2.1.14) we have $\nu_{\pm,l}(\cdot)\in L^0_{m-1}(\mathbb{R}_\pm)$. This implies

$$u_{\pm,n+1}(x,\cdot),\ \frac{\partial}{\partial x}u_{\pm,n+1}(x,\cdot)\in L^0_{m-1}(\mathbb{R}_\pm). \tag{2.3.7}$$

Comparing (2.3.1) with (2.3.5) gives

$$\begin{aligned}U_{\pm,n}(\lambda,x)=u_{\pm,n+1}(x)+\int_0^{\pm\infty}&\left(\frac{\partial}{\partial x}u_{\pm,n+1}(x,y)\right.\\ &\left.\pm\int_x^{\pm\infty}q_\pm(\alpha)u_{\pm,n+1}(\alpha,y)d\alpha\right)\mathrm{e}^{\pm 2\mathrm{i}k_\pm y}dy,\end{aligned} \tag{2.3.8}$$

where the function $u_{\pm,n+1}(x,y)$, defined by (2.3.6), satisfies $u_{\pm,n+1}(x,0)=u_{\pm,n+1}(x)$. From (2.3.2) it follows that the representation for $u_{l,\pm}(x)$ involves

$q_\pm^{(l-2)}(x)$ and lower order derivatives of the potential. Thus $u_{\pm,n+1}(x)$ can be differentiated only one more time with respect to x. But we cannot differentiate the right-hand side of (2.3.8) directly under the integral. To avoid this let us first integrate by parts the first summand in this integral. By (2.3.6) we have $\frac{\partial}{\partial y}u_{\pm,n}(x,y) = -u_{\pm,n+1}(x,y)$. Taking the derivative with respect to x outside the integral we get

$$\int_0^{\pm\infty} \frac{\partial}{\partial x}u_{\pm,n+1}(x,y)\mathrm{e}^{\pm 2\mathrm{i}k_\pm y}dy = \frac{d}{dx}\left(u_{\pm,n}(x) \mp 2\mathrm{i}k_\pm \int_0^{\pm\infty} u_{\pm,n}(x,y)\mathrm{e}^{\pm 2\mathrm{i}k_\pm y}dy\right).$$

According to (2.3.2) we have $u'_{\pm,n+1}(x) + u''_{\pm,n}(x) = q_\pm(x)u_{\pm,n}(x)$, therefore

$$\frac{\partial}{\partial x}U_{\pm,n}(\lambda,x) = 2\mathrm{i}k_\pm\left(\frac{q_\pm(x)u_{\pm,n}(x)}{(2\mathrm{i}k_\pm)} \mp \int_0^{\pm\infty}\frac{\partial^2}{\partial x^2}u_{\pm,n}(x,y)\mathrm{e}^{\pm 2\mathrm{i}k_\pm y}dy\right) - \\ \mp\int_0^{\pm\infty} u_{\pm,n+1}(x,y)q_\pm(x)\mathrm{e}^{\pm 2\mathrm{i}k_\pm y}dy,$$

which together with (2.3.7) proves (2.3.3). $\square$

Our next step is to specify an asymptotic expansion for the Weyl functions

$$m_\pm(\lambda,x) = \frac{\phi'_\pm(\lambda,x)}{\phi_\pm(\lambda,x)} \tag{2.3.9}$$

for the Schrödinger equation. Note that due to estimate (2.1.7) and continuity of $\hat\sigma(x)$ for any $b > 0$ there exist some $k_0 > 0$ such that for all real $k_\pm$ with $|k_\pm| > k_0$ the function $\phi_\pm(\lambda,x)$ does not have zeros for $|x| < b$. Therefore $m_\pm(k_\pm,x)$ is well-defined for all large real $k_\pm$ and x in any compact set $\mathcal{K} \subset \mathbb{R}$.

Lemma 2.3.3. *Let $q \in \mathcal{L}_m^n(c_+,c_-)$. Then for large $\lambda \in \mathbb{R}_+$ the Weyl functions* (2.3.9) *admit the asymptotic expansion*

$$m_\pm(k,x) = \pm\mathrm{i}\sqrt{\lambda} + \sum_{j=1}^n \frac{m_j(x)}{(\pm 2\mathrm{i}\sqrt{\lambda})^j} + \frac{m_{\pm,n}(\lambda,x)}{(\pm 2\mathrm{i}\sqrt{\lambda})^n}, \tag{2.3.10}$$

where

$$m_1(x) = q(x), \quad m_{l+1}(x) = -\frac{d}{dx}m_l(x) - \sum_{j=1}^{l-1} m_{l-j}(x)m_j(x), \tag{2.3.11}$$

and the functions $m_{\pm,n}(\lambda,x)$ are $m-1$ times differentiable with respect to $k_\pm$ with

$$\frac{\partial^s}{\partial k_\pm^s}m_n(\lambda,x) \in L^2(\infty), \quad s \le m-1,\ \forall x \in \mathcal{K}. \tag{2.3.12}$$

Remark 2.3.4. *The recurrence relations* (2.3.11) *are well-known for the case of the Schrödinger operator with smooth potentials and are usually proven via the Riccati equation satisfied by the Weyl functions. Our point here is the fact that* (2.3.10) *is* $m-1$ *times differentiable with respect to* $k_\pm$ *together with* (2.3.12).

Proof. We follow the proof of [59], Lemma 1.4.2, adapting it for the steplike case. From (2.3.9) and (2.0.1) we have $m_\pm(\lambda,x) = \mathrm{i}k_\pm + \kappa_\pm(\lambda,x)$, where $\kappa_\pm(\lambda,x)$ satisfy the equations

$$\kappa_\pm'(\lambda,x) \pm 2\mathrm{i}k_\pm\kappa_\pm(\lambda,x) + \kappa_\pm^2(\lambda,x) - q_\pm(x) = 0, \quad \kappa_\pm(\lambda,x) = o(1), \quad \lambda\to\infty.$$

Introduce notations $\phi_\pm(\lambda,x) = \mathrm{e}^{\pm\mathrm{i}k_\pm x}Q_{\pm,n}(\lambda,x)$, where (cf. Lemma 2.3.2)

$$Q_{\pm,n}(\lambda,x) := P_{\pm,n}(\lambda,x) + \frac{U_{\pm,n}(\lambda,x)}{(\pm 2\mathrm{i}k_\pm)^{n+1}}, \tag{2.3.13}$$

$$P_{\pm,n}(\lambda,x) := 1 + \frac{u_{\pm,1}(x)}{(\pm 2\mathrm{i}k)} + \cdots + \frac{u_{\pm,n}(x)}{(\pm 2\mathrm{i}k)^n}. \tag{2.3.14}$$

Then

$$\kappa_\pm(\lambda,x) = \frac{P_{\pm,n}'(\lambda,x)}{P_{\pm,n}(\lambda,x)} + \frac{U_{\pm,n}'(\lambda,x)P_{\pm,n}(\lambda,x) - U_{\pm,n}(\lambda,x)P_{\pm,n}'(\lambda,x)}{(\pm 2\mathrm{i}k_\pm)^{n+1}P_{\pm,n}(\lambda,x)Q_{\pm,n}(\lambda,x)}.$$

Decompose the first fraction in a series with respect to $(2\mathrm{i}k_\pm)^{-1}$ using (2.3.14). Since $P_{\pm,n}(\lambda,x)\neq 0$ for $x\in\mathcal{K}$ and sufficiently large λ we get

$$\frac{P_{\pm,n}'(\lambda,x)}{P_{\pm,n}(\lambda,x)} = \sum_{j=1}^n \frac{\kappa_{\pm,j}(x)}{(\pm 2\mathrm{i}k_\pm)^j} + \frac{f_{\pm,n}(\lambda,x)}{(\pm 2\mathrm{i}k_\pm)^n},$$

where $\kappa_{\pm,j}(x)$ are polynomials of $u_{\pm,l}$, $l\le j$, and the function $f_{\pm,n}(\lambda,x)$ is infinitely many times differentiable with respect to $k_\pm$ for sufficiently big $k_\pm$ and

$$\frac{\partial^l}{\partial k_\pm^l} f(\lambda,x) \in L^2(\infty),\ l = 0,1,\dots \tag{2.3.15}$$

Correspondingly,

$$\kappa_\pm(\lambda,x) = \sum_{j=1}^n \frac{\kappa_{\pm,j}(x)}{(\pm 2\mathrm{i}k_\pm)^j} + \frac{\kappa_{\pm,n}(\lambda,x)}{(2\mathrm{i}k_\pm)^n}, \tag{2.3.16}$$

where

$$\kappa_{\pm,n}(\lambda,x) = f_{\pm,n}(k,x) + \frac{U_{\pm,n}'(\lambda,x)}{2\mathrm{i}k_\pm Q_{\pm,n}(\lambda,x)} - \frac{U_{\pm,n}(\lambda,x)P_{\pm,n}'(\lambda,x)}{2\mathrm{i}k_\pm P_{\pm,n}(\lambda,x)Q_{\pm,n}(\lambda,x)}.$$

Taking into account (2.3.2), (2.3.7), (2.3.3), (2.3.13), (2.3.14), and (2.3.15) we get

$$\frac{\partial^s}{\partial k_\pm^s}\kappa_{\pm,n}(\lambda,x)\in L^2(\infty),\quad s\le m-1,\ \forall x\in\mathcal{K}.$$

Next, due to (2.3.2) the functions $u_l(x)$ depend on $q^{(l-2)}(x)$ and lower order derivatives of the potential, and can be differentiated at least twice more with respect to x for $l\le n$. Since the function $\phi_\pm(\lambda,x)$ itself is also twice differentiable with respect to x, the same is valid for $U_{\pm,n}(\lambda,x)$ and $\kappa_\pm(\lambda,x)$. Hence each summand of (2.3.14) can be differentiated twice and we conclude that all $\kappa_{\pm,j}(x)$, $j\le n$, in (2.3.16) are differentiable with respect to x, and so is $\kappa_{\pm,n}(\lambda,x)$.

Next, for large λ we can expand $k_\pm$ with respect to $\sqrt{\lambda}$ and represent $m_\pm(\lambda,x)$ using (2.3.16) as $m_\pm(\lambda,x)=\pm\mathrm{i}\sqrt{\lambda}+\tilde\kappa_\pm(\lambda,x)$, where

$$\tilde\kappa_\pm(\lambda,x)=\sum_{j=1}^{n}\frac{\tilde\kappa_{\pm,j}(x)}{(\pm 2\mathrm{i}\sqrt{\lambda})^j}+\frac{m_{\pm,n}(\lambda,x)}{(2\mathrm{i}\sqrt{\lambda})^n}.$$

Here $\tilde\kappa_{\pm,j}(x)$ are some other coefficients, but they also depend on the potential and its derivatives up to order $n-1$, i.e. one time differentiable together with $\tilde\kappa_{\pm,n}(\lambda,x)$ with respect to x. Moreover, $m_{\pm,n}(\lambda,x)$ satisfies the same estimates as in (2.3.12). But $\tilde\kappa_\pm(\lambda,x)$ satisfies the Riccati equation

$$\tilde\kappa_\pm'(\lambda,x)\pm 2\mathrm{i}\sqrt{\lambda}\kappa_\pm(\lambda,x)+\kappa_\pm^2(\lambda,x)-q(x)=0,$$

and therefore $\tilde\kappa_{+,l}(x)=\tilde\kappa_{-,l}(x)=m_l(x)$, where $m_l(x)$ satisfies (2.3.11). $\square$

Corollary 2.3.5. *Let $q\in\mathcal{L}_m^n(c_+,c_-)$ with $n\ge 1$ and $m\ge 1$. Then for any $\mathcal{K}\subset\mathbb{R}$, $x\in\mathcal{K}$ and sufficiently large $\lambda>\bar c$ the function*

$$f_{\pm,n}(\lambda,x):=k_\pm^n\left(\overline{m_\pm(\lambda,x)}-m_\mp(\lambda,x)\right)$$

is $m-1$ times differentiable with respect to $k_\pm$ with

$$\frac{\partial^s}{\partial k_\pm^s}f_{\pm,n}(\lambda,x)\in L^2(\infty),\quad 0\le s\le m-1.$$

The claim of Theorem 2.3.1 follows immediately from (2.1.24), evaluated for $x\in\mathcal{K}$, (2.1.9), (2.3.9), Lemma 2.3.3, and Corollary 2.3.5.

Chapter 3

Rarefaction Waves of the Korteweg–de Vries Equation via Nonlinear Steepest Descent

In this section we investigate the Cauchy problem for the Korteweg–de Vries (KdV) equation with steplike initial data $q(x,0) = q(x)$ satisfying

$$\begin{cases} q(x) \to 0, & \text{as } \ x \to +\infty, \\ q(x) \to c^2, & \text{as } \ x \to -\infty. \end{cases} \tag{3.0.1}$$

This case is known as rarefaction problem. The corresponding long-time asymptotics of $q(x,t)$ as $t \to \infty$ are well understood on a physical level of rigour ([68, 56, 64]) and can be split into three main regions:

- In the region $x < -6c^2 t$ the solution is asymptotically close to the background c^2.
- In the region $-6c^2 t < x < 0$ the solution can asymptotically be described by $-\frac{x}{6t}$.
- In the region $0 < x$ the solution is asymptotically given by a sum of solitons.

This is illustrated in Figure 3.1. The application of the inverse scattering transform to the problem (1.1.1),(3.0.1) (see [25], [30]) imply that the solution $q(x,t)$ of the Cauchy problem exists in the classical sense and is unique in the class

$$\int_0^{+\infty} |x|(|q(x,t)| + |q(-x,t) - c^2|)dx < \infty, \qquad \forall t \in \mathbb{R}, \tag{3.0.2}$$

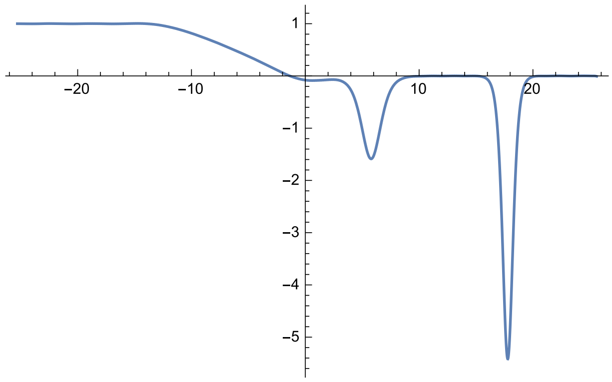

Figure 3.1: Numerically computed solution $q(x,t)$ of the KdV equation at time $t = 1.5$, with initial condition $q(x,0) = \frac{1}{2}(1 - \operatorname{erf}(x)) - 4\operatorname{sech}(x-1)$.

provided the initial data satisfy the following conditions: $q \in \mathcal{C}^8(\mathbb{R})$ and

$$\int_0^\infty x^4 \left(|q(x)| + |q(-x) - c^2| + |q^{(i)}(x)|\right) dx < \infty, \quad i = 1, \dots, 8. \tag{3.0.3}$$

To simplify considerations, we will additionally suppose that the initial condition decays exponentially fast to the asymptotics:

$$\int_0^{+\infty} \mathrm{e}^{\varepsilon x}(|q(x)| + |q(-x) - c^2| dx < \infty, \tag{3.0.4}$$

for some $\varepsilon > 0$. We remark that by [65] the solution will be even real analytic under this assumption, but we will not need this fact.

The chapter is organized as follows: Section 3.1 provides some necessary information about the inverse scattering transform with steplike backgrounds and formulates the initial vector RH problems both on the plane and on a Riemann surfaces. In Section 3.2 we study the soliton region. In Section 3.3 the initial RH problem is reduced to a "model" problem in the domain $-6c^2 t < x < 0$ which is then solved. In Section 3.5 we establish the asymptotics in the oscillatory/dispersive region $-6c^2 t > x$.

Finally, we should remark that our results do not cover the two transitional regions: $0 \approx x$ near the leading wave front, and $x \approx -6c^2 t$ near the back wave front. Since the error bounds obtained from the RH method break down near the edges, a rigorous justification is beyond the scope of the present thesis.

3.1 Statement of the RH problem and the first conjugation step

Let $q(x,t)$ be the solution of the Cauchy problem (1.1.1), (3.0.4) and consider the underlying spectral problem

$$(L(t)f)(x) := -\frac{d^2}{dx^2}f(x) + q(x,t)f(x) = \lambda f(x), \qquad x \in \mathbb{R}.$$

In order to set up the respective Riemann–Hilbert (RH) problems we recall some facts from scattering theory with steplike backgrounds. We refer to [24] for proofs and further details and to [66] for general background.

Throughout this chapter we will use the following notations: Denote by $\mathcal{D} := \mathbb{C} \setminus \Sigma$, where $\Sigma = \Sigma_u \cup \Sigma_l$ with $\Sigma_u = \{\lambda^u = \lambda + \mathrm{i}0, \lambda \in [0,\infty)\}$ and $\Sigma_l = \{\lambda^l = \lambda - \mathrm{i}0, \lambda \in [0,\infty)\}$. That is, we treat the boundary of the domain $\mathcal{D}$ as consisting of the two sides of the cut along the interval $[0,\infty)$, with different points λ^u and λ^l on different sides. In equation (2.0.1) the spectral parameter λ belongs to the set $\mathrm{clos}(\mathcal{D})$, where $\mathrm{clos}(\mathcal{D}) = \mathcal{D} \cup \Sigma$. Along with λ we will use two more spectral parameters

$$k = \sqrt{\lambda}, \quad k_1 = \sqrt{\lambda - c^2}, \quad \text{where } k > 0 \text{ and } k_1 > 0, \quad \text{for } \lambda^u > c^2. \tag{3.1.1}$$

The functions $k_1(\lambda)$ and $k(\lambda)$ conformally map the domain $\mathcal{D}$ onto $\mathfrak{D}_1 := \mathbb{C}^+ \setminus (0, \mathrm{i}c]$ and $\mathfrak{D} := \mathbb{C}^+$, respectively. Since there is a bijection between the closed domains $\mathrm{clos}\,\mathcal{D}$, $\mathrm{clos}\,\mathfrak{D} = \mathbb{C}^+ \cup \mathbb{R}$ and $\mathrm{clos}\,\mathfrak{D}_1 = \mathfrak{D}_1 \cup \mathbb{R} \cup [0, \mathrm{i}c]_r \cup [0, \mathrm{i}c]_l$, we will use the ambiguous notation $f(k)$ or $f(k_1)$ or $f(\lambda)$ as the same value of an arbitrary function $f(\lambda)$ in these respective coordinates. Here $[0, \mathrm{i}c]_{r,l}$ are the right and left sides of the respective cut. Thus, if $k > 0$ corresponds to λ^u then $-k$ corresponds to λ^l, and for functions defined on the set Σ we will sometimes use the notation $f(k)$ and $f(-k)$ to indicate the values at symmetric points λ^u and λ^l.

Since the potential $q(x,t)$ satisfies (3.0.2) the following facts are valid for the operator $L(t)$ ([24]):

- The spectrum of $L(t)$ consists of an absolutely continuous part $\mathbb{R}_+$, plus a finite number of negative eigenvalues $\lambda_1 < ... < \lambda_N < 0$. Moreover, the (absolutely) continuous spectrum consists of a part $[0, c^2]$ of multiplicity one and a part $[c^2, \infty)$ of multiplicity two. In terms of k, k_1 the continuous spectrum corresponds to $k \in \mathbb{R}$ and the spectrum of multiplicity two to $k_1 \in \mathbb{R}$.

- Equation (2.0.1) has two Jost solutions $\phi(\lambda) = \phi(\lambda, x, t)$ and $\phi_1(\lambda) = \phi_1(\lambda, x, t)$, satisfying the conditions

$$\lim_{x\to+\infty} \mathrm{e}^{-\mathrm{i}kx}\phi(\lambda, x, t) = \lim_{x\to-\infty} \mathrm{e}^{\mathrm{i}k_1 x}\phi_1(\lambda, x, t) = 1, \quad \text{for} \quad \lambda \in \operatorname{clos}\mathcal{D}.$$

- The Jost solutions satisfy the scattering relations

$$T(\lambda, t)\phi_1(\lambda, x, t) = \overline{\phi(\lambda, x, t)} + R(\lambda, t)\phi(\lambda, x, t), \qquad k \in \mathbb{R}, \tag{3.1.2}$$
$$T_1(\lambda, t)\phi(\lambda, x, t) = \overline{\phi_1(\lambda, x, t)} + R_1(\lambda, t)\phi_1(\lambda, x, t), \quad k_1 \in \mathbb{R}, \tag{3.1.3}$$

  where $T(\lambda, t)$, $R(\lambda, t)$ (resp., $T_1(\lambda, t)$, $R_1(\lambda, t)$) are the right (resp., the left) transmission and reflection coefficients. The right coefficients are given by formulas

$$T(\lambda, t) = \frac{2\mathrm{i}k}{W(\lambda, t)}, \qquad R(\lambda, t) = -\frac{\tilde{W}(\lambda, t)}{W(\lambda, t)},$$

  where

$$\begin{aligned}\tilde{W}(\lambda, t) =& \phi_1(\lambda, x, t)\overline{\phi'(\lambda, x, t)} - \phi_1'(\lambda, x, t)\overline{\phi(\lambda, x, t)},\\ W(\lambda, t) =& \phi_1(\lambda, x, t)\phi'(\lambda, x, t) - \phi_1'(\lambda, x, t)\phi(\lambda, x, t).\end{aligned}$$

- The Wronskian of the Jost solutions $W(\lambda, t)$ has simple zeros at the points of the discrete spectrum and the only other possible zero is 0, which is known as the resonant case. In this case $W(\lambda) = \mathrm{i}\gamma k(1+o(1))$, $\gamma > 0$.

- The solutions $\phi(\lambda_j, x, t)$ and $\phi_1(\lambda_j, x, t)$ are the corresponding (linearly dependent) eigenfunctions of $L(t)$. We use the notations

$$\phi(\lambda_j, x, t) = \mu_j(t)\phi_1(\lambda_j, x, t), \quad \gamma_j(t) = \left(\int_{\mathbb{R}} \phi^2(\lambda_j, x, t)dx\right)^{-1}.$$

- There is a symmetry $T(\lambda^u, t) = \overline{T(\lambda^l, t)}$, $R(\lambda^u, t) = \overline{R(\lambda^l, t)}$ for $k \in \mathbb{R}$, i.e. $\lambda \in \Sigma$. The same is valid for $\phi(\lambda, x, t)$ and $\phi_1(\lambda, x, t)$. Moreover, $\phi_1(\lambda, x, t) = \overline{\phi_1(\lambda, x, t)}$ for $k \in [-c, c]$.

- The following identities are valid on the continuous spectrum[1]:

$$\frac{T(k, t)}{\overline{T(k, t)}} = R(k, t), \text{ for } k \in [-c, c], and \tag{3.1.4}$$

[1]Recall that according to our agreement everywhere in this dissertation the notation $f(k)$ means $f(\lambda(k))$

$$
\begin{aligned}
&1-|R(k,t)|^2 = T_1(k,t)\overline{T(k,t)}, \\
&R_1(k,t)\overline{T(k,t)} + \overline{R(k,t)}T(k,t) = 0, \ k_1 \in \mathbb{R}.
\end{aligned} \tag{3.1.5}
$$

- The time evolution of the right reflection coefficient and the right norming constant is given by ([30, 48, 50])

$$
R(\lambda,t) = R(\lambda)\mathrm{e}^{8\mathrm{i}k^3 t}, \quad k \in \mathbb{R}, \quad \gamma_j(t) = \gamma_j \mathrm{e}^{8\kappa_j^3 t}, \tag{3.1.6}
$$

  where $R(\lambda) = R(\lambda,0)$, $\gamma_j = \gamma_j(0)$, and $0 < \kappa_j = \sqrt{-\lambda_j}$.

- Under condition (3.0.4) the solution $\overline{\phi(\lambda,x,0)}$ has an analytical continuation to a subdomain $\mathcal{D}_\varepsilon \subseteq \mathcal{D}$, which is the image under the map $\lambda(k)$ of the strip $0 \le \operatorname{Im} k < \varepsilon$. Correspondingly, the function $R(\lambda)$ also has an analytical continuation to this domain. Since the transmission coefficient always has an analytical continuation identity (3.1.2) remains valid for this analytical continuation.

- The potential $q(x)$ can be uniquely recovered from the right scattering data

$$
\{R(k), \quad k \in \mathbb{R}; \quad \lambda_j = -\kappa_j^2, \ \gamma_j > 0, \ j = 1, \dots, N\}. \tag{3.1.7}
$$

The properties, listed above, allow us to introduce a vector RH problem. To this end we introduce a vector-function

$$
m(\lambda,x,t) = \big(T(\lambda,t)\phi_1(\lambda,x,t)\mathrm{e}^{\mathrm{i}kx}, \quad \phi(\lambda,x,t)\mathrm{e}^{-\mathrm{i}kx}\big) \tag{3.1.8}
$$

on $\operatorname{clos}\mathcal{D}$. Evidently, this function is meromorphic in $\mathcal{D}$ with simple poles at the points $\mathrm{i}\kappa_j$, and continuous up to the boundary Σ. We treat this function as a function of $k \in \mathfrak{D}$, keeping x and t as parameters, $m(k) := m(\lambda(k),x,t)$. It has the following asymptotical behaviour as $k \to \infty$ (cf. [23]):

$$
m(k) = (1,1) - \frac{1}{2\mathrm{i}k}\int_x^{+\infty} q(y,t)dy(-1,1) + O\left(\frac{1}{k^2}\right). \tag{3.1.9}
$$

Moreover, we extend the definition of $m(k)$ to $\mathbb{C}^-$ using the *symmetry condition*

$$
m(k) = m(-k)\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}. \tag{3.1.10}
$$

Evidently this vector function has a jump along the real axis. We consider the real axis as a contour with the natural orientation from minus to plus infinity and denote by $m_+(k)$ (resp. $m_-(k)$) the (nontangential) limiting values of $m(k)$ from the upper (resp. lower) half-plane.

Theorem 3.1.1. *Let* (3.1.7) *be the right scattering data for a potential* $q(x)$ *satisfying condition* (3.0.4). *Let* $q(x,t)$ *be the unique solution of the Cauchy problem* (1.1.1), (3.0.3). *Then the vector-valued function* $m(k)$ *defined by* (3.1.8) *and* (3.1.10) *is the unique solution of the following vector Riemann–Hilbert problem:*

Find a vector-valued function $m(k) = \big(m_1(k), m_2(k)\big)$, *which is meromorphic away from the contour* $\mathbb{R}$ *and satisfies:*

I. The jump condition $m_+(k) = m_-(k)v(k)$, *where*

$$v(k) := v(\lambda(k), x, t) = \begin{pmatrix} 1 - |R(k)|^2 & -\overline{R(k)}\mathrm{e}^{-2t\Phi(k)} \\ R(k)\mathrm{e}^{2t\Phi(k)} & 1 \end{pmatrix}, \quad k \in \mathbb{R}; \tag{3.1.11}$$

II. the pole conditions

$$\begin{aligned} \operatorname{Res}_{\mathrm{i}\kappa_j} m(k) &= \lim_{k\to \mathrm{i}\kappa_j} m(k) \begin{pmatrix} 0 & 0 \\ \mathrm{i}\gamma_j \mathrm{e}^{2t\Phi(\mathrm{i}\kappa_j)} & 0 \end{pmatrix}, \\ \operatorname{Res}_{-\mathrm{i}\kappa_j} m(k) &= \lim_{k\to -\mathrm{i}\kappa_j} m(k) \begin{pmatrix} 0 & -\mathrm{i}\gamma_j \mathrm{e}^{2t\Phi(\mathrm{i}\kappa_j)} \\ 0 & 0 \end{pmatrix}, \end{aligned} \tag{3.1.12}$$

III. the symmetry condition (3.1.10)*;*

IV. the normalization condition

$$m(k) = (1, 1) + O(k^{-1}), \quad k \to \infty; \tag{3.1.13}$$

V. $m_2(k) \in \mathbb{R}$ *for* $k \in \mathrm{i}\mathbb{R}_+$.

The phase $\Phi(k) = \Phi(\lambda(k), x, t)$ *in* (3.1.11) *is given by*

$$\Phi(k) = (4\lambda + \frac{x}{t})\sqrt{-\lambda} = 4\mathrm{i}k^3 + \mathrm{i}k\frac{x}{t}. \tag{3.1.14}$$

Remark 3.1.2. *By property* (3.1.4) $|R(k)| = 1$ *for* $k \in [-c, c]$ *and hence*

$$v(k) = \begin{pmatrix} 0 & -\overline{R(k)}\mathrm{e}^{-2t\Phi(k)} \\ R(k)\mathrm{e}^{2t\Phi(k)} & 1 \end{pmatrix}, \quad k \in [-c, c].$$

Proof of Theorem 3.1.1. We will omit the variables x, t in our notation below whenever possible. Let $m(k)$ be defined by (3.1.8). It is a meromorphic function the upper half-plane, its first component $m_1(k)$ has simple poles at the points $\mathrm{i}\kappa_j$ while the second component $m_2(k)$ is holomorphic. Both components have continuous limits up to the boundary $\mathbb{R}$. Moreover, for $k \in \mathbb{R}$ we have $m_+(-k) = \overline{m_+(k)}$. To compute the jump we observe that if $m_+ = (T\phi_1 z, \phi z^{-1})$, where $z = \mathrm{e}^{\mathrm{i}kx}$, $k \in \mathbb{R}$, then by the symmetry condition $m_- = (\overline{\phi} z, \overline{T\phi_1} z^{-1})$ at the same point $k \in \mathbb{R}$. Let $\left(\begin{smallmatrix} \alpha(k) & \beta(k) \\ \gamma(k) & \delta(k) \end{smallmatrix}\right)$ be the unknown jump matrix. Then

$$T\phi_1 z = \overline{\phi}\, z\alpha + \overline{T\phi_1} z^{-1}\gamma, \qquad \phi\, z^{-1} = \overline{\phi}\, z\beta + \overline{T\phi_1} z^{-1}\delta.$$

Multiply the first equality by z^{-1}, the second one by z, and then conjugate both of them. We finally get

$$\overline{\alpha}\phi = \overline{T\phi_1} - T\overline{\gamma}\phi_1 z^2, \qquad T\overline{\delta}\phi_1 = \overline{\phi} - \overline{\beta}\phi z^{-2}. \tag{3.1.15}$$

Now divide the first of these equalities by $\overline{T}$ and compare it with (3.1.3) for $k_1 \in \mathbb{R}$. From (3.1.5) it follows that $\alpha = \overline{T}_1 T = 1 - |R|^2$, $\gamma z^{-2} = R$ for $k_1 \in \mathbb{R}$. For $k \in [-c, c]$ we use the first equality of (3.1.15) taking into account that $\overline{\phi}_1 = \phi_1$. Then by (3.1.4) $\overline{\alpha}\phi = \phi_1\overline{T}(1 - \overline{\gamma} z^2 R)$ and therefore $\alpha = 0$, $\gamma z^{-2} = R$ as $k \in [-c, c]$. Taking into account (3.1.6) and $z = \mathrm{e}^{\mathrm{i}kx}$ we finally establish the 11 and 21 entries of the jump matrix (3.1.11). Comparing now the second equality of (3.1.15) with (3.1.2) gives $\delta = 1$ and $-\overline{\beta} z^{-2} = R$. This establishes the 12 and 22 entries.

The pole condition (3.1.12) is proved in [40] or in Appendix A of [23]. The symmetry condition holds by definition and the normalization condition follows from (3.1.9).

Finally we turn to uniqueness. Let $\tilde{m}(k)$ be another solution. Then $\hat{m}(k) = m(k) - \tilde{m}(k)$ satisfies I–III. Note, that condition II does not guarantee that $\hat{m}$ is a holomorphic solution! Condition IV for $\hat{m}$ reads $\hat{m}(k) = O(k^{-1})$ and therefore the function

$$F(k) := \hat{m}_1(k)\overline{\hat{m}_1(\overline{k})} + \hat{m}_2(k)\overline{\hat{m}_2(\overline{k})}, \quad \operatorname{Im} k > 0,$$

is a meromorphic function in $\mathbb{C}^+$ with simple poles at the points $\mathrm{i}\kappa_j$ and asymptotical behaviour $F(k) = O(k^{-2})$ at infinity. Since $-\overline{k} = k$ for $k \in \mathrm{i}\mathbb{R}$ conditions II and V imply

$$\operatorname{Res}_{\mathrm{i}\kappa_j} F(k) = 2\mathrm{i}\gamma_j \left(\hat{m}_2(\mathrm{i}\kappa_j)\right)^2 \mathrm{e}^{2t\Phi(\mathrm{i}\kappa_j)} \in \mathrm{i}\mathbb{R}_+. \tag{3.1.16}$$

Moreover, $F(k)$ has continuous limiting values $F_+(k)$ on $\mathbb{R}$, which can be represented due to condition III as $F_+(k) = \hat{m}_{1,+}(k)\overline{\hat{m}_{1,-}(k)} + \hat{m}_{2,+}(k)\overline{\hat{m}_{2,-}(k)}$. From condition I we then get

$$F_+(k) = (\mathcal{S}\hat{m}_{1,-} + \mathcal{R}\hat{m}_{2,-})\overline{\hat{m}_{1,-}} + (\hat{m}_{2,-} - \overline{\mathcal{R}}\hat{m}_{1,-})\overline{\hat{m}_{2,-}},$$

where $\mathcal{S}(k) = 1 - |R(k)|^2 \geq 0$, $\mathcal{R}(k) = R(k)\mathrm{e}^{2t\Phi(k)}$. Thus

$$F_+(k) = G_1(k) + \mathrm{i}G_2(k), \quad G_1(k) \geq 0, \quad G_2(k) \in \mathbb{R}, \tag{3.1.17}$$

with $G_1(k) = (1 - |R(k)|^2)|\hat{m}_{1,-}(k)|^2 + |\hat{m}_{2,-}(k)|^2$. Let now $\rho > \kappa_1 > \kappa_N$ be an arbitrary large number. Consider the semicircle

$$\mathcal{C}_\rho = \{k : k \in [-\rho, \rho], \text{or } k = \rho\mathrm{e}^{\mathrm{i}\theta}, \quad 0 < \theta < \pi\}$$

as a contour, oriented counterclockwise. By the Cauchy theorem and (3.1.16)

$$\oint_{\mathcal{C}_\rho} F(k)dk = 2\pi\mathrm{i}\sum_{j=1}^N \operatorname{Res}_{\mathrm{i}\kappa} F(k) = -4\pi \sum_{j=1}^N \gamma_j \left(\hat{m}_2(\mathrm{i}\kappa_j)\right)^2 \mathrm{e}^{2t\Phi(\mathrm{i}\kappa_j)}.$$

Since $F(k) = O(k^{-2})$ as $k \to \infty$ then $\lim_{\rho\to\infty} \int_0^\pi F(\rho\mathrm{e}^{\mathrm{i}\theta})\rho\mathrm{e}^{\mathrm{i}\theta}d\theta = 0$. Therefore,

$$\int_{\mathbb{R}} F_+(k)dk + 4\pi \sum_{j=1}^N \gamma_j \left(\hat{m}_2(\mathrm{i}\kappa_j)\right)^2 \mathrm{e}^{2t\Phi(\mathrm{i}\kappa_j)} = 0.$$

By (3.1.17) and (3.1.16) the real part of this expression is positive and vanishes if

$$\hat{m}_{2,-}(k) = 0 \text{ for } k \in (\mathbb{R} \cup_j \{\mathrm{i}\kappa_j\}), \text{ and } \hat{m}_{1,-}(k) = 0 \text{ for } k_1 \in \mathbb{R}.$$

Thus $F(k)$ is holomorphic and due to (3.1.17) has no jump on the real axis. Taking into account its behaviour at infinity we conclude that it is zero. This proves uniqueness for the RH problem under consideration. □

For our further analysis we rewrite the pole condition as a jump condition and hence turn our meromorphic Riemann–Hilbert problem into a holomorphic Riemann–Hilbert problem following literally [40]. Choose $\delta > 0$ so small that the discs $|k - \mathrm{i}\kappa_j| < \delta$ lie inside the upper half-plane and do not intersect any of the other contours, moreover $\kappa_1 - \delta > \varepsilon$, where ε is from estimate (3.0.4). Redefine $m(k)$ in a neighborhood of $\mathrm{i}\kappa_j$(respectively $-\mathrm{i}\kappa_j$) according to

$$m(k) = \begin{cases} m(k)\begin{pmatrix} 1 & 0 \\ -\frac{\mathrm{i}\gamma_j \mathrm{e}^{2t\Phi(\mathrm{i}\kappa_j)}}{k - \mathrm{i}\kappa_j} & 1 \end{pmatrix}, & |k - \mathrm{i}\kappa_j| < \delta, \\ m(k)\begin{pmatrix} 1 & \frac{\mathrm{i}\gamma_j \mathrm{e}^{2t\Phi(\mathrm{i}\kappa_j)}}{k + \mathrm{i}\kappa_j} \\ 0 & 1 \end{pmatrix}, & |k + \mathrm{i}\kappa_j| < \delta, \\ m(k), & \text{else}. \end{cases} \tag{3.1.18}$$

Denote the boundaries of these small discs as $\mathbb{T}^{j,U}$ and $\mathbb{T}^{j,L}$ (U and L for "upper" and "lower"). Set also

$$h^U(k,j) := -\frac{\mathrm{i}\gamma_j \mathrm{e}^{2t\Phi(\mathrm{i}\kappa_j)}}{k-\mathrm{i}\kappa_j}, \quad h^L(k,j) := -\frac{\mathrm{i}\gamma_j \mathrm{e}^{2t\Phi(\mathrm{i}\kappa_j)}}{k+\mathrm{i}\kappa_j}. \tag{3.1.19}$$

Then a straightforward calculation using $\operatorname{Res}_{\mathrm{i}\kappa} m(k) = \lim_{k\to\mathrm{i}\kappa}(k-\mathrm{i}\kappa)m(k)$ shows the following well-known result:

Lemma 3.1.3 ([40]). *Suppose $m(k)$ is redefined as in* (3.1.18). *Then $m(k)$ is holomorphic in $\mathbb{C}\setminus\big(\mathbb{R}\cup\cup_{j=1}^N(\mathbb{T}^{j,U}\cup\mathbb{T}^{j,L})\big)$. Furthermore it satisfies conditions I, III, IV and*

$$m_+(k) = m_-(k)\begin{cases}\begin{pmatrix}1 & 0\\ h^U(k,j) & 1\end{pmatrix}, & k\in\mathbb{T}^{j,U}\\ \begin{pmatrix}1 & h^L(k,j)\\ 0 & 1\end{pmatrix}, & k\in\mathbb{T}^{j,L},\end{cases} \tag{3.1.20}$$

where the small circles $\mathbb{T}^{j,U}$ around the points $\mathrm{i}\kappa_j$ are oriented counterclockwise, and the circles $\mathbb{T}^{j,L}$ around $-\mathrm{i}\kappa_j$ are oriented clockwise.

This holomorphic RH problem is equivalent to the initial one, given by conditions I–V and hence also has a unique solution. We use it everywhere except in the vicinities of the rays $x = 4\kappa_j^2 t$ along which the solitons travel. In what follows we will denote this RH problem as RH-k problem since it is formulated in the k variable. This problem is convenient for investigating the soliton region $x > 0$. In the other regions it will be more convenient to use other two formulations of this RH problem, which we denote as RH-k_1 problem and as RH-λ problem. In particular, let $\mathfrak{D}_1 = \mathbb{C}^+\setminus(0,\mathrm{i}c]$ be the domain corresponding to the k_1 variable, which is the one-to-one image of the domain $\mathcal{D}$ with respect to λ and the upper half-plane with respect to k. We consider the same vector function $m(\lambda)$, defined by (3.1.8), as a function of k_1 with poles at points $\mathrm{i}\kappa_{1,j} = \sqrt{c^2-\lambda_j}$. Extend this definition to the lower half-plane using the same symmetry condition (the point $-k_1$ corresponds to the point $-k$ in $\mathfrak{D}_1$). Then m will satisfy condition V for $k_1\in[\mathrm{i}c,+\mathrm{i}\infty)$ and conditions III and IV. On the k_1 plane introduce the contour consisting of: (1) the real axis, oriented naturally from the left to the right; (2) the interval $[-\mathrm{i}c,\mathrm{i}c]$, oriented from top to bottom; (3) the images of circles $\mathbb{T}^{j,U}$ and $\mathbb{T}^{j,L}$, which we denote as $\mathbb{T}_1^{j,U}$ and $\mathbb{T}_1^{j,L}$, with the orientation induced from their preimages.

To make a statement about the RH-k_1 problem we will need the following notation: Let $k\in[0,c]$ be the point which corresponds to a point k_1 on the

+ (right) side of the cut along the interval $[\mathrm{i}c, 0]$. Set

$$\chi(k_1) = \chi(k_1, x, t) =: R(k)\mathrm{e}^{2t\Phi(k)}, \ k \in [0, c], \quad \chi(-k_1) := \chi^{-1}(k_1).$$

Lemma 3.1.4. *The function $m(k_1)$, defined by* (3.1.8) *and* (3.1.10)*, is the unique solution of the following RH problem: to find a vector-function m holomorphic in the domain $\mathbb{C}\setminus\left(\mathbb{R}\cup[-\mathrm{i}c,\mathrm{i}c]\cup_j(\mathbb{T}_1^{j,U}\cup\mathbb{T}_1^{j,L})\right)$ and satisfying:*

1. The jump condition $m_+(k_1) = m_-(k_1)v(k_1)$, where

$$v(k_1) := \begin{cases} \begin{pmatrix} 1-|R(k)|^2 & -\overline{R(k)}\mathrm{e}^{-2t\Phi(k)} \\ R(k)\mathrm{e}^{2t\Phi(k)} & 1 \end{pmatrix}, & k_1 \in \mathbb{R}; \\ \begin{pmatrix} \chi(k_1) & 1 \\ 0 & -\chi^{-1}(k_1) \end{pmatrix}, & k_1 \in [\mathrm{i}c, 0] \\ \begin{pmatrix} -\chi^{-1}(k_1) & 0 \\ 1 & \chi(k_1) \end{pmatrix}, & k_1 \in [0, -\mathrm{i}c] \end{cases}$$

$$v(k_1) = \begin{cases} \begin{pmatrix} 1 & 0 \\ h^U(k,j) & 1 \end{pmatrix}, & k_1 \in \mathbb{T}_1^{j,U}, \\ \begin{pmatrix} 1 & h^L(k,j) \\ 0 & 1 \end{pmatrix}, & k \in \mathbb{T}_1^{j,L}, \end{cases}$$

2. the symmetry condition (3.1.10)*;*

3. the normalization condition $m(k_1) = (1,1) + O(k_1^{-1})$, $k_1 \to \infty$.

Proof. We have to verify the jump condition over the interval $[-\mathrm{i}c, \mathrm{i}c]$ only. Let $z = \mathrm{e}^{\mathrm{i}kx}$ and $T = T(k,t)$, $\phi = \phi(k,x,t)$ and $\phi_1 = \phi_1(k,x,t)$ are taken in this point k, and so does the reflection coefficient $R = R(k,t)$ and the phase function $\Phi(k) = \Phi(k,x,t)$. Then $m_+(k_1) = (T\phi_1 z, \phi z^{-1})$ and $m_-(k_1) = (\overline{T}\phi_1 z^{-1}, \overline{\phi} z)$. Let $\left(\begin{smallmatrix} \alpha(k_1) & \beta(k_1) \\ \gamma(k_1) & \delta(k_1) \end{smallmatrix}\right)$ be the unknown jump matrix. Then

$$T\phi_1 z = \alpha\overline{T}\phi_1 z^{-1} + \gamma\overline{\phi}z, \quad \phi z^{-1} = \beta\overline{T}\phi_1 z^{-1} + \delta\overline{\phi}z.$$

Multiplying the first formula on z^{-1}, the second one on z and taking into account (3.1.4) we get

$$\phi_1\overline{T}\left(R - \alpha z^{-2}\right) = \gamma\overline{\phi}, \quad -\frac{\beta\overline{T}}{\delta z^2}\phi_1 = -\frac{1}{\delta z^2}\phi + \overline{\phi}.$$

The first formula gives

$$\alpha = Rz^2 = R(k)\mathrm{e}^{8\mathrm{i}k^3t+2\mathrm{i}kx} = R(k)\mathrm{e}^{2t\Phi(k)}, \quad \gamma = 0.$$

Comparing the second formula with (3.1.2) we get $-\frac{1}{\delta z^2} = R$, that is $-\delta z^2 = \overline{R} = \frac{\overline{T}}{T}$. Respectively, $T = -\frac{\beta\overline{T}}{\delta z^2} = \frac{\beta\overline{T}T}{\overline{T}}$, i.e.

$$\beta = 1, \quad \delta = -\overline{R(k)\mathrm{e}^{2t\Phi(k)}}.$$

This justifies the jump along the contour $[\mathrm{i}c, 0]$. To get the jump over the contour $[0, -\mathrm{i}c]$ we use the symmetry condition. Namely, since the contour $[0, -\mathrm{i}c]$ oriented in the same way as contour $[\mathrm{i}c, 0]$, i.e. from in top to down, then by symmetry condition $m_\pm(-k_1) = m_\mp(k_1)\sigma_1$, where $\sigma_1 = \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$ is the first Pauli matrix. If $v(k_1)$ is the jump matrix on $[\mathrm{i}c, 0]$ then

$$m_+(-k_1) = m_+(k_1)v^{-1}(k_1)\sigma_1 = m_-(-k_1)\sigma_1 v^{-1}(k_1)]\sigma_1,$$

that is $v(-k_1) = \sigma_1 v^{-1}(k_1)\sigma_1$. □

This RH-k_1 problem is convenient for studying in the domain behind the back wave front, i.e. when $x < -6c^2t$. In the middle region $-6c^2t < x < 0$ we use the RH-λ problem, formulated on the Riemann surface of the function $\sqrt{-\lambda}$. For this square root we keep the same meaning as in (3.1.1), that is $\sqrt{-\lambda} = \mathrm{i}k$ for $\lambda \in \Sigma^u$ and $k > 0$. Let $\mathbb{M}$ be the Riemann surface associated with $\sqrt{-\lambda}$, with two sheets, upper Π_U, and lower Π_L, glued via the sides of cut along $\mathbb{R}_+$. A point on $\mathbb{M}$ is denoted by $p = (\lambda, \pm)$, $\lambda \in \operatorname{clos}\mathcal{D}$. We associate the domain $\mathcal{D}$ (or $\mathbb{C}^+$ in the variable k), with the upper sheet of $\mathbb{M}$. Thus, all functions which we introduced above, can be considered as function of point p on $\operatorname{clos}\Pi_U$. In particular, the vector function $m(\lambda, x, t)$ given by (3.1.8), is naturally defined on $\operatorname{clos}\Pi_U$ and we keep for it the notation $m(p)$. The sheet exchange map on $\mathbb{M}$ is given by $p^* = (\lambda, \mp)$ for $p = (\lambda, \pm)$, and if p corresponds to k, then p^* corresponds to $-k$. Therefore, we use without confusion the symmetry condition on $\mathbb{M}$

$$m(p^*) = m(p)\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

to continue $m(p)$ on Π_U, and we have $m(-k) = m(p^*)$ in our formal notations. Correspondingly, the lower sheet Π_L of $\mathbb{M}$ is associated with the half-plane $\mathbb{C}^-$ of the k- plane. Using the same notations as above for

$$\Sigma_u = \{p = (\lambda + \mathrm{i}0, +) | \lambda \in \mathbb{R}_+\}, \quad \Sigma_\ell = \{p = (\lambda - \mathrm{i}0, +) | \lambda \in \mathbb{R}_+\},$$

and $\Sigma = \Sigma_u \cup \Sigma_\ell$,we consider Σ as clockwise oriented contour, and associate its points with points of contour $\mathbb{R}$ from RH-k problem. Then the jump matrix $v(k)$ from RH-k problem has evident values on Σ. Let also $\mathbb{T}^{j,U}_{\{\lambda\}}$ and $\mathbb{T}^{j,L}_{\{\lambda\}}$ be the images of circles $\mathbb{T}^{j,U}$ and $\mathbb{T}^{j,L}$ from the k plane on the upper and lower sheets of $\mathbb{M}$. Let us replace also on these contours the jump matrices from (3.1.20).

Lemma 3.1.5. *The function $m(p)$ is the unique solution of the following problem: Find a vector-valued function $m(p)$ which is holomorphic away from the contour Σ on $\mathbb{M}$ and satisfies:*

I. The jump condition $m_+(p,x,t) = m_-(p,x,t)v(p,x,t)$, where

$$v(p) = \begin{cases} \begin{pmatrix} 1-|R(p)|^2 & -\overline{R(p)}\mathrm{e}^{-2t\Phi(p)} \\ R(p)\mathrm{e}^{2t\Phi(p)} & 1 \end{pmatrix}, & p \in \Sigma, \\ \begin{pmatrix} 1 & 0 \\ h^U(k,j) & 1 \end{pmatrix}, & p \in \mathbb{T}^{j,U}_{\{\lambda\}} \\ \begin{pmatrix} 1 & h^L(k,j) \\ 0 & 1 \end{pmatrix}, & p \in \mathbb{T}^{j,L}_{\{\lambda\}}, \end{cases}$$

II. the symmetry condition

$$m(p^*) = m(p)\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad p \in \mathbb{M}, \tag{3.1.21}$$

III. the normalization condition

$$\lim_{p\in\Pi_U\to\infty_+} m(p) = (1,1). \tag{3.1.22}$$

Here the phase $\Phi(p) = \Phi(p,\xi)$ is given by

$$\Phi(p) = (4\lambda + 12\xi)\sqrt{-\lambda}, \qquad \xi = \frac{x}{12t}.$$

Our aim is to reduce these RH problems to model problems which can be solved explicitly. To this end we record the following well-known result for easy reference.

Lemma 3.1.6 (Conjugation)**.** *Let m be the solution of the RH problem $m_+(p) = m_-(p)v(p)$, $p \in \hat{\Sigma}$, on a Riemann surface $\hat{\mathbb{M}}$ which satisfies the symmetry and normalization conditions. Let $\tilde{\Sigma}$ be a contour on $\hat{\mathbb{M}}$ with the* *same orientation as* $\hat{\Sigma}$ *on the common part of these contours and suppose that* $\hat{\Sigma}$ *and* $\tilde{\Sigma}$ *contain with each point* p *also* p^**. Let* D *be a matrix of the form*

$$D(p) = \begin{pmatrix} d(p)^{-1} & 0 \\ 0 & d(p) \end{pmatrix}, \tag{3.1.23}$$

where $d : \hat{\mathbb{M}} \setminus \tilde{\Sigma} \to \mathbb{C}$ *is a sectionally analytic function with* $d(p) \neq 0$ *except for a finite number of points on* $\hat{\Sigma}$*. Set*

$$\tilde{m}(p) = m(p)D(p), \tag{3.1.24}$$

then the jump matrix of the problem $\tilde{m}_+ = \tilde{m}_- \tilde{v}$ *is*

$$\tilde{v} = \begin{cases} \begin{pmatrix} v_{11} & v_{12}d^2 \\ v_{21}d^{-2} & v_{22} \end{pmatrix}, & p \in \hat{\Sigma}\backslash(\tilde{\Sigma} \cap \hat{\Sigma}), \\ \begin{pmatrix} v_{11}d_+^{-1}d_- & v_{12}d_+d_- \\ v_{21}d_+^{-1}d_-^{-1} & v_{22}d_-^{-1}d_+ \end{pmatrix}, & p \in \tilde{\Sigma} \cap \hat{\Sigma}, \\ \begin{pmatrix} d_+^{-1}d_- & 0 \\ 0 & d_-^{-1}d_+ \end{pmatrix}, & p \in \tilde{\Sigma} \setminus (\tilde{\Sigma} \cap \hat{\Sigma}). \end{cases} \tag{3.1.25}$$

If d *satisfies* $d(p^*) = d(p)^{-1}$ *for* $p \in \hat{\mathbb{M}} \setminus \tilde{\Sigma}$*, then the transformation* (3.1.24) *respects the symmetry condition* (3.1.21).

Note that in general, for an oriented contour $\hat{\Sigma}$, the value $f_+(p_0)$ (resp. $f_-(p_0)$) will denote the nontangential limit of the function $f(p)$ as $p \to p_0 \in \hat{\Sigma}$ from the positive (resp. negative) side of $\hat{\Sigma}$, where the positive side is the one which lies to the left as one traverses the contour in the direction of its orientation.

In addition to this lemma in Sections 3.3 and 3.5 we will apply the so called g-function technique [21] in a form proposed in [53]. This method is very successful and has been used in [12] and several other papers. These g-functions are in fact Abel integrals on modified Riemann surfaces which are "slightly truncated" with respect to the initial one and depend on a parameter ξ. These Abel integrals approximate the phase functions at infinity and transform the jump matrices in a way that allows us to factorize them such that one gets asymptotically constant jump matrices. These RH problems with constant jumps are the corresponding model problems which will be solved explicitly. In the next section we briefly discuss the soliton region, where we work with the RH-k problem and do not require a g-function.

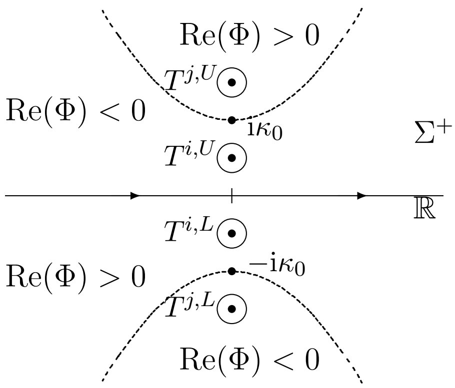

Figure 3.2: Signature table for $\Phi(k)$ in the soliton region.

3.2 Asymptotics in the soliton domain $x > 0$

Here we use the holomorphic RH-k problem with jump given by (3.1.11), (3.1.20), and (3.1.19). We consider x and t as parameters, which change in a way that the value $\xi = \frac{x}{12t}$ evolves slowly when x and t are sufficiently large in the region $\xi > 0$ under consideration. To reduce the RH-k problem to a model problem that can be solved explicitly, we will use the well-known conjugation and deformation techniques (see e.g. [40], [23]).

The signature table of the phase function $\Phi(k) = 4\mathrm{i}k^3 + 12\mathrm{i}\xi k$ in this region is shown in Figure 3.2. Namely, $\operatorname{Re}\Phi(k) = 0$ if $\operatorname{Im} k = 0$ or $(\operatorname{Im} k)^2 - 3(\operatorname{Re} k)^2 = 3\xi$. The second curve consists of two hyperbolas which cross the imaginary axis at the points $\pm\mathrm{i}\sqrt{3\xi}$. Set

$$\kappa_0 = \sqrt{\frac{x}{4t}} > 0.$$

Then we have $\operatorname{Re}(\Phi(\mathrm{i}\kappa_j)) > 0$ for all $\kappa_j > \kappa_0$ and $\operatorname{Re}(\Phi(\mathrm{i}\kappa_j)) < 0$ for all $\kappa_j < \kappa_0$. Hence, in the first case the off-diagonal entries of our jump matrices (3.1.20) are exponentially growing, and we need to turn them into exponentially decaying ones. We set

$$\Lambda(k, \xi) := \Lambda(k) = \prod_{\kappa_j > \kappa_0} \frac{k + \mathrm{i}\kappa_j}{k - \mathrm{i}\kappa_j},$$

and introduce the matrix

$$D(k) = \begin{cases} \begin{pmatrix} 1 & \left(h^U(k,j)\right)^{-1} \\ -h^U(k,j) & 0 \end{pmatrix} D_0(k), & |k - \mathrm{i}\kappa_j| < \delta, \quad j = 1, \dots, N, \\ \begin{pmatrix} 0 & h^L(k,j) \\ -\left(h^L(k,j)\right)^{-1} & 1 \end{pmatrix} D_0(k), & |k + \mathrm{i}\kappa_j| < \delta, \quad j = 1, \dots, N, \\ D_0(k), & \text{else}, \end{cases} \tag{3.2.1}$$

where

$$D_0(k) = \begin{pmatrix} \Lambda(k)^{-1} & 0 \\ 0 & \Lambda(k) \end{pmatrix}. \tag{3.2.2}$$

Observe that by $\Lambda(-k) = \Lambda^{-1}(k)$ we have

$$D(-k) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} D(k) \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}. \tag{3.2.3}$$

Now we set

$$\tilde{m}(k) = m(k) D(k). \tag{3.2.4}$$

Note that by (3.2.3) this conjugation preserves the properties III and IV.

Then (for details see Lemma 4.2 of [40]) the jump corresponding to $\kappa_0 < \kappa_j$ is given by

$$\tilde{v}(k) = \begin{pmatrix} 1 & \frac{\Lambda^2(k)}{h^U(k,j)} \\ 0 & 1 \end{pmatrix}, \qquad k \in \mathbb{T}^{j,U},$$

$$\tilde{v}(k) = \begin{pmatrix} 1 & 0 \\ -\frac{1}{h^L(k,j)\Lambda^2(k)} & 1 \end{pmatrix}, \qquad k \in \mathbb{T}^{j,L},$$

and the jumps corresponding to $\kappa_0 > \kappa_j$ (if any) by

$$\tilde{v}(k) = \begin{pmatrix} 1 & 0 \\ h^U(k,j)\Lambda^{-2}(k) & 1 \end{pmatrix}, \qquad k \in \mathbb{T}^{j,U},$$

$$\tilde{v}(k) = \begin{pmatrix} 1 & h^L(k,j)\Lambda^2(k) \\ 0 & 1 \end{pmatrix}, \qquad k \in \mathbb{T}^{j,L}.$$

In particular, all jumps corresponding to poles, except for possibly one if $\kappa_j = \kappa_0$, are exponentially close to the identity for $t \to \infty$. In the latter case we will keep the pole condition for $\kappa_j = \kappa_0$ which now reads

$$\operatorname{Res}_{\mathrm{i}\kappa_j} \tilde{m}(k) = \lim_{k \to \mathrm{i}\kappa_j} \tilde{m}(k) \begin{pmatrix} 0 & 0 \\ \mathrm{i}\gamma_j \mathrm{e}^{2t\Phi(\mathrm{i}\kappa_j)} \Lambda(\mathrm{i}\kappa_j)^{-2} & 0 \end{pmatrix},$$

$$\operatorname{Res}_{-\mathrm{i}\kappa_j} \tilde{m}(k) = \lim_{k \to -\mathrm{i}\kappa_j} \tilde{m}(k) \begin{pmatrix} 0 & -\mathrm{i}\gamma_j \mathrm{e}^{2t\Phi(\mathrm{i}\kappa_j)} \Lambda(\mathrm{i}\kappa_j)^{-2} \\ 0 & 0 \end{pmatrix}.$$

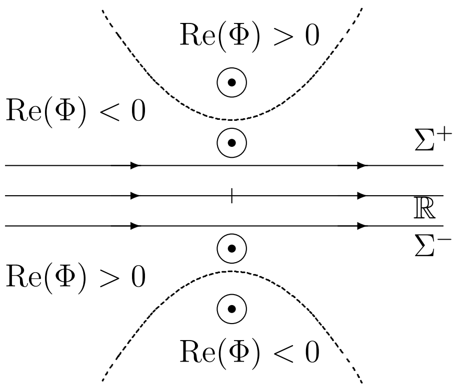

Figure 3.3: Contour deformation in the soliton region.

Furthermore, the jump along $\mathbb{R}$ now reads

$$\tilde{v}(k) = \begin{pmatrix} 1-|R(k)|^2 & -\Lambda^2(k)\overline{R(k)}\mathrm{e}^{-2t\Phi(k)} \\ \Lambda^{-2}(k)R(k)\mathrm{e}^{2t\Phi(k)} & 1 \end{pmatrix}, \qquad k\in\mathbb{R}.$$

The new Riemann–Hilbert problem

$$\tilde{m}_+(k) = \tilde{m}_-(k)\tilde{v}(k)$$

for the vector $\tilde{m}$ preserves its asymptotics (3.1.13) as well as the symmetry condition (3.1.10). In particular, after conjugation all jumps corresponding to poles are now exponentially close to the identity as $t\to\infty$. To turn the remaining jump along $\mathbb{R}$ into this form as well, we chose two contours $\Sigma^\pm = \mathbb{R}\pm\mathrm{i}\epsilon/2$, where $\epsilon = \min\{\varepsilon, \kappa_1-\delta\}$ and ε is from (3.0.4). This condition guarantees that the reflection coefficient can be continued analytically to the domain $0<\operatorname{Im} k<\epsilon$ and that Σ^+ does not intersect with $\mathbb{T}^{1,U}$. Since by definition $\overline{R(k)} = R(-k)$, the function $\overline{R}$ is analytic in the domain $-\epsilon < \operatorname{Im} k < 0$ and thus up to Σ^-.

Now we factorize the jump matrix along $\mathbb{R}$ according to

$$\tilde{v} = b_L^{-1}b_U = \begin{pmatrix} 1 & -\Lambda^2(k)\overline{R(k)}\mathrm{e}^{-2t\Phi(k)} \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ \Lambda^{-2}(k)R(k)\mathrm{e}^{2t\Phi(k)} & 1 \end{pmatrix}$$

and set

$$\hat{m}(k) = \begin{cases} \tilde{m}(k)b_U^{-1}(k), & 0<\operatorname{Im} k<\epsilon/2, \\ \tilde{m}(k)b_L^{-1}(k), & -\epsilon/2<\operatorname{Im} k<0, \\ \tilde{m}(k), & \text{else}, \end{cases}$$

such that the jump along $\mathbb{R}$ is moved 3.3 to $\Sigma^+ \cup \Sigma^-$ and is given by

$$\hat{v}(k) = \begin{cases} \begin{pmatrix} 1 & 0 \\ \Lambda^{-2}(k) R(k) \mathrm{e}^{2t\Phi(k)} & 1 \end{pmatrix}, & k \in \Sigma^+, \\ \\ \begin{pmatrix} 1 & -\Lambda^2(k)\overline{R(k)}\mathrm{e}^{-2t\Phi(k)} \\ 0 & 1 \end{pmatrix}, & k \in \Sigma^-. \end{cases}$$

Hence, all jumps $\hat{v}$ are exponentially close to the identity as $t \to \infty$ and one can use Theorem A.6 from [54] or Theorem B.6 [55] to obtain (repeating literally the proof of Theorem 4.4 in [40]) the following result:

Theorem 3.2.1. *Assume* (3.0.4)–(3.0.3) *and abbreviate by $c_j = 4\kappa_j^2$ the velocity of the j'th soliton determined by* $\mathrm{Re}(\Phi(\mathrm{i}\kappa_j)) = 0$. *Then the asymptotics in the soliton region, $x/t \geq \epsilon$ for some small $\epsilon > 0$, are as follows:*

Let $\delta > 0$ be sufficiently small such that the intervals $[c_j - \delta, c_j + \delta]$, $1 \leq j \leq N$, are disjoint and $c_1 - \delta > 0$.

If $|\frac{x}{t} - c_j| < \delta$ for some j, one has

$$q(x,t) = \frac{-4\kappa_j \gamma_j(x,t)}{(1 + (2\kappa_j)^{-1}\gamma_j(x,t))^2} + O(t^{-l})$$

for any $l \in \mathbb{N}$, where

$$\gamma_j(x,t) = \gamma_j \mathrm{e}^{-2\kappa_j x + 8\kappa_j^3 t} \prod_{i=j+1}^{N} \left(\frac{\kappa_i - \kappa_j}{\kappa_i + \kappa_j} \right)^2.$$

If $|\frac{x}{t} - c_j| \geq \delta$, for all j, one has

$$q(x,t) = O(t^{-l})$$

for any $l \in \mathbb{N}$.

3.3 Asymptotics in the domain $-6c^2t < x < 0$

In this section we work with RH-λ problem. We suppose that the circles $\mathbb{T}^{j,U}$, which are the preimages of the curves $\mathbb{T}^{j,U}_{\{\lambda\}}$, have radii δ satisfying the inequalities

$$(\kappa_1 - \delta)^3 > 3\delta \left((\kappa_N + \delta)^2 + \frac{c^2}{2} \right) \tag{3.3.1}$$

and $\delta < \kappa_1 - \varepsilon$, where ε is from (1.1.3).

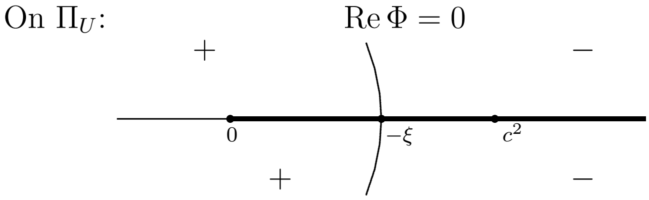

Figure 3.4: Signature table of $\mathrm{Re}\,\Phi(p)$ for $\xi < 0$

We start by collecting some properties of the phase function $\Phi(p) = (4p + 12\xi)\sqrt{-p}$ for $-\frac{c^2}{2} < \xi < 0$. First let $p = (\lambda, +)$, $\lambda \in \operatorname{clos} D$. The curves which describe the set $\mathrm{Re}\,\Phi(p) = 0$ consist of the contour Σ and the hyperbola $(\mathrm{Im}\, k)^2 - 3(\mathrm{Re}\, k)^2 = 3\xi$. This curve intersect the contour Σ at the point $-\xi$. For the signature table of $\Phi(p)$ on the upper sheet see Figure 3.4. The signature table on the lower sheet follows from symmetry $\Phi(p^*) = -\Phi(p)$. Since now all the contours $\mathbb{T}^{j,U}_{\{\lambda\}}$ are situated in the domain $\mathrm{Re}\,\Phi > 0$, i.e. the functions (3.1.19) grow exponentially with respect to t, then our first step consists in the transformations, which are the direct analogues on $\mathbb{M}$ of (3.2.4), (3.2.1), and (3.2.2) with

$$\Lambda(p) = \prod_{j=1}^{N} \frac{\sqrt{-\lambda} + \sqrt{-\lambda_j}}{\sqrt{-\lambda} - \sqrt{-\lambda_j}}, \quad p = (\lambda, +), \quad \Lambda(p^*) = \Lambda(p)^{-1}. \tag{3.3.2}$$

STEP 1. Denote by

$$\tilde{h}^U(p, j) = \gamma_j^{-1}\Lambda^2(p)(\sqrt{-p} - \sqrt{-p_j})\mathrm{e}^{-2t\Phi(p_j)},$$

$$\tilde{h}^L(p, j) = \gamma_j^{-1}\Lambda^{-2}(p)(\sqrt{-p} + \sqrt{-p_j})\mathrm{e}^{-2t\Phi(p_j)},$$

where $p_j = (\lambda_j, +)$. Performing the same transformation as in (3.2.4), (3.2.1) we get the following RH-λ problem for the function $\tilde{m} = m^1$, obtained as above: $m^1(p)$ is a holomorphic function in $\mathbb{M} \setminus (\Sigma \cup \cup_{j=1}^N (\mathbb{T}^{j,U}_{\{\lambda\}} \cup \mathbb{T}^{j,L}_{\{\lambda\}}))$, which satisfies the jump condition $m^1_+(p) = m^1_-(p) v^1(p)$, with

$$v^1(p) = \begin{cases} \begin{pmatrix} 1 - |R(p)|^2 & -\overline{R(p)}\Lambda^2(p)\mathrm{e}^{-2t\Phi(p)} \\ \Lambda^{-2}(p)R(p)\mathrm{e}^{2t\Phi(p)} & 1 \end{pmatrix}, & p \in \Sigma, \\ \begin{pmatrix} 1 & \tilde{h}^U(p, j) \\ 0 & 1 \end{pmatrix}, & p \in \mathbb{T}^{j,U}_{\{\lambda\}}, \\ \begin{pmatrix} 1 & 0 \\ -\tilde{h}^L(p, j) & 1 \end{pmatrix}, & p \in \mathbb{T}^{j,L}_{\{\lambda\}}, \end{cases} \tag{3.3.3}$$

as well as conditions (3.1.21) and (3.1.22).
STEP 2. Introduce the function

$$g(p) := g(p, x, t) = 4(\lambda + 2\xi)\sqrt{-\lambda - 2\xi}, \quad p = (\lambda, +), \quad g(p^*) = -g(p). \tag{3.3.4}$$

We observe that this function is real-valued for $\lambda < -2\xi$, and has equal limiting values from the upper sheet to symmetric points of $\Sigma_1^u(\xi) = [0, -2\xi] + \mathrm{i}0$ and $\Sigma_1^\ell(\xi) = [0, -2\xi] - \mathrm{i}0$. Here we choose the same orientation as on the contour Σ. Hence the function g has a jump from Π^U to Π^L in this part of the contour, namely

$$g_+(p) + g_-(p) = 0, \quad p \in \Sigma_1(\xi). \tag{3.3.5}$$

We also observe that for $k^2 = p \in \Pi^U$, $k \to \infty$,

$$\Phi(p) - g(p) = 4\mathrm{i}\left(k^3 + 3\xi k - (k^3 + 2\xi k)\sqrt{1 + \frac{2\xi}{k^2}}\right) = \frac{12\xi^2}{2\mathrm{i}k}(1 + O(k^{-1})). \tag{3.3.6}$$

The signature table for g is given in Figure 3.5.

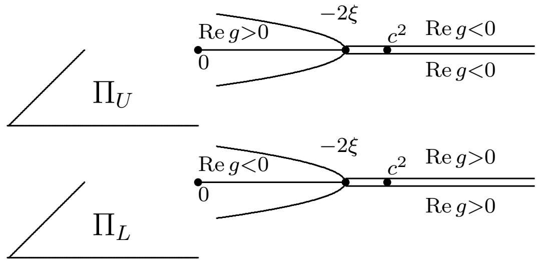

Figure 3.5: Signature table for the function $g(p)$.

Set

$$d(p) := \mathrm{e}^{t(\Phi(p) - g(p))}, \quad D(p) = \begin{pmatrix} d(p)^{-1} & 0 \\ 0 & d(p) \end{pmatrix}, \quad m^2(p) = m^1(p)D(p). \tag{3.3.7}$$

Due to the oddness of $\Phi(p)$ and $g(p)$ we have $d(p^*) = d^{-1}(p)$. From (3.3.6) it follows that $d(p) \to 1$ as $p \to \infty$. Therefore one can apply Lemma 3.1.6. In doing so note that the function $d(p)$ has no jump on $\Sigma_2(\xi) = \Sigma \setminus \Sigma_1(\xi)$, that it has a jump on $\Sigma_1(\xi)$ because of the jump of g and that using (3.3.5) we have $d_+ = d_-^{-1}$ on $\Sigma_1(\xi)$. Thus, the jump matrix for $m_+^2 = m_-^2 v^2$ is

$$v^2(p) = \begin{cases} \begin{pmatrix} 0 & -\overline{R(p)}\Lambda^2(p) \\ R(p)\Lambda^{-2}(p) & \mathrm{e}^{-2tg_+(p)} \end{pmatrix}, & p \in \Sigma_1(\xi), \\ \begin{pmatrix} 1-|R(p)|^2 & -\overline{R(p)}\Lambda^2(p)\mathrm{e}^{-2tg(p)} \\ R(p)\Lambda^{-2}(p)\mathrm{e}^{2tg(p)} & 1 \end{pmatrix}, & p \in \Sigma_2(\xi), \\ \begin{pmatrix} 1 & \tilde{h}^U(p,j)\mathrm{e}^{2t(\Phi(p)-g(p))} \\ 0 & 1 \end{pmatrix}, & p \in \mathbb{T}^{j,U}_{\{\lambda\}}, \\ \begin{pmatrix} 1 & 0 \\ -\tilde{h}^L(p,j)\mathrm{e}^{-2t(\Phi(p)-g(p))} & 1 \end{pmatrix}, & p \in \mathbb{T}^{j,L}_{\{\lambda\}}, \end{cases}$$

One has to check that this transformation does not chance the exponential decay of the off diagonal entries of the jump matrix on $\mathbb{T}^{j,U}_{\{\lambda\}}$. To this end have to estimate the value of $\mathrm{e}^{2t(\Phi(k)-g(k)-\Phi(\mathrm{i}\kappa_j))}$ on the circles $\mathbb{T}^{j,U}$ in the k variable plane. It suffices to check that for sufficiently small δ we have $\mathrm{Re}(\Phi(k)-g(k)-\Phi(\mathrm{i}\kappa_j)) < 0$ as $|k-\mathrm{i}\kappa_j| = \delta$. The rough estimates

$$|\Phi(k)-\Phi(\mathrm{i}\kappa_j)| \le 12\left((\kappa_N+\delta)^2+|\xi|\right)\delta \le 12\delta\left((\kappa_N+\delta)^2+\frac{c^2}{2}\right),$$

and $\mathrm{Re}\, g(k) \ge 4(\kappa_1-\delta)^3$, show that it is sufficient to choose δ for Step 1 satisfying (3.3.1). Denote by

$$\mathbb{T}_\delta = \cup_{j=1}^N \left(\mathbb{T}^{j,U}_{\{\lambda\}} \cup \mathbb{T}^{j,L}_{\{\lambda\}}\right)$$

and let $\mathbb{I}$ be the unit matrix. We observe that the matrix v^2 admits the following representation on $\mathbb{T}_\delta$:

$$v^2(p,x,t) = \mathbb{I} + A(p,\xi,t), \ \|A(p,\xi,t)\| \le C_1(\delta)\mathrm{e}^{-C(\delta)t}, \quad C(\delta), C_1(\delta) > 0, \tag{3.3.8}$$

where the estimate for A is uniform with respect to $p \in \mathbb{T}_\delta$ and $\xi \in [0, -\frac{c^2}{2}]$.

Next, the function $\Lambda(p)$ possesses the property

$$\overline{\Lambda(p)} = \Lambda^{-1}(p), \quad |\Lambda(p)| = 1, \quad p \in \Sigma.$$

Abbreviating $\mathcal{R}(p) := R(p)\Lambda^{-2}(p)$ we finally represent v^2 as

$$v^2(p) = \begin{cases} \begin{pmatrix} 0 & -\overline{\mathcal{R}(p)} \\ \mathcal{R}(p) & \mathrm{e}^{-2tg_+(p)} \end{pmatrix}, & p \in \Sigma_1(\xi), \\ \begin{pmatrix} 1-|\mathcal{R}(p)|^2 & -\overline{\mathcal{R}(p)}\mathrm{e}^{-2tg(p)} \\ \mathcal{R}(p)\mathrm{e}^{2tg(p)} & 1 \end{pmatrix}, & p \in \Sigma_2(\xi), \\ \mathbb{I} + A(p,\xi,t), & p \in \mathbb{T}_\delta. \end{cases} \tag{3.3.9}$$

STEP 3. It is convenient to treat this jump problem as an RH problem on the Riemann surface $\mathbb{M}(\xi)$ associated with the function $\sqrt{-\lambda - 2\xi}$ (see figure 3.6). The sheets $\Pi^U(\xi)$ and $\Pi^L(\xi)$ are glued along the contour $\Sigma_2(\xi)$. To take into account the influence of the remaining part of the jump contour we introduce two contours $I^U(\xi)$ and $I^L(\xi)$ on the upper and lower sheets, respectively. These contours project onto the interval $[0, -2\xi]$. The upper contour is oriented from 0 to -2ξ and the lower one is orientated from -2ξ to 0. Set $m^3(\hat{p}) = m^2(p)$ for $\hat{p} \in \mathbb{M}(\xi) \setminus \big((I^U(\xi) \cup I^L(\xi) \cup \Sigma_2(\xi) \cup \mathbb{T}_\delta\big)$. Here

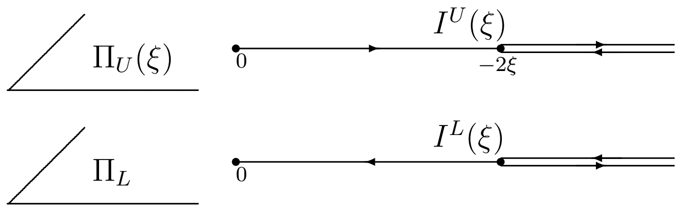

Figure 3.6: Riemann surface $\mathbb{M}(\xi)$ with cuts

the points $\hat{p} \in \mathbb{M}(\xi)$ and $p \in \mathbb{M}$ have the same projections λ, and have to be distinguished only along the intervals $I^U(\xi)$ and $I^L(\xi)$. We will use the same symbol p for both surfaces for notational simplicity. Obviously the jumps of m^3 along $\Sigma_2(\xi)$ and $\mathbb{T}_\delta$ remain the same as for m^2. To compute the jumps along $I^U(\xi)$ and $I^L(\xi)$ we identify the $+$ side of $\Sigma_1^u(\xi)$ with the $+$ side of $I^U(\xi)$ and the $-$ side of $I^L(\xi)$ with the $-$ side of $\Sigma_1^\ell(\xi)$. Thus $m_+^3(p) = m_+^2(p)$ for $p \in I^U(\xi)$ and by the symmetry condition we have $m_-^3(p) = m_-^2(p^*)\sigma_1$, where $\sigma_1 = \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$ is the first Pauli matrix.

Since the function g, considered as a function on $\mathbb{M}(\xi)$, has no jump along $I^U(\xi)$, we have $g_+(p) = g(p)$ on $I^U(\xi)$. On the other hand, the function $\mathcal{R}(p)$ was continuous in a vicinity of Σ_1^u on the Riemann surface $\mathbb{M}$, but it has a jump along $I(\xi)$ on the Riemann surface $\mathbb{M}(\xi)$. Set

$$\tilde{\mathcal{R}}(p) := \mathcal{R}((\lambda + \mathrm{i}0, +)), \quad p = (\lambda, +) \in I^U(\xi); \quad \tilde{\mathcal{R}}(p^*) := \overline{\tilde{\mathcal{R}}(p)}. \tag{3.3.10}$$

Note, that this function has the values of $\mathcal{R}$ from the jump matrix v^2 along $\Sigma_1^u(\xi)$. Thus

$$v^3(p) = \begin{pmatrix} \tilde{\mathcal{R}}(p) & \mathrm{e}^{-2tg(p)} \\ 0 & -\overline{\tilde{\mathcal{R}}(p)} \end{pmatrix}, \quad p \in I^U(\xi). \tag{3.3.11}$$

Now, by the symmetry condition, we have $m_\pm^3(p) = m_\mp^3(p^*)\sigma_1$. Therefore, for $p \in I^L(\xi)$

$$m_+^3(p) = m_-^3(p^*)\sigma_1 = m_+^3(p^*)\left(v^3(p^*)\right)^{-1}\sigma_1 = m_-^3(p)\sigma_1\left(v^3(p^*)\right)^{-1}\sigma_1.$$

Thus

$$v^3(p) = \sigma_1 \left(v^3(p^*)\right)^{-1} \sigma_1 = \begin{pmatrix} -\tilde{\mathcal{R}}(p^*) & 0 \\ \mathrm{e}^{-2t(g(p^*))} & \frac{}{\tilde{\mathcal{R}}(p^*)} \end{pmatrix}, \quad p \in I^L(\xi).$$

Recall that by (3.3.10) we have $\tilde{\mathcal{R}}(p^*) = \overline{\tilde{\mathcal{R}}(p)}$ on $I^U(\xi) \cup I^L(\xi)$ and that g is an odd function, $g(p^*) = -g(p)$. Therefore

$$v^3(p) = \begin{pmatrix} -\overline{\tilde{\mathcal{R}}(p)} & 0 \\ \mathrm{e}^{2t(g(p))} & \tilde{\mathcal{R}}(p) \end{pmatrix}, \quad p \in I^L(\xi). \tag{3.3.12}$$

Note, that to get this jump we could also be obtained as (3.3.11), that is by comparing the limiting values of m^2 and m^3. We will get the same result because on the contour $\Sigma_1^\ell(\xi)$ the entries of v^2 are connected with $\tilde{\mathcal{R}}(p)$ by the relation $\tilde{\mathcal{R}}(p) = \overline{\mathcal{R}(p)}$, where $p = (\lambda - \mathrm{i}0, +) \in I^U(\xi)$.

Thus the new RH problem is equivalent to the previous one and consists of finding a holomorphic function on $\mathbb{M}(\xi) \setminus \left(I^U(\xi) \cup I^L(\xi) \cup \Sigma_2(\xi) \cup \mathbb{T}_\delta\right)$, which satisfies the jump condition $m_+^3(p) = m_-^3(p)v^3(p)$, where $v^3(p) = v^2(p)$ for $p \in \Sigma_2(\xi) \cup \mathbb{T}_\delta$, and $v^3(p)$ is defined by formulas (3.3.11), (3.3.12), and (3.3.10) for $p \in I^U(\xi) \cup I^L(\xi)$. The vector function m^3 also satisfy conditions II–III of Lemma 3.1.5, and therefore this RH problem also has a unique solution.

STEP 4. Now we perform the upper-lower factorization of the jump matrix v^3 on $\Sigma_2(\xi)$:

$$v^3(p) = B^L(B^U)^{-1}, \quad p \in \Sigma_2(\xi),$$

with

$$B^L(p) = \begin{pmatrix} 1 & -\overline{\mathcal{R}}(p)\mathrm{e}^{-2tg(p)} \\ 0 & 1 \end{pmatrix}, \quad (B^U)^{-1}(p) = \begin{pmatrix} 1 & 0 \\ \mathcal{R}(p)\mathrm{e}^{2tg(p)} & 1 \end{pmatrix}. \tag{3.3.13}$$

Introduce two contours $\mathcal{C}^U$ and $\mathcal{C}^L$ surrounding the contour $\Sigma_2(\xi)$ (remaining

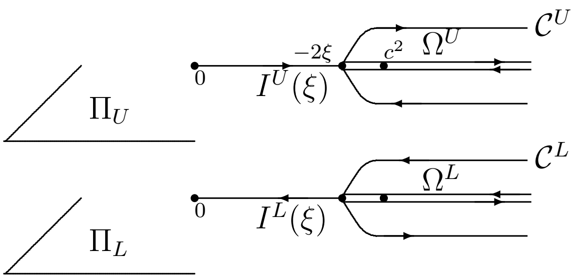

Figure 3.7: Riemann surface $\mathbb{M}(\xi)$, contour deformations

close to it), and redefine m^3 in the enclosed domains Ω^U and Ω^L (see Figure 3.7) by

$$m^4(p) = \begin{cases} m^3(p)B^U(p), & p \in \Omega^U, \\ m^3(p)B^L(p), & p \in \Omega^L, \\ m^3(p), & \text{else.} \end{cases}$$

This leads to an equivalent RH problem with the jump matrix

$$v^4(p) = \begin{cases} (B^U(p))^{-1}, & p \in \mathcal{C}^U, \\ B^L(p), & p \in \mathcal{C}^L, \\ v^3(p), & p \in I^U(\xi) \cup I^L(\xi) \cup \mathbb{T}_\delta, \end{cases}$$

where the oscillatory jump along the contour $\Sigma_2(\xi)$ disappeared and the jump matrices along the contours $\mathcal{C}^U$ and $\mathcal{C}^L$ are close to the identity matrix with the exponentially small errors except for a small vicinity of the point -2ξ.
STEP 5. This step will allow us to simplify jump the matrices on $I(\xi)$ and $I(\xi)^*$ by conjugating them with diagonal matrices. To this end consider the following scalar conjugation problem:
Find a holomorphic function $d(p)$ on $\mathbb{M}(\xi) \setminus I^U(\xi) \cup I^L(\xi)$ satisfying the jump condition

$$d_+(p) = \mathrm{i}\, d_-(p)\tilde{\mathcal{R}}(p) \quad \text{as } p \in I^U(\xi), \tag{3.3.14}$$

the symmetry condition $d(p^*) = d^{-1}(p)$, and the normalization $d(p) \to 1$ as $p \to \infty$.

Note that by definition $|\tilde{\mathcal{R}}(p)| = |R(p)\Lambda(p)| = 1$ for $p \in \Sigma_1(\xi)$. Therefore, $\overline{\tilde{\mathcal{R}}(p)} = \tilde{\mathcal{R}}^{-1}(p)$ which, together with our symmetry $d_\pm(p^*) = d_\mp(p)^{-1}$ and (3.3.10), implies

$$d_+(p) = d_-^{-1}(p^*) = \mathrm{i}\tilde{\mathcal{R}}(p^*)d_+^{-1}(p^*) = \mathrm{i}\overline{\tilde{\mathcal{R}}(p)}d_-(p), \quad p \in I^L(\xi). \tag{3.3.15}$$

Lemma 3.3.1. *The solution of this scalar RH problem is given by*

$$d(p) = \exp\left(-\frac{1}{2\pi}\int_{I^U(\xi)} \frac{(\arg T(\lambda) + \arg T_1(\lambda) - 2\arg\Lambda(\lambda) + \pi)\sqrt{-p-2\xi}\, d\lambda}{(\lambda - \pi(p))\sqrt{-\lambda - 2\xi}}\right) \tag{3.3.16}$$

with Λ from (3.3.2).

Proof. Formula (3.1.4) implies that on the spectrum of multiplicity one we have $\log R = \mathrm{i}\arg R = 2\mathrm{i}\arg T$. Hence (3.3.14) is equivalent to

$$\begin{aligned}\log d_+ - \log d_- &= \log(\mathrm{i}) + \log \mathcal{R} = \mathrm{i}\left(\frac{\pi}{2} + \arg R - 2\arg\Lambda\right) \\ &= \mathrm{i}\left(\frac{\pi}{2} + 2\arg T - 2\arg\Lambda\right) = \mathrm{i}\big(\arg T + \arg T_1 - 2\arg\Lambda + \pi\big),\end{aligned}$$

since $\arg T_1 + \frac{\pi}{2} = \arg T$ on I_1 according to $T = T_1 \dfrac{\sqrt{-\lambda}}{\sqrt{c^2-\lambda}}$. The rest follows from the Sokhotski–Plemelj formula. Note that the representation (3.3.16) satisfies the symmetry condition $d(p^*) = d^{-1}(p)$ since $\sqrt{-p^* - 2\xi} = -\sqrt{-p-2\xi}$. The normalization condition is evident. □

Now consider the matrix $D(p)$ constructed from (3.3.16) as in (3.1.23) and set $m^5(p) = m^4(p)D(p)$. Applying(3.1.25) and taking into account (3.3.14), (3.3.15), (3.3.11), (3.3.12), and (3.3.13) we get the jump problem $m^5_+(p) = m^5_-(p)v^5(p)$ with

$$v^5(p) = \begin{cases} \begin{pmatrix} -\mathrm{i} & d_+(p)d_-(p)\mathrm{e}^{-2tg(p)} \\ 0 & -\mathrm{i} \end{pmatrix}, & p \in I^U(\xi), \\ \begin{pmatrix} \mathrm{i} & 0 \\ d_+^{-1}(p)d_-^{-1}(p)\mathrm{e}^{2tg(p)} & \mathrm{i} \end{pmatrix}, & p \in I^L(\xi), \\ \begin{pmatrix} 1 & 0 \\ d^2(p)\mathcal{R}(p)\mathrm{e}^{2tg(p)} & 1 \end{pmatrix}, & p \in C^U(\xi), \\ \begin{pmatrix} 1 & -d^{-2}(p)\overline{\mathcal{R}}(p)\mathrm{e}^{-2tg(p)} \\ 0 & 1 \end{pmatrix}, & p \in C^L(\xi), \\ \mathbb{I} + D^{-1}(p)A(p,\xi,t)D(p), & p \in \mathbb{T}_\delta. \end{cases}$$

Of course $m^5(p)$ also satisfies the standard symmetry and normalization conditions. $m^5(p)$ also satisfy the standard symmetry and normalization conditions. We observe that our jump matrix has the structure

$$v^5(p) = -\mathrm{i}\,\mathbb{I} + O(\mathrm{e}^{-2tg(p)}),\ p \in I^U(\xi), \quad v^5(p) = \mathbb{I} + O(\mathrm{e}^{-2tg(p)}),\ p \in \mathcal{C}^L(\xi),$$

on the contours in the domain $\operatorname{Re} g > 0$, neglecting the small contribution of the point $p = -2\xi$ where $\operatorname{Re} g = 0$;

$$v^5(p) = \mathrm{i}\,\mathbb{I} + O(\mathrm{e}^{2tg(p)}),\ p \in I^L(\xi); \quad v^5(p) = \mathbb{I} + O(\mathrm{e}^{2tg(p)}),\ p \in \mathcal{C}^U(\xi),$$

for $\operatorname{Re} g < 0$. For $p \in \mathbb{T}_\delta$ the matrix $D^{-1}AD - \mathbb{I}$ is uniformly exponentially small. Hence we may suppose, that the solution of the RH problem for m^5 can be approximated by the solution of the following model RH problem: *Find an holomorphic function $m^{mod}(p)$ in $\mathbb{M}(\xi) \setminus \big(I^U(\xi) \cup I^L(\xi)\big)$ satisfying the jump condition*

$$m^{mod}_+(p) = m^{mod}_-(p)v^{mod}(p)$$

with

$$v^{mod}(p) = \begin{cases} \begin{pmatrix} -\mathrm{i} & 0 \\ 0 & -\mathrm{i} \end{pmatrix}, & p \in I^U(\xi), \\ \begin{pmatrix} \mathrm{i} & 0 \\ 0 & \mathrm{i} \end{pmatrix}, & p \in I^L(\xi), \end{cases}$$

the symmetry condition

$$m^{mod}(p^*) = m^{mod}(p)\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad p \in \mathbb{M}(\xi),$$

and the normalization condition

$$\lim_{p\to\infty} m^{mod}(p) = (1,1).$$

To describe the solution of this model problem on $\Pi^U(\xi)$ introduce the function

$$\Delta(p) = \sqrt[4]{\frac{\pi(p)}{\pi(p)+2\xi}},$$

where $\sqrt[4]{\cdot}$ is chosen with positive values for $\lambda > -2\xi$. This function solves the jump problem $\Delta_+(p) = -\mathrm{i}\Delta_-(p)$ for $p \in I^U(\xi)$. Extend this function to $\Pi^L(\xi)$ via $\Delta(p^*) = \Delta(p)$. Since this function does not have jumps on the sets $\lambda < 0$ and $\lambda > -2\xi$ for $p = (\lambda, +)$, the function $\Delta(p)$ is holomorphic as a function on $\mathbb{M}(\xi) \setminus (I^U(\xi) \cup I^L(\xi))$. Thus the solution of the model problem can be given by $m^{mod}(p) = (\Delta(p), \Delta(p^*))$, $p \in \mathbb{M}(\xi)$. Evidently, the normalization and symmetry conditions are fulfilled.

Now we discuss how to get the asymptotics for $q(x,t)$ from this model solution assuming that $m^5(p,\xi,t) = m^{mod}(p,\xi)(1+o(1))$ with respect to large t. Since $\Delta(p) = 1 + O(p^{-1})$ as $p \to \infty$, this function does not contribute to the term of order $p^{-1/2}$. Reversing our chain of transformations we observe, that the only transformations which changed this asymptotics were the multiplication by the diagonal matrices (3.1.23), constructed from (3.3.16), (3.3.7), and (3.3.2), respectively. Therefore, $m_1(p,\xi,t) = \Lambda(p,\xi)d(p,\xi)\mathrm{e}^{t(\Phi(p,\xi)-g(p\xi))}(1+O(p^{-1})$ as $p \to \infty$. But

$$q(x,t) = -\frac{\partial}{\partial x}\lim_{p\to\infty} 2\sqrt{-p}\,(m_1(p,\xi,t)-1),$$

and $\frac{\partial \xi}{\partial x} = O(t^{-1})$. That is why the multiplier $\Lambda(p,\xi)d(p,\xi)$ does not contribute to the leading term for $q(x,t)$ too. From (3.1.9) and (3.3.6) we see that

$$-q(x,t) = t\cdot\frac{\partial}{\partial x}(12\xi^2)(1+o(1)) = \frac{x}{6t}(1+o(1)).$$

Theorem 3.3.2. *In the domain $(-6c^2+\epsilon)t < x < -\epsilon_1 t$ the following asymptotics is valid:*

$$q(x,t) = -\frac{x}{6t}(1+o(1)), \quad \text{as } t \to +\infty.$$

Note that this asymptotics matches the leading asymptotics 0 of the right background along $x = 0$ (near the leading front) as well as the leading asymptotics c^2 of the left background along $x = -6c^2t$ (near the back front).

3.4 Asymptotics in the domain $x < -6c^2 t$ via the left scattering data

It turns out that in this region a formulation of our RH problem in terms of the left scattering data will be more convenient. We will outline how to solve the original RH problem in the next section. Again we will work in the k_1 plane.

Now the discrete spectrum of $L(t)$ is located at the points $\mathrm{i}\kappa_{1,j}$, $\kappa_{1,j} = \sqrt{\kappa_j^2 + c^2}$. Denote the squares of norms of the left eigenfunctions by $\gamma_{1,j}$ and let $R_1(k_1, t)$, $T_1(k_1, t)$ be the left reflection and transmission coefficients. Abbreviate

$$\begin{aligned} \mathcal{T}(k_1) &= -\overline{T_1(k_1, 0)}T(k_1, 0), \text{ for } k_1 \in [0, \mathrm{i}c]_r, \text{ i.e. } k \in [0, c]; \\ \mathcal{T}(-k_1) &: = -\mathcal{T}(k_1). \end{aligned} \tag{3.4.1}$$

Introduce the vector-valued function

$$m(k_1, x, t) = \left(T_1(k_1, t)\phi(k_1, x, t)\mathrm{e}^{-\mathrm{i}k_1 x}, \ \phi_1(k_1, x, t)\mathrm{e}^{\mathrm{i}k_1 x}\right), \ k_1 \in \mathbb{C}^+ \setminus (0, \mathrm{i}c], \tag{3.4.2}$$

$$m(-k_1) = m(k_1) \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}. \tag{3.4.3}$$

This function has the following asymptotical behavior as $k_1 \to +\mathrm{i}\infty$.

$$m(k_1, x, t) = (1, 1) + \frac{1}{2\mathrm{i}k_1}\left(\int_{-\infty}^{x} (q(y, t) - c^2)dy\right)(1, -1) + O\left(\frac{1}{k_1^2}\right). \tag{3.4.4}$$

Theorem 3.4.1. *Let $\{R_1(k_1),\ k_1 \in \mathbb{R};\ \mathcal{T}(k_1),\ k_1 \in [0, \mathrm{i}c];\ (\kappa_{1,j}, \gamma_{1,j}),\ 1 \le j \le N\}$ be the left scattering data of the operator $L(0)$. Let $\mathbb{T}_j^U$ (resp. $\mathbb{T}_j^L$) be circles with centers in $\mathrm{i}\kappa_{1,j}$ (resp., $-\mathrm{i}\kappa_{1,j}$) and radii $0 < \varepsilon < \frac{1}{4}\min_{j=1}^N |\kappa_{1,j} - \kappa_{1,j-1}|$, $\kappa_{1,0} = 0$. Then $m(k_1) = m(k_1, x, t)$ defined in (3.4.2) is the unique solution of the following vector RH problem: Find a function $m(k_1)$ which is holomorphic away from the contour $\cup_{j=1}^N(\mathbb{T}_j^U \cup \mathbb{T}_j^L) \cup \mathbb{R} \cup [-\mathrm{i}c, \mathrm{i}c]$ and satisfies:*

(i) The jump condition $m_+(k_1) = m_-(k_1) v(k_1)$

$$v(k_1) = \begin{cases} \begin{pmatrix} 1 - |R_1(k_1)|^2 & -\overline{R_1(k_1)} \mathrm{e}^{-2t\Phi_1(k_1)} \\ R_1(k_1)\mathrm{e}^{2t\Phi_1(k_1)} & 1 \end{pmatrix}, & k_1 \in \mathbb{R}, \\ \begin{pmatrix} 1 & 0 \\ \mathcal{T}(k_1)\mathrm{e}^{2t\Phi_1(k_1)} & 1 \end{pmatrix}, & k_1 \in [\mathrm{i}c, 0], \\ \begin{pmatrix} 1 & \mathcal{T}(k_1)\mathrm{e}^{-2t\Phi_1(k_1)} \\ 0 & 1 \end{pmatrix}, & k_1 \in [0, -\mathrm{i}c], \\ \begin{pmatrix} 1 & 0 \\ -\frac{\mathrm{i}\gamma_{1,j}\mathrm{e}^{t\Phi_1(\mathrm{i}\kappa_{1,j})}}{k_1 - \mathrm{i}\kappa_{1,j}} & 1 \end{pmatrix}, & k_1 \in \mathbb{T}_j^U, \\ \begin{pmatrix} 1 & -\frac{\mathrm{i}\gamma_{1,j}\mathrm{e}^{-t\Phi_1(-\mathrm{i}\kappa_{1,j})}}{k_1 + \mathrm{i}\kappa_{1,j}} \\ 0 & 1 \end{pmatrix}, & k_1 \in \mathbb{T}_j^L, \end{cases}$$

(ii) the symmetry condition (3.4.3),

(iii) and the normalization condition $\lim_{\kappa \to \infty} m(\mathrm{i}\kappa) = (1, 1)$.

Here the phase $\Phi_1(k) = \Phi_1(k_1, x, t)$ *is given by*

$$\Phi_1(k_1) = -4\mathrm{i}k_1^3 - 6\mathrm{i}c^2 k_1 - 12\mathrm{i}\xi k_1, \quad \xi = \frac{x}{12t},$$

and the function $\mathcal{T}(k_1)$ *is defined by* (3.4.1). *The contours are oriented in the same way as in Lemma 3.1.4.*

Proof. The proof is analogous to Theorem 3.1.1 and Lemma 3.1.3 using the scattering relations (3.1.2), (3.1.3), formula (3.4.1), and oddness of the phase Φ_1. Also one has to take into account that $\phi_1(k_1, t) \in \mathbb{R}$ has no jump along $k_1 \in [0, \mathrm{i}c]$ and use the relation (cf. [24])

$$R(k,t) = -\frac{T_1(k,t)}{\overline{T_1(k,t)}}, \text{ as } k \in [-c, c] \text{ (that is } k_1 \in [0, \mathrm{i}c]_r \text{ and } k_1 \in [0, \mathrm{i}c]_l),$$

as well as the time evolution (cf. [25])

$$\overline{T_1(k_1,t)} T(k_1,t) = \overline{T_1(k_1,0)} T(k_1,0) \mathrm{e}^{-8\mathrm{i}tk_1^3 - 12\mathrm{i}tk_1c^2}, \quad k \in [-c, c],$$

$$R_1(k_1,t) = R_1(k_1,0)\mathrm{e}^{-8\mathrm{i}tk_1^2 - 12\mathrm{i}tk_1c^2}, \ k_1 \in \mathbb{R},$$

$$\gamma_{1,j}(t) = \gamma_{1,j}(0)\mathrm{e}^{-8\kappa_{1,j}^3 t + 12c^2\kappa_{1,j}t}. \qquad \square$$

Let $k_1^\pm = \pm\sqrt{-\frac{c^2}{2} - \xi}$ be the stationary phase points of Φ_1. The signature table for $\operatorname{Re}\Phi_1$ in the present domain $\xi < -\frac{c^2}{2}$ is shown in Figure 3.8. Therefore the jump matrix $v(k_1)$ is exponentially close to the identity matrix as $t \to \infty$ except for $k_1 \in \mathbb{R}$. Now, following the usual procedure [22], [40], we

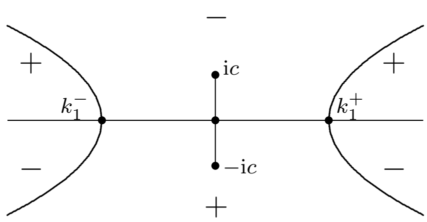

Figure 3.8: Sign of $\operatorname{Re}(\Phi_1(k_1))$

let $d(k_1)$ be an analytic function in the domain $\mathbb{C} \setminus \big(\mathbb{R} \setminus [k_1^-, k_1^+]\big)$ satisfying

$\tilde d_+(k_1) = \tilde d_-(k_1)(1-|R_1(k_1)|^2)$ for $k_1 \in \mathbb{R}\setminus[k_1^-, k_1^+]$ and $d(k_1) \to 1,\ k_1 \to \infty.$

Then by the Sokhotski–Plemelj formulas

$$\tilde d(k_1) = \exp\left(\frac{1}{2\pi\mathrm{i}} \int_{(-\infty,k_1^-)\cup(k_1^+,\infty)} \frac{\log(1-|R_1(s)|^2)}{s-k_1} ds\right). \tag{3.4.5}$$

Note that this integral is well defined since $R_1(k_1) = O(k_1^{-1})$ and $|R_1(k_1)| < 1$ for $k_1 \neq 0$ (cf. [24]). As the domain of integration is even and the function $\log(1-|R_1|^2)$ is also even, we obtain $\tilde d(-k_1) = \tilde d^{-1}(k_1)$ and the matrix

$$D(k_1) = \begin{pmatrix} \tilde d^{-1}(k_1) & 0 \\ 0 & \tilde d(k_1) \end{pmatrix}$$

satisfies the symmetry conditions of Lemma 3.1.6. Now set $\tilde m(k_1) = m(k_1)D(k_1)$ and the new RH problem will read $\tilde m_+(k_1) = \tilde m_-(k_1)\tilde v(k_1)$, where $\tilde m(k_1) \to (1,1)$ as $k \to \infty$, $\tilde m(-k) = \tilde m(k)\left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$, and

$$\tilde v(k) = \begin{cases} A_L(k_1)A_U(k_1), & k_1 \in \mathbb{R}\setminus[k_1^-, k_1^+] \\[2ex] B_L(k_1)B_U(k_1), & k_1 \in [k_1^-, k_1^+] \\[2ex] D^{-1}(k_1)v(k_1)D(k_1), & k_1 \in [\mathrm{i}c, -\mathrm{i}c] \cup_j (\mathbb{T}_j^U \cup \mathbb{T}_j^L), \end{cases}$$

where

$$A_L(k_1) = \begin{pmatrix} 1 & 0 \\ \frac{R_1(k_1)\mathrm{e}^{t\Phi_1(k_1)}}{(1-|R_1(k_1)|^2)\tilde d^2(k_1)} & 1 \end{pmatrix}, \quad k \in \Omega_l^U \cup \Omega_r^L,$$

$$A_U(k_1) = \begin{pmatrix} 1 & -\frac{\tilde{d}^2(k_1)\overline{R_1(k_1)}\mathrm{e}^{-t\Phi_1(k_1)}}{(1-|R_1(k_1)|^2)} \\ 0 & 1 \end{pmatrix}, \quad k_1 \in \Omega_l^L \cup \Omega_r^U,$$

$$B_L(k_1) = \begin{pmatrix} 1 & -\tilde{d}^2(k_1)\overline{R_1(k_1)}\mathrm{e}^{-t\Phi_1(k_1)} \\ 0 & 1 \end{pmatrix}, \quad k_1 \in \Omega_c^L,$$

$$B_U(k_1) = \begin{pmatrix} 1 & 0 \\ \tilde{d}^{-2}(k_1)R_1(k_1)\mathrm{e}^{t\Phi_1(k_1)} & 1 \end{pmatrix}, \quad k_1 \in \Omega_c^U.$$

Here the domains Ω_l^L, Ω_l^U, Ω_r^L, Ω_r^U, Ω_c^L and Ω_c^U together with their boundaries $\mathcal{C}_l^L$, $\mathcal{C}_l^U$, $\mathcal{C}_r^L$, $\mathcal{C}_r^U$, $\mathcal{C}_c^L$ and $\mathcal{C}_c^U$ are shown in Figure 3.9. Evidently, the matrix B_U

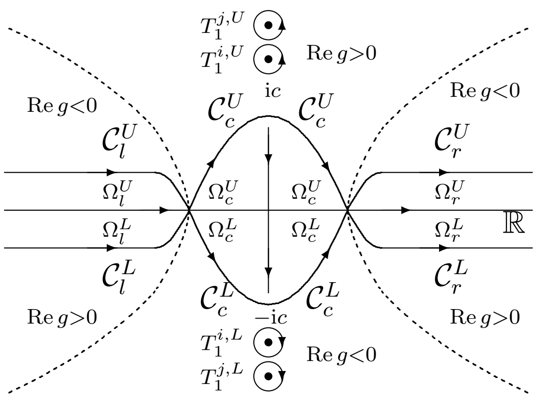

Figure 3.9: Contour deformation in the domain $x < -6c^2 t$

(resp. B_L) has a jump along the contour $[\mathrm{i}c, 0]$ (resp. $[0, -\mathrm{i}c]$). All contours are oriented from left to right. They are chosen to respect the symmetry $k_1 \mapsto -k_1$ and are inside a set, where $R_1(k_1)$ has an analytic continuation. We also set $\overline{R_1(k_1)} = R_1(-k_1)$ in these domains.

Lemma 3.4.2. *The following formula is valid*

$$(B_U)_- \,\tilde{v}\, (B_U)_+^{-1} = \mathbb{I}, \quad k_1 \in [\mathrm{i}c, 0]; \quad (B_L)_-^{-1} \,\tilde{v}\, (B_L)_+ = \mathbb{I}, \quad k_1 \in [0, -\mathrm{i}c].$$

Proof. By virtue of the Plücker identity (cf. [66], [23]). □

Now redefine $\tilde{m}(k_1)$ according to

$$\hat{m}(k_1) = \begin{cases} \tilde{m}(k_1)A_L(k_1), & k_1 \in \Omega_l^L \cup \Omega_r^L, \\ \tilde{m}(k_1)A_U(k_1)^{-1}, & k_1 \in \Omega_l^U \cup \Omega_r^U, \\ \tilde{m}(k_1)B_L(k_1), & k_1 \in \Omega_c^L, \\ \tilde{m}(k_1)B_U(k_1)^{-1}, & k_1 \in \Omega_c^U, \\ \tilde{m}(k_1), & \text{else.} \end{cases}$$

Then the vector function $\hat{m}(k_1)$ has no jump for $k_1 \in \mathbb{R}$ and (by Lemma 3.4.2) also for $k_1 \in [\mathrm{i}c, -\mathrm{i}c]$. All remaining jumps on the contours $\mathcal{C}_l^L$, $\mathcal{C}_l^U$, $\mathcal{C}_c^L$, $\mathcal{C}_c^U$, $\mathcal{C}_r^L$, $\mathcal{C}_r^U$, and $\cup_{j=1}^N(\mathbb{T}_j^U \cup \mathbb{T}_j^L)$ are close to identity matrix up to exponentially small errors except for small vicinities of the stationary phase points k_1^- and k_1^+. Thus, the model problem has the trivial solution $\hat{m}(k_1) = (1,1)$. For large imaginary k_1 with $|k_1| > \kappa_{1,N} + 1$ we have $\tilde{m}(k_1) = \hat{m}(k_1)$ and consequently

$$m(k_1) = \tilde{m}(k_1) D^{-1}(k_1) = (\tilde{d}(k_1), \tilde{d}^{-1}(k_1))$$

for sufficiently large k_1. By (3.4.5)

$$d(k_1) = 1 + \frac{1}{2\mathrm{i}k_1}\left(-\frac{1}{\pi}\int_{(-\infty,k_1^-)\cup(k_1^+,\infty)} \log(1-|R_1(s)|^2)ds\right) + O\left(\frac{1}{k_1^2}\right)$$

and comparing this formula with formula (3.4.4) we conclude the expected leading asymptotics in the region $x < -c^2 t$:

$$q(x,t) = c^2(1 + O(t^{-1/2})).$$

Moreover, the contribution from the small crosses at $k_1^{\pm}$ can be computed using the usual techniques [22], [40] giving our final result:

Theorem 3.4.3. *In the domain $x < (-6c^2 - \epsilon)t$ the following asymptotics is valid:*

$$q(x,t) = c^2 + \sqrt{\frac{4\nu(k_1^+)k_1^+}{3t}} \sin(16t(k_1^+)^3 - \nu(k_1^+)\log(192t(k_1^+)^3) + \delta(k_1^+)) + o(t^{-\alpha})$$

for any $1/2 < \alpha < 1$. Here $k_1^+ = \sqrt{-\frac{c^2}{2} - \xi}$ and

$$\begin{aligned}
\nu(k_1^+) =& -\frac{1}{2\pi}\log\left(1 - |R_1(k_1^+)|^2\right),\\
\delta(k_1^+) =& \frac{\pi}{4} - \arg(R_1(k_1^+)) + \arg(\Gamma(\mathrm{i}\nu(k_1^+)))\\
& - \frac{1}{\pi}\int_{(-\infty,-k_1^+)\cup(k_1^+,\infty)} \log\left(\frac{\log(1-|R_1(s)|^2)}{\log(1-|R_1(k_1^+)|^2)}\right)\frac{1}{s-k_1^+}ds.
\end{aligned}$$

3.5 Asymptotics in the domain $x < -6c^2 t$

In this section we provide an alternate approach for the region $x < -6c^2 t$. We start again from the holomorphic RH problem formulated in Lemma 3.1.5. In the region $x < -6c^2 t$ the curves separating the domains with different sign of $\operatorname{Re}\Phi(p)$ cross the real axis at the point $-\xi > c^2/2 > 0$.

STEP 1 is identical as in the rarefaction section after which we arrived at the RH problem with jump (3.3.3). We use the same notation m^1 for its solution.
STEP 2. In the rarefaction section, when the point ξ moved from 0 to $-c^2/2$, the crossing points of the lines $\operatorname{Re} g(p) = 0$ moved from 0 to c^2 where they eventually reached the lower edge of the spectrum of multiplicity two. Here the g function from Section 3.3 is not helpful more. As a new proper g function we choose the phase, which corresponds to the evolution of the left scattering data, but with the opposite sign. Recall that the left phase, considered as a function of k_1, is given by

$$\Phi_1(k_1) = -4\mathrm{i}k_1^3 - 6\mathrm{i}c^2 k_1 - 12\mathrm{i}\xi k_1,$$

with $\xi = \frac{x}{12t}$ as before. For $\xi < -c^2/2$ the hyperbola $\operatorname{Re}\Phi_1 = 0$ cross the line $\operatorname{Im} k_1 = 0$ at the points

$$k_1^{\pm} = \pm\sqrt{-\xi - c^2/2}. \tag{3.5.1}$$

Since $\lambda = k_1^2 - c^2$ the images of the lines $\operatorname{Re}\Phi_1 = 0$ cross the real line at the point $\eta = c^2/2 - \xi$. Therefore, in the domain under consideration, the point η satisfies $\eta > c^2$ and increases as $\xi \to -\infty$. Now set

$$g(p) := g(p,x,t) = (4\lambda + 2c^2 + 12\xi)\sqrt{c^2 - \lambda},\ p = (\lambda, +), \quad g(p^*) = -g(p) \tag{3.5.2}$$

and giving us signature table 3.10. First of all, we observe that

$$g(p) = 4\lambda\sqrt{-p} + 12\xi\sqrt{-p} + \frac{12\xi c^2 + c^4}{2\sqrt{-p}} + O(p^{-1}),$$

that is

$$\Phi(p) - g(p) = \frac{12\xi c^2 + c^4}{2\sqrt{-p}}(1 + o(1)).$$

Next split the contour Σ into two pieces: Σ_1 with its projection on the interval $[0, c^2]$ and $\Sigma_2 = \Sigma \setminus \Sigma_1$. On the contour Σ_1 (3.5.2) possesses the same property (3.3.5) as (3.3.4) on $\Sigma_1(\xi)$. Set $m^2(p) = m^1(p)D(p)$, where $D(p)$ is the matrix (3.1.23) constructed from $d(p) = \mathrm{e}^{t(\Phi(p)-g(p))}$. Abbreviate $\mathcal{R}(p) := R(p)\Lambda^{-2}(p)$ and apply Lemma 3.1.6 plus the same arguments as in Step 2 of Section 3.3. Then we obtain

$$v^2(p) = \begin{cases} \begin{pmatrix} 0 & -\overline{\mathcal{R}(p)} \\ \mathcal{R}(p) & \mathrm{e}^{-2tg_+(p)} \end{pmatrix}, & p \in \Sigma_1, \\ \begin{pmatrix} 1 - |\mathcal{R}(p)|^2 & -\overline{\mathcal{R}(p)}\mathrm{e}^{-2tg(p)} \\ \mathcal{R}(p)\mathrm{e}^{2tg(p)} & 1 \end{pmatrix}, & p \in \Sigma_2, \\ \mathbb{I} + A(p,\xi,t), & p \in \mathbb{T}_\delta. \end{cases}$$

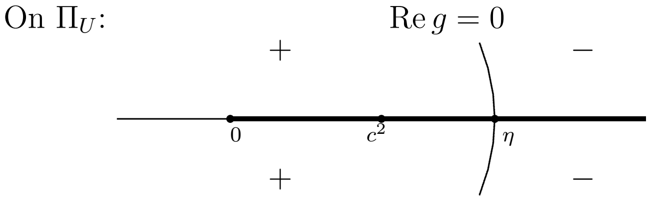

Figure 3.10: Signature table of $\mathrm{Re}\, g(p)$ for $\xi < -c^2/2$

Here the matrix $A(p, \xi, t)$ has the same form as in (3.3.9) and its norm will satisfy the estimate (3.3.8) provided that we choose a $\delta > 0$ such that the estimate $\mathrm{Re}(\Phi(p) - \Phi(p_j) - g(p)) < -C(\delta) < 0$ is valid uniformly with respect to $\xi > c^2$ when $p \in \mathbb{T}^{j,U}_{\{\lambda\}}$. To get this estimate, consider $\Phi(p) - \Phi(p_j) - g(p)$ as a function of ξ. We observe that the difference $\Phi(k) - \Phi(\mathrm{i}\kappa_j)$ grows like $|\xi\delta|$ when $\xi \to -\infty$ with $k \in \mathbb{T}^{j,U}$. On the other hand, $\mathrm{Re}\, g$ grows on this contour approximately like $|\xi\sqrt{(\kappa_j + \delta)^2 + c^2}|$ as $\xi \to -\infty$. A rough elementary estimate shows that it suffices to choose (starting from the very beginning, when formulating the initial holomorphic RH problem)

$$\delta = \min\Big\{\frac{\kappa_1}{2},\ \frac{\kappa_1^2 c}{3(\kappa_N + \kappa_1)^2}\Big\}.$$

STEP 3 will be a reformulation of the problem for m^2 on the k_1 plane. Let $\mathbb{R} \cup [-\mathrm{i}c, \mathrm{i}c] \cup_j (\mathbb{T}_1^{j,U} \cup \mathbb{T}_1^{j,L})$ be the contour on this plane, oriented as in Lemma 3.1.4. We denote by $m^3(k_1) = m^2(p)$ and keep the notations $\mathcal{R}(k_1) = \mathcal{R}(p)$ and $g(k_1) = g(p)$ for $p \in \Sigma_2$, i.e., $k_1(p) = k_1 \in \mathbb{R}$. Moreover, let $A(k_1, \xi, t)$ be the values of $A(p, \xi, t)$ on the contours $\mathbb{T}_1^{j,U} \cup \mathbb{T}_1^{j,L}$. Since the + side of the contour $[\mathrm{i}c, 0]$ is the image of Σ_1^u, we again use $\tilde{\mathcal{R}}(k_1)$, analogous to (3.3.10), for the entries of the jump matrix on $[\mathrm{i}c, 0]$. Extend this function to $[0, -\mathrm{i}c]$ by $\tilde{\mathcal{R}}(-k_1) = \overline{\tilde{\mathcal{R}}(k_1)} = \tilde{\mathcal{R}}^{-1}(k_1)$. Now proceeding analogous to Step 3 of the rarefaction section, and taking into account that the function (3.5.2) has no jump on $[\mathrm{i}c, -\mathrm{i}c]$, we conclude that m^3 is holomorphic in the domain $\mathbb{C} \setminus \Big(\mathbb{R} \cup [-\mathrm{i}c, \mathrm{i}c] \cup_j (\mathbb{T}_1^{j,U} \cup \mathbb{T}_1^{j,L})\Big)$ and solves the RH problem $m^3_+ = m^3_- v^3$

with

$$v^3(p) = \begin{cases} \begin{pmatrix} 1-|\mathcal{R}(k_1)|^2 & -\overline{\mathcal{R}(k_1)}\mathrm{e}^{-2tg(k_1)} \\ \mathcal{R}(k_1)\mathrm{e}^{2tg(k_1)} & 1 \end{pmatrix}, & k_1 \in \mathbb{R}, \\ \begin{pmatrix} \tilde{\mathcal{R}}(k_1) & \mathrm{e}^{-2tg(k_1)} \\ 0 & -\overline{\tilde{\mathcal{R}}(k_1)} \end{pmatrix}, & k_1 \in [\mathrm{i}c, 0], \\ \begin{pmatrix} -\overline{\tilde{\mathcal{R}}(k_1)} & 0 \\ \mathrm{e}^{2t(g(k_1))} & \tilde{\mathcal{R}}(k_1) \end{pmatrix}, & k_1 \in [0, -\mathrm{i}c], \\ \mathbb{I} + A(k_1, \xi, t), & k_1 \in \cup_j (\mathbb{T}_1^{j,U} \cup \mathbb{T}_1^{j,L}). \end{cases}$$

STEP 4. Let $k_1^{\pm}$ be defined by (3.5.1). Consider a part of the contour $\mathbb{R}$ by restricting to the interval $[k_1^-, k_1^+]$. This interval divides the domains $\operatorname{Re} g > 0$ for $\operatorname{Im} k_1 > 0$ and $\operatorname{Re} g < 0$ for $\operatorname{Im} k_1 < 0$. In this case the usual lower-upper factorization of the jump matrix will be used. We combine steps 4 and 5 from the rarefaction section, following the standard procedure [22], [40] together with the use of the jump problems (3.3.14), (3.3.15).
Namely, let $d(k_1)$ be an analytic function in the domain $\mathbb{C}\setminus\big([\mathrm{i}c, -\mathrm{i}c] \cup [k_1^-, k_1^+]\big)$ satisfying the jump

$$d_+(k_1) = d_-(k_1) \begin{cases} (1-|\mathcal{R}(k_1)|^2) & k_1 \in [k_1^-, k_1^+], \\ \mathrm{i}\tilde{\mathcal{R}(k_1)} & k_1 \in [\mathrm{i}c, 0], \\ \mathrm{i}\overline{\tilde{\mathcal{R}(k_1)}} & k_1 \in [0, -\mathrm{i}c], \end{cases}$$

and the normalization condition $d(k_1) \to 1,\ k_1 \to \infty$. It can be expressed by the Sokhotski–Plemelj formula as

$$d(k_1) = \exp\left\{ \frac{1}{2\pi\mathrm{i}} \left(\int_{k_1^-}^{k_1^+} \frac{\log(1-|\mathcal{R}(s)|^2)}{s-k_1} ds - \int_{-\mathrm{i}c}^{\mathrm{i}c} \frac{f(s)\,ds}{s-k_1} \right) \right\}, \qquad (3.5.3)$$

where $f(s) = \mathrm{i} \arg \mathcal{R}(s)$ as $s \in [0, \mathrm{i}c]$ and $f(-s) = f(s)$. Note that since the function $\log(1-|\mathcal{R}(s)|^2)$ is even, the function $d(k_1)$ possesses the required symmetry property $d(-k_1) = d(k_1)^{-1}$. Also the required normalization $d(k_1) \to 1$ as $k_1 \to \infty$ is evident. Now let $D(k_1)$ be the matrix (3.1.23) generated by this function $d(k_1)$. Set $m^4(k_1) = m^3(k_1)D(k_1)$, and the new RH problem will read $m^4_+(k_1) = m^4_-(k_1)v^4(k_1)$, where $m^4(k) \to (1,1)$ as

$k_1 \to \infty$, $m^4(-k_1) = m^4(k_1)\sigma_1$ and

$$v^4(k) = \begin{cases} A^L(k_1)A^U(k_1), & k_1 \in [k_1^-, k_1^+] \\ B^L(k_1)B^U(k_1), & k_1 \in \mathbb{R} \setminus [k_1^-, k_1^+] \\ \begin{pmatrix} -\mathrm{i} & d_+(k_1)d_-(k_1)\mathrm{e}^{-2tg(k_1)} \\ 0 & -\mathrm{i} \end{pmatrix}, & k_1 \in [\mathrm{i}c, 0], \\ \begin{pmatrix} \mathrm{i} & 0 \\ d_+^{-1}(k_1)d_-^{-1}(k_1)\mathrm{e}^{2tg(k_1)} & \mathrm{i} \end{pmatrix}, & k_1 \in [0, -\mathrm{i}c], \\ \mathbb{I} + D^{-1}(k_1)A(k_1\xi, t)D(k_1), & k_1 \in \cup_j(\mathbb{T}_1^{j,U} \cup \mathbb{T}_1^{j,L}), \end{cases} \tag{3.5.4}$$

where

$$A^L(k_1) = \begin{pmatrix} 1 & 0 \\ \frac{\mathcal{R}(k_1)\mathrm{e}^{tg(k_1)}}{(1-|\mathcal{R}(k_1)|^2)d^2(k_1)} & 1 \end{pmatrix}, \quad k_1 \in \Omega_c^L,$$

$$A^U(k_1) = \begin{pmatrix} 1 & -\frac{d^2(k_1)\overline{\mathcal{R}(k_1)}\mathrm{e}^{-tg(k_1)}}{(1-|\mathcal{R}(k_1)|^2)} \\ 0 & 1 \end{pmatrix}, \quad k_1 \in \Omega_c^U,$$

$$B^L(k_1) = \begin{pmatrix} 1 & -d^2(k_1)\overline{\mathcal{R}(k_1)}\mathrm{e}^{-tg(k_1)} \\ 0 & 1 \end{pmatrix}, \quad k_1 \in \Omega_l^U \cup \Omega_r^L,$$

$$B^U(k) = \begin{pmatrix} 1 & 0 \\ d^{-2}(k_1)\mathcal{R}(k_1)\mathrm{e}^{tg(k_1)} & 1 \end{pmatrix}, \quad k_1 \in \Omega_l^L \cup \Omega_r^U.$$

Here the domains Ω_l^L, Ω_l^U, Ω_c^L, Ω_c^U, Ω_r^L, Ω_r^U together with their boundaries $\mathcal{C}_l^L$, $\mathcal{C}_l^U$, $\mathcal{C}_c^L$, $\mathcal{C}_c^U$, $\mathcal{C}_r^L$, $\mathcal{C}_r^U$ are shown in Figure 3.11. We set $\overline{\mathcal{R}(k_1)} = \mathcal{R}(-k_1)$ in these domains. These functions have an analytical continuation as we choose our contours sufficiently close to the real axis, avoiding the loops the discrete spectrum.

Now redefine $m^4(k_1)$ according to

$$m^5(k) = \begin{cases} m^4(k_1)A^L(k_1), & k_1 \in \Omega_c^L, \\ m^4(k_1)A^U(k_1)^{-1}, & k_1 \in \Omega_c^U, \\ m^4(k_1)B^L(k_1), & k_1 \in \Omega_l^L \cup \Omega_r^L, \\ m^4(k_1)B^U(k_1)^{-1}, & k_1 \in \Omega_l^U \cup \Omega_r^U, \\ m^4(k_1), & \text{else.} \end{cases}$$

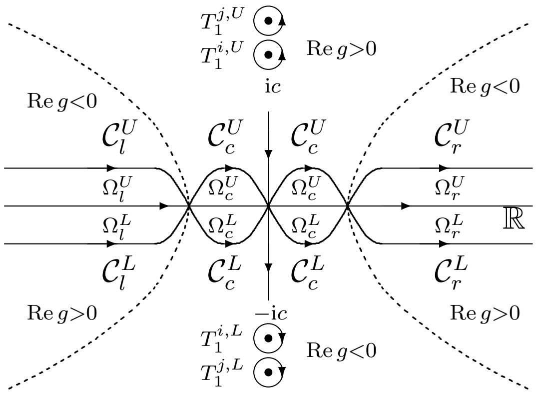

Figure 3.11: Contour deformation in the domain $x < -6c^2t$

Introduce the matrix

$$\Delta(k_1) = \begin{cases} \begin{pmatrix} 0 & d_+(k_1)d_-(k_1)\mathrm{e}^{-2tg(k_1)} \\ 0 & 0 \end{pmatrix}, & k_1 \in [\mathrm{i}c, 0]; \\ \begin{pmatrix} 0 & 0 \\ d_+^{-1}(k_1)d_-^{-1}(k_1)\mathrm{e}^{2tg(k_1)} & 0 \end{pmatrix}, & k_1 \in [0, -\mathrm{i}c], \\ A^U(k_1) - \mathbb{I}, & k_1 \in \mathcal{C}_c^U; \\ A^L(k_1) - \mathbb{I}, & k_1 \in \mathcal{C}_c^L, \\ B^U(k_1) - \mathbb{I}, & k_1 \in \mathcal{C}_l^U \cup \mathcal{C}_r^U; \\ B^L(k_1) - \mathbb{I}, & k_1 \in \mathcal{C}_l^L \cup \mathcal{C}_r^L, \\ D^{-1}(k_1)A(k_1,\xi,t)D(k_1), & k_1 \in \cup_j(\mathbb{T}_1^{j,U} \cup \mathbb{T}_1^{j,L}), \end{cases} \tag{3.5.5}$$

where the last matrix is defined in (3.5.4). Abbreviate also

$$\mathcal{C} := \mathcal{C}_l^U \cup \mathcal{C}_c^U \cup \mathcal{C}_r^U \cup \mathcal{C}_l^L \cup \mathcal{C}_c^L \cup \mathcal{C}_r^L \cup_j (\mathbb{T}_1^{j,U} \cup \mathbb{T}_1^{j,L}).$$

Lemma 3.5.1. *In the domain $\xi < -c^2/2$ the initial RH problem, formulated in Lemma*
3.1.5, is equivalent to the following RH problem formulated in k_1: Find a function $m^5(k_1)$ holomorphic in $\mathbb{C} \setminus (\mathcal{C} \cup [\mathrm{i}c, -\mathrm{i}c])$ and satisfying 1) the jump condition $m^5_+ = m^5_- v^5$ with

$$v^5(k_1) = \begin{cases} -\mathrm{i}\mathbb{I} + \Delta(k_1) & k_1 \in [\mathrm{i}c, 0], \\ \mathrm{i}\mathbb{I} + \Delta(k_1), & k_1 \in [0, -\mathrm{i}c], \\ \mathbb{I} + \Delta(k_1), & k_1 \in \mathcal{C}; \end{cases}$$

2) the symmetry condition $m^5(-k_1) = m^5(k_1)\sigma_1$*; 3) normalization condition* $m^5(k_1) \to (1,1)$ *as* $k \to \infty$*.*

Here the matrix $\Delta(k_1)$ *is defined by* (3.5.5) *and admits an estimate* $|\Delta(k_1)| \leq C(\epsilon)\mathrm{e}^{-C(\epsilon)t}$ *outside of* ϵ *vicinities of the points* $k_1^{\pm} = \pm\sqrt{-\xi - c^2/2}$ *and the point* $k_1 = 0$*.*

In the domain $(\operatorname{Im} k_1)^2 > \kappa_n^2 + c^2 + 1$ *we have*

$$m^5(k_1) = d^{-1}(k_1)\Lambda^{-1}(k_1))\mathrm{e}^{-(\Phi(k_1)-g(k_1))t}\, m(k_1),$$

where the functions $g(k_1)$*,* $d(k_1)$*, and* $\Phi(k_1)$ *are defined by formulas* (3.5.2)*,* (3.5.3)*, and* (3.1.14) *respectively.*

Thus one can expect that the solution of the following model problem $m_+^{mod}(k_1) = m_-^{mod}(k_1)v^{mod}(k_1)$ where

$$v^{mod}(p) = \begin{cases} \begin{pmatrix} -\mathrm{i} & 0 \\ 0 & -\mathrm{i} \end{pmatrix}, & k_1 \in [\mathrm{i}c, 0], \\ \begin{pmatrix} \mathrm{i} & 0 \\ 0 & \mathrm{i} \end{pmatrix}, & k_1 \in [0, -\mathrm{i}c], \end{cases}$$

satisfying $m^{mod}(-k_1) = m^{mod}(k_1)\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, $k_1 \in \mathbb{C}$ and $\lim_{k_1\to\infty} m^{mod}(k_1) = (1,1)$, will be asymptotically close to solution $m^5(k_1)$ as $t \to \infty$, at least outside of the aforementioned vicinities of $k_1^{\pm}$ and 0. The model problem has the unique solution, given by formula

$$m^{mod}(k_1) = \left(\sqrt[4]{\frac{k_1^2 + c^2}{k_1^2}}, \sqrt[4]{\frac{k_1^2 + c^2}{k_1^2}} \right), \quad k_1 \in \mathbb{C}, \quad \text{where } \sqrt[4]{\infty} = 1.$$

As this solution has square root singularities at the edge of spectrum of multiplicity two, which corresponds to the point $k_1 = 0$ it requires a more delicate analysis for the approximation step and hence we will not provide further details here. We expect this problem to be solvable by using a method described in [44].

Chapter 4

Numerics for the steplike KdV Cauchy problem

We close the thesis by giving some known information about the numerics for the KdV equation. For solving nonlinear evolution equations there are pseudospectral methods and difference methods. Regarding the difference methods there are different versions of the standard Euler method. We will discus here the leap–frog method which has been developed by Zabusky and Kruskal [69].

We discretize the space and time by $x = mh$ and $t = nk$ with $m = 0, 1 \dots, M$ and $n = 0, 1, \dots$. h and k are the step sizes in space and time. Since the domain is of finite length the step size is chosen to be $h = 2\pi/M$. The discretization of space and time leads to an approximation of the continuous solution $q(x,t)$ for each step by $q_m^n(x,t)$. The difference of the various Euler methods comes from the formulation of the derivative. The studied leap–frog method for the KdV equation

$$q_t - 6qq_x + q_{xxx} = 0$$

is given by

$$\begin{aligned} q_m^{n+1} =& q_m^{n-1} + \frac{6k}{3h}(q_{m+1}^n + q_m^n + q_{m-1}^n)(q_{m+1}^n - q_{m-1}^n) \\ &- \frac{k}{h^3}(q_{m+2}^n - 2q_{m+1}^n + 2q_{m-1}^n - q_{m-2}^n). \end{aligned} \tag{4.0.1}$$

On the right handside the first summand depends to the time derivative, the second summand represents the nonlinearity and the last term approximates the dispersion e.g. the third derivative in x. This representations has two advantages: On the one hand the mass of $\sum_{m=0}^{M-1}$ is conserved, on the other hand the special form of the nonlinear term $\frac{1}{3}(q_{m+1}^n + q_m^n + q_{m-1}^n)$ gives conservation of energy up to the second order

$$\frac{1}{2}\sum_{m=0}^{M-1}(q_n^m)^2 - \frac{1}{2}\sum_{m=0}^{M-1}(q_m^{n-1})^2 = \mathcal{O}(k^3) \quad \text{for} \quad k \to 0,$$

if q is periodic or decays fast enough at the end of the interval. Since the method is of second order and uses not only q_n^m but also q_{n-1}^m we need another method for the first step, an Euler method

$$\begin{aligned} q_m^{n+1} =& q_m^n + \frac{6k}{3h}(q_{m+1}^n + q_m^n + q_{m-1}^n)(q_{m+1}^n - q_{m-1}^n) \\ &- \frac{k}{h^3}(q_{m+2}^n - 2q_{m+1}^n + 2q_{m-1}^n - q_{m-2}^n). \end{aligned} \tag{4.0.2}$$

Thus the procedure is the following

- Produce the initial condition.
- Use method (4.0.2) for the first step.
- Iterate by using (4.0.1).

By this method we created the picture 3.1. Therefore we implemented the described method into Mathematica,using Code from [5].

Code *Mathematica* 4.1: *Mathematica Sourcecode*

```
KdVNIntegrate[initial_, dx_, dt_, M_, T_] := Block[
{uPresent, uPast, uFuture, initialh, h = dx, k = dt, m, n,  out = {}},
(* --- stability condition --- *)
    Print[k, "\[LessEqual]", h^3/(4 + h^2 10), " is ",
          If[k <= h^3/(4 + h^2 10),
     "OK", "NOT OK"]];
    Print["x \[Element](", dx*M/2, ",", -dx*M/2, ") and tmax=", dt*T];
(* --- transform the initial conditions on the grid --- *)
    initialh = initial /. x -> (m - M/2) dx;
(* --- calculate the initial solutions on the grid --- *)
    uPast = Table[initialh, {m, 1, M}];
(* --- initialization of the lists containing the grid points
        uPresent = present     (m)
        uFuture = future      (m+1)
        uPast = past          (m-1)          --- *)
    uPresent = uPast;
    uFuture = uPresent;
    out = Table[{{(m - M/2) dx, 0 }, uPresent[[m]]}, {m, 3, M - 2}];
(* --- iterate the time --- *)
    Do[
(* --- iterate the space points --- *)
        Do[
           uFuture[[m]] = uPast[[m]] + 6 k (uPresent[[m + 1]] +uPresent[[m]]
```

```
                                + uPresent[[m - 1]])(uPresent[[m + 1]]
                                - uPresent[[m - 1]])/(3 h)
                                - k (uPresent[[m + 2]] - 2 uPresent[[m + 1]]
                                + 2 uPresent[[m - 1]] -uPresent[[m - 2]])/h^3,
        {m, 3, M - 2}];
(* --- exchange lists --- *)
        uPast = uPresent;
        uPresent = uFuture;
        out = Join[out, Table[{{(m - M/2) dx, n dt}, uPresent[[m]]},
        {m, 3, M - 2}]],
    {n, T}];
    Interpolation[out]
];


u = KdVNIntegrate[-((Erf[x] - 1)/2) - 5 Sech[x - 1], 0.1, 0.0002, 512,5000];
```

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Appendix A

Zusammenfassung

Die Korteweg–de Vries Gleichung ist eine Wellengleichung, die Flachwasser Wellen modelliert und eine der bekanntesten Solitonen Gleichungen ist. Das dazugehörige Cauchy–Problem wurde von Gardner, Green, Kruskal und Miura mit Hilfe der Inversen Streutheorie gelöst. Im klassischen Fall verschwindet der Anfangswert asymptotisch und ist gut ausgearbeitet. Ein weiterer Fall, mit dem Schock und Verdünnung der Wellen modelliert wird, ist wenn die Anfangsbedingungen asymptotisch gegen verschiedene Konstanten konvergieren und wird stufenartige Anfangsbedingung genannt.

Im ersten Teil der Arbeit untersuchen wir das zu Grunde liegende direkte und inverse Streuproblem für die eindimensionale Schrödinger Gleichung mit stufenartigem Potential. Wir geben notwendige und hinreichende Bedingungen für die Streudaten an, um zu einem Potential mit vorgegebener Glattheit und räumlichen Zerfall zu gehören. Dieses Problem wurde zuvor betrachtet, allerdings verallgemeinert unser Ergebnis alle vorherigen Ergebnisse. Im zweiten Teil wenden wir die Ergebnisse auf das Cauchy–Problem der KdV Gleichung mit stufenartigen, genauer gesagt Verdünnungswellen produzierenden Anfangsbedingungen, an. Dazu formulieren wir das inverse Problem als ein oszillatorisches Riemann–Hilbert Problem und wenden die Methode des nichtlinearen schnellsten Abstieges an, um das Langzeitverhalten der Lösung zu erhalten. Um das Problem untersuchen zu können, muss eine neue Phasenfunktion, die sogenannte $g-$ Funktion, eingeführt werden, die von einer langsamen Variable $\xi = \frac{x}{t}$ abhängt. Danach kann das Problem zu einem explizit lösbaren Modellproblem umgeformt werden. In Abhängigkeit von dem Wert von ξ gibt es drei Hauptregionen, wenn t gegen unendlich geht: Für $\xi < -\xi_0$ ist die Lösung nahezu die linke Konstante. Für $-\xi_0 < \xi < 0$ gibt es eine Verdünnungsregion, wo die Lösung sich wie $\frac{x}{t}$ verhält. Für $0 < \xi$ gibt es die Solitonenregion, in der die Lösung durch eine Summe von Solitonen gegeben ist.

Abschließend vergleichen wir die analytisch erlangte Lösung mit einer numerisch errechneten.

Appendix B

Curriculum Vitae

CV

Personal Details

Name:	Till Luc Lange
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Place of birth:	24211 Preetz, Deutschland

Education

10/2013 – 07/2015	PhD in Mathematics University of Vienna, Vienna
10/2011 – 09/2013	Field of Study: Mathematics (Master of Science), Minor Subject: Computer Science, TU Braunschweig, Braunschweig, Graduation: Master of Science
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Publications and Preprints

Inverse scattering theory for Schrödinger operators with steplike potentials, together with I. Egorova, Z. Gladka, G. Teschl, Zh. Mat. Fiz. Anal. Geom. **11**, 123–158 (2015).

Rarefaction Waves of the Korteweg–de Vries Equation via Nonlinear Steepest Descent, together with K. Andreiev, I. Egorova and G. Teschl, in preparation.

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