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Dispersive Estimates for Spherical Schrödinger Equations

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MARKUS HOLZLEITNER, MSc

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Abstract

This thesis is concerned with dispersive estimates for a class of radially symmetric one-dimensional Schrödinger equations on the positive halfline, that are important in many physical applications. We generalize some already existing results and investigate certain borderline cases: In the first part we look at the case of general boundary conditions; the second part deals with a critical case, that has not been treated before, because of additional technical difficulties; the third part proves properties for transformation operators related to this equations and aims at improving some previous results.

Zusammenfassung

Diese Arbeit beschäftigt sich mit dispersiven Abschätzungen für eine Klasse radialsymmetrischer eindimensionaler Schrödingeroperatoren, definiert auf der positiven Halbachse, welche zahlreiche Anwendungen in der mathematischen Physik besitzen. Wir verallgemeinern einige bereits existierende Resultate und beschäftigen uns mit einigen bisher noch nicht behandelten Grenzfällen: Im ersten Teil betrachten wir den Fall beliebiger Randbedingungen; der zweite Artikel behandelt einen kritischen Fall, welcher bis jetzt noch nicht im Detail analysiert wurde, da er in einigen technischen Details um einiges anspruchsvoller ist; im dritten Teil beschäftigen wir uns mit einigen Eigenschaften der dazugehörigen Transformationsoperatoren, wobei der Fokus darauf liegt, einige bisher erhaltene Resultate zu verbessern.

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Contents

Introduction

This work is concerned with the one-dimensional Schrödinger equation

$$\mathrm{i}\dot{\psi}(t,x) = H\psi(t,x), \quad H := -\frac{d^2}{dx^2} + \frac{l(l+1)}{x^2} + q(x), \quad (t,x) \in \mathbb{R}\times\mathbb{R}_+, \tag{0.1}$$

with real integrable potential q and with the angular momentum $l \geq -\frac{1}{2}$. The operators H naturally arise in many physically relevant models, e.g. as the radial part after a separation of variables in higher dimensional Schrödinger equations(cf. [21, Section 17.F]), as the most prominent example of strongly singular Schrödinger operators (see e.g. [8–11]), or also in some wave propagation models(cf. [15, Section 3.7]). Our main interest here lies in dispersive estimates for the Schrödinger equation associated to H. These are estimates of the form

$$\left\|\mathrm{e}^{-\mathrm{i}tH} P_c(H)\right\|_{L^1(\mathbb{R}_+)\to L^\infty(\mathbb{R}_+)} = \mathcal{O}(|t|^{-1/2}), \quad t \to \infty, \tag{0.2}$$

where $\mathrm{e}^{-\mathrm{i}tH}$ denotes the Schrödinger evolution obtained by the Spectral Theorem, and by $P_c(H)$ we denote the projection in $L^2(\mathbb{R}_+)$ on the continuous subspace of H. In order for (0.2) to even make sense, we first of all will need to discuss spectral properties of H, especially self-adjoint realizations and their spectral properties. Before digging deeper into the details, let us briefly discuss, why dispersive estimates are interesting in general: On the one hand, they lead to Strichartz estimates for the linear Schrödinger equations and thus, on the other hand, can provide for a useful tool to obtain properties for associated nonlinear equations, especially when it comes to proving (in)stabilty of solitons. On the whole line, there are already many results available in this direction, we refer e.g. to the reviews [7, 16] for an overview. On the half line the case $l = 0$ was investigated by Weder [20]. The case for general l but with $q = 0$ was considered in [13]. As mentioned above, let us now get back to focus on details regarding H. Let us use τ to describe the formal Sturm–Liouville differential expression corresponding to H. By $H_{\max}$ we denote the maximal operator associated with τ, i.e.

$$\mathrm{dom}(H^{\max}) = \{f \in L^2(\mathbb{R}_+) : Hf \in L^2(\mathbb{R}_+), f, f' \in AC_{loc}(\mathbb{R}_+)\}.$$

We further assume, that q satisfies some further integrability conditions. Let's therefore first introduce the notion of weighted L^p-spaces: for any set $K \subset \mathbb{R}_+$, $L^p(K, w(x))$ denotes the usual weighted L^p space with weight $w(x)$, i.e. the associated norm is given by

$$\|f\|_{L^p(K,w(x))} := \left(\int_K |f(x)^p w(x)|\, dz\right)^{\frac{1}{p}}.$$

Now we assume that q should belong to the so called Marchenko class:

$$q \in \begin{cases} L^1(\mathbb{R}_+, x), & l > -\frac{1}{2}, \\ L^1(\mathbb{R}_+, x(1+|\log(x)|)), & l = -\frac{1}{2}. \end{cases}$$

Then we end up with:

- For $l \in [-\frac{1}{2}, \frac{1}{2})$: There is a one-parameter family H_α, $\alpha \in [0, \pi)$ of self-adjoint restrictions of H_{max}. The case $\alpha = 0$ corresponds to the Friedrichs extension.
- For $l \geq \frac{1}{2}$: $H = H^*_{\max}$ is already self-adjoint.

For further details, proofs, and explicit formulas for the boundary conditions, we refer e.g. to [2] and [1]. A short summary is also contained in [5, Section 2]. We continue with the spectral properties of H resp. H_α: it has a purely absolutely continuous spectrum on $(0,\infty)$ plus a finite number of eigenvalues in $(-\infty, 0]$, cf. [18, Sect. 9.7] for details. At the edge of the continuous spectrum there could be a resonance (or an eigenvalue if $l > \frac{1}{2}$). The precise meaning of a resonance will be explained later. For the reader's convenience, we will now briefly outline our main strategy for proving estimates of the type (0.2):

1) The starting point is Stone's formula: Suppose $g \in C(\mathbb{R})$ is bounded. Then
$$\frac{1}{2\pi\mathrm{i}} \int_a^b g(k)(R_H(k+\mathrm{i}\varepsilon) - R_H(k-\mathrm{i}\varepsilon))dk \xrightarrow{s} \frac{g(H)}{2}\left(P_H([a,b]) + P_H((a,b))\right)$$
strongly. $R_H(z) = (H-z)^{-1}$ denotes usual resolvent, and the (operator valued) integral is understood in the sense of Riemann. For a proof and more information, see e.g. [18, Section 4.1]. It gives a convenient way to find formulas for functions $g(H)$. In our situation we set $g(k) = \mathrm{e}^{-\mathrm{i}tk}$.
2) Next we have a closer look at the resolvent. To this end let us denote by $\phi(k^2, x)$ a solution of $\tau f = k^2 f$, which satisfies the boundary condition near 0, and by $f(k,x)$ the so called Jost solution of $\tau f = k^2 f$, i.e. a solution that satisfies $f(k,x) \sim \mathrm{e}^{\mathrm{i}kx}$ as $x \to \infty$. Let furthermore $f(k) := W(f(k,.), \phi(k^2,.))$ and $F(k) := C_l k^l W(f,\phi)$, where $W(f,g) = fg' - f'g$ is the usual Wronskian. The function F is usually called the Jost function. A well known result from the theory of Sturm–Liouville operators(see e.g. [18, Section 9.2.]) says that the resolvent can be expressed as an integral operator, and the integral kernel $G(k^2,x,y)$(or also called Green's function) is given by the following expression:
$$G(k,x,y) := \frac{\phi(k^2,x)f(k,y)}{W(\phi(k^2,x), f(k,x))}$$
for $x \leq y$(if $y \leq x$: the positions of ϕ and f are reversed). Now applying Stone's formula in combination with the structure of the spectrum and the Green's function of H, yields the following expression for the integral kernel of the propagator appearing in (0.2):
$$[\mathrm{e}^{-\mathrm{i}tH} P_c(H)](x,y) = \frac{2}{\pi}\int_{-\infty}^{\infty} \mathrm{e}^{-\mathrm{i}tk^2} \frac{\phi(k^2,x)\phi(k^2,y)k^{2(l+1)}}{|F(k)|^2}\, dk. \tag{0.3}$$
3) To properly analyze the oscillatory integral in (0.3), we make use of the famous van der Corput lemma, which in our situation can be formulated in the following way: Let $I(t) = \int_a^b \mathrm{e}^{-\mathrm{i}tk^2} g(k)dk$. If the Fourier transform $\hat{g}$ of g is contained in $L^1(a,b)$, then $|I(t)| \leq Ct^{-1/2}\|\hat{g}\|_1$. Sometimes we need different versions, for a short summary we refer e.g. to [5, Appendix A]. So in order to apply this lemma to (0.3) and therefore get the desired result (0.2), we need to establish satisfying estimates for the solutions ϕ and f, and also for the Jost function F, such that the Fourier transform of the integrand can be controlled properly. This is the main part of the whole analysis.

We continue with some further technical remarks concerning the investigation of (0.3):

i) To deduce useful estimates for ϕ and f and their derivatives, one starts with the behavior of the corresponding solutions to the free equation(i.e. $q = 0$) and then uses perturbation theory. The free solutions can be expressed in terms of Bessel and Hankel functions, which are of course much harder to handle than the trigonometric functions appearing e.g. in the case $l = 0$.
ii) It can be the case, that near the edge of the continuous spectrum(i.e. near 0) the Jost function F vanishes. In this case, we say that there is resonance (or an eigenvalue if $l > \frac{1}{2}$), and this situation usually imposes further technical difficulties. The analysis of F near 0 is in general the hardest part in all the computations.

iii) Also due to the lack of satisfying estimates for F, one has to use different strategies to investigate (0.3) near $k = 0$ and $k = \infty$, and therefore uses cutoffs to focus on the so called low and high energy parts. Near 0 we utilize transformation operators and near ∞ the situation is handled with the aid of a Born series expansion of the resolvent. The details are contained e.g. in [6, 12, Chapter 3].

Let us now discuss the main results of this thesis. The starting point was the article [12], where (0.2) was proven in the nonresonant case for $l > -\frac{1}{2}$, with Friedrichs boundary condition and under some further integrability assumptions on the potential q(which come from the transformation operators). The main task was to relax these conditions and give some generalizations. This was partially done in the three articles of the present work:

1) The first article [5] considers general boundary conditions $\alpha \in [0, \pi)$ for H, but only in the free case $q = 0$ and if $-\frac{1}{2} < l < \frac{1}{2}$. We observed the interesting fact, that the for negative l, we get the usual decay (0.2), but for positive $0 < l < \frac{1}{2}$, the dispersive behavior might change(either we have a worse time decay or we need to consider weighted L^p-spaces). The analysis basically follows the steps explained above, but in order to apply the van der Corput lemma, we needed some more advanced results from Fourier analysis(cf. [14] or [5, Appendix A]), that have not been linked before with this topic. It also provides a useful condition, under which functions of Schrödinger operators can be expressed as integral operators, cf. [5, Appendix C].
2) The second article [6] treats the critical case $l = -\frac{1}{2}$, under the Friedrichs condition, in the nonresonant case, but for general potential $q \neq 0$. This case usually leads to additional technical difficulties. We derive several new estimates for solutions of the underlying differential equation and investigate in detail the behavior of the Jost function near the edge of the continuous spectrum. Again in combination with rather recent results from Fourier analysis(cf. [6, Appendix A]), we could establish (0.2) under certain conditions on q.
3) The third contribution [4] is concerned with transformation operators for H. Let us briefly explain this notion, as an example near ∞(similar considerations of course also make sense near 0). The intention is to construct an operator, that maps the Jost solution $f_l(k, x)$ of the free equation to $f(k, x)$, such that the properties of f_l near ∞ are preserved. This operator K should then be expressed as an integral operator of the following form:

$$f(k, x) = f_l(k, x) + \int_x^\infty K(x, y) f_l(k, y) dy = (I + K) f_l(k, x).$$

The aim is to show existence of K and derive good estimates for the kernel $K(x, y)$, e.g. s.t. K is bounded operator. This usually leads to further restrictions on the potential q. Especially the boundedness of K is crucial for our computations regarding the dispersive estimates(see e.g. [6, Theorem 3.2.]). In principle, the approach to establish existence and estimates for K is well known, and there are also some rather old results available, that deal with proving these properties(cf. [3,17,19]). Unfortunately, during this project, we realized, that these results don't cover all the situations that are considered in [6, 12]; thus the aim of the present work is to fill this gap. With the techniques available so far, we needed to impose stronger conditions on q than the ones mentioned in [6, 12], but we were at least able to slightly improve the previous results and also give a rigorous and more detailed presentation of the material. Probably there is a possibility to further relax the conditions on q, but therefore one might need new methods. It's also worthwhile mentioning, that transformation operators for H play an important role in other recent research directions, cf. [4, Introduction] for more information on the corresponding literature.

To conclude the introduction, we want to discuss some further open questions, that are connected with our results and which would be interesting to work on:

- Of course the resonant case is definitely an open task, that would be nice to resolve. The new approach in [6] already gave us a hint on how we could find out more about the behavior of the Jost function F near 0.
- It would also be interesting, if our results could be applied to multi-dimensional Schrödinger equations, especially in dimension $n = 2$, where the half integer values of l(and thus also the case $l = -\frac{1}{2}$) play an important role.

References

[1] A. Ananieva and V. Budika, *To the spectral theory of the Bessel operator on a finite interval and half-line*, Ukrainian Mat. Visnyk, **12**, no. 2, 160–199 (2015), (in Russian); *English transl.:* J. Math. Sci. **211**, 624–645 (2015).

[2] L. Bruneau, J. Dereziński, and V. Georgescu, *Homogeneous Schrödinger operators on half-line*, Ann. Henri Poincaré **12**, no. 3, 547–590 (2011).

[3] M. Coz and C. Coudray, *The Riemann solution and the inverse quantum mechanical problem*, J. Math. Phys. **17**, no. 6, 888–893 (1976).

[4] M. Holzleitner, *Transformation operators for spherical Schrödinger operators*, in preparation.

[5] M. Holzleitner, A. Kostenko, and G. Teschl, *Dispersion estimates for spherical Schrödinger equations: The effect of boundary conditions*, Opuscula Math. **36**, no. 6, 769–786 (2016).

[6] M. Holzleitner, A. Kostenko, and G. Teschl, *Dispersion estimates for spherical Schrödinger equations with critical angular momentum*, in Partial Differential Equations, Mathematical Physics, and Stochastic Analysis, F. Gesztesy et al. (eds), EMS Congress Reports (to appear).

[7] E. Kopylova, *Dispersion estimates for Schrödinger and Klein–Gordon equation*, Russian Math. Surveys, **65**, no. 1, 95–142 (2010).

[8] A. Kostenko, A. Sakhnovich, and G. Teschl, *Inverse eigenvalue problems for perturbed spherical Schrödinger operators*, Inverse Problems **26**, 105013, 14pp (2010).

[9] A. Kostenko, A. Sakhnovich, and G. Teschl, *Weyl–Titchmarsh theory for Schrödinger operators with strongly singular potentials*, Int. Math. Res. Not. **2012**, 1699–1747 (2012).

[10] A. Kostenko and G. Teschl, *On the singular Weyl–Titchmarsh function of perturbed spherical Schrödinger operators*, J. Differential Equations **250**, 3701–3739 (2011).

[11] A. Kostenko and G. Teschl, *Spectral asymptotics for perturbed spherical Schrödinger operators and applications to quantum scattering*, Comm. Math. Phys. **322**, 255–275 (2013).

[12] A. Kostenko, G. Teschl and J. H. Toloza, *Dispersion estimates for spherical Schrödinger equations*, Ann. Henri Poincaré **17**, no. 11, 3147–3176 (2016).

[13] H. Kovařík and F. Truc, *Schrödinger operators on a half-line with inverse square potentials*, Math. Model. Nat. Phenom. **9**, no. 5, 170–176 (2014).

[14] E. Liflyand, S. Samko and R. Trigub, *The Wiener algebra of absolutely convergent Fourier integrals: an overview*, Anal. Math. Phys. **2**, 1–68, (2012).

[15] K. Okamoto, *Fundamentals of optical waveguides*, San Diego: Academic Press, 2000.

[16] W. Schlag, *Dispersive estimates for Schrödinger operators: a survey*, in "Mathematical aspects of nonlinear dispersive equations", 255–285, Ann. Math. Stud. **163**, Princeton Univ. Press, Princeton, NJ, 2007.

[17] A. S. Sohin, *On a class of transformation operators*, Trudy Fiz.-Teh. Inst. Nizkih Temp. AN USSR, Mat. Fiz., Funkts. Analiz, no. 1, 117–125 (1969) (in Russian); *English transl.* in Sel. Math. Sov. **3**, no. 3, 301–308 (1983).

[18] G. Teschl, *Mathematical Methods in Quantum Mechanics; With Applications to Schrödinger Operators*, 2nd ed., Amer. Math. Soc., Rhode Island, 2014.

[19] V. Ya. Volk, *On inversion formulas for a differential equation with a singularity at $x = 0$*, Uspehi Matem. Nauk (N.S.) **8**, 141–151 (1953).

[20] R. Weder, *The $L^p - L^{\dot{p}}$ estimates for the Schrödinger equation on the half-line*, J. Math. Anal. Appl. **281**, 233–243 (2003).

[21] J. Weidmann, *Spectral Theory of Ordinary Differential Operators*, Lecture Notes in Mathematics **1258**, Springer, Berlin, 1987.

Dispersion Estimates for Spherical Schrödinger Equations: The Effect of Boundary Conditions

by Markus Holzleitner, Aleksey Kostenko and Gerald Teschl

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DISPERSION ESTIMATES FOR SPHERICAL SCHRÖDINGER EQUATIONS: THE EFFECT OF BOUNDARY CONDITIONS

MARKUS HOLZLEITNER, ALEKSEY KOSTENKO, AND GERALD TESCHL

Dedicated with great pleasure to Petru A. Cojuhari on the occasion of his 65th birthday

Abstract. We investigate the dependence of the $L^1 \to L^\infty$ dispersive estimates for one-dimensional radial Schrödinger operators on boundary conditions at 0. In contrast to the case of additive perturbations, we show that the change of a boundary condition at zero results in the change of the dispersive decay estimates if the angular momentum is positive, $l \in (0, 1/2)$. However, for nonpositive angular momenta, $l \in (-1/2, 0]$, the standard $O(|t|^{-1/2})$ decay remains true for all self-adjoint realizations.

1. Introduction

We are concerned with the one-dimensional Schrödinger equation

$$\mathrm{i}\dot{\psi}(t,x) = H_\alpha \psi(t,x), \quad H_\alpha := -\frac{d^2}{dx^2} + \frac{l(l+1)}{x^2}, \quad (t,x) \in \mathbb{R} \times \mathbb{R}_+, \tag{1.1}$$

with the angular momentum $|l| < \frac{1}{2}$ and self-adjoint boundary conditions at $x = 0$ parameterized by a parameter $\alpha \in [0, \pi)$ (the definition is given in Section 2, see (2.1)–(2.2) — for recent discussion of this family of operators see [1, 4]). More precisely, we are interested in the dependence of the $L^1 \to L^\infty$ dispersive estimates associated to the evolution group $\mathrm{e}^{-\mathrm{i}tH_\alpha}$ on the parameters $\alpha \in [0, \pi)$ and $l \in (-1/2, 1/2)$.

On the whole line such results have a long tradition and we refer to Weder [22], Goldberg and Schlag [9], Egorova, Kopylova, Marchenko and Teschl [5], as well as the reviews [10, 18]. On the half line, the case $l = 0$ with a Dirichlet boundary condition was treated by Weder [23]. The case of general l and the Friedrichs boundary condition at 0 ($\alpha = 0$ in our notation)

$$\lim_{x \to 0} x^l((l+1)f(x) - xf'(x)) = 0, \qquad l \in \Big(-\frac{1}{2}, \frac{1}{2}\Big), \tag{1.2}$$

was recently considered in Kovařík and Truc [14] and they proved (see Theorem 2.4 in [14]) that

$$\|\mathrm{e}^{-\mathrm{i}tH_0}\|_{L^1(\mathbb{R}_+) \to L^\infty(\mathbb{R}_+)} = \mathcal{O}(|t|^{-1/2}), \quad t \to \infty. \tag{1.3}$$

It was proved in [13] that this estimate remains true under additive perturbations. More precisely (see [13, Theorem 1.1]), let $H = H_0 + q$, where the potential q is a

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real integrable on $\mathbb{R}_+$ function. If in addition

$$\int_0^1 |q(x)|dx < \infty \quad \text{and} \quad \int_1^\infty x^{\max(2,l+1)}|q(x)|dx < \infty, \tag{1.4}$$

and there is neither a resonance nor an eigenvalue at 0, then

$$\big\|\mathrm{e}^{-\mathrm{i}tH}P_c(H)\big\|_{L^1(\mathbb{R}_+)\to L^\infty(\mathbb{R}_+)} = \mathcal{O}(|t|^{-1/2}), \quad t\to\infty. \tag{1.5}$$

Here $P_c(H)$ is the orthogonal projection in $L^2(\mathbb{R}_+)$ onto the continuous spectrum of H.

The main result of the present paper shows that the decay estimates (1.3) and (1.5) are no longer true for $\alpha\in(0,\pi)$ if $l\in(0,1/2)$. In other words, this means that singular rank one perturbations destroy these decay estimates if $l\in(0,1/2)$ (since the change of a boundary condition can be considered as a rank one perturbation in the resolvent sense). Namely, consider first the operator $H_{\pi/2}$, which is associated with the following boundary condition at $x=0$:

$$\lim_{x\to 0} x^{-l-1}(lf(x)+xf'(x)) = 0, \qquad l\in\Big(-\frac{1}{2},\frac{1}{2}\Big). \tag{1.6}$$

Theorem 1.1. *Let $|l|<1/2$. Then*

$$\|\mathrm{e}^{-\mathrm{i}tH_{\pi/2}}\|_{L^1(\mathbb{R}_+)\to L^\infty(\mathbb{R}_+)} = \mathcal{O}(|t|^{-1/2}), \quad t\to\infty, \tag{1.7}$$

for all $l\in(-1/2,0]$, and

$$\|\mathrm{e}^{-\mathrm{i}tH_{\pi/2}}\|_{L^1(\mathbb{R}_+,\max(x^{-l},1))\to L^\infty(\mathbb{R}_+,\min(x^l,1))} = \mathcal{O}(|t|^{-1/2+l}), \quad t\to\infty, \tag{1.8}$$

whenever $l\in(0,1/2)$. The last estimate is sharp.

In the remaining case $\alpha\in(0,\pi/2)\cup(\pi/2,\pi)$, the decay estimate is given by the the next theorem.

Theorem 1.2. *Let $|l|<1/2$ and $\alpha\in(0,\pi/2)\cup(\pi/2,\pi)$. Then*

$$\|\mathrm{e}^{-\mathrm{i}tH_\alpha}P_c(H_\alpha)\|_{L^1(\mathbb{R}_+)\to L^\infty(\mathbb{R}_+)} = \mathcal{O}(|t|^{-1/2}), \quad t\to\infty, \tag{1.9}$$

for all $l\in(-1/2,0]$, and

$$\|\mathrm{e}^{-\mathrm{i}tH_\alpha}P_c(H_\alpha)\|_{L^1(\mathbb{R}_+,\max(x^{-l},1))\to L^\infty(\mathbb{R}_+,\min(x^l,1))} = \mathcal{O}(|t|^{-1/2}), \quad t\to\infty, \tag{1.10}$$

whenever $l\in(0,1/2)$.

Notice that in the case $l\in(0,1/2)$ we need to consider weighted L^1 and L^∞ spaces since functions contained in the domain of H_α might be unbounded near 0.

Finally, let us briefly outline the content of the paper. In the next section we define the operator H_α and collect its basic spectral properties. Section 3 contains the proof of Theorem 1.1. In particular, we compute explicitly the kernel of the evolution group $\mathrm{e}^{-\mathrm{i}tH_{\pi/2}}$ and this enables us to prove (1.7) and (1.8) by using the estimates for Bessel functions J_ν (all necessary facts on Bessel functions are contained in Appendix A). Theorem 1.2 is proved in Section 4. Its proof is based on the use of a version of the van der Corput lemma, which is given in Appendix B. Also Appendix B contains necessary facts about the Wiener algebras $\mathcal{W}_0(\mathbb{R})$ and $\mathcal{W}(\mathbb{R})$. In the final section we formulate some sufficient conditions for a function $f(H)$ of a 1-D Schrödinger operator H to be an integral operator.

2. Self-adjoint realizations and their spectral properties

Let $l \in (-1/2, 1/2)$ and denote by $H_{\max}$ the maximal operator associated with

$$\tau = -\frac{d^2}{dx^2} + \frac{l(l+1)}{x^2}$$

in $L^2(\mathbb{R}_+)$. Note that τ is limit point at infinity and limit circle at $x = 0$ since $|l| < 1/2$. Therefore, self-adjoint restrictions of $H_{\max}$ (or in other words, self-adjoint realizations of τ in $L^2(\mathbb{R}_+)$) form a 1-parameter family. More precisely (see, e.g., [7] and also [1]), the following limits

$$\Gamma_0 f := \lim_{x\to 0} W_x(f, x^{l+1}), \quad \Gamma_1 f := \frac{-1}{2l+1} \lim_{x\to 0} W_x(f, x^{-l}) \tag{2.1}$$

exist and are finite for all $f \in \operatorname{dom}(H_{\max})$. Self-adjoint restrictions H_α of $H_{\max}$ are parameterized by the following boundary conditions at $x = 0$:

$$\operatorname{dom}(H_\alpha) = \{f \in \operatorname{dom}(H_{\max})\colon \sin(\alpha)\,\Gamma_1 f = \cos(\alpha)\,\Gamma_0 f\}, \quad \alpha \in [0, \pi). \tag{2.2}$$

Note that the case $\alpha = 0$ corresponds to the Friedrichs extension of $H_{\min} = H_{\max}^*$.

Let $\phi(z, x)$ and $\theta(z, x)$ be the fundamental system of solutions of $\tau u = zu$ given by

$$\begin{aligned} \phi(z,x) &= C_l^{-1}\sqrt{\frac{\pi x}{2}}\, z^{-\frac{2l+1}{4}} J_{l+\frac{1}{2}}(\sqrt{z}x), \\ \theta(z,x) &= C_l\sqrt{\frac{\pi x}{2}}\, \frac{z^{\frac{2l+1}{4}}}{\sin((l+\frac{1}{2})\pi)} J_{-l-\frac{1}{2}}(\sqrt{z}x), \end{aligned} \tag{2.3}$$

where J_ν is the Bessel function of order ν (see Appendix A) and

$$C_l = \frac{\sqrt{\pi}}{\Gamma(l+\frac{3}{2})2^{l+1}}. \tag{2.4}$$

The Weyl solution normalized by $\Gamma_0\psi = 1$ is given by

$$\psi(z,x) = \theta(z,x) + m(z)\phi(z,x) = C_l \mathrm{i} z^{\frac{2l+1}{4}} \sqrt{\frac{\pi x}{2}} H^{(1)}_{l+1/2}(\sqrt{z}x) \in L^2(0,\infty), \tag{2.5}$$

where $H^{(1)}_\nu$ is the Hankel function of the first kind [17, Chapter X.2], and

$$m(z) = -C_l^2 \frac{(-z)^{l+1/2}}{\sin((l+\frac{1}{2})\pi)}, \quad z \in \mathbb{C}\setminus\mathbb{R}_+, \tag{2.6}$$

is the Weyl function associated with H_0. Here the branch cut of the root is taken along the negative real axis. Notice that

$$d\rho(\lambda) = \frac{C_l^2}{\pi}\mathbb{1}_{[0,\infty)}(\lambda)\lambda^{l+\frac{1}{2}}d\lambda \tag{2.7}$$

is the corresponding spectral measure. It follows from (A.1) that

$$\phi(z,x) = x^{l+1}(1+o(1)), \quad \theta(z,x) = \frac{x^{-l}}{2l+1}(1+o(1)),$$

as $x \to 0$ and, moreover,

$$\Gamma_0\theta = \Gamma_1\phi = 1, \quad \Gamma_1\theta = \Gamma_0\phi = 0.$$

Set

$$\begin{aligned}\phi_\alpha(z,x) &:= \cos(\alpha)\phi(z,x) + \sin(\alpha)\theta(z,x),\\ \theta_\alpha(z,x) &:= \cos(\alpha)\theta(z,x) - \sin(\alpha)\phi(z,x),\end{aligned} \tag{2.8}$$

for all $z \in \mathbb{C}$. Therefore, $W(\theta_\alpha,\phi_\alpha) = 1$ and

$$\psi_\alpha(z,x) := \theta_\alpha(z,x) + m_\alpha(z)\phi_\alpha(z,x), \quad m_\alpha(z) = \frac{m(z)\cos(\alpha)+\sin(\alpha)}{\cos(\alpha) - m(z)\sin(\alpha)}, \tag{2.9}$$

is a Weyl solution normalized by $W(\psi_\alpha,\phi_\alpha) = 1$. Hence

$$G_\alpha(z;x,y) = \begin{cases}\phi_\alpha(z,x)\psi_\alpha(z,y), & x \le y,\\ \phi_\alpha(z,x)\psi_\alpha(z,y), & x \ge y,\end{cases} \tag{2.10}$$

is the Green's function of H_α. The absolutely continuous spectrum remains unchanged, $\sigma_{\mathrm{ac}}(H_\alpha) = [0,\infty)$, but there is one additional eigenvalue

$$E_\alpha = -\left(\frac{\cot(\alpha)\cos(l\pi)}{C_l^2}\right)^{\frac{2}{2l+1}} \tag{2.11}$$

if $\frac{\pi}{2} < \alpha < \pi$. Finally, since

$$\operatorname{Im} m_\alpha(z) = \frac{\operatorname{Im} m(z)}{|\cos(\alpha) - m(z)\sin(\alpha)|^2}, \tag{2.12}$$

we get the absolutely continuous part of the corresponding spectral measure of the operator H_α:

$$\begin{aligned}\rho_\alpha'(\lambda)d\lambda &= \frac{1}{\pi}\operatorname{Im} m_\alpha(\lambda+\mathrm{i}0)d\lambda\\ &= \frac{1}{\pi}\frac{C_l^2\lambda^{l+1/2}\mathbb{1}_{[0,\infty)}(\lambda)}{(\cos(\alpha) - C_l^2\sin(\alpha)\tan(\pi l)\lambda^{l+1/2})^2 + C_l^4\sin^2(\alpha)\lambda^{2l+1}}d\lambda.\end{aligned} \tag{2.13}$$

3. Proof of Theorem 1.1

Similar to the case $\alpha = 0$ (see [14]), the kernel of the evolution group $\mathrm{e}^{-\mathrm{i}tH_{\pi/2}}$ can be computed explicitly.

Lemma 3.1. *Let $|l| < 1/2$. Then the evolution group $\mathrm{e}^{-\mathrm{i}tH_{\pi/2}}$ is an integral operator for all $t \neq 0$ and its kernel is given by*

$$[\mathrm{e}^{-\mathrm{i}tH_{\pi/2}}](x,y) = \frac{\mathrm{i}^{l-1/2}}{2t}\mathrm{e}^{\mathrm{i}\frac{x^2+y^2}{4t}}\sqrt{xy}\,J_{-l-1/2}\left(\frac{xy}{2t}\right), \tag{3.1}$$

for all x, $y > 0$ and $t \neq 0$.

Proof. First, notice that

$$\phi_{\pi/2}(z,x) = \theta(z,x), \quad m_{\pi/2}(z) = -1/m(z),$$

and then define the spectral transformation $U\colon L^2(\mathbb{R}_+) \to L^2(\mathbb{R}_+;\rho_{\pi/2})$ by

$$U\colon f \mapsto \hat{f}, \quad \hat{f}(\lambda) := \int_{\mathbb{R}_+}\theta(\lambda,x)f(x)dx,$$

for every $f \in L^2_c(\mathbb{R}_+)$. Notice that U extends to an isometry on $L^2(\mathbb{R}_+)$ and its inverse $U^{-1}\colon L^2(\mathbb{R}_+;\rho_{\pi/2}) \to L^2(\mathbb{R}_+)$ is given by

$$U^{-1}\colon g \mapsto \check{g}, \quad \check{g}(x) := \int_{\mathbb{R}_+}\theta(\lambda,x)g(\lambda)d\rho_{\pi/2}(\lambda),$$

for all $g \in L^2_c(\mathbb{R}_+;\rho_{\pi/2})$. Therefore, we get by using (2.3) and (2.13)

$$\begin{aligned}(\mathrm{e}^{-(\mathrm{i}t+\varepsilon)H_{\pi/2}}f)(x) =&(U^{-1}\mathrm{e}^{-(\mathrm{i}t+\varepsilon)\lambda}Uf)(x) = (U^{-1}\mathrm{e}^{-(\mathrm{i}t+\varepsilon)\lambda}\check{f})(x)\\ &= \int_{\mathbb{R}_+}\theta(\lambda,x)\mathrm{e}^{-(\mathrm{i}t+\varepsilon)\lambda}\int_{\mathbb{R}_+}\theta(\lambda,y)f(y)\,dy\,d\rho_{\pi/2}(\lambda)\\ &= \int_{\mathbb{R}_+}\int_{\mathbb{R}_+}\mathrm{e}^{-(\mathrm{i}t+\varepsilon)\lambda}\frac{\sqrt{xy}}{2}J_{-l-\frac12}(\sqrt{\lambda}x)J_{-l-\frac12}(\sqrt{\lambda}y)f(y)\,dy\,d\lambda.\end{aligned}$$

Since $|l|<1/2$, (A.1) implies that

$$|J_{-l-1/2}(k)| \le \frac{2^{l+1/2}}{\Gamma(1/2-l)k^{l+1/2}}(1+\mathcal{O}(k)) \tag{3.2}$$

as $k\to 0$. Noting that $f\in L^2_c(\mathbb{R}_+)$ and using (3.2), Fubini's theorem implies

$$(\mathrm{e}^{-(\mathrm{i}t+\varepsilon)H_{\pi/2}}f)(x) = \int_{\mathbb{R}_+}f(y)\int_{\mathbb{R}_+}\mathrm{e}^{-(\mathrm{i}t+\varepsilon)\lambda}\frac{\sqrt{xy}}{2}J_{-l-\frac12}(\sqrt{\lambda}x)J_{-l-\frac12}(\sqrt{\lambda}y)d\lambda\,dy. \tag{3.3}$$

The integral

$$[\mathrm{e}^{-(\mathrm{i}t+\varepsilon)H_{\pi/2}}](x,y) := \frac{\sqrt{xy}}{2}\int_0^\infty \mathrm{e}^{-\mathrm{i}t\lambda}J_{-l-\frac12}(\sqrt{\lambda}x)J_{-l-\frac12}(\sqrt{\lambda}y)d\lambda \tag{3.4}$$

is known as Weber's second exponential integral [21, §13.31] (cf. also [6, (4.14.39)]) and hence

$$(\mathrm{e}^{-(\mathrm{i}t+\varepsilon)H_{\pi/2}}f)(x) = \frac{1}{\varepsilon+\mathrm{i}t}\int_0^\infty \mathrm{e}^{-\frac{x^2+y^2}{4(\varepsilon+\mathrm{i}t)}}\frac{\sqrt{xy}}{2}I_{-l-\frac12}\Big(\frac{xy}{2(\varepsilon+\mathrm{i}t)}\Big)f(y)dy,$$

where I_ν is the modified Bessel function (see [17, Chapter X] and in particular formula (10.27.6) there)

$$I_\nu(z) = \sum_{n=0}^\infty \frac{(z/2)^{\nu+2n}}{n!\Gamma(\nu+m+1)} = \mathrm{e}^{\mp\mathrm{i}\nu\pi/2}J_\nu(\pm\mathrm{i}z), \quad -\pi\le\arg(z)\le\pi/2. \tag{3.5}$$

The estimate (A.2) implies

$$|J_{-l-1/2}(k)| \le k^{-1/2}(1+\mathcal{O}(k^{-1})) \tag{3.6}$$

as $k\to\infty$. Therefore, there is $C>0$ which depends only on l and such that

$$|\sqrt{k}J_{-l-1/2}(k)| \le C\left(\frac{1+k}{k}\right)^l, \quad k>0. \tag{3.7}$$

By (3.7) we deduce

$$\frac{\sqrt{xy}}{2|\varepsilon+\mathrm{i}t|}\left|\mathrm{e}^{-\frac{x^2+y^2}{4(\varepsilon+\mathrm{i}t)}}I_{-l-\frac12}\Big(\frac{xy}{2(\varepsilon+\mathrm{i}t)}\Big)\right| \le C\sqrt{\frac{1}{|\varepsilon+\mathrm{i}t|}}\left|1+\frac{2(\varepsilon+\mathrm{i}t)}{xy}\right|^l,$$

which is uniformly (wrt. ε) bounded on compact sets $K\subset\subset\mathbb{R}_+\times\mathbb{R}_+$. Thus we can apply dominated convergence and hence the claim follows. $\square$

In particular, we immediately arrive at the following estimate.

Corollary 3.2. *Let $|l|<1/2$. Then there is a constant $C>0$ which depends only on l and such that the inequality*

$$\left|[\mathrm{e}^{-\mathrm{i}tH_{\pi/2}}](x,y)\right| \le \frac{C}{\sqrt{2t}}\left(\frac{2t+xy}{xy}\right)^{l} \tag{3.8}$$

holds for all x, $y>0$ and $t>0$.

Proof. Applying (3.7) to (3.1), we arrive at (3.8). □

Remark 3.3. *For any fixed x and $y\in\mathbb{R}_+$, we get from* (A.1)

$$\left|\mathrm{e}^{-\mathrm{i}tH_{\pi/2}}(x,y)\right| \sim \frac{\sqrt{xy}}{2t}\left(\frac{xy}{4t}\right)^{-l-1/2} = \frac{1}{t^{1/2-l}}\left(\frac{xy}{2}\right)^{-l} \tag{3.9}$$

Moreover, in view of (A.1) *one can see that*

$$\left|\mathrm{e}^{-\mathrm{i}tH_{\pi/2}}(x,y)\right| \ge c_l\, t^{l-1/2}\left(\frac{xy}{2}\right)^{-l}, \tag{3.10}$$

whenever $xy<t$ with some constant $c_l>0$, which depends only on l.

Now we are ready to prove our first main result.

Proof of Theorem 1.1. If $l\in(-1/2,0]$, then

$$\left(\frac{2t+xy}{xy}\right)^{l} \le 1$$

for all x,$y>0$ and $t\ge 0$. This immediately implies (1.7).

Assume now that $l\in(0,1/2)$. Clearly,

$$\frac{2t+xy}{xy} = 1+2\frac{t}{xy} \le 3t\max(x^{-1},1)\max(y^{-1},1)$$

for all $t\ge 1$ and x, $y>0$. Indeed, the latter follows from the weaker estimate

$$\frac{t}{xy} \le t\max(x^{-1},1)\max(y^{-1},1), \quad t\ge 1,\ x,y>0,$$

which is equivalent to $1\le \max(x,1)\max(y,1)$ for all x, $y>0$. Therefore,

$$\left(\frac{2t+xy}{xy}\right)^{l} \le 3t^{l}\max(x^{-l},1)\max(y^{-l},1), \quad t\ge 1,\ x,y>0,$$

which proves (1.8). Remark 3.3 shows that (1.8) is sharp. □

4. Proof of Theorem 1.2

Let us consider the following improper integrals:

$$I_1(t;x,y) := \sqrt{xy}\int_{\mathbb{R}_+} \mathrm{e}^{-\mathrm{i}tk^2} J_{l+\frac12}(kx)J_{l+\frac12}(ky)\,\mathrm{Im}\, m_\alpha(k^2)\, k^{-2l}dk, \tag{4.1}$$

$$I_2(t;x,y) := \sqrt{xy}\int_{\mathbb{R}_+} \mathrm{e}^{-\mathrm{i}tk^2} J_{l+\frac12}(kx)J_{-l-\frac12}(ky)\,\mathrm{Im}\, m_\alpha(k^2)\, kdk, \tag{4.2}$$

$$I_3(t;x,y) := \sqrt{xy}\int_{\mathbb{R}_+} \mathrm{e}^{-\mathrm{i}tk^2} J_{-l-\frac12}(kx)J_{-l-\frac12}(ky)\,\mathrm{Im}\, m_\alpha(k^2)\, k^{2l+2}dk, \tag{4.3}$$

where x, $y>0$ and $t\neq 0$. Moreover, here and below we shall use the convention $\mathrm{Im}\, m_\alpha(k^2) := \mathrm{Im}\, m_\alpha(k^2+\mathrm{i}0) = \lim_{\varepsilon\downarrow 0}\mathrm{Im}\, m_\alpha(k^2+\mathrm{i}\varepsilon)$ for all $k\in\mathbb{R}$. Denote the corresponding integrand by A_j, that is, $I_j(t) = \int_{\mathbb{R}_+}\mathrm{e}^{-\mathrm{i}tk^2}A_j(k;x,y)dk$. Our aim is to use Lemma B.2 (plus the remarks after this lemma) and hence we need to show that each A_j belongs to the Wiener algebra $\mathcal{W}(\mathbb{R})$, that is, coincide with a function which is the Fourier transform of a finite measure.

We also need the following estimates, which follow from (2.13)

$$\operatorname{Im} m_\alpha(k^2) = \begin{cases} C_l^2 |k|^{2l+1}, & \alpha = 0, \\ \frac{\cos^2(\pi l)}{C_l^2 \sin^2(\alpha)} |k|^{-2l-1} + \mathcal{O}(|k|^{-4l-2}), & \alpha \neq 0, \end{cases} \quad k \to \infty, \tag{4.4}$$

and

$$\operatorname{Im} m_\alpha(k^2) = \begin{cases} \frac{C_l^2}{\cos(\alpha)^2} |k|^{2l+1} + \mathcal{O}(|k|^{4l+2}), & \alpha \neq \pi/2, \\ C_l^{-2} \cos^2(\pi l) |k|^{-2l-1}, & \alpha = \pi/2, \end{cases} \quad k \to 0. \tag{4.5}$$

4.1. **The integral I_1.** Consider the function

$$J(r) := \sqrt{r}\, J_{l+\frac{1}{2}}(r) = \frac{r^{l+1}}{2^{l+1/2}} \sum_{n=0}^{\infty} \frac{(-r^2/4)^n}{n!\Gamma(\nu+n+1)}, \quad r \geq 0.$$

Note that $J(r) \sim r^{l+1}$ as $r \to 0$ and $J(r) = \sqrt{\frac{2}{\pi}} \sin(r - \frac{l\pi}{2}) + O(r^{-1})$ as $r \to +\infty$ (see (A.2)). Moreover, $J'(r) \sim r^l$ as $r \to 0$ and $J'(r) = \sqrt{\frac{2}{\pi}} \cos(r - \frac{l\pi}{2}) + O(r^{-1})$ as $r \to +\infty$ (see (A.4)). In particular, $\tilde{J}(r) := J(r) - \sqrt{\frac{2}{\pi}} \sin(r - \frac{l\pi}{2})$ is in $H^1(\mathbb{R}_+)$. Moreover, we can define $J(r)$ for $r < 0$ such that it is locally in H^1 and $J(r) = \sqrt{\frac{2}{\pi}} \sin(r - \frac{l\pi}{2})$ for $r < -1$. By construction we then have $\tilde{J} \in H^1(\mathbb{R})$ and thus $\tilde{J}$ is the Fourier transform of an integrable function (see Lemma B.3). Moreover, $\sin(r - \frac{l\pi}{2})$ is the Fourier transform of the sum of two Dirac delta measures and so J is the Fourier transform of a finite measure. By scaling, the total variation of the measures corresponding to $J(kx)$ is independent of x.

Next consider the function

$$F(k) := \frac{\operatorname{Im} m_\alpha(k^2)}{|k|^{2l+1}} = \frac{C_l^2}{(\cos(\alpha) - C_l^2 \sin(\alpha)\tan(\pi l)|k|^{2l+1})^2 + C_l^4 \sin^2(\alpha)|k|^{4l+2}}.$$

By Corollary B.6, F is in the Wiener algebra $\mathcal{W}_0(\mathbb{R})$.

Now it remains to note that

$$I_1(t) = \int_{\mathbb{R}_+} \mathrm{e}^{-\mathrm{i}tk^2} A_1(k^2; x, y) dk = \int_{\mathbb{R}_+} \mathrm{e}^{-\mathrm{i}tk^2} J(kx) J(ky) F(k) dk, \tag{4.6}$$

and applying Lemma B.2 we end up with the estimate

$$|I_1(t; x, y)| \leq C t^{-1/2}, \quad t > 0, \tag{4.7}$$

with a positive constant $C > 0$ independent of x, $y > 0$.

4.2. **The integral I_2.** Assume first that $l \in (0, 1/2)$ and write

$$A_2(k^2; x, y) = J(kx) Y(ky) \frac{\chi_l(k)}{\chi_l(ky)} \frac{\operatorname{Im} m_\alpha(k^2)}{\chi_l(k)},$$

where

$$J(r) = \sqrt{r}\, J_{l+\frac{1}{2}}(r), \quad Y(r) = \chi_l(r) \sqrt{r}\, J_{-l-\frac{1}{2}}(r), \quad \chi_l(r) = \frac{|r|^l}{1 + |r|^l}.$$

The asymptotic behavior (4.4) and (4.5) of $\operatorname{Im} m_\alpha$ shows that

$$M(k) = \frac{\operatorname{Im} m_\alpha(k^2)}{\chi_l(k)} = \begin{cases} |k|^{1+l}, & k \to 0, \\ |k|^{-2l-1}, & |k| \to \infty, \end{cases}$$

and hence $M \in H^1(\mathbb{R})$, which implies that M is in the Wiener algebra $\mathcal{W}_0(\mathbb{R})$.

We continue $J(r)$, $Y(r)$ to the region $r < 0$ such that they are continuously differentiable and satisfy

$$J(r) = \sqrt{\frac{2}{\pi}} \sin\left(r - \frac{\pi l}{2}\right), \quad Y(r) = \sqrt{\frac{2}{\pi}} \cos\left(r + \frac{\pi l}{2}\right),$$

for $r < -1$. Then $\tilde{J}(r) := J(r) - \sqrt{\frac{2}{\pi}} \sin(r - \frac{\pi l}{2})$ and $\tilde{Y}(r) := Y(r) - \sqrt{\frac{2}{\pi}} \cos\left(r + \frac{\pi l}{2}\right)$ are in $H^1(\mathbb{R})$. In fact, they are continuously differentiable and hence it suffices to look at their asymptotic behavior. For $r < -1$ they are zero and for $r > 1$ they are $O(r^{-1})$ and their derivative is $O(r^{-1})$ as can be seen from the asymptotic behavior of Bessel functions (see Appendix A). Hence both J and Y are Fourier transforms of finite measures. By scaling the total variation of the measures corresponding to $J(kx)$ and $Y(ky)$ are independent of x and y, respectively.

It remains to consider the function $\chi_l(k)/\chi_l(ky)$. Observe that

$$h_{y,l}(k) := 1 - \frac{\chi_l(k)}{\chi_l(ky)} = 1 - \frac{1 + |ky|^l}{y^l + |ky|^l} = \frac{1 - y^{-l}}{1 + |k|^l} = (1 - y^{-l})(1 - \chi_l(k)).$$

By Corollary B.6, $1 - \chi_l \in \mathcal{W}_0(\mathbb{R})$. Therefore, applying Lemma B.2, we obtain the following estimate

$$|I_2(t; x, y)| \le C t^{-1/2} \max(1, y^{-l}), \quad t > 0, \tag{4.8}$$

whenever $l \in (0, 1/2)$.

Consider now the remaining case $l \in (-1/2, 0]$. Write

$$A_2(k^2; x, y) = J(kx) Y(ky) \operatorname{Im} m_\alpha(k^2),$$

where

$$J(r) = \sqrt{r}\, J_{l+\frac{1}{2}}(r), \quad Y(r) = \sqrt{r}\, J_{-l-\frac{1}{2}}(r).$$

Noting that $Y(r) \sim r^{-l}$ as $r \to 0$ and using Lemma B.3, we can continue J and Y to the region $r < 0$ such that both J and Y are Fourier transforms of finite measures.

It remains to consider $\operatorname{Im} m_\alpha(k^2)$ given by (2.13). However, by Corollary B.6, this function is in the Wiener algebra $\mathcal{W}_0(\mathbb{R})$ and hence applying Lemma B.2, we end up with the estimate

$$|I_2(t; x, y)| \le C t^{-1/2}, \quad t > 0, \tag{4.9}$$

whenever $l \in (-1/2, 0]$.

4.3. **The integral I_3.** Again let us consider two cases. Assume first that $l \in (-1/2, 0]$ and then write

$$A_3(k^2; x, y) = Y(kx) Y(ky) \operatorname{Im} m_\alpha(k^2) k^{2l+1},$$

where

$$Y(r) = \sqrt{r}\, J_{-l-\frac{1}{2}}(r), \quad r > 0.$$

Notice that

$$|k|^{2l+1} \operatorname{Im} m_\alpha(k^2) = \frac{C_l^2 k^{4l+2}}{(\cos(\alpha) - C_l^2 \sin(\alpha) \tan(\pi l) k^{2l+1})^2 + C_l^4 \sin^2(\alpha) k^{4l+2}},$$

which is the sum of a constant and a function of the form (B.5), and hence it belongs to the Wiener algebra $\mathcal{W}(\mathbb{R})$ by Corollary B.6. Arguing as in the previous subsection and applying Lemma B.2, we arrive at the following estimate

$$|I_3(t;x,y)| \le Ct^{-1/2}, \quad t>0, \tag{4.10}$$

whenever $l \in (-1/2, 0]$.

If $l \in (0, 1/2)$, write

$$A_3(k^2;x,y) = Y(kx)Y(ky)\frac{\chi_l(k)}{\chi_l(kx)}\frac{\chi_l(k)}{\chi_l(ky)}\frac{\operatorname{Im} m_\alpha(k^2)}{\chi_l^2(k)},$$

where

$$Y(r) = \chi_l(r)\sqrt{r}\, J_{-l-\frac{1}{2}}(r), \quad \chi_l(r) = \frac{|r|^l}{1+|r|^l}.$$

Notice that

$$\begin{aligned} M(k) :=& \frac{\operatorname{Im} m_\alpha(k^2)|k|^{2l+1}}{\chi_l^2(k)} \\ =& \frac{C_l^2|k|^{2l+2}(1+k^l)^2}{(\cos(\alpha) - C_l^2\sin(\alpha)\tan(\pi l)|k|^{2l+1})^2 + C_l^4\sin^2(\alpha)|k|^{4l+2}} \end{aligned}$$

Clearly, by Corollary B.6, $M \in \mathcal{W}(\mathbb{R})$. Therefore, similar to the previous subsection, we end up with the estimate

$$|I_3(t;x,y)| \le Ct^{-1/2}\max(1,x^{-l})\max(1,y^{-l}), \quad t>0, \tag{4.11}$$

whenever $l \in (0, 1/2)$.

4.4. **Proof of Theorem 1.2.** We begin with the representation of the integral kernel of the evolution group.

Lemma 4.1. *Let $|l| < 1/2$ and $\alpha \in [0,\pi)$. Then the evolution group $\mathrm{e}^{-\mathrm{i}tH_\alpha}P_c(H_\alpha)$ is an integral operator and its kernel is given by*

$$[\mathrm{e}^{-\mathrm{i}tH_\alpha}P_c(H_\alpha)](x,y) = \frac{2}{\pi}\int_{\mathbb{R}_+}\mathrm{e}^{-\mathrm{i}tk^2}\,\phi_\alpha(k^2,x)\phi_\alpha(k^2,y)\operatorname{Im} m_\alpha(k^2)k\,dk, \tag{4.12}$$

where the integral is to be understood as an improper integral.

Proof. By (2.3) and (2.8),

$$\begin{aligned} \phi_\alpha(k^2,x) &= \cos(\alpha)\phi(k^2,x) + \sin(\alpha)\theta(k^2,x) \\ &= \sqrt{\frac{\pi x}{2}}\left(C_l^{-1}\cos(\alpha)k^{-l-1/2}J_{l+\frac{1}{2}}(kx) + C_l k^{l+1/2}\frac{\sin(\alpha)}{\cos(\pi l)}J_{-l-\frac{1}{2}}(kx)\right), \end{aligned}$$

and hence

$$\phi_\alpha(k^2,x)\phi_\alpha(k^2,y) = \frac{\pi}{2}\sqrt{xy}\left(\frac{\cos^2(\alpha)}{C_l^2}k^{-2l-1}J_{l+\frac{1}{2}}(kx)J_{l+\frac{1}{2}}(ky)\right. \tag{4.13}$$

$$+\frac{\sin(2\alpha)}{2\cos(\pi l)}(J_{l+\frac{1}{2}}(kx)J_{-l-\frac{1}{2}}(ky) + J_{-l-\frac{1}{2}}(kx)J_{l+\frac{1}{2}}(ky)) \tag{4.14}$$

$$\left.+C_l^2k^{2l+1}\frac{\sin^2(\alpha)}{\cos^2(\pi l)}J_{-l-\frac{1}{2}}(kx)J_{-l-\frac{1}{2}}(ky)\right). \tag{4.15}$$

By our considerations in the previous subsections, we have

$$\phi_\alpha(k^2,x)\phi_\alpha(k^2,y)\operatorname{Im} m_\alpha(k^2)k \in \mathcal{W}(\mathbb{R})$$

with norm uniformly bounded for x, y restricted to any compact subset of $(0,\infty)$. Moreover, we have $\mathrm{e}^{-\mathrm{i}(t-\mathrm{i}\varepsilon)H_\alpha}P_c(H_\alpha) \to \mathrm{e}^{-\mathrm{i}tH_\alpha}P_c(H_\alpha)$ as $\varepsilon \downarrow 0$ in the strong operator topology. By Lemma C.1, $\mathrm{e}^{-\mathrm{i}(t-\mathrm{i}\varepsilon)H_\alpha}P_c(H_\alpha)$ is an integral operator for all $\varepsilon > 0$ and, moreover, the kernel converges uniformly on compact sets by Lemma C.2. Hence $\mathrm{e}^{-\mathrm{i}tH_\alpha}P_c(H_\alpha)$ is an integral operator whose kernel is given by the limits of the kernels of the approximating operators, that is, by (4.12). □

Proof of Theorem 1.2. Combining (4.7), (4.8), (4.9), (4.10) and (4.11), we arrive at the following decay estimate for the kernel of the evolution group

$$\Big|[\mathrm{e}^{-\mathrm{i}tH_\alpha}P_c(H_\alpha)](x,y)\Big| \le Ct^{-1/2}\times\begin{cases}1, & l\in(-1/2,0],\\ \max(1,x^{-l})\max(1,y^{-l}), & l\in(0,1/2).\end{cases} \tag{4.16}$$

This completes the proof of Theorem 1.2. □

Appendix A. Bessel functions

Here we collect basic formulas and information on Bessel functions (see, e.g., [17, 21]). We start with the definition:

$$J_\nu(z) = \Big(\frac{z}{2}\Big)^\nu \sum_{n=0}^\infty \frac{(-z^2/4)^n}{n!\Gamma(\nu+n+1)}. \tag{A.1}$$

The asymptotic behavior as $|z|\to\infty$ is given by

$$J_\nu(z) = \sqrt{\frac{2}{\pi z}}\Big(\cos(z-\nu\pi/2-\pi/4) + \mathrm{e}^{|\operatorname{Im} z|}\mathcal{O}(|z|^{-1})\Big), \quad |\arg z|<\pi. \tag{A.2}$$

Noting that

$$J_\nu'(z) = -J_{\nu+1}(z) + \frac{\nu}{z}J_\nu(z) = J_{\nu-1}(z) - \frac{\nu}{z}J_\nu(z), \tag{A.3}$$

one can show that the derivative of the reminder satisfies

$$\Big(\sqrt{\frac{\pi z}{2}}J_\nu(z) - \cos(z-\frac{1}{2}\nu\pi-\frac{1}{4}\pi)\Big)' = \mathrm{e}^{|\operatorname{Im} z|}\mathcal{O}(|z|^{-1}), \quad |z|\to\infty. \tag{A.4}$$

Appendix B. The van der Corput Lemma and the Wiener algebra

We will need the classical van der Corput lemma (see, e.g., [19, page 334]):

Lemma B.1. *Consider the oscillatory integral*

$$I(t) = \int_a^b \mathrm{e}^{\mathrm{i}tk^2+\mathrm{i}ck}A(k)dk.$$

If $A\in \mathrm{AC}(a,b)$, then

$$|I(t)| \le C_2\,|t|^{-1/2}\,(\|A\|_\infty + \|A'\|_1), \quad |t|\ge 1,$$

where $C_2 \le 2^{8/3}$ is a universal constant.

Note that we can apply the above result with $(a,b)=(-\infty,\infty)$ by considering the limit $(-a,a)\to(-\infty,\infty)$.

Our proof will be based on the following variant of the van der Corput lemma (see, e.g., [13, Lemma A.2]).

Lemma B.2. *Let $(a,b) \subseteq \mathbb{R}$ and consider the oscillatory integral*

$$I(t) = \int_a^b \mathrm{e}^{\mathrm{i}tk^2} A(k)dk.$$

If $A \in \mathcal{W}(\mathbb{R})$, i.e., A is the Fourier transform of a signed measure

$$A(k) = \int_{\mathbb{R}} \mathrm{e}^{\mathrm{i}kp} d\alpha(p),$$

then the above integral exists as an improper integral and satisfies

$$|I(t)| \le C_2 \, |t|^{-1/2} \, \|A\|_{\mathcal{W}}, \quad |t| > 0.$$

where $\|A\|_{\mathcal{W}} := \|\alpha\| = |\alpha|(\mathbb{R})$ denotes the total variation of α and C_2 is the constant from the van der Corput lemma.

In this respect we note that if A_1 and A_2 are two such functions, then (cf. p. 208 in [2])

$$(A_1 A_2)(k) = \frac{1}{(2\pi)^2} \int_{\mathbb{R}} \mathrm{e}^{\mathrm{i}kp} d(\alpha_1 * \alpha_2)(p)$$

is associated with the convolution

$$\alpha_1 * \alpha_2(\Omega) = \iint \mathbb{1}_\Omega(x+y) d\alpha_1(x) d\alpha_2(y),$$

where $\mathbb{1}_\Omega$ is the indicator function of a set Ω. Note that

$$\|\alpha_1 * \alpha_2\| \le \|\alpha_1\| \|\alpha_2\|.$$

Let $\mathcal{W}_0(\mathbb{R})$ be the Wiener algebra of functions $C(\mathbb{R})$ which are Fourier transforms of L^1 functions,

$$\mathcal{W}_0(\mathbb{R}) = \Big\{ f \in C(\mathbb{R}) : \; f(k) = \int_{\mathbb{R}} \mathrm{e}^{\mathrm{i}kx} g(x)dx, \; g \in L^1(\mathbb{R}) \Big\}.$$

Clearly, $\mathcal{W}_0(\mathbb{R}) \subset \mathcal{W}(\mathbb{R})$. Moreover, by the Riemann–Lebesgue lemma, $f \in C_0(\mathbb{R})$, that is, $f(k) \to 0$ as $k \to \infty$ if $f \in \mathcal{W}_0(\mathbb{R})$. A comprehensive survey of necessary and sufficient conditions for $f \in C(\mathbb{R})$ to be in the Wiener algebras $\mathcal{W}_0(\mathbb{R})$ and $\mathcal{W}(\mathbb{R})$ can be found in [15], [16]. We need the following statements.

Lemma B.3. *If $f \in L^2(\mathbb{R})$ is locally absolutely continuous and $f' \in L^p(\mathbb{R})$ with $p \in (1,2]$, then f is in the Wiener algebra $\mathcal{W}_0(\mathbb{R})$ and*

$$\|f\|_{\mathcal{W}} \le C_p \big(\|f\|_{L^2(\mathbb{R})} + \|f'\|_{L^p(\mathbb{R})} \big), \tag{B.1}$$

where $C_p > 0$ is a positive constant, which depends only on p.

Proof. Since the Fourier transform is unitary on $L^2(\mathbb{R})$, it suffices to show that $\hat{f} \in L^1(\mathbb{R})$. First of all, the Cauchy–Schwarz inequality implies $\hat{f} \in L^1_{\mathrm{loc}}(\mathbb{R})$ and, in particular,

$$\int_{-1}^1 |\hat{f}(\lambda)| d\lambda \le \sqrt{2} \left(\int_{-1}^1 |\hat{f}(\lambda)|^{1/2} d\lambda \right)^2 \le \sqrt{2} \|f\|_{L^2(\mathbb{R})}. \tag{B.2}$$

On the other hand, $f' \in L^p(\mathbb{R})$ and hence the Hausdorff–Young inequality implies $\lambda \hat{f}(\lambda) \in L^q(\mathbb{R})$ with $1/p + 1/q = 1$. Applying the Hölder inequality and then the

Hausdorff–Young inequality once again, we get

$$\begin{aligned}\int_{|\lambda|>1}|\hat{f}(\lambda)|d\lambda &\le 2\int_{|\lambda|>1}\frac{1}{1+|\lambda|}|\lambda\hat{f}(\lambda)|d\lambda\\ &\le 2\left(\int_{\mathbb{R}}\frac{1}{(1+|\lambda|)^p}d\lambda\right)^{1/p}\left(\int_{\mathbb{R}}|\lambda\hat{f}(\lambda)|^q d\lambda\right)^{1/q}\le C_p'\|f'\|_{L^p(\mathbb{R})},\end{aligned}$$

which completes the proof. □

Remark B.4. *The case $p=2$ is due to Beurling* [15, Theorem 5.3]. *A similar result was obtained by S. G. Samko. Namely, if $f\in L^1(\mathbb{R})\cap AC_{\mathrm{loc}}(\mathbb{R})$ is such that $f,\ f'\in L^p(\mathbb{R})$ with some $p\in(1,2]$, then $f\in\mathcal{W}_0(\mathbb{R})$ (see Theorem 6.8 in* [15]*).*

The next result is also due to Beurling (see, e.g., Theorem 5.4 in [15]).

Theorem B.5 (Beurling)**.** *Let $f\in C_0(\mathbb{R})$ be even and $f,\ f'\in AC_{\mathrm{loc}}(\mathbb{R})$. If*

$$C:=\int_{\mathbb{R}_+}k|f''(k)|dk<\infty, \tag{B.3}$$

then $f\in\mathcal{W}_0(\mathbb{R})$ and $\|f\|_{\mathcal{W}}\le C$.

Consider the following functions, which appear in Section 4:

$$\chi_l(k)=\frac{|k|^l}{1+|k|^l},\quad l>0, \tag{B.4}$$

$$f_{l,p}(k)=\frac{|k|^p}{a+b|k|^l+|k|^{2l}},\quad 2l>p\ge 0, \tag{B.5}$$

where $a,\ b\in\mathbb{R}$ are such that $a+b|k|^p+|k|^{2p}>0$ for all $k\in\mathbb{R}$. As an immediate corollary of Beurling's result we get

Corollary B.6. *$\chi_l\in\mathcal{W}(\mathbb{R})$, $1-\chi_l\in\mathcal{W}_0(\mathbb{R})$, and $f_{l,p}\in\mathcal{W}_0(\mathbb{R})$.*

Appendix C. Integral kernels

There are various criteria for operators in L^p spaces to be integral operators (see, e.g., [3]). Below we present a simple sufficient condition on a function K for $K(H)$ to be an integral operator, where H is a one-dimensional Schrödinger operator. More precisely, let H be a singular Schrödinger operator on $L^2(a,b)$ as in [11] or [12] with corresponding entire system of solutions $\theta(z,x)$ and $\phi(z,x)$. Recall

$$(H-z)^{-1}f(x)=\int_a^b G(z,x,y)f(y)dy, \tag{C.1}$$

where

$$G(z,x,y)=\begin{cases}\phi(z,x)\psi(z,y), & y\ge x,\\ \phi(z,y)\psi(z,x), & y\le x,\end{cases} \tag{C.2}$$

is the Green function of H and $\psi(z,x)$ is the Weyl solution normalized by $W(\theta,\psi)=1$ (cf. [20, Lem. 9.7]). We start with a simple lemma ensuring that a function $K(H)$ is an integral operator. To this end recall that $K(H)$ is defined as $U^{-1}KU$ with K the multiplication operator in $L^2(\mathbb{R},d\rho)$, ρ the associates spectral measure, and $U: L^2(a,b)\to L^2(\mathbb{R},d\rho)$ the spectral transformation

$$(Uf)(\lambda)=\int_a^b\phi(\lambda,x)f(x)dx. \tag{C.3}$$

Lemma C.1. *Suppose H is bounded from below and $|K(\lambda)| \le C(1+|\lambda|)^{-1}$ or otherwise $|K(\lambda)| \le C(1+|\lambda|)^{-2}$. Then $K(H)$ is an integral operator*

$$(K(H)f)(x) = \int_a^b K(x,y)f(y)dy, \tag{C.4}$$

with kernel

$$K(x,y) = \int_{\mathbb{R}} K(\lambda)\phi(\lambda,x)\phi(\lambda,y)d\rho(\lambda). \tag{C.5}$$

In particular, $(1+|.|)^{-1/2}\phi(.,x) \in L^2(\mathbb{R},d\rho)$ and $K(x,.) \in L^2(a,b)$ for every $x \in (a,b)$.

Proof. Note that (cf. [11, Lemma 3.6])

$$(UG(z;x,.))(\lambda) = \frac{\phi(\lambda,x)}{z-\lambda}.$$

If H is bounded from below then $G(z;x,.)$ is in the form domain of H for fixed x and every $z \in \mathbb{C}\setminus\sigma(H)$ (cf. [8, (A.6)]) and we obtain from [11, Lemma 3.6] that $(1+|\lambda|)^{-1/2}\phi(\lambda,x) \in L^2(\mathbb{R},d\rho)$. In the general case we at least have $G(z;x,.) \in L^2(a,b)$ and thus $(1+|\lambda|)^{-1}\phi(\lambda,x) \in L^2(\mathbb{R},d\rho)$. Hence we can use Fubini's theorem to evaluate

$$\begin{aligned} K(H)f(x) = U^{-1}KUf(x) &= \int_{\mathbb{R}} \phi(x,\lambda)K(\lambda)\left(\int_a^b \phi(\lambda,y)f(y)dy\right)d\rho(\lambda) \\ &= \int_a^b K(x,y)f(y)dy. \end{aligned}$$ □

As a consequence we obtain that (4.12) holds at least for $\mathrm{Im}(t) < 0$. To take the limit $\mathrm{Im}(t) \to 0$ we need the following result which follows from [5, Lemma 3.1].

Lemma C.2. *Consider the improper integral*

$$F(\varepsilon) = \int_{-\infty}^{\infty} \mathrm{e}^{-\mathrm{i}(t+\mathrm{i}\varepsilon)k^2} f(k)dk, \qquad \varepsilon \le 0,$$

where

$$f(k) = \int_{\mathbb{R}} \mathrm{e}^{\mathrm{i}kp} d\alpha(p), \qquad |\alpha|(\mathbb{R}) < \infty.$$

Then

$$F(\varepsilon) = \frac{1}{\sqrt{4\pi\mathrm{i}(t+\mathrm{i}\varepsilon)}} \int_{\mathbb{R}} \mathrm{e}^{-\frac{p^2}{4(t+\mathrm{i}\varepsilon)}} d\alpha(p).$$

References

[1] A. Ananieva and V. Budika, *To the spectral theory of the Bessel operator on a finite interval and half-line*, Ukrainian Mat. Visnyk, **12**, no. 2, 160–199 (2015), (in Russian); *English transl.:* J. Math. Sci. **211**, 624–645 (2015).

[2] V. I. Bogachev, *Measure Theory. I*, Springer-Verlag, Berlin, Heidelberg, 2007.

[3] A. V. Bukhvalov, *Application of methods of the theory of order-bounded operators to the theory of operators in L^p-spaces*, Russ. Math. Surveys **38**, no. 6, 43–98 (1983).

[4] J. Dereziński and S. Richard, *On almost homogeneous Schrödinger operators*, arXiv:1604.03340

[5] I. Egorova, E. Kopylova, V. Marchenko, and G. Teschl, *Dispersion estimates for one-dimensional Schrödinger and Klein–Gordon equations revisited*, Russ. Math. Surveys **71**, 391–415 (2016).

[6] A. Erdelyi, *Tables of Integral Transforms, Vol. 1*, McGraw-Hill, New York, 1954.

[7] W. N. Everitt and H. Kalf, *The Bessel differential equation and the Hankel transform*, J. Comp. Appl. Math. **208**, 3–19 (2007).

[8] F. Gesztesy, B. Simon, and G. Teschl, *Zeros of the Wronskian and renormalized oscillation theory*, Amer. J. Math. **118**, 571–594 (1996).

[9] M. Goldberg and W. Schlag, *Dispersive estimates for Schrödinger operators in dimensions one and three*, Comm. Math. Phys. **251**, 157–178 (2004).

[10] E. Kopylova, *Dispersion estimates for Schrödinger and Klein–Gordon equation*, Russ. Math. Surveys, **65**, no. 1, 95–142 (2010).

[11] A. Kostenko, A. Sakhnovich, and G. Teschl, *Weyl–Titchmarsh theory for Schrödinger operators with strongly singular potentials*, Int. Math. Res. Not. **2012**, 1699–1747 (2012).

[12] A. Kostenko and G. Teschl, *Spectral asymptotics for perturbed spherical Schrödinger operators and applications to quantum scattering*, Comm. Math. Phys. **322**, 255–275 (2013).

[13] A. Kostenko, G. Teschl and J. H. Toloza, *Dispersion estimates for spherical Schrödinger equations*, Ann. Henri Poincaré **17**, 3147–3176 (2016).

[14] H. Kovařík and F. Truc, *Schrödinger operators on a half-line with inverse square potentials*, Math. Model. Nat. Phenom. **9**, no. 5, 170–176 (2014).

[15] E. Liflyand, S. Samko and R. Trigub, *The Wiener algebra of absolutely convergent Fourier integrals: an overview*, Anal. Math. Phys. **2**, 1–68, (2012).

[16] E. Liflyand and R. Trigub, *Conditions for the absolute convergence of Fourier integrals*, J. Approx. Theory **163**, 438–459 (2011).

[17] F. W. J. Olver et al., *NIST Handbook of Mathematical Functions*, Cambridge University Press, Cambridge, 2010.

[18] W. Schlag, *Dispersive estimates for Schrödinger operators: a survey*, in "Mathematical aspects of nonlinear dispersive equations", 255–285, Ann. Math. Stud. **163**, Princeton Univ. Press, Princeton, NJ, 2007.

[19] E. M. Stein, *Harmonic analysis: real-variable methods, orthogonality, and oscillatory integrals*, Princeton Math. Series **43**, Princeton University Press, Princeton, NJ, 1993.

[20] G. Teschl, *Mathematical Methods in Quantum Mechanics; With Applications to Schrödinger Operators*, 2nd ed., Amer. Math. Soc., Rhode Island, 2014.

[21] G. N. Watson, *A Treatise on the Theory of Bessel Functions*, Cambridge Univ. Press, 1944.

[22] R. Weder, *$L^p - L^{\dot{p}}$ estimates for the Schrödinger equation on the line and inverse scattering for the nonlinear Schrödinger equation with a potential*, J. Funct. Anal. **170**, 37–68 (2000).

[23] R. Weder, *The $L^p - L^{\dot{p}}$ estimates for the Schrödinger equation on the half-line*, J. Math. Anal. Appl. **281**, 233–243 (2003).

Dispersion Estimates for Spherical Schrödinger Equations with Critical Angular Momentum

by Markus Holzleitner, Aleksey Kostenko and Gerald Teschl

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DISPERSION ESTIMATES FOR SPHERICAL SCHRÖDINGER EQUATIONS WITH CRITICAL ANGULAR MOMENTUM

MARKUS HOLZLEITNER, ALEKSEY KOSTENKO, AND GERALD TESCHL

To Helge Holden, inspiring colleague and friend, on the occasion of his 60th birthday

Abstract. We derive a dispersion estimate for one-dimensional perturbed radial Schrödinger operators, where the angular momentum takes the critical value $l = -\frac{1}{2}$. We also derive several new estimates for solutions of the underlying differential equation and investigate the behavior of the Jost function near the edge of the continuous spectrum.

1. Introduction

The stationary one-dimensional radial Schrödinger equation

$$\mathrm{i}\dot{\psi}(t,x) = H\psi(t,x), \quad H := -\frac{d^2}{dx^2} + \frac{l(l+1)}{x^2} + q(x), \quad (t,x) \in \mathbb{R} \times \mathbb{R}_+, \tag{1.1}$$

is a well-studied object in quantum mechanics. Starting from the Schrödinger equation with a spherically symmetric potential in three dimensions, one obtains (1.1) with l a nonnegative integer. However, other dimensions will lead to different values for l (see e.g. [34, Sect. 17.F]). In particular, the half-integer values arise in two dimensions and hence are equally important. Moreover, the integer case is typically not more difficult than the case $l > -\frac{1}{2}$ but the borderline case $l = -\frac{1}{2}$ usually imposes additional technical problems. For example in [19] we investigated the dispersive properties of the associated radial Schrödinger equation, but were not able to cover the case $l = -\frac{1}{2}$. This was also partly due to the fact that several results we relied upon were only available for the case $l > -\frac{1}{2}$. The present paper aims at filling this gap by investigating

$$\mathrm{i}\dot{\psi}(t,x) = H\psi(t,x), \quad H := -\frac{d^2}{dx^2} - \frac{1}{4x^2} + q(x), \quad (t,x) \in \mathbb{R} \times \mathbb{R}_+, \tag{1.2}$$

with real locally integrable potential q. We will use τ to describe the formal Sturm–Liouville differential expression and H the self-adjoint operator acting in $L^2(\mathbb{R}_+)$ and given by τ together with the Friedrichs boundary condition at $x = 0$:

$$\lim_{x\to 0} W(\sqrt{x}, f(x)) = 0. \tag{1.3}$$

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More specifically, our goal is to provide dispersive decay estimates for these equations. To this end we recall that under the assumption

$$\int_0^\infty x(1+|\log(x)|)|q(x)|dx < \infty$$

the operator H has a purely absolutely continuous spectrum on $[0,\infty)$ plus a finite number of eigenvalues in $(-\infty,0)$ (see, e.g., [25, Theorem 5.1] and [29, Sect. 9.7]).

Then our main result reads as follows:

Theorem 1.1. *Assume that*

$$\int_0^1 |q(x)|dx < \infty \quad \textit{and} \quad \int_1^\infty x\log^2(1+x)|q(x)|dx < \infty, \tag{1.4}$$

and suppose there is no resonance at 0 *(see Definition 2.17). Then the following decay holds*

$$\left\|\mathrm{e}^{-\mathrm{i}tH}P_c(H)\right\|_{L^1(\mathbb{R}_+)\to L^\infty(\mathbb{R}_+)} = \mathcal{O}(|t|^{-1/2}), \quad t\to\infty. \tag{1.5}$$

Here $P_c(H)$ is the orthogonal projection in $L^2(\mathbb{R}_+)$ onto the continuous spectrum of H.

Such dispersive estimates for Schrödinger equations have a long tradition and here we refer to a brief selection of articles [4, 5, 8, 10, 11, 14, 19, 20, 24, 32, 33], where further references can be found. We will show this result by establishing a corresponding low energy result, Theorem 3.2 (see also Theorem 3.1), and a corresponding high energy result, Theorem 3.3. Our proof is based on the approach proposed in [19], however, the main technical difficulty is the analysis of the low and high energy behavior of the corresponding Jost function. Let us also mention that the potential $q\equiv 0$ does not satisfy the conditions of Theorem 1.1, that is, there is a resonance at 0 in this case. However, it is known that the dispersive decay (1.5) holds true if $q\equiv 0$ [17] and hence Theorem 1.1 states that the corresponding estimate remains true under additive non-resonant perturbations. For related results on scattering theory for such operators we refer to [2, 3].

Finally, let us briefly describe the content of the paper. Section 2 is of preliminary character, where we collect and derive some necessary estimates for solutions, the Green's function and the high and low energy behavior of the Jost function (2.29). However, we would like to emphasize that the behavior of the Jost function near the bottom of the essential spectrum is still not understood satisfactorily, and for this very reason the resonant case had to be excluded from our main theorem. The proof of Theorem 1.1 is given in Section 3. In order to make the exposition self-contained, we gathered the appropriate version of the van der Corput lemma and necessary facts on the Wiener algebra in Appendix A. Appendix B contains relevant facts about Bessel and Hankel functions.

2. Properties of solutions

In this section we will collect some properties of the solutions of the underlying differential equation required for our main results.

2.1. **The regular solution.** Suppose that

$$q \in L^1_{\mathrm{loc}}(\mathbb{R}_+) \quad \text{and} \quad \int_0^1 x\big(1-\log(x)\big)|q(x)|dx < \infty. \tag{2.1}$$

Then the ordinary differential equation

$$\tau f = zf, \qquad \tau := -\frac{d^2}{dx^2} - \frac{1}{4x^2} + q(x),$$

has a system of solutions $\phi(z,x)$ and $\theta(z,x)$ which are real entire with respect to z and such that

$$\phi(z,x) = \sqrt{\frac{\pi x}{2}}\tilde{\phi}(z,x), \quad \theta(z,x) = -\sqrt{\frac{2x}{\pi}}\log(x)\tilde{\theta}(z,x), \tag{2.2}$$

where $\tilde{\phi}(z,\cdot) \in W^{1,1}[0,1]$, $\tilde{\theta}(z,\cdot) \in C[0,1]$ and $\tilde{\phi}(z,0) = \tilde{\theta}(z,0) = 1$. Moreover, we can choose $\theta(z,x)$ such that $\lim_{x\to 0} W(\sqrt{x}\log(x), \theta(z,x)) = 0$ for all $z \in \mathbb{C}$. Here $W(u,v) = u(x)v'(x) - u'(x)v(x)$ is the usual Wronski determinant. For a detailed construction of these solutions we refer to, e.g., [17].

We start with two lemmas containing estimates for the Green's function of the unperturbed equation

$$G_{-\frac{1}{2}}(z,x,y) = \phi_{-\frac{1}{2}}(z,x)\theta_{-\frac{1}{2}}(z,y) - \phi_{-\frac{1}{2}}(z,y)\theta_{-\frac{1}{2}}(z,x)$$

and the regular solution $\phi(z,x)$ (see, e.g., [15, Lemmas 2.2, A.1, and A.2]). Here

$$\phi_{-\frac{1}{2}}(z,x) = \sqrt{\frac{\pi x}{2}} J_0(\sqrt{z}x), \quad \theta_{-\frac{1}{2}}(z,x) = \sqrt{\frac{\pi x}{2}}\left(\frac{1}{\pi}\log(z)J_0(\sqrt{z}x) - Y_0(\sqrt{z}x)\right), \tag{2.3}$$

where J_0 and Y_0 are the usual Bessel and Neumann functions (see Appendix B). All branch cuts are chosen along the negative real axis unless explicitly stated otherwise.

The first two results are essentially from [15, Appendix A]. However, since the focus there was on a finite interval, some small adaptions are necessary to cover the present case of a half-line.

Lemma 2.1 ([15])**.** *The following estimates hold:*

$$\left|\phi_{-\frac{1}{2}}(k^2,x)\right| \le C\left(\frac{x}{1+|k|\,x}\right)^{\frac{1}{2}} \mathrm{e}^{|\mathrm{Im}\ k|x}, \tag{2.4}$$

$$\left|\theta_{-\frac{1}{2}}(k^2,x)\right| \le C\left(\frac{x}{1+|k|\,x}\right)^{\frac{1}{2}} \left(1 + \left|\log\left(\frac{1+|k|\,x}{x}\right)\right|\right) \mathrm{e}^{|\mathrm{Im}\ k|x}, \tag{2.5}$$

for all $x > 0$, and

$$\left|G_{-\frac{1}{2}}(k^2,x,y)\right| \le C\left(\frac{x}{1+|k|x}\right)^{\frac{1}{2}}\left(\frac{y}{1+|k|y}\right)^{\frac{1}{2}}\left(1+\log\left(\frac{x}{y}\right)\right)\mathrm{e}^{|\mathrm{Im}\ k|(x-y)} \tag{2.6}$$

for all $0 < y \le x < \infty$.

Proof. The first two estimates are clear from the asymptotic behavior of the Bessel function J_0 and the Neumann function Y_0 (see (B.1), (B.2) and (B.4), (B.5)).

To consider the third one, first of all we have

$$\begin{aligned} G_{-\frac{1}{2}}(k^2,x,y) &= -\frac{\pi}{2}\sqrt{xy}\big[J_0(kx)Y_0(ky) - J_0(ky)Y_0(kx)\big] \\ &= -\frac{\mathrm{i}\pi}{4}\sqrt{xy}\left[H_0^{(1)}(kx)H_0^{(2)}(ky) - H_0^{(1)}(ky)H_0^{(2)}(kx)\right]. \end{aligned} \tag{2.7}$$

We divide the proof of (2.6) in three steps.

Step (i): $|ky| \le |kx| \le 1$. Using the first equality in (2.7) and employing (B.1) and (B.2), we get

$$\left|G_{-\frac{1}{2}}(k^2,x,y)\right| \le C\sqrt{xy}\left(1+\log\left(\frac{|k|x}{|k|y}\right)\right) = C\sqrt{xy}\left(1+\log\left(\frac{x}{y}\right)\right),$$

which immediately implies (2.6).

Step (ii): $|ky| \le 1 \le |kx|$. Using the asymptotics (B.1)–(B.5) from Appendix B, we get

$$\left|G_{-\frac{1}{2}}(k^2,x,y)\right| \le C\sqrt{xy}\sqrt{\frac{1}{|k|x}}\mathrm{e}^{|\mathrm{Im}\ k|(x-y)}\left(1-\log(|k|y)\right).$$

We arrive at (2.6) by noting that

$$0 < -\log(|k|y) \le \log(x/y)$$

since $|k|y \le 1 \le |k|x$.

Step (iii): $1 \le |ky| \le |kx|$. For the remaining case it suffices to use the second equality in (2.7) and (B.6)–(B.7) to arrive at

$$\left|G_{-\frac{1}{2}}(k^2,x,y)\right| \le C\sqrt{xy}\sqrt{\frac{1}{|k|x|k|y}}\mathrm{e}^{|\mathrm{Im}\ k|(x-y)} = \frac{C}{|k|}\mathrm{e}^{|\mathrm{Im}\ k|(x-y)},$$

which implies the claim. □

Lemma 2.2 ([15])**.** *Assume* (2.1)*. Then $\phi(z,x)$ satisfies the integral equation*

$$\phi(z,x) = \phi_{-\frac{1}{2}}(z,x) + \int_0^x G_{-\frac{1}{2}}(z,x,y)\phi(z,y)q(y)dy. \tag{2.8}$$

Moreover, $\phi(\cdot,x)$ is entire for every $x>0$ and satisfies the estimate

$$\begin{aligned}\left|\phi(k^2,x) - \phi_{-\frac{1}{2}}(k^2,x)\right| \le & C\left(\frac{x}{1+|k|\,x}\right)^{\frac{1}{2}}\mathrm{e}^{|\mathrm{Im}\ k|x} \\ & \times \int_0^x \frac{y}{1+|k|y}\left(1+\log\left(\frac{x}{y}\right)\right)|q(y)|dy\end{aligned} \tag{2.9}$$

for all $x>0$ and $k\in\mathbb{C}$.

Proof. The proof is based on the successive iteration procedure. As in the proof of Lemma 2.2 in [15], set

$$\phi = \sum_{n=0}^{\infty}\phi_n, \quad \phi_0 = \phi_{-\frac{1}{2}}, \quad \phi_n(k^2,x) := \int_0^x G_{-\frac{1}{2}}(k^2,x,y)\phi_{n-1}(k^2,y)q(y)dy$$

for all $n\in\mathbb{N}$. The series is absolutely convergent since

$$\begin{aligned}\left|\phi_n(k^2,x)\right| \le & \frac{C^{n+1}}{n!}\left(\frac{x}{1+|k|x}\right)^{\frac{1}{2}}\mathrm{e}^{|\mathrm{Im}\,k|x} \\ & \times\left(\int_0^x \frac{y}{1+|k|y}\left(1+\log\left(\frac{x}{y}\right)\right)|q(y)|dy\right)^n, \quad n\in\mathbb{N}.\end{aligned} \tag{2.10}$$

This is all we need to finish the proof of this lemma. □

We also need the estimates for derivatives.

Lemma 2.3. *The following estimates hold*

$$|\partial_k \phi_{-\frac{1}{2}}(k^2,x)| \le C|k|x\left(\frac{x}{1+|k|\,x}\right)^{\frac{3}{2}} \mathrm{e}^{|\mathrm{Im}\,k|x} \tag{2.11}$$

for all $x>0$, and

$$\begin{aligned}\Big|\partial_k G_{-\frac{1}{2}}(k^2,x,y)\Big| \le C|k|x\left(\frac{x}{1+|k|x}\right)^{\frac{3}{2}}\left(\frac{y}{1+|k|y}\right)^{\frac{1}{2}}&\\ \times\left(1+\log\left(\frac{x}{y}\right)\right)\mathrm{e}^{|\mathrm{Im}\ k|(x-y)}&,\end{aligned} \tag{2.12}$$

for all $0<y\le x<\infty$.

Proof. The first inequality follows from the identity (see [23, (10.6.3)])

$$\partial_k \phi_{-\frac{1}{2}}(k^2,x) = -x\sqrt{\frac{\pi x}{2}}J_1(kx)$$

along with the asymptotic behavior of the Bessel function J_1 (cf. [19, Lemma 2.1]).

To prove (2.12), we first calculate

$$\begin{aligned}\partial_k G_{-\frac{1}{2}}(k^2,x,y) &= \frac{\pi}{2}\sqrt{xy}\Big[xJ_1(kx)Y_0(ky) - yJ_1(ky)Y_0(kx)\\ &\qquad - xJ_0(ky)Y_1(kx) + yJ_0(kx)Y_1(ky)\Big]\\ &= \frac{\mathrm{i}\pi}{4}\sqrt{xy}\,\Big[xH_1^{(1)}(kx)H_0^{(2)}(ky) - yH_1^{(1)}(ky)H_0^{(2)}(kx)\\ &\qquad +xH_0^{(1)}(ky)H_1^{(2)}(kx) - yH_0^{(1)}(kx)H_1^{(2)}(ky)\Big],\end{aligned} \tag{2.13}$$

where we have used formulas (2.7) and the identities for derivatives of Bessel and Hankel functions (cf. Appendix B).

Step (i): $|ky|\le|kx|\le 1$. Employing the series expansions (B.1)–(B.2) we get from the first equality in (2.13)

$$\begin{aligned}\partial_k G_{-\frac{1}{2}}(k^2,x,y) &= \frac{\pi}{2}\sqrt{xy}\,\Big[x\frac{kx}{4}\frac{2\log(ky)}{\pi} - y\frac{ky}{4}\frac{2\log(kx)}{\pi}\\ &\quad - x\Big(\frac{1}{2\pi kx}+\frac{2\log(kx)}{\pi}\frac{kx}{4}\Big) + y\Big(\frac{1}{2\pi ky}+\frac{2\log(ky)}{\pi}\frac{ky}{4}\Big)\Big](1+\mathcal{O}(1))\\ &= \frac{\pi}{2}\sqrt{xy}\big(kx^2+ky^2\big)\big(\log(ky)-\log(kx)\big)(1+\mathcal{O}(1))\\ &= \frac{\pi}{2}\sqrt{xy}kx^2\log(y/x)(1+\mathcal{O}(1)).\end{aligned}$$

This immediately implies the desired claim.

Step (ii): $|ky| \le 1 \le |kx|$. Again we employ the asymptotics (B.1)–(B.5) from Appendix B to get:

$$\begin{aligned}\partial_k G_{-\frac{1}{2}}(k^2,x,y) = \frac{\pi\sqrt{xy}}{2}\Bigg[&\sqrt{\frac{2x}{\pi k}}\cos\Big(kx-\frac{3\pi}{4}\Big)\frac{2\log(ky)}{\pi} - yky\sqrt{\frac{2}{\pi kx}}\cos\Big(kx-\frac{\pi}{4}\Big)\\ &-\sqrt{\frac{2x}{\pi k}}\cos\Big(kx-\frac{3\pi}{4}\Big)+y\sqrt{\frac{2}{\pi kx}}\cos\Big(kx-\frac{\pi}{4}\Big)\frac{1}{2\pi ky}\Bigg](1+\mathcal{O}(1))\\ =\frac{\pi\sqrt{xy}}{2}\Bigg[&\sqrt{\frac{2x}{\pi k}}\cos\Big(kx-\frac{3\pi}{4}\Big)\Big(\frac{2}{\pi}\log(ky)-1\Big)\\ &+\sqrt{\frac{2}{\pi kx}}\cos\Big(kx-\frac{\pi}{4}\Big)\Big(\frac{1}{2\pi k}-yky\Big)\Bigg](1+\mathcal{O}(1)).\end{aligned}$$

This gives the desired estimate, where we have to use $\frac{1}{|k|} \le x$ to estimate the second summand and the logarithmic expression appropriately (cf. step (ii) of 2.1).

Step (iii): $1 \le |ky| \le |kx|$. To deal with the remaining case we shall use the second equality in (2.13) and the asymptotic expansions of Hankel functions (B.6)–(B.7):

$$\begin{aligned}\partial_k G_{-\frac{1}{2}}(k^2,x,y) = \frac{\mathrm{i}\pi\sqrt{xy}}{4}\Bigg[&x\frac{2}{\pi k\sqrt{xy}}\mathrm{e}^{\mathrm{i}k(x-y)-\mathrm{i}\pi/2} - y\frac{2}{\pi k\sqrt{xy}}\mathrm{e}^{\mathrm{i}k(y-x)-\mathrm{i}\pi/2}\\ &+x\frac{2}{\pi k\sqrt{xy}}\mathrm{e}^{\mathrm{i}k(y-x)+\mathrm{i}\pi/2} - y\frac{2}{\pi k\sqrt{xy}}\mathrm{e}^{\mathrm{i}k(x-y)+\mathrm{i}\pi/2}\Bigg](1+\mathcal{O}(1))\\ =\frac{x+y}{2\mathrm{i}k}&\sin(k(x-y))(1+\mathcal{O}(1)).\end{aligned}$$

This again immediately implies (2.12). □

Lemma 2.4. *Assume* (2.1). *Then* $\partial_k\phi(k^2,x)$ *is a solution to the integral equation*

$$\begin{aligned}\partial_k\phi(k^2,x) &= \partial_k\phi_{-\frac{1}{2}}(k^2,x)\\ &+\int_0^x[\partial_k G_{-\frac{1}{2}}(k^2,x,y)]\phi(k^2,y)+G_{-\frac{1}{2}}(k^2,x,y)\partial_k\phi(k^2,y)]q(y)dy \qquad (2.14)\end{aligned}$$

and satisfies the estimate

$$\begin{aligned}\Big|\partial_k\phi(k^2,x)-\partial_k\phi_{-\frac{1}{2}}(k^2,x)\Big| \le\, &C|k|x\left(\frac{x}{1+|k|\,x}\right)^{\frac{3}{2}}\mathrm{e}^{|\mathrm{Im}\ k|x} \qquad (2.15)\\ &\times\int_0^x\frac{y}{1+|k|y}\left(1+\log\left(\frac{x}{y}\right)\right)|q(y)|dy.\end{aligned}$$

Proof. Let us show that $\partial_k\phi(k^2,x)$ given by

$$\partial_k\phi=\sum_{n=0}^{\infty}\beta_n,\quad \beta_0(k,x)=\partial_k\phi_{-\frac{1}{2}}(k^2,x), \qquad (2.16)$$

$$\begin{aligned}\beta_n(k,x)=\int_0^x&\partial_k G_{-\frac{1}{2}}(k^2,x,y)\,\phi_{n-1}(k^2,y)q(y)dy\\ &+\int_0^x G_{-\frac{1}{2}}(k^2,x,y)\beta_{n-1}(k,y)q(y)dy,\quad n\in\mathbb{N}, \qquad (2.17)\end{aligned}$$

satisfies (2.14). Here ϕ_n is defined in Lemma 2.2. Using (2.10) and (2.11), we can bound the first summand in (2.17) as follows

$$\begin{aligned}|\text{1st term}| \le \frac{C^{n+1}}{(n-1)!}|k|x\left(\frac{x}{1+|k|x}\right)^{\frac{3}{2}}\mathrm{e}^{|\mathrm{Im}\,k|x}\\ \int_0^x\left(1+\log\left(\frac{x}{y}\right)\right)\frac{y\,|q(y)|}{1+|k|y}\left(\int_0^y\left(1+\log\left(\frac{y}{t}\right)\right)\frac{t\,|q(t)|}{1+|k|t}dt\right)^{n-1}dy\\ \le \frac{C^{n+1}}{n!}|k|x\left(\frac{x}{1+|k|x}\right)^{\frac{3}{2}}\mathrm{e}^{|\mathrm{Im}\,k|x}\left(\int_0^x\left(1+\log\left(\frac{x}{y}\right)\right)\frac{y|q(y)|}{1+|k|y}dy\right)^n.\end{aligned}$$

Next, using induction, one can show that the second summand admits a similar bound and hence we finally get

$$|\beta_n(k,x)| \le \frac{C^{n+1}}{n!}|k|x\left(\frac{x}{1+|k|x}\right)^{\frac{3}{2}}\mathrm{e}^{|\mathrm{Im}\,k|x}\left(\int_0^x\left(1+\log\left(\frac{x}{y}\right)\right)\frac{y|q(y)|}{1+|k|y}dy\right)^n.$$

This immediately implies the convergence of (2.16) and, moreover, the estimate

$$|\partial_k\phi(k^2,x)-\partial_k\phi_{-\frac{1}{2}}(k^2,x)| \le \sum_{n=1}^{\infty}|\beta_n(k,x)|,$$

from which (2.15) follows under the assumption (2.1). □

Furthermore, by [9, 7, 30] (see also [12]), the regular solution ϕ admits a representation by means of transformation operators preserving the behavior of solutions at $x=0$ (see also [6, Chap. III] for further details and historical remarks).

Lemma 2.5. *Suppose $q\in L^1_{\mathrm{loc}}([0,\infty))$. Then*

$$\phi(z,x)=\phi_{-\frac{1}{2}}(z,x)+\int_0^x B(x,y)\phi_{-\frac{1}{2}}(z,y)dy=(I+B)\phi_{-\frac{1}{2}}(z,x), \tag{2.18}$$

where the so-called Gelfand–Levitan kernel $B\colon\mathbb{R}^2_+\to\mathbb{R}$ satisfies the estimate

$$|B(x,y)|\le\frac{1}{2}\sigma_0\left(\frac{x+y}{2}\right)\mathrm{e}^{\sigma_1(x)},\quad \sigma_j(x)=\int_0^x s^j|q(s)|ds, \tag{2.19}$$

for all $0<y<x$ and $j\in\{0,1\}$.

In particular, this lemma immediately implies the following useful result.

Corollary 2.6. *Suppose $q\in L^1((0,1))$. Then B is a bounded operator on $L^\infty((0,1))$.*

Proof. If $f\in L^\infty((0,1))$, then using the estimate (2.19) we get

$$\begin{aligned}|(Bf)(x)|=\Big|\int_0^x B(x,y)f(y)dy\Big|\le\|f\|_\infty\int_0^x|B(x,y)|dy\\ \le\frac{1}{2}\|f\|_\infty\mathrm{e}^{\sigma_1(1)}\int_0^x\sigma_0\Big(\frac{x+y}{2}\Big)dy\le\frac{1}{2}\|f\|_\infty\mathrm{e}^{\sigma_1(1)}\sigma_0(1),\end{aligned}$$

which proves the claim. □

Remark 2.7. *Note that B is a bounded operator on $L^2((0,a))$ for all $a>0$. However, the estimate (2.19) allows to show that its norm behaves like $\mathcal{O}(a)$ as $a\to\infty$ and hence B might not be bounded on $L^2(\mathbb{R}_+)$.*

2.2. **The Jost solution and the Jost function.** In this subsection, we assume that the potential q belongs to *the Marchenko class*, i.e., in addition to (2.1), q also satisfies

$$\int_1^\infty x\log(1+x)|q(x)|dx < \infty. \tag{2.20}$$

Recall that under these assumptions on q the spectrum of H is purely absolutely continuous on $(0,\infty)$ with an at most finite number of eigenvalues $\lambda_n \in (-\infty, 0)$. A solution $f(k,\cdot)$ to $\tau y = k^2 y$ with $k \neq 0$ satisfying the following asymptotic normalization

$$f(k,x) = \mathrm{e}^{\mathrm{i}kx}(1+o(1)), \qquad f'(k,x) = \mathrm{i}k\mathrm{e}^{\mathrm{i}kx}(1+o(1)) \tag{2.21}$$

as $x\to\infty$, is called *the Jost solution*. In the case $q\equiv 0$, we have (cf. (B.6))

$$f_{-\frac12}(k,x) = \mathrm{e}^{\mathrm{i}\frac{\pi}{4}}\sqrt{\frac{\pi x k}{2}}H_0^{(1)}(kx), \tag{2.22}$$

which is analytic in $\mathbb{C}_+$ and continuous in $\overline{\mathbb{C}_+}\setminus\{0\}$. Here $H_\nu^{(1)}$ is the Hankel function of the first kind (see Appendix B). Using the estimates for Hankel functions we obtain

$$\left|f_{-\frac12}(k,x)\right| \le C\left(\frac{|k|\,x}{1+|k|\,x}\right)^{\frac12}\mathrm{e}^{-|\mathrm{Im}\ k|x}\left(1-\log\left(\frac{|k|x}{1+|k|x}\right)\right) \le C\mathrm{e}^{-|\mathrm{Im}\ k|x} \tag{2.23}$$

for all $x>0$. Notice that for the second inequality in (2.23) we have to use the fact that the function $x\mapsto\sqrt{\frac{x}{x+1}}\log\left(\frac{x}{x+1}\right)$ is bounded on $\mathbb{R}_+$.

Lemma 2.8. *Assume* (2.20)*. Then the Jost solution satisfies the integral equation*

$$f(k,x) = f_{-\frac12}(k,x) - \int_x^\infty G_{-\frac12}(k^2,x,y)f(k,y)q(y)dy. \tag{2.24}$$

For all $x>0$, $f(\cdot,x)$ is analytic in the upper half plane and can be continuously extended to the real axis away from $k=0$ and

$$\begin{aligned}|f(k,x)-f_{-\frac12}(k,x)| \le C&\left(\frac{x}{1+|k|\,x}\right)^{\frac12}\mathrm{e}^{-|\mathrm{Im}\ k|\,x} \\ &\times\int_x^\infty\left(\frac{y}{1+|k|\,y}\right)^{\frac12}\left(1+\log\left(\frac{y}{x}\right)\right)|q(y)|dy.\end{aligned} \tag{2.25}$$

Proof. The proof is based on the successive iteration procedure. Set

$$f=\sum_{n=0}^\infty f_n, \quad f_0 = f_{-\frac12}, \quad f_n(k,x) = -\int_x^\infty G_{-\frac12}(k^2,x,y)f_{n-1}(k,y)q(y)dy$$

for all $n\in\mathbb{N}$. The series is absolutely convergent since

$$\begin{aligned}|f_n(k,x)| \le \frac{C^{n+1}}{n!}&\left(\frac{x}{1+|k|x}\right)^{\frac12}\mathrm{e}^{-|\mathrm{Im}\,k|x} \\ &\times\left(\int_x^\infty\left(\frac{y}{1+|k|\,y}\right)^{\frac12}\left(1+\log\left(\frac{y}{x}\right)\right)|q(y)|dy\right)^n\end{aligned}$$

holds for all $n\in\mathbb{N}$. The latter also proves (2.25). □

Furthermore, by [9, 7, 26, 27] (see also [12]), the Jost solution f admits a representation by means of transformation operators preserving the behavior of solutions at infinity.

Lemma 2.9 ([26, 27]). *Assume* (2.20) *and let* $k \neq 0$. *Then*

$$f(k,x) = f_{-\frac{1}{2}}(k,x) + \int_x^\infty K(x,y) f_{-\frac{1}{2}}(k,y)dy = (I+K)f_{-\frac{1}{2}}(k,x), \qquad (2.26)$$

where the so-called Marchenko kernel $K\colon \mathbb{R}^2 \to \mathbb{R}$ *satisfies the estimate*

$$|K(x,y)| \le \frac{c_0}{2}\tilde{\sigma}_0\left(\frac{x+y}{2}\right) \mathrm{e}^{c_0\tilde{\sigma}_1(x) - \tilde{\sigma}_1(\frac{x+y}{2})}, \quad \tilde{\sigma}_j(x) = \int_x^\infty s^j |q(s)|ds, \qquad (2.27)$$

for all $x < y < \infty$. *Here* c_0 *is a positive constant given by*

$$c_0 := \sup_{s\in(0,1)} (1-s)^{1/2}\, {}_2F_1\left(\begin{matrix}1/2, 1/2\\ 1\end{matrix}; s\right) = \sup_{s\in(0,1)} (1-s)^{1/2} \sum_{n=0}^\infty \frac{((1/2)_n)^2}{(n!)^2} s^n.$$

Notice that c_0 is finite in view of [23, (15.4.21)]. Moreover, this lemma immediately implies the following useful result.

Corollary 2.10. *If* (2.20) *holds, then* K *is a bounded operator on* $L^\infty((1,\infty))$.

Proof. If $f \in L^\infty((1,\infty))$, then using the estimate (2.27) we get

$$\begin{aligned} |(Kf)(x)| &= \Big|\int_x^\infty K(x,y)f(y)dy\Big| \le \|f\|_\infty \int_x^\infty |K(x,y)|dy \\ &\le \frac{c_0}{2}\|f\|_\infty \mathrm{e}^{c_0\tilde{\sigma}_1(x)} \int_1^\infty \tilde{\sigma}_0\Big(\frac{1+y}{2}\Big)dy \\ &\le c_0\|f\|_\infty \mathrm{e}^{c_0\tilde{\sigma}_1(1)} \int_1^\infty \tilde{\sigma}_0(s)ds = c_0\|f\|_\infty \big(\tilde{\sigma}_1(1) - \tilde{\sigma}_0(1)\big)\mathrm{e}^{c_0\tilde{\sigma}_1(1)}, \end{aligned}$$

which proves the claim. □

By Lemma 2.8, the Jost solution is analytic in the upper half plane and can be continuously extended to the real axis away from $k = 0$. We can extend it to the lower half plane by setting $f(k,x) = f(-k,x) = f(k^*,x)^*$ for $\mathrm{Im}(k) < 0$ (here and below we denote the complex conjugate of z by z^*). For $k \in \mathbb{R}\setminus\{0\}$ we obtain two solutions $f(k,x)$ and $f(-k,x) = f(k,x)^*$ of the same equation whose Wronskian is given by (cf. (2.21))

$$W(f(-k,.), f(k,.)) = 2\mathrm{i}k. \qquad (2.28)$$

The Jost function is defined as

$$f(k) := W(f(k,.), \phi(k^2,.)) \qquad (2.29)$$

and we also set

$$g(k) := W(f(k,.), \theta(k^2,.))$$

such that

$$f(k,x) = f(k)\theta(k^2,x) - g(k)\phi(k^2,x). \qquad (2.30)$$

In particular, the function given by

$$m(k^2) := -\frac{g(k)}{f(k)}, \quad k \in \mathbb{C}_+,$$

is called *the Weyl m-function* (we refer to [16, 18] for further details). Note that both $f(k)$ and $g(k)$ are analytic in the upper half plane and $f(k)$ has simple zeros at $\mathrm{i}\kappa_n = \sqrt{\lambda_n} \in \mathbb{C}_+$.

Since $f(k,x)^* = f(-k,x)$ for $k \in \mathbb{R}\setminus\{0\}$, we obtain $f(k)^* = f(-k)$ and $g(k)^* = g(-k)$. Moreover, (2.28) shows

$$\phi(k^2,x) = \frac{f(-k)}{2\mathrm{i}k} f(k,x) - \frac{f(k)}{2\mathrm{i}k} f(-k,x), \quad k \in \mathbb{R}\setminus\{0\}, \tag{2.31}$$

and by (2.30) we get

$$2\mathrm{i}\,\mathrm{Im}(f(k)g(k)^*) = f(k)g(k)^* - f(k)^*g(k) = W(f(-k,\cdot), f(k,\cdot)) = 2\mathrm{i}k.$$

Moreover,

$$\mathrm{Im}\ m(k^2) = -\frac{\mathrm{Im}\big(f(k)^*g(k)\big)}{|f(k)|^2} = \frac{k}{|f(k)|^2}, \quad k \in \mathbb{R}\setminus\{0\}. \tag{2.32}$$

Note that

$$f_{-\frac{1}{2}}(k) = W(f_{-\frac{1}{2}}(k,.), \phi_{-\frac{1}{2}}(k^2,.)) = \sqrt{k}\mathrm{e}^{-\mathrm{i}\frac{\pi}{4}}, \quad 0 \le \arg(k) < \pi.$$

Thus, by [18, Theorem 2.1] (see also Eq. (5.15) in [18] or [13]), on the real line we have

$$|f(k)| = \sqrt{|k|}(1 + o(1)), \quad k \to \infty. \tag{2.33}$$

2.3. **High and low energy behavior of the Jost function.** Consider the following function

$$F(k) = \frac{f(k)}{f_{-\frac{1}{2}}(k)} = \mathrm{e}^{\mathrm{i}\frac{\pi}{4}}\, k^{-\frac{1}{2}} f(k) = \mathrm{e}^{\mathrm{i}\frac{\pi}{4}}\, k^{-\frac{1}{2}} W(f(k,.), \phi(k^2,.)), \quad \mathrm{Im}\ k \ge 0. \tag{2.34}$$

Let us summarize the basic properties of F.

Lemma 2.11. *The function F defined by* (2.34) *is analytic in $\mathbb{C}_+$ and continuous in $\overline{\mathbb{C}_+}\setminus\{0\}$. Moreover, $F(k)^* = F(-k) \neq 0$ for all $k \in \mathbb{R}\setminus\{0\}$ and*

$$|F(k)| = 1 + o(1) \tag{2.35}$$

as $k \in \mathbb{R}$ tends to ∞.

Proof. The first claim follows from the corresponding properties of the Jost function. Next, (2.31) implies that $f(k) \neq 0$ for all $k \in \mathbb{R}\setminus\{0\}$. Finally, (2.35) follows from (2.33). □

The analysis of the behavior of F near zero is much more delicate. We start with the following integral representation.

Lemma 2.12 ([18]). *Assume* (2.1) *and* (2.20)*. Then the function F admits the integral representation*

$$\begin{aligned} F(k) &= 1+\mathrm{e}^{\mathrm{i}\frac{\pi}{4}}\, k^{-\frac{1}{2}} \int_0^\infty f_{-\frac{1}{2}}(k,x)\phi(k^2,x)q(x)dx \\ &= 1 + \mathrm{e}^{\mathrm{i}\frac{\pi}{4}}\, k^{-\frac{1}{2}} \int_0^\infty f(k,x)\phi_{-\frac{1}{2}}(k^2,x)q(x)dx \end{aligned} \tag{2.36}$$

for all $k \in \overline{\mathbb{C}_+}\setminus\{0\}$.

Proof. To prove the integral representations (2.36), we need to replace ϕ and f in (2.34) by (2.8) and (2.24), respectively, use the asymptotic estimates for ϕ, f and $G_{-\frac{1}{2}}$, and then take the limits $x \to +\infty$ and $x \to 0$. $\square$

Corollary 2.13. *Assume in addition that q satisfies*

$$\int_1^\infty x \log^2(1+x)|q(x)|dx < \infty. \tag{2.37}$$

Then for $k > 0$ the integral representation (2.36) *can be rewritten as follows*

$$\begin{aligned} F(k) = 1+ &\int_0^\infty \theta_{-\frac{1}{2}}(k^2,x)\phi(k^2,x)q(x)dx \\ &+ \Big(\mathrm{i} - \frac{1}{\pi}\log(k^2)\Big)\int_0^\infty \phi_{-\frac{1}{2}}(k^2,x)\phi(k^2,x)q(x)dx. \end{aligned} \tag{2.38}$$

Proof. Indeed, the integrals converge for all $k \in \mathbb{R}\setminus\{0\}$ due to (2.4), (2.5) and (2.9). Then (2.38) follows from the first formula in (2.36) since (cf. (2.3) and (2.22))

$$\theta_{-\frac{1}{2}}(k^2,x) - \frac{1}{\pi}\log(-k^2)\phi_{-\frac{1}{2}}(k^2,x) = \mathrm{e}^{\mathrm{i}\frac{\pi}{4}} k^{-\frac{1}{2}} f_{-\frac{1}{2}}(k,x).$$

Notice also that it suffices to consider only positive $k > 0$ since $F(-k) = F(k)^*$ by Lemma 2.12. $\square$

Before proceed further, we need the following simple facts.

Lemma 2.14. *Suppose that q satisfies* (2.1) *and* (2.37). *Then*

$$\int_0^\infty \phi_{-\frac{1}{2}}(0,s)\phi(0,s)q(s)ds = \sqrt{\frac{\pi}{2}}\lim_{x\to\infty} W(\sqrt{x},\phi(0,x)), \tag{2.39}$$

$$\int_0^\infty \theta_{-\frac{1}{2}}(0,s)\phi(0,s)q(s)ds = -1 - \sqrt{\frac{2}{\pi}}\lim_{x\to\infty} W(\sqrt{x}\log(x),\phi(0,x)). \tag{2.40}$$

Proof. First observe that the integrals on the left-hand side are finite since

$$\phi_{-\frac{1}{2}}(0,x) = \sqrt{\frac{\pi x}{2}}, \quad \theta_{-\frac{1}{2}}(0,x) = -\sqrt{\frac{2x}{\pi}}\log(x),$$

and q satisfies (2.1) and (2.37). Now notice that

$$\int_0^x \phi_{-\frac{1}{2}}(0,s)\phi(0,s)q(s)ds = \int_0^x \phi_{-\frac{1}{2}}(0,s)(\phi''(0,s) + \frac{1}{4s^2}\phi(0,s))ds$$

since $\tau\phi = 0$. Integrating by parts and noting that $\phi_{-\frac{1}{2}}(0,x)$ solves $y'' + \frac{1}{4x^2}y = 0$, we get

$$\int_0^x \phi_{-\frac{1}{2}}(0,s)\phi(0,s)q(s)ds = \sqrt{\frac{\pi}{2}}W(\sqrt{x},\phi(0,x))$$

since $W(\sqrt{x},\phi(0,x)) \to 0$ as $x \to 0$. Passing to the limit as $x \to \infty$, we arrive at (2.39). The proof of (2.40) is analogous. $\square$

Lemma 2.15. *Assume the conditions of Lemma 2.14. Then the equation*

$$\tau y = -y'' - \frac{1}{4x^2}y + q(x)y = 0$$

has two linearly independent solution y_1 and y_2 such that

$$y_1(x) = \sqrt{x}(1+o(1)), \quad y_1'(x) = \frac{1}{2\sqrt{x}}(1+o(1)) \tag{2.41}$$

and

$$y_2(x) = \sqrt{x}\log(x)(1+o(1)), \quad y_2'(x) = \frac{\log(\sqrt{x})}{\sqrt{x}}(1+o(1)) \tag{2.42}$$

as $x \to \infty$.

Proof. The proof is based on successive iteration. Namely, each solution to $\tau y = 0$ solves the integral equation

$$f(x) = a\sqrt{x} + b\sqrt{x}\log(x) - \int_x^\infty \sqrt{xs}\log(x/s)f(s)q(s)ds.$$

Since the argument is fairly standard we only provide some details for $y_2(x)$; the calculations for $y_1(x)$ are similar. For simplicity we set $x > \mathrm{e}$, which is no restriction since we only need estimates for large x anyway. As in, e.g., Lemma 2.2 we set

$$y_2(x) = \sum_{n=0}^\infty \phi_n, \quad \phi_0(x) := \sqrt{x}\log(x), \quad \phi_n(x) := -\int_x^\infty \sqrt{xs}\log(x/s)\phi_{n-1}(s)q(s)ds.$$

Since $\log(s/x) \le \log(x)\log(s)$ for all $\mathrm{e} \le x \le s < \infty$, we immediately get

$$|\phi_1(x)| \le \int_x^\infty \sqrt{xs}\log(s/x)\sqrt{s}\log(s)|q(s)|ds \le \sqrt{x}\log(x)\int_x^\infty s\log^2(s)|q(s)|ds$$

and then inductively we obtain that

$$|\phi_n(x)| \le \frac{\sqrt{x}\log(x)}{n!}\left(\int_x^\infty s\log^2(s)|q(s)|ds\right)^n,$$

for all $n \in \mathbb{N}$ and $x \ge \mathrm{e}$. Therefore, we end up with the following estimate

$$|y_2(x) - \sqrt{x}\log(x)| \le C\sqrt{x}\log(x)\int_x^\infty s\log^2(s)|q(s)|ds, \quad x \ge \mathrm{e}. \tag{2.43}$$

The derivative $y_2'(x)$ has to satisfy

$$y_2'(x) = \frac{1}{\sqrt{x}}\left(1+\log(\sqrt{x})\right) - \int_x^\infty \sqrt{\frac{s}{x}}\left(1+\log(\sqrt{x/s})\right)y_2(s)q(s)ds.$$

Employing the same procedure as before we set

$$y_2'(x) = \sum_{n=0}^\infty \beta_n, \quad \beta_0(x) := \frac{1+\log(\sqrt{x})}{\sqrt{x}},$$

$$\beta_n(x) := -\int_x^\infty \sqrt{\frac{s}{x}}\left(1+\log(\sqrt{x/s})\right)\beta_{n-1}(s)q(s)ds.$$

Iteration then gives

$$|\beta_n(x)| \le \frac{C^{n+1}}{n!}\frac{1+\log(\sqrt{x})}{\sqrt{x}}\left(\int_x^\infty s\log^2(s)|q(s)|ds\right)^n$$

for all $n \in \mathbb{N}$ and $x \ge \mathrm{e}$ since

$$1+\log(x/s) \le (1+\log(x))(1+\log(s)) \le 2\log(s)(1+\log(x)),$$

for all $\mathrm{e} \le x \le s < \infty$. Thus we end up with the estimate

$$\left|y_2'(x) - \frac{1+\log(\sqrt{x})}{\sqrt{x}}\right| \le C\frac{1+\log(\sqrt{x})}{\sqrt{x}}\int_x^\infty s\log^2(s)|q(s)|ds, \quad x \ge \mathrm{e}, \tag{2.44}$$

which completes the proof. $\square$

Now we are in position to characterize the behavior of F near 0.

Lemma 2.16. *Suppose that* $k > 0$ *and* q *satisfies* (2.1) *and* (2.37)*. Then*

$$F(k) = F_1(k) + \Big(\mathrm{i} - \frac{1}{\pi}\log(k^2)\Big)F_2(k), \quad k \neq 0, \tag{2.45}$$

where F_1 *and* F_2 *are continuous real-valued functions on* $\mathbb{R}$*. Moreover,*

$$F_2(0) = \sqrt{\frac{\pi}{2}}\lim_{x\to\infty} W(\sqrt{x}, \phi(0,x)) = 0 \tag{2.46}$$

precisely when $\phi(0,x) = \mathcal{O}(\sqrt{x})$ *as* $x \to \infty$*. In the latter case*

$$F(k) = F_1(0) + \mathcal{O}(k^2\log(-k^2)), \quad k \to 0, \tag{2.47}$$

with

$$F_1(0) = -\sqrt{\frac{2}{\pi}}\lim_{x\to\infty} W(\sqrt{x}\log(x), \phi(0,x)) \neq 0. \tag{2.48}$$

Proof. The first claim follows from the integral representation (2.38) since the corresponding integrals are continuos in k by the dominated convergence theorem. Moreover, $\phi(k^2,x)$ and $\theta(k^2,x)$ are real if $k \in \mathbb{R}$ and hence so are F_1 and F_2.

By Lemma 2.15, $\phi(0,x) = ay_1(x) + by_2(x)$, where the asymptotic behavior of y_1 and y_2 is given by (2.41) and (2.42), respectively. Combining Lemma 2.14 with the representation (2.38), we conclude that $F_2(0) = b\sqrt{\pi/2} \neq 0$ in (2.45) precisely when $b \neq 0$ and hence the second claim follows.

Assume now that $F_2(0) = 0$, which is equivalent to the equality $\phi(0,x) = ay_1(x)$ with $a = \sqrt{\pi/2}F_1(0) \neq 0$. Noting that both $\phi_{-\frac{1}{2}}(\cdot,x)$ and $\phi(\cdot,x)$ are analytic for each $x > 0$ and applying the dominated convergence theorem once again, we conclude that

$$\int_0^\infty \phi_{-\frac{1}{2}}(k^2,x)\phi(k^2,x)q(x)dx = \mathcal{O}(k^2), \quad k \to 0.$$

This immediately proves (2.47). ☐

Definition 2.17. *We shall say that there is a resonance at* 0 *if* $\phi(0,x) = \mathcal{O}(\sqrt{x})$ *as* $x \to \infty$*.*

Let us mention that there is a resonance at 0 if $q \equiv 0$ since in this case $\phi(0,x) = \phi_{-\frac{1}{2}}(0,x) = \sqrt{\pi x/2}$.

We finish this section with the following estimate.

Lemma 2.18. *Assume that* q *satisfies* (2.1) *and* (2.20)*. Then* F *is differentiable for all* $k \neq 0$ *and*

$$|F'(k)| \leq \frac{C}{|k|}, \quad k \neq 0.$$

Proof. Setting

$$\tilde f_{-\frac{1}{2}}(k,x) := \frac{f_{-\frac{1}{2}}(k,x)}{f_{-\frac{1}{2}}(k)} = \mathrm{e}^{\mathrm{i}\frac{\pi}{4}}k^{-\frac{1}{2}}f_{-\frac{1}{2}}(k,x),$$

we find that its derivative is given by (cf. [23, (10.6.3)])

$$\partial_k \tilde f_{-\frac{1}{2}}(k,x) = -\mathrm{i}x\sqrt{\frac{\pi x}{2}}H_1^{(1)}(kx).$$

Similar to (2.23) we obtain the estimate

$$\left|\partial_k \tilde{f}_{-\frac{1}{2}}(k,x)\right| \le C\frac{\sqrt{x(1+|k|x)}}{|k|}\mathrm{e}^{-|\mathrm{Im}\ k|x} \tag{2.49}$$

which holds for all $x>0$. Using (2.36), we get

$$F'(k) = \int_0^\infty \big(\partial_k \tilde{f}_{-\frac{1}{2}}(k,x)\phi(k^2,x) + \tilde{f}_{-\frac{1}{2}}(k,x)\partial_k\phi(k^2,x)\big)q(x)dx.$$

The integral converges absolutely for all $k\neq 0$. Indeed, we have

$$1+\log\left(\frac{x}{y}\right) \le (1+|\log(x)|)(1+|\log(y)|), \quad 0<y\le x. \tag{2.50}$$

By (2.15), (2.23) and also (2.50), we obtain

$$\begin{aligned}\left|\int_0^\infty \tilde{f}_{-\frac{1}{2}}(k,x)\partial_k\phi(k^2,x)q(x)dx\right| &\le C\int_0^\infty \sqrt{|k|}x\left(\frac{x}{1+|k|x}\right)^{\frac{3}{2}}(1+|\log(x)|)|q(x)|dx\\ &\le \frac{C}{|k|}\int_0^\infty x(1+|\log(x)|)|q(x)|dx.\end{aligned}$$

Using (2.9) and (2.49) (again in combination with (2.50)), we get the following estimates for the first summand:

$$\left|\int_0^\infty \partial_k\tilde{f}_{-\frac{1}{2}}(k,x)\phi(k^2,x)q(x)dx\right| \le \frac{C}{|k|}\int_0^\infty x(1+|\log(x)|)|q(x)|dx.$$

Now the claim follows. □

3. Dispersive decay

In this section we prove the dispersive decay estimate (1.5) for the Schrödinger equation (1.2). In order to do this, we divide the analysis into a low and high energy regimes. In the analysis of both regimes we make use of variants of the van der Corput lemma (see Appendix A), combined with a Born series approach for the high energy regime suggested in [10] and adapted to our setting in [19].

3.1. **The low energy part.**

For the low energy regime, it is convenient to use the following well-known representation of the integral kernel of $\mathrm{e}^{-\mathrm{i}tH}P_c(H)$,

$$\begin{aligned}[\mathrm{e}^{-\mathrm{i}tH}P_c(H)](x,y) &= \frac{2}{\pi}\int_{-\infty}^\infty \mathrm{e}^{-\mathrm{i}tk^2}\,\phi(k^2,x)\phi(k^2,y)\,\mathrm{Im}\,m(k^2)k\,dk\\ &= \frac{2}{\pi}\int_{-\infty}^\infty \mathrm{e}^{-\mathrm{i}tk^2}\,\frac{\phi(k^2,x)\phi(k^2,y)k^2}{|f(k)|^2}\,dk\\ &= \frac{2}{\pi}\int_{-\infty}^\infty \mathrm{e}^{-\mathrm{i}tk^2}\,\frac{\tilde\phi(k,x)\tilde\phi(k,y)}{|F(k)|^2}\,dk,\end{aligned} \tag{3.1}$$

where the integral is to be understood as an improper integral. In fact, adding an additional energy cut-off (which is all we will need below) the formula is immediate from the spectral transformation [16, §3] and the general case can then be established taking limits (see [19] for further details).

In the last equality we have used

$$\tilde\phi(k,x) := |k|^{\frac{1}{2}}\phi(k^2,x), \quad k\in\mathbb{R}. \tag{3.2}$$

Note that

$$|\tilde\phi(k,x)| \le C\left(\frac{|k|x}{1+|k|x}\right)^{\frac12} \mathrm{e}^{|\,\mathrm{Im}\,k|x}\left(1+\int_0^x\left(1+\log\left(\frac{x}{y}\right)\right)\frac{y|q(y)|}{1+|k|y}dy\right), \quad (3.3)$$

$$|\partial_k\tilde\phi(k,x)| \le Cx\left(\frac{|k|x}{1+|k|x}\right)^{-\frac12} \mathrm{e}^{|\,\mathrm{Im}\,k|x}\left(1+\int_0^x\left(1+\log\left(\frac{x}{y}\right)\right)\frac{y|q(y)|}{1+|k|y}dy\right), \quad (3.4)$$

which follow from (2.4), (2.9) and the equality

$$\partial_k\tilde\phi(k,x) = \frac12 \mathrm{sgn}(k)|k|^{-\frac12}\phi(k^2,x) + |k|^{\frac12}\partial_k\phi(k^2,x)$$

together with (2.11), (2.15).

We begin with the following estimate.

Theorem 3.1. *Assume* (2.1) *and* (2.37). *Let* $\chi\in C_c^\infty(\mathbb{R})$ *with* $\mathrm{supp}(\chi)\subset(-k_0,k_0)$. *Then*

$$\left|[\mathrm{e}^{-\mathrm{i}tH}\chi(H)P_c(H)](x,y)\right| \le C\sqrt{xy}|t|^{-\frac12} \quad (3.5)$$

for all $x,y\le 1$.

Proof. We want to apply the van der Corput Lemma A.1 to the integral

$$I(t,x,y) := [\mathrm{e}^{-\mathrm{i}tH}\chi(H)P_c(H)](x,y) = \frac{2}{\pi}\int_{-\infty}^{\infty} \mathrm{e}^{-\mathrm{i}tk^2}\chi(k^2)\,\frac{\tilde\phi(k,x)\tilde\phi(k,y)}{|F(k)|^2}\,dk.$$

Denote

$$A(k)=\chi(k^2)A_0(k), \qquad A_0(k)=\frac{\tilde\phi(k,x)\tilde\phi(k,y)}{|F(k)|^2}.$$

Note that

$$\|A\|_\infty \le \|\chi\|_\infty\|A_0\|_\infty, \qquad \|A'\|_1 \le \|\chi'\|_1\|A_0\|_\infty + \|\chi\|_1\|A_0'\|_\infty.$$

By Lemma 2.11, $F(k)\ne 0$ for all $k\in\mathbb{R}\setminus\{0\}$. Moreover, combining (2.35) with Lemma 2.16, we conclude that $\|1/F\|_\infty<\infty$. Using (3.3) and noting that $\log(x/y)\le \log(1/y)$ for all $0<y\le x\le 1$, we get

$$|\tilde\phi(k,x)| \le C\left(\frac{|k|x}{1+|k|x}\right)^{\frac12}\mathrm{e}^{|\,\mathrm{Im}\,k|x}, \quad x\in(0,1]. \quad (3.6)$$

Therefore,

$$\sup_{k\in[-k_0,k_0]}|A_0(k)| \le C\,\|1/F\|_\infty^2|k_0|\sqrt{xy}, \quad (3.7)$$

which holds for all $x,y\in(0,1]$ with some uniform constant $C>0$.

Next, we get

$$A_0'(k) = \frac{\partial_k\tilde\phi(k,x)\tilde\phi(k,y)+\tilde\phi(k,x)\partial_k\tilde\phi(k,y)}{|F(k)|^2} - A_0(k)\,\mathrm{Re}\,\frac{F'(k)}{F(k)}.$$

To consider the second term, we infer from (3.6), Lemma 2.16 and Lemma 2.18 that

$$\left|A_0(k)\,\mathrm{Re}\,\frac{F'(k)}{F(k)}\right| \le \frac{|\tilde\phi(k,x)\tilde\phi(k,y)|}{|F(k)|^2}\left|\frac{F'(k)}{F(k)}\right| \le C\sqrt{xy}.$$

The estimate for the first term follows from (3.6) and (3.4) since

$$\begin{aligned}&\Big|\partial_k\tilde\phi(k,x)\tilde\phi(k,y)+\tilde\phi(k,x)\partial_k\tilde\phi(k,y)\Big|\\ &\le C\left(\frac{|k|x}{1+|k|x}\right)^{\frac12}\left(\frac{|k|y}{1+|k|y}\right)^{\frac12}\left(\frac{1+|k|x}{|k|}+\frac{1+|k|y}{|k|}\right)\\ &\le C\sqrt{xy}\frac{1+|k|x+1+|k|y}{\sqrt{(1+|k|x)(1+|k|y)}}\le 2C(1+|k|)\sqrt{xy},\quad x,y\in(0,1].\end{aligned}$$

The claim now follows by applying the classical van der Corput Lemma (see [28, page 334]) or by noting that $A\in\mathcal{W}_0(\mathbb{R})$ in view of Lemma A.2 and then it remains to apply Lemma A.1. □

Theorem 3.2. *Assume*

$$\int_0^1|q(x)|dx<\infty\quad\textit{and}\quad\int_1^\infty x\log^2(1+x)|q(x)|dx<\infty. \tag{3.8}$$

Let also $\chi\in C_c^\infty(\mathbb{R})$ *with* $\operatorname{supp}(\chi)\subset(-k_0,k_0)$*. If* $\phi(0,x)/\sqrt{x}$ *is unbounded near* ∞*, then*

$$\big|[\mathrm{e}^{-\mathrm{i}tH}\chi(H)P_c(H)](x,y)\big|\le C|t|^{-\frac12}, \tag{3.9}$$

whenever $\max(x,y)\ge1$*.*

Proof. Assume that $0<x\le1\le y$. We proceed as in the previous proof but use Lemma 2.5 and Lemma 2.9 to write

$$A(k)=\chi(k^2)\frac{(I+B_x)\tilde\phi_{-\frac12}(k,x)\cdot(I+K_y)\tilde\phi_{-\frac12}(k,y)}{|F(k)|^2},\quad k\ne0.$$

Indeed, for all $k\in\mathbb{R}\setminus\{0\}$, $\phi(k^2,\cdot)$ admits the representation (2.31). Therefore, by Lemma 2.9, $\tilde\phi(k,y)=(I+K_y)\tilde\phi_{-\frac12}(k,y)$ for all $k\in\mathbb{R}\setminus\{0\}$.

By symmetry $A(k)=A(-k)$ and hence our integral reads

$$I(t,x,y)=\frac{4}{\pi}\int_0^\infty \mathrm{e}^{-\mathrm{i}tk^2}A(k)dk.$$

Let us show that the individual parts of $A(k)$ coincide with a function which is the Fourier transform of a finite measure. Clearly, we can redefine $A(k)$ for $k<0$. To this end note that $\tilde\phi_{-\frac12}(k^2,x)=J(|k|x)$, where $J(r)=\sqrt{r}J_0(r)$. Note that $J(r)\sim\sqrt{r}$ as $r\to0$ and $J(r)=\sqrt{\frac{2}{\pi}}\cos(r-\frac{\pi}{4})+O(r^{-1})$ as $r\to+\infty$ (see (B.4)). Moreover, $J'(r)\sim\frac{1}{2\sqrt{r}}$ as $r\to0$ and $J'(r)=\sqrt{\frac{2}{\pi}}\cos(r+\frac{\pi}{4})+O(r^{-1})$ as $r\to+\infty$ (see (B.8)). Moreover, we can define $J(r)$ for $r<0$ such that it is locally in H^1 and $J(r)=\sqrt{\frac{2}{\pi}}\cos(r-\frac{\pi}{4})$ for $r<-1$. By construction we then have $\tilde J\in L^2(\mathbb{R})$ and $\tilde J\in L^p(\mathbb{R})$ for all $p\in(1,2)$. By Lemma A.2, $\tilde J\in\mathcal{W}_0$ and hence $\tilde J$ is the Fourier transform of an integrable function. Moreover, $\cos(r-\frac{\pi}{4})$ is the Fourier transform of the sum of two Dirac delta measures and so J is the Fourier transform of a finite measure. By scaling, the total variation of the measures corresponding to $J(kx)$ is independent of x.

Let us show that $\chi(k^2)|F(k)|^{-2}$ belongs to the Wiener algebra $\mathcal{W}_0(\mathbb{R})$. As in Lemma A.3, we define the functions f_0 and f_1. Since $\phi(0,x)/\sqrt{x}$ is unbounded near

∞, by Lemma 2.16 we conclude that $F(k) = \log(k^2)(c + o(1))$ as $k \to 0$ with some $c \neq 0$. Hence Lemma 2.18 yields

$$\left|\frac{d}{dk}\frac{1}{|F(k)|^2}\right| = \left|-\frac{1}{|F(k)|^2} 2\,\mathrm{Re}\left(\frac{F'(k)}{F(k)^*}\right)\right| \leq 2\frac{|F'(k)|}{|F(k)|^3} \leq \frac{C}{|k||\log(k)|^3}$$

for k near zero, which implies that

$$f_1(k) \leq C\frac{1}{k\log^3(2/k)}, \quad k \in (0,1).$$

Therefore, we get

$$\int_0^1 \log\big(2/k\big) f_1(k)dk \leq C\int_0^1 \frac{dk}{k\log^2(2/k)} = C\int_0^{1/2}\frac{dk}{k\log^2(k)} = \frac{C}{\log 2} < \infty.$$

Noting that the second condition in (A.3) is satisfied since χ has compact support and hence so are f_0 and f_1. Therefore Lemma A.3 implies that $\chi(k^2)|F(k)|^{-2}$ belongs to the Wiener algebra $\mathcal{W}_0(\mathbb{R})$.

Lemma A.1 then shows

$$|\tilde{I}(t,x,y)| \leq \frac{C}{\sqrt{t}}, \qquad \tilde{I}(t,x,y) := \frac{4}{\pi}\int_0^\infty \mathrm{e}^{-\mathrm{i}tk^2}\chi(k^2)\frac{\tilde{\phi}_{-\frac12}(k,x)\tilde{\phi}_{-\frac12}(k,y)}{|F(k)|^2}dk.$$

But by Fubini we have $I(t,x,y) = (1+B_x)(1+K_y)\tilde{I}(t,x,y)$ and the claim follows since both $B\colon L^\infty((0,1)) \to L^\infty((0,1))$ and $K\colon L^\infty((1,\infty)) \to L^\infty((1,\infty))$ are bounded in view of Corollary 2.6 and Corollary 2.10, respectively.

By symmetry, we immediately obtain the same estimate if $0 < y \leq 1 \leq x$. The case $\min(x,y) \geq 1$ can be proved analogously, we only need to write

$$A(k) = \chi(k^2)\frac{(I+K_x)\tilde{\phi}_{-\frac12}(k,x)\cdot(I+K_y)\tilde{\phi}_{-\frac12}(k,y)}{|F(k)|^2}, \quad k \neq 0. \qquad \square$$

3.2. **The high energy part.** For the analysis of the high energy regime we use the following —also well-known— alternative representation:

$$\begin{aligned}\mathrm{e}^{-\mathrm{i}tH}P_c(H) &= \frac{1}{2\pi\mathrm{i}}\int_0^\infty \mathrm{e}^{-\mathrm{i}t\omega}\left[\mathcal{R}_H(\omega+\mathrm{i}0) - \mathcal{R}_H(\omega-\mathrm{i}0)\right]d\omega \\ &= \frac{1}{\pi\mathrm{i}}\int_{-\infty}^\infty \mathrm{e}^{-\mathrm{i}tk^2}\mathcal{R}_H(k^2+\mathrm{i}0)\,k\,dk, \end{aligned} \tag{3.10}$$

where $\mathcal{R}_H(\omega) = (H-\omega)^{-1}$ is the resolvent of the Schrödinger operator H and the limit is understood in the strong sense (see, e.g., [29]). We recall that for $k \in \mathbb{R}\setminus\{0\}$ the Green's function is given by

$$[\mathcal{R}_H(k^2 \pm \mathrm{i}0)](x,y) = [\mathcal{R}_H(k^2\pm\mathrm{i}0)](y,x) = \phi(k^2,x)\frac{f(\pm k,y)}{f(\pm k)}, \quad x \leq y. \tag{3.11}$$

Fix $k_0 > 0$ and let $\chi\colon \mathbb{R} \to [0,\infty)$ be a C^∞ function such that

$$\chi(k^2) = \begin{cases} 0, & |k| < 2k_0, \\ 1, & |k| > 3k_0. \end{cases} \tag{3.12}$$

The purpose of this section is to prove the following estimate.

Theorem 3.3. *Suppose $q \in L^1(\mathbb{R}_+)$ satisfies* (2.20). *Then*

$$\left|[\mathrm{e}^{-\mathrm{i}tH}\chi(H)P_c(H)](x,y)\right| \leq C|t|^{-\frac12}.$$

Our starting point is the fact that the resolvent $\mathcal{R}_H$ of H can be expanded into the Born series

$$\mathcal{R}_H(k^2 \pm \mathrm{i}0) = \sum_{n=0}^{\infty} \mathcal{R}_{-\frac{1}{2}}(k^2 \pm \mathrm{i}0)(-q\,\mathcal{R}_{-\frac{1}{2}}(k^2 \pm \mathrm{i}0))^n, \tag{3.13}$$

where $\mathcal{R}_{-\frac{1}{2}}$ stands for the resolvent of the unperturbed radial Schrödinger operator. To this end we begin by collecting some facts about $\mathcal{R}_{-\frac{1}{2}}$. Its kernel is given

$$\mathcal{R}_{-\frac{1}{2}}(k^2 \pm \mathrm{i}0, x, y) = \frac{1}{k} r_{-\frac{1}{2}}(\pm k, x, y),$$

where

$$r_{-\frac{1}{2}}(k; x, y) = r_{-\frac{1}{2}}(k; y, x) = k\sqrt{xy}\, J_0(kx) H_0^{(1)}(ky), \quad x \le y.$$

Lemma 3.4. *The function $r_{-\frac{1}{2}}(k, x, y)$ can be written as*

$$r_{-\frac{1}{2}}(k, x, y) = \chi_{(-\infty,0]}(k) \int_{\mathbb{R}} \mathrm{e}^{\mathrm{i}kp} d\rho_{x,y}(p) + \chi_{[0,\infty)}(k) \int_{\mathbb{R}} \mathrm{e}^{-\mathrm{i}kp} d\rho^*_{x,y}(p)$$

with a measure whose total variation satisfies

$$\|\rho_{x,y}\| \le C.$$

Here ρ^* is the complex conjugated measure.

Proof. Let $x \le y$ and $k \ge 0$. Write

$$r_{-\frac{1}{2}}(k, x, y) = J(kx)H(ky),$$

where

$$J(r) = \sqrt{r}\, J_0(r), \quad H(r) = \sqrt{r}\, H_0^{(1)}(r).$$

We continue $J(r)$, $H(r)$ to the region $r < 0$ such that they are continuously differentiable and satisfy

$$J(r) = \sqrt{\frac{2}{\pi}} \cos\left(r - \frac{\pi}{4}\right), \quad H(r) = \sqrt{\frac{2}{\pi}} \mathrm{e}^{\mathrm{i}\left(r - \frac{\pi}{4}\right)},$$

for $r < -1$. It's enough to show that

$$\tilde{J}(r) = J(r) - \sqrt{\frac{2}{\pi}} \cos(r - \frac{\pi}{4}) \quad \text{and} \quad \tilde{H}(r) = H(r) - \sqrt{\frac{2}{\pi}} \mathrm{e}^{\mathrm{i}(r - \frac{\pi}{4})}$$

are elements of the Wiener Algebra $\mathcal{W}_0(\mathbb{R})$. In fact, they are continuously differentiable and hence it suffices to look at their asymptotic behavior. To do this, we need the results about Bessel and Hankel functions, collected in Appendix B. For $r < -1$ both $\tilde{J}(r)$ and $\tilde{H}(r)$ are zero. $\tilde{J}$ is integrable near 0 and for $r > 1$ it behaves like $O(r^{-1})$ and $O(r^{-1})$ for the derivative. So $\tilde{J}$ is contained in $H^1(\mathbb{R})$ and therefore in $\mathcal{W}_0$ by Lemma A.2. As for $\tilde{H}$, near 0 it behaves like $\sqrt{r}\log r$ and hence its derivative belongs to L^p for all $p \in (1, 2)$ near zero. Since $\tilde{H}(r)$ and its derivative also behave like $O(r^{-1})$ for $r > 1$, Lemma A.2 applies and thus we also have $\tilde{H} \in \mathcal{W}_0$. As a consequence, both J and H are Fourier transforms of finite measures. By scaling the total variation of the measures corresponding to $J(kx)$, $H(ky)$, are independent of x and y, respectively. This finishes the proof. □

Now we are in position to finish the proof of the main result.

Proof of Theorem 3.3. As a consequence of Lemma 3.4 we note

$$|\mathcal{R}_{-\frac{1}{2}}(k^2 \pm \mathrm{i}0, x, y)| \le \frac{C}{|k|}$$

and hence the operator $q\,\mathcal{R}_{-\frac{1}{2}}(k^2 \pm \mathrm{i}0)$ is bounded on L^1 with

$$\|q\,\mathcal{R}_{-\frac{1}{2}}(k^2 \pm \mathrm{i}0)\|_{L^1} \le \frac{C}{|k|}\|q\|_{L^1}.$$

Thus we get

$$\begin{aligned}\left|\left\langle \mathcal{R}_{-\frac{1}{2}}(k^2 \pm \mathrm{i}0)(-q\,\mathcal{R}_{-\frac{1}{2}}(k^2 \pm \mathrm{i}0))^n f, g\right\rangle\right| &= \left|\left\langle -q\,\mathcal{R}_{-\frac{1}{2}}(k^2 \pm \mathrm{i}0))^n f, \mathcal{R}_{-\frac{1}{2}}(k^2 \mp \mathrm{i}0) g\right\rangle\right| \\ &\le \left\|(-q\,\mathcal{R}_{-\frac{1}{2}}(k^2 \pm \mathrm{i}0))^n f\right\|_{L^1} \left\|\mathcal{R}_{-\frac{1}{2}}(k^2 \mp \mathrm{i}0) g\right\|_{L^\infty} \\ &\le \frac{C^{n+1}\|q\|_{L^1}^n}{|k|^{n+1}} \|f\|_{L^1}\,\|g\|_{L^1}\,.\end{aligned}$$

This estimate holds for all L^1 functions f and g and hence the series (3.13) weakly converges whenever $|k| > k_0 = C(l)\|q\|_{L^1}$. Namely, for all L^1 functions f and g we have

$$\left\langle \mathcal{R}_H(k^2 \pm \mathrm{i}0) f, g\right\rangle = \sum_{n=0}^{\infty} \left\langle \mathcal{R}_{-\frac{1}{2}}(k^2 \pm \mathrm{i}0)(-q\,\mathcal{R}_{-\frac{1}{2}}(k^2 \pm \mathrm{i}0))^n f, g\right\rangle. \tag{3.14}$$

Using the estimates (2.9), (2.25), (2.34), and (2.35) for the Green's function (3.11), one can see that

$$\mathcal{R}_H(k^2 \pm \mathrm{i}0)\, g \in L^\infty$$

whenever $g \in L^1$ and $|k| > 0$. Therefore, we get

$$\begin{aligned}&\left|\left\langle \mathcal{R}_H(k^2 \pm \mathrm{i}0)(-q\,\mathcal{R}_{-\frac{1}{2}}(k^2 \pm \mathrm{i}0))^n f, g\right\rangle\right| \\ &\qquad = \left|\left\langle (-q\,\mathcal{R}_{-\frac{1}{2}}(k^2 \pm \mathrm{i}0))^n f, \mathcal{R}_H(k^2 \mp \mathrm{i}0) g\right\rangle\right| \\ &\qquad \le \left\|(-q\,\mathcal{R}_{-\frac{1}{2}}(k^2 \pm \mathrm{i}0))^n f\right\|_{L^1} \left\|\mathcal{R}_H(k^2 \mp \mathrm{i}0) g\right\|_{L^\infty} \\ &\qquad \le \left(\frac{C\,\|q\|_{L^1}}{k}\right)^n \left\|\mathcal{R}_H(k^2 \mp \mathrm{i}0) g\right\|_{L^\infty}\,,\end{aligned}$$

which means that $\mathcal{R}_H(k^2 \pm \mathrm{i}0)(-q\,\mathcal{R}_{-\frac{1}{2}}(k^2 \pm \mathrm{i}0))^n$ weakly tends to 0 whenever $|k| > k_0$.

Let us consider again a function χ as in (3.12) with $k_0 = C\|q\|_1$. Since $\mathrm{e}^{\mathrm{i}tH}\chi(H)P_c = \mathrm{e}^{\mathrm{i}tH}\chi(H)$, we get from (3.10)

$$\left\langle \mathrm{e}^{-\mathrm{i}tH}\chi(H) f, g\right\rangle = \frac{1}{\pi \mathrm{i}} \int_{-\infty}^{\infty} \mathrm{e}^{-\mathrm{i}tk^2}\chi(k^2) k \left\langle \mathcal{R}_H(k^2 + \mathrm{i}0) f, g\right\rangle dk.$$

Using (3.14) and noting that we can exchange summation and integration, we get

$$\begin{aligned}&\left\langle \mathrm{e}^{-\mathrm{i}tH}\chi(H) f, g\right\rangle \\ &\qquad = \frac{1}{\pi \mathrm{i}} \sum_{n=0}^{\infty} \int_{-\infty}^{\infty} \mathrm{e}^{-\mathrm{i}tk^2}\chi(k^2) k \left\langle \mathcal{R}_{-\frac{1}{2}}(k^2 + \mathrm{i}0)(-q\,\mathcal{R}_{-\frac{1}{2}}(k^2 + \mathrm{i}0))^n f, g\right\rangle dk.\end{aligned}$$

The kernel of the operator $\mathcal{R}_{-\frac{1}{2}}(k^2+\mathrm{i}0)(-q\,\mathcal{R}_{-\frac{1}{2}}(k^2+\mathrm{i}0))^n$ is given by

$$\frac{1}{k^{n+1}}\int_{\mathbb{R}_+^n} r_{-\frac{1}{2}}(k;x,y_1)\prod_{i=1}^n q(y_i)\prod_{i=1}^{n-1} r_{-\frac{1}{2}}(k;y_i,y_{i+1})r_{-\frac{1}{2}}(k;y_n,y)dy_1\cdots dy_n.$$

Applying Fubini's theorem, we can integrate in k first and hence we need to obtain a uniform estimate of the oscillatory integral

$$I_n(t;u_0,\dots,u_{n+1})=\int_{\mathbb{R}}\mathrm{e}^{-\mathrm{i}tk^2}\chi(k^2)\left(\frac{k}{2k_0}\right)^{-n}\prod_{i=0}^n r_{-\frac{1}{2}}(k;u_i,u_{i+1})\,dk$$

since, recalling that $k_0=C(l)\|q\|_{L^1}$, one obtains

$$\left|\left\langle \mathrm{e}^{-\mathrm{i}tH}\chi(H)f,g\right\rangle\right|\le\frac{1}{\pi}\sum_{n=0}^\infty\frac{1}{(2C)^n}\sup_{\{u_i\}_{i=0}^{n+1}}|I_n(t;u_0,\dots,u_{n+1})|\,\|f\|_{L^1}\,\|g\|_{L^1}\,.$$

Consider the function $f_n(k)=\chi(k^2)\left(\frac{k}{2k_0}\right)^{-n}$. Clearly, f_0 is the Fourier transform of a measure ν_0 satisfying $\|\nu_0\|\le C_1$. For $n\ge 1$, f_n belongs to $H^1(\mathbb{R})$ with $\|f_n\|_{H^1}\le\pi^{-1/2}C_1(1+n)$. Hence by Lemma A.1 and Lemma 3.4 we obtain

$$|I_n(t;u_0,\dots,u_{n+1})|\le\frac{2C_vC_1}{\sqrt{t}}(1+n)C^{n+1}$$

implying

$$\left|\left\langle \mathrm{e}^{-\mathrm{i}tH}\chi(H)f,g\right\rangle\right|\le\frac{2C_vC_1\,C}{\sqrt{t}}\,\|f\|_{L^1}\,\|g\|_{L^1}\sum_{n=0}^\infty\frac{1+n}{2^n}.$$

This proves Theorem 3.3. □

Appendix A. The van der Corput Lemma

We will need the the following variant of the van der Corput lemma (see, e.g., [19, Lemma A.2] and [28, page 334]).

Lemma A.1. *Let $(a,b)\subseteq\mathbb{R}$ and consider the oscillatory integral*

$$I(t)=\int_a^b \mathrm{e}^{\mathrm{i}tk^2}A(k)dk.$$

If $A\in\mathcal{W}(\mathbb{R})$, i.e., A is the Fourier transform of a signed measure

$$A(k)=\int_{\mathbb{R}}\mathrm{e}^{\mathrm{i}kp}d\alpha(p),$$

then the above integral exists as an improper integral and satisfies

$$|I(t)|\le C_2\,|t|^{-\frac{1}{2}}\,\|A\|_{\mathcal{W}}\,,\quad |t|>0.$$

where $\|A\|_{\mathcal{W}}:=\|\alpha\|=|\alpha|(\mathbb{R})$ denotes the total variation of α and $C_2\le 2^{8/3}$ is a universal constant.

Note that if A_1, $A_2\in\mathcal{W}(\mathbb{R})$, then (cf. p. 208 in [1])

$$(A_1A_2)(k)=\frac{1}{(2\pi)^2}\int_{\mathbb{R}}\mathrm{e}^{\mathrm{i}kp}d(\alpha_1*\alpha_2)(p)$$

is associated with the convolution

$$\alpha_1*\alpha_2(\Omega)=\iint\mathbb{1}_\Omega(x+y)d\alpha_1(x)d\alpha_2(y),$$

where $\mathbb{1}_\Omega$ is the indicator function of a set Ω. Note that

$$\|\alpha_1 * \alpha_2\| \le \|\alpha_1\|\|\alpha_2\|.$$

Let $\mathcal{W}_0(\mathbb{R})$ be the Wiener algebra of functions $C(\mathbb{R})$ which are Fourier transforms of L^1 functions,

$$\mathcal{W}_0(\mathbb{R}) = \Big\{ f \in C(\mathbb{R})\colon f(k) = \int_{\mathbb{R}} \mathrm{e}^{\mathrm{i}kx} g(x)dx,\ g \in L^1(\mathbb{R}) \Big\}.$$

Clearly, $\mathcal{W}_0(\mathbb{R}) \subset \mathcal{W}(\mathbb{R})$. Moreover, by the Riemann–Lebesgue lemma, $f \in C_0(\mathbb{R})$, that is, $f(k) \to 0$ as $k \to \infty$ if $f \in \mathcal{W}_0(\mathbb{R})$. A comprehensive survey of necessary and sufficient conditions for $f \in C(\mathbb{R})$ to be in the Wiener algebras $\mathcal{W}_0(\mathbb{R})$ and $\mathcal{W}(\mathbb{R})$ can be found in [21], [22]. We need the following statement, which extends the well-known Beurling condition (see [11, Lemma B.3]).

Lemma A.2. *If $f \in L^2(\mathbb{R})$ is locally absolutely continuous and $f' \in L^p(\mathbb{R})$ with $p \in (1, 2]$, then f is in the Wiener algebra $\mathcal{W}_0(\mathbb{R})$ and*

$$\|f\|_{\mathcal{W}} \le C_p\big(\|f\|_{L^2(\mathbb{R})} + \|f'\|_{L^p(\mathbb{R})}\big), \tag{A.1}$$

where $C_p > 0$ is a positive constant, which depends only on p.

We also need the following result from [22].

Lemma A.3. *Let $f \in C_0(\mathbb{R})$ be locally absolutely continuous on $\mathbb{R} \setminus \{0\}$. Set*

$$f_0(x) := \sup_{|y|\ge|x|} |f(y)|, \quad f_1(x) := \operatorname{ess\,sup}_{|y|\ge|x|} |f'(y)|, \tag{A.2}$$

for all $x \neq 0$. If

$$\int_0^1 \log\big(2/x\big) f_1(x)dx < \infty, \quad \int_1^\infty \bigg(\int_x^\infty f_0(y) f_1(y) dy \bigg)^{1/2} dx < \infty, \tag{A.3}$$

then $f \in \mathcal{W}_0(\mathbb{R})$.

Appendix B. Bessel functions

Here we collect basic formulas and information on Bessel and Hankel functions (see, e.g., [23, 31]). First of all assume $m \in \mathbb{N}_0$. We start with the definitions:

$$J_m(z) = \Big(\frac{z}{2}\Big)^m \sum_{n=0}^\infty \frac{(\frac{-z^2}{4})^n}{n!(n+m+1)!}, \tag{B.1}$$

$$\begin{aligned} Y_m(z) = &-\frac{\big(\frac{-z}{2}\big)^{-m}}{\pi} \sum_{n=0}^{m-1} \frac{(m-n-1)!(\frac{z^2}{4})^n}{n!} + \frac{2}{\pi}\log(z/2) J_m(z) \\ &+ \frac{\big(\frac{z}{2}\big)^m}{\pi} \sum_{n=0}^\infty \left(\psi(n+1) + \psi(n+m+1)\right) \frac{(\frac{-z^2}{4})^n}{n!(n+m+1)!}, \end{aligned} \tag{B.2}$$

$$H_m^{(1)}(z) = J_m(z) + \mathrm{i}Y_m(z), \qquad H_m^{(2)}(z) = J_m(z) - \mathrm{i}Y_m(z). \tag{B.3}$$

Here ψ is the digamma function [23, (5.2.2)]. The asymptotic behavior as $|z| \to \infty$ is given by

$$J_m(z) = \sqrt{\frac{2}{\pi z}} \Big(\cos(z - \pi m/2 - \pi/4) + \mathrm{e}^{|\operatorname{Im} z|}\mathcal{O}(|z|^{-1})\Big), \quad |\arg z| < \pi, \tag{B.4}$$

$$Y_m(z) = \sqrt{\frac{2}{\pi z}} \Big(\sin(z - \pi m/2 - \pi/4) + \mathrm{e}^{|\operatorname{Im} z|}\mathcal{O}(|z|^{-1})\Big), \quad |\arg z| < \pi, \tag{B.5}$$

$$H_m^{(1)}(z) = \sqrt{\frac{2}{\pi z}} \mathrm{e}^{i(z - \frac{2m+1}{4}\pi)} \big(1 + \mathcal{O}(|z|^{-1})\big), \quad -\pi < \arg z < 2\pi, \tag{B.6}$$

$$H_m^{(2)}(z) = \sqrt{\frac{2}{\pi z}} \mathrm{e}^{-i(z - \frac{2m+1}{4}\pi)} \big(1 + \mathcal{O}(|z|^{-1})\big), \quad -2\pi < \arg z < \pi. \tag{B.7}$$

Using [23, (10.6.2)], one can show that the derivative of the reminder satisfies

$$\left(\sqrt{\frac{\pi z}{2}} J_0(z) - \cos(z - \pi/4)\right)' = \mathrm{e}^{|\operatorname{Im} z|}\mathcal{O}(|z|^{-1}), \tag{B.8}$$

as $|z| \to \infty$. The same is true for Y_m, $H_m^{(1)}$ and $H_m^{(2)}$.

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References

[1] V. I. Bogachev, *Measure Theory. I*, Springer-Verlag, Berlin, Heidelberg, 2007.

[2] D. Bollé and F. Gesztesy, *Scattering observables in arbitrary dimension $n \geq 2$*, Phys. Rev. A **30**, no. 2, 1279–1293 (1984).

[3] D. Bollé and F. Gesztesy, *Low-energy parametrization of scattering in n-dimensional quantum systems*, Phys. Rev. Lett. **52**, no. 17, 1469–1472 (1984).

[4] N. Burq, F. Planchon, J. Stalker, and S. Tahvildar-Zadeh, *Strichartz estimates for the wave and Schrödinger equations with the inverse-square potential*, J. Funct. Anal. **203**, 519–549 (2003).

[5] N. Burq, F. Planchon, J. Stalker, and S. Tahvildar-Zadeh, *Strichartz estimates for the wave and Schrödinger equations with potentials of critical decay*, Indiana Univ. Math. J. **53**, 1665–1680 (2004).

[6] K. Chadan and P. C. Sabatier, *Inverse Problems in Quantum Scattering Theory*, 2nd ed., Springer-Verlag, 1989.

[7] M. Coz and C. Coudray, *The Riemann solution and the inverse quantum mechanical problem*, J. Math. Phys. **17**, no. 6, 888–893 (1976).

[8] I. Egorova, E. Kopylova, V. Marchenko, and G. Teschl, *Dispersion estimates for one-dimensional Schrödinger and Klein–Gordon equations revisited*, Russian Math. Surveys **71**, 3–26 (2016).

[9] L. Faddeev, *The inverse problem in quantum scattering theory*, J. Math. Phys. **4**, 72–104 (1963).

[10] M. Goldberg and W. Schlag, *Dispersive estimates for Schrödinger operators in dimensions one and three*, Comm. Math. Phys. **251**, 157–178 (2004).

[11] M. Holzleitner, A. Kostenko, and G. Teschl, *Dispersion estimates for spherical Schrödinger equations: The effect of boundary conditions*, Opuscula Math. **36**, no. 6, 769–786 (2016).

[12] M. Holzleitner, A. Kostenko, and G. Teschl, *Transformation operators for spherical Schrödinger operators*, in preparation.

[13] I. S. Kac, *On the behavior of spectral functions of second-order differential systems*, Dokl. Akad. Nauk SSSR **106**, 183–186 (1956). [in Russian]

[14] E. Kopylova, *Dispersion estimates for Schrödinger and Klein–Gordon equation*, Russian Math. Surveys, **65**, no. 1, 95–142 (2010).

[15] A. Kostenko, A. Sakhnovich, and G. Teschl, *Inverse eigenvalue problems for perturbed spherical Schrödinger operators*, Inverse Problems **26**, 105013, 14pp (2010).
[16] A. Kostenko, A. Sakhnovich, and G. Teschl, *Weyl–Titchmarsh theory for Schrödinger operators with strongly singular potentials*, Int. Math. Res. Not. **2012**, 1699–1747 (2012).
[17] A. Kostenko and G. Teschl, *On the singular Weyl–Titchmarsh function of perturbed spherical Schrödinger operators*, J. Differential Equations **250**, 3701–3739 (2011).
[18] A. Kostenko and G. Teschl, *Spectral asymptotics for perturbed spherical Schrödinger operators and applications to quantum scattering*, Comm. Math. Phys. **322**, 255–275 (2013).
[19] A. Kostenko, G. Teschl and J. H. Toloza, *Dispersion estimates for spherical Schrödinger equations*, Ann. Henri Poincaré **17**, no. 11, 3147–3176 (2016).
[20] H. Kovařík and F. Truc, *Schrödinger operators on a half-line with inverse square potentials*, Math. Model. Nat. Phenom. **9**, no. 5, 170–176 (2014).
[21] E. Liflyand, S. Samko and R. Trigub, *The Wiener algebra of absolutely convergent Fourier integrals: an overview*, Anal. Math. Phys. **2**, 1–68, (2012).
[22] E. Liflyand and R. Trigub, *Conditions for the absolute convergence of Fourier integrals*, J. Approx. Theory **163**, 438–459 (2011).
[23] F. W. J. Olver et al., *NIST Handbook of Mathematical Functions*, Cambridge University Press, Cambridge, 2010.
[24] W. Schlag, *Dispersive estimates for Schrödinger operators: a survey*, in "Mathematical aspects of nonlinear dispersive equations", 255–285, Ann. Math. Stud. **163**, Princeton Univ. Press, Princeton, NJ, 2007.
[25] N. Setô, *Bargmann's inequalities in spaces of arbitrary dimension*, Publ. RIMS, Kyoto Univ. **9**, 429–461 (1974).
[26] A. S. Sohin, *On a class of transformation operators*, Trudy Fiz.-Teh. Inst. Nizkih Temp. AN USSR, Mat. Fiz., Funkts. Analiz, no. 1, 117–125 (1969) (in Russian); *English transl.* in Sel. Math. Sov. **3**, no. 3, 301–308 (1983).
[27] A. S. Sohin, *The inverse scattering problem for an equation with a singularity*, Trudy Fiz.-Teh. Inst. Nizkih Temp. AN USSR, Mat. Fiz., Funkts. Analiz, no. 2, 182–235 (1971) (in Russian).
[28] E. M. Stein, *Harmonic Analysis: Real-Variable Methods, Orthogonality, and Oscillatory Integrals*, Princeton Math. Series **43**, Princeton University Press, Princeton, NJ, 1993.
[29] G. Teschl, *Mathematical Methods in Quantum Mechanics; With Applications to Schrödinger Operators*, 2nd ed., Amer. Math. Soc., Rhode Island, 2014.
[30] V. Ya. Volk, *On inversion formulas for a differential equation with a singularity at* $x = 0$, Uspehi Matem. Nauk (N.S.) **8**, 141–151 (1953).
[31] G. N. Watson, *A Treatise on the Theory of Bessel Functions*, Cambridge Univ. Press, 1944.
[32] R. Weder, $L^p - L^{\dot p}$ *estimates for the Schrödinger equation on the line and inverse scattering for the nonlinear Schrödinger equation with a potential*, J. Funct. Anal. **170**, 37–68 (2000).
[33] R. Weder, *The* $L^p - L^{\dot p}$ *estimates for the Schrödinger equation on the half-line*, J. Math. Anal. Appl. **281**, 233–243 (2003).
[34] J. Weidmann, *Spectral Theory of Ordinary Differential Operators*, Lecture Notes in Mathematics **1258**, Springer, Berlin, 1987.

Transformation Operators for Spherical Schrödinger Operators

by Markus Holzleitner

Final stages of preparation

TRANSFORMATION OPERATORS FOR SPHERICAL SCHRÖDINGER OPERATORS

MARKUS HOLZLEITNER

Abstract. The present work aims at obtaining estimates for transformation operators for one-dimensional perturbed radial Schrödinger operators. It provides more details and suitable extensions to already existing results, that are needed in other recent contributions dealing with these kinds of operators.

1. Introduction

In general, transformation and transmutation for one dimensional Schrödinger or Sturm-Liouville operators on the whole or the half line have a long history due to their importance in inverse spectral theory, see e.g. [2, Page 145–163] for an overview. The present work deals with transformation properties for the radial Schrödinger operators

$$H := -\frac{d^2}{dx^2} + \frac{l(l+1)}{x^2} + q(x) := H_l + q, \quad x \in \mathbb{R}_+, \tag{1.1}$$

where $l \geq -\frac{1}{2}$ and q should satisfy some further integrability conditions mentioned later. Operators of the form (1.1) appear naturally after a separation of variables, and therefore have received considerable attention (see, e.g., [3], [4], [5], [16], [17], [18], [21], [22], [26, Section 3.7] and [31]). It's also worthwhile mentioning, that one field of recent research is concerned about proving dispersive estimates for the related Schrödinger equations, c.f. [11], [12], [19] and [20]. In many of these contributions the existence and precise estimates for transformation operators for H are crucial. There are some rather old publications available, that aim at proving these properties for H: [30] is concerned with transformation operators near 0, and [28] with the situation near ∞, cf. also [9, 13]. Unfortunately, we realized, that these results don't cover all the situations that are considered in the recent articles mentioned before; thus the aim of the present work is to fill this gap, i.e. to give full and detailed proofs and also to provide appropriate extensions. The work should also be seen as a useful tool to stimulate further research for topics that deal with Bessel operators of the form H. Now let us discuss the main theorems that we want to establish. By τ, τ_l let us denote the differential symbols corresponding to H, H_l respectively. We first focus on transformation near 0: The intention is to construct a transformation operator, that maps a solution $\phi_l(z,x)$, $z \in \mathbb{C}_+$, of the equation

$$\tau_l \phi_l(z,x) = z\phi_l(z,x), \tag{1.2}$$

to a solution $\phi(z,x)$ of

$$\tau\phi(z,x) = z\phi(z,x), \tag{1.3}$$

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such that the properties of ϕ_l near 0 are preserved. Concerning the asymptotic behavior of these solutions ϕ_l, we refer e.g. to [12, 19, Section 2]. We want to express this transformation operator as an integral operator and prove an estimate for it. To fix some notation, for any compact set $A \subset\subset \mathbb{R}_+ := (0,\infty)$, $L^p(A, w(k))$ denotes the usual weighted L^p space with positive weight $w(k)$, i.e. the associated norm is given by

$$\|f\|_{L^p(A,w(k))} = \begin{cases} \left(\int_A w(k)|f(k)|^p dk\right)^{1/p}, & 1 \le p < \infty, \\ \sup_{k\in A} w(k)|f(k)|, & p = \infty. \end{cases}$$

Furthermore, by p' we denote the corresponding dual index, i.e. $\frac{1}{p} + \frac{1}{p'} = 1$. The main theorem of the first section now reads as follows:

Theorem 1.1. *Let $L > 0$ fixed and $0 < y < x \le L$. Then*

$$\phi(z,x) = \phi_l(z,x) + \int_0^x B(x,y)\phi_l(z,y)dy =: (I+B)\phi_l(z,x), \tag{1.4}$$

where $B : \mathbb{R}_+^2 \to \mathbb{R}$ is the so-called Gelfand–Levitan kernel. Concerning the conditions on the potential q for (1.4) *to hold, and the estimates for B, we need to distinguish three cases:*

(i) If $l \ge 0$, $p > 1$ and $q \in L^p_{\mathrm{loc}}([0,\infty))$, then (1.4) *is valid and B satisfies the following estimate:*

$$|B(x,y)| \le \frac{(xy)^{\frac{1}{2p'}-\alpha}}{\alpha} \|q\|_{L^p((0,x])} x^{2\alpha} \exp\left(\frac{\tilde{C}x^{1+\frac{1}{p'}}}{\alpha} \|q\|_{L^p((0,x])}\right), \tag{1.5}$$

where $0 < \alpha < \frac{1}{2p'}$.

(ii) If $-\frac{1}{2} < l < 0$, $p > \frac{1}{2l+1}$ and $q \in L^p_{\mathrm{loc}}([0,\infty))$, then again (1.4) *and* (1.5) *are valid for $0 < \alpha < l + \frac{1}{2p'}$.*

(iii) If $l = -\frac{1}{2}$, $p > 2$ and $q \in L^p_{\mathrm{loc}}([0,\infty), k^{-\frac{p}{p'}})$, then (1.4) *and*

$$\begin{aligned} |B(x,y)| \le {} & \frac{(xy)^{\frac{1}{p'}-\alpha}}{\alpha} \|q\|_{L^p(0,x],k^{-\frac{p}{p'}})} x^{2\alpha}(\max(1,L))^{\frac{1}{2p'}} \\ & \times \exp\left(\frac{\tilde{C}(\max(1,L))^{\frac{1}{2p'}} x^{1+\frac{1}{p'}}}{\alpha} \|q\|_{L^p((0,x],z^{-\frac{p}{p'}})}\right) \end{aligned} \tag{1.6}$$

hold, where $0 < \alpha < -\frac{1}{2} + \frac{1}{p'}$.

An important conclusion of this theorem is, that the closer the parameter l is to $-\frac{1}{2}$, the more we need to restrict our assumptions on the potential q. Moreover, in the case $l = -\frac{1}{2}$, not even boundedness of q seems to be enough to guarantee the desired estimates. The proof of this result will be discussed in the first section. The aim of the second section is to verify a similar result near ∞, i.e. establishing the following theorem, where $f(k^2,x)$, $f_l(k^2,x)$ denote the Jost solutions of the corresponding equations (1.2), (1.3) respectively, which satisfy $f(k,x) \sim \mathrm{e}^{\mathrm{i}kx}$ as $x \to \infty$ near ∞:

Theorem 1.2 ([9]). *Suppose* $\int_1^\infty (x+x^l)|q(x)|dx < \infty$. *Then*

$$f(k^2,x) = f_l(k^2,x) + \int_x^\infty K(x,y) f_l(k^2,y)dy =: (I+K)f_l(k^2,x), \tag{1.7}$$

where the so-called Marchenko kernel $K : \mathbb{R}^2 \to \mathbb{R}$ *satisfies the estimate*

$$|K(x,y)| \le C_l \left(\frac{2}{x}\right)^l \tilde{\sigma}_0\left(\frac{x+y}{2}\right) \mathrm{e}^{\tilde{\sigma}_1(x)}, \quad \tilde{\sigma}_j(x) := \int_x^\infty y^j |q(y)|dy, \tag{1.8}$$

for all $x < y < \infty$.

Here we end up with a similar situation as in the case of Theorem 1.1: The bigger the parameter l, the more restrictive the assumptions on q need to be. The approach we use to obtain our results is in principle well known: first of all, one establishes a second order equation for the kernel, which can be solved using Riemann's method in combination with successive approximation. The crucial points are the estimates for the Riemann function and the iterates, which we improve at some points. We will go through the details later. Let us finish the introduction by briefly explaining the main novelties of this article: concerning the transformation operators near 0, we are able to generalize the previous results from [10, 9, 13, 30], where only continuous potentials $q \in C[0,L]$ were considered. Moreover we are able to fix some technical inconsistencies in the proofs of the estimates for B and provide further details to make the presentation more accessible. For the transformation operators near ∞, the results in [28] only consider estimates for the case $l > 0$, which we are able to generalize to $-\frac{1}{2} \le l$.

2. Transformation Operators near 0

As a starting point, we want to obtain an equation for the kernel $B(x,y)$ on a finite interval $0 < y \le x \le L$. To this end we assume first that B is $C^2(\mathbb{R}^2_+)$ and satisfies the estimates from Theorem 1.1, which leads to

$$B(x,y) = \begin{cases} \mathcal{O}(y^{\frac{1}{2p'}-\alpha}), & l > -\frac{1}{2} \\ \mathcal{O}(y^{\frac{1}{p'}-\alpha}), & l = -\frac{1}{2} \end{cases} \quad \text{as } y \to 0. \tag{2.1}$$

We will see later that these asymptotics are indeed valid. We start by differentiating (1.4) twice wrt. x to obtain

$$\begin{aligned} \phi''(z,x) =& \phi_l''(z,x) + \frac{\partial B(x,x)}{\partial x}\phi_l(z,x) + B(x,x)\phi_l'(z,x) \\ &+ \frac{\partial B(x,y)}{\partial x}\Big|_{y=x}\phi_l(z,x) + \int_0^x \frac{\partial^2 B(x,y)}{\partial x^2}\phi_l(z,y)dy. \end{aligned} \tag{2.2}$$

On the other hand, using the facts that ϕ satisfies (1.3), ϕ_l satisfies (1.2) and plugging in (1.4) for ϕ , we also get

$$\phi''(z,x) = \phi_l''(z,x) + q(x)\phi_l(z,x) + \int_0^x B(x,y)(\frac{l(l+1)}{x^2} + q(x) - z)\phi_l(z,y)dy. \tag{2.3}$$

Once more applying (1.2) and integrating by parts twice leads to

$$\begin{aligned} z\int_0^x B(x,y)\phi_l(z,y)dy = \int_0^x & B(x,y)\frac{l(l+1)}{y^2}\phi_l(z,y)dy + B(x,0)\phi_l'(z,0) \\ & - B(x,x)\phi_l'(z,x) + \frac{\partial B(x,y)}{\partial y}\Big|_{y=x}\phi_l(z,x) \\ & - \frac{\partial B(x,y)}{\partial y}\Big|_{y=0}\phi_l(z,0) - \int_0^x \frac{\partial^2 B(x,y)}{\partial y^2}\phi_l(z,y)dy. \end{aligned} \tag{2.4}$$

Now plugging in (2.4) into (2.3) and setting (2.2) equal to (2.3) gives us the following identity for the kernel $B(x,y)$:

$$\begin{aligned} & \frac{\partial B(x,x)}{\partial x}\phi_l(z,x) + \left(\frac{\partial B(x,y)}{\partial x} + \frac{\partial B(x,y)}{\partial y}\right)\Big|_{y=x}\phi_l(z,x) - q(x)\phi_l(z,x) \\ & + B(x,0)\phi_l'(z,0) - \frac{\partial B(x,y)}{\partial y}\Big|_{y=0}\phi_l(z,0) \\ & + \int_0^x \left(\frac{\partial^2 B}{\partial x^2} - \frac{\partial^2 B}{\partial y^2} + \frac{l(l+1)}{y^2}B - \frac{l(l+1)}{x^2}B - q(x)\right)\phi_l(z,y)dy = 0 \end{aligned} \tag{2.5}$$

Hence, in order to ensure that the right-hand side of (1.4) satisfies equation (1.3), it's sufficient that B solves the following problem:

$$\left(\frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial y^2} + \frac{l(l+1)}{y^2} - \frac{l(l+1)}{x^2} - q(x)\right)B(x,y) = 0, \quad 0 < y < x \tag{2.6}$$

$$\frac{\partial B(x,x)}{\partial x} = \frac{q(x)}{2}, \quad \lim_{y\to 0} B(x,y)\phi_l'(x,y) = 0 = \lim_{y\to 0} B(x,y)y^l. \tag{2.7}$$

The term $\frac{\partial B(x,y)}{\partial y}\Big|_{y=0}\phi_l(z,0)$ will disappear, since $\phi_l(z,y) = \mathcal{O}(y^{l+1})$ by the properties of ϕ_l mentioned e.g. in [12, 19, Section 2], which will go to 0 as $y \to 0$, and $\frac{\partial B(x,y)}{\partial y}$ can be assumed to be bounded(cf. Lemma 2.13). The next step is to bring this equation into a simpler form. As the resulting transformed equations in [30](cf. formulas (1.4)–(1.6) there) contain a small error, we provide the details of these calculations in the following lemma:

Lemma 2.1. *Let*

$$z = \frac{(x+y)^2}{4}, \quad s = \frac{(x-y)^2}{4}, \quad u(z,s) = (z-s)^l B(x,y). \tag{2.8}$$

Then $u(z,s)$ satisfies the equation

$$\frac{\partial^2 u}{\partial z \partial s} + \frac{l}{z-s}\frac{\partial u}{\partial z} - \frac{l}{z-s}\frac{\partial u}{\partial s} = \frac{1}{4\sqrt{zs}}q(\sqrt{z}+\sqrt{s})u, \tag{2.9}$$

whereas the boundary conditions (2.7) *transform according to*

$$\frac{\partial u}{\partial z} + \frac{lu}{z} = \frac{z^{l-\frac{1}{2}}q(\sqrt{z})}{4}, \quad u(z,z) = \begin{cases} \mathcal{O}(z^{l+\frac{1}{2p'}-\alpha}), & l > -\frac{1}{2} \\ \mathcal{O}(z^{l+\frac{1}{p'}-\alpha}), & l = -\frac{1}{2} \end{cases} \quad \text{as } z \to 0. \tag{2.10}$$

Proof. Let us first evaluate the second derivatives of $B(x,y) = (xy)^{-l}u\left(\frac{(x+y)^2}{4}, \frac{(x-y)^2}{4}\right)$ in formula (2.8):

$$\frac{\partial B}{\partial x} = -lx^{-l-1}y^{-l}u + (xy)^{-l}\left(\frac{\partial u}{\partial z}\cdot\frac{x+y}{2} + \frac{\partial u}{\partial s}\cdot\frac{x-y}{2}\right)$$

$$\begin{aligned}\frac{\partial^2 B}{\partial x^2} &= (-l-1)(-l)x^{-l-2}y^{-l}u + 2(-l)x^{-l-1}y^{-l}\left(\frac{\partial u}{\partial z}\cdot\frac{x+y}{2} + \frac{\partial u}{\partial s}\cdot\frac{x-y}{2}\right)\\ &+(xy)^{-l}\frac{1}{2}\left(\frac{\partial u}{\partial z} + \frac{\partial u}{\partial s}\right) + (xy)^{-l}\frac{x+y}{2}\left(\frac{\partial^2 u}{\partial z^2}\cdot\frac{x+y}{2} + \frac{\partial^2 u}{\partial z\partial s}\cdot\frac{x-y}{2}\right)\\ &+(xy)^{-l}\frac{x-y}{2}\left(\frac{\partial^2 u}{\partial z\partial s}\cdot\frac{x+y}{2} + \frac{\partial^2 u}{\partial s^2}\cdot\frac{x-y}{2}\right)\end{aligned}$$

and similarly

$$\begin{aligned}\frac{\partial^2 B}{\partial y^2} &= (-l-1)(-l)x^{-l}y^{-l-2}u + 2(-l)x^{-l}y^{-l-1}\left(\frac{\partial u}{\partial z}\cdot\frac{x+y}{2} - \frac{\partial u}{\partial s}\cdot\frac{x-y}{2}\right)\\ &+(xy)^{-l}\frac{1}{2}\left(\frac{\partial u}{\partial z} + \frac{\partial u}{\partial s}\right) + (xy)^{-l}\frac{x+y}{2}\left(\frac{\partial^2 u}{\partial z^2}\cdot\frac{x+y}{2} + \frac{\partial^2 u}{\partial z\partial s}\cdot\frac{x-y}{2}\right)\\ &-(xy)^{-l}\frac{x-y}{2}\left(\frac{\partial^2 u}{\partial z\partial s}\cdot\frac{x+y}{2} + \frac{\partial^2 u}{\partial s^2}\cdot\frac{x-y}{2}\right).\end{aligned}$$

Thus we get

$$\begin{aligned}\frac{\partial^2 B}{\partial x^2} - \frac{\partial^2 B}{\partial y^2} &= (xy)^{-l}l(l+1)(x^{-2}-y^{-2})u + (xy)^{-l}(x+y)(x-y)\frac{\partial^2 u}{\partial z\partial s}\\ &+(xy)^{-l}\left(l(x+y)(y^{-1}-x^{-1})\frac{\partial u}{\partial z} - l(x-y)(x^{-1}+y^{-1})\frac{\partial u}{\partial s}\right)\\ &= l(l+1)B(x^{-2}-y^{-2}) + (xy)^{-l}(x-y)(x+y)\left(\frac{\partial^2 u}{\partial z\partial s} + \frac{l}{xy}\frac{\partial u}{\partial z} - \frac{l}{xy}\frac{\partial u}{\partial s}\right).\end{aligned}$$

Since B satisfies (2.6) and $(x+y)(x-y) = 4\sqrt{zs}$, we end up with (2.9). To get the equation for the boundary conditions we first note that $B(x,x) = u(x^2,0)x^{-2l}$ and then perform the following calculation:

$$\begin{aligned}\frac{\partial B}{\partial x}(x,x) &= \left(\frac{\partial u}{\partial z}(x^2,0)2x + \frac{\partial u}{\partial s}(x^2,0)0\right)x^{-2l} - 2lx^{-2l-1}u(x^2,0)\\ &= 2x^{-2l+1}\left(\frac{\partial u}{\partial z}(x^2,0) - lx^{-2}u(x^2,0)\right)\end{aligned}$$

Because of (2.7), (2.10) immediately follows. The last claim is clear. □

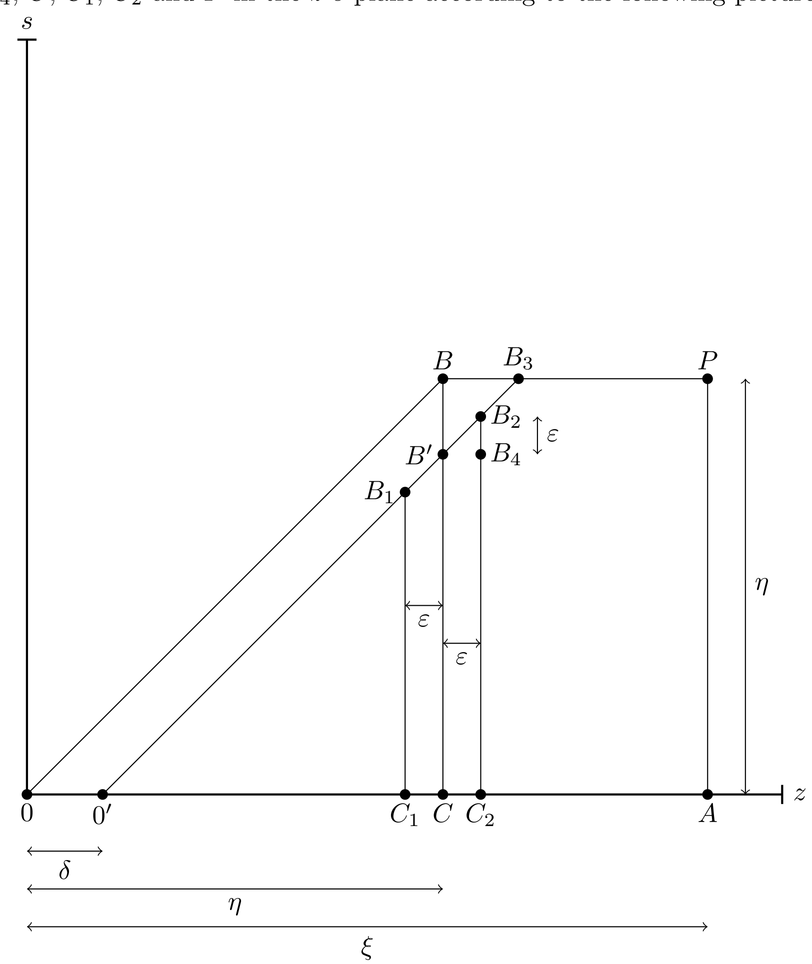
To solve the equation (2.9)–(2.10), we will use Riemann's method, a well known approach to solve linear hyperbolic partial differential equations of the second order in two independent variables. We will go through this procedure carefully in the subsequent calculations, for further information and applications we refer to the huge amount of literature, e.g. [6], [8], [15], [23]. For the time being let us assume that our potential q and our function u from Lemma 2.1 are smooth enough, i.e. $C^2(\mathbb{R}_+)$(or $C^2(\mathbb{R}_+^2)$ resp.). Moreover let us continue by introducing the following operator defined on $C^2(\mathbb{R}_+^2)$:

$$Lu := \frac{\partial^2 u}{\partial z\partial s} + \frac{l}{z-s}\frac{\partial u}{\partial z} - \frac{l}{z-s}\frac{\partial u}{\partial s}$$

and its formal adjoint, which can be computed, using integration by parts, as

$$Mv := L^*v = \frac{\partial^2 v}{\partial z \partial s} - \frac{l}{z-s}\frac{\partial v}{\partial z} + \frac{l}{z-s}\frac{\partial v}{\partial s} - \frac{2l}{(z-s)^2}v.$$

Next let $0 < \eta \le \xi \le L$ and ε, $\delta > 0$. We define the points $0'$, A, B, B', B_1, B_2, B_3, B_4, C, C_1, C_2 and P in the z-s-plane according to the following picture:



By G we define the region enclosed by the segment $\Gamma := \overline{APB0}$. If all the appearing functions are smooth enough and well defined on $\overline{G}$, applying Green's Theorem leads to:

$$\begin{aligned} 2\iint_G (vLu - uMv)\, dz ds = \oint_\Gamma \left(u\frac{\partial v}{\partial z} - v\frac{\partial u}{\partial z} + \frac{2l}{z-s}uv \right) dz \\ - \left(u\frac{\partial v}{\partial s} - v\frac{\partial u}{\partial s} - \frac{2l}{z-s}uv \right) ds. \end{aligned} \tag{2.11}$$

However, the problem is, that $z = s$ and $\eta = z$ might lead to singularities for u and v. So we have to be careful and continue as follows: we divide G into the parts $APB_3B_2C_2$ and $0'B_1C_1$ and investigate these regions separately, thus isolating the singularities at $z = s$ and $\eta = z$, and afterward let ε and δ tend to 0. Let's begin with $APB_3B_2C_2$ and apply Green's Theorem to get:

$$
\begin{aligned}
2\iint_{APB_3B_2C_2}&\Big(vLu-uMv\Big)dzds=\\
&\int_{C_2A}\left(u\frac{\partial v}{\partial z}-v\frac{\partial u}{\partial z}+\frac{2l}{z}uv\right)dz-\int_{AP}\left(u\frac{\partial v}{\partial s}-v\frac{\partial u}{\partial s}-\frac{2l}{\xi-s}uv\right)ds\\
&+\int_{PB_3}\left(u\frac{\partial v}{\partial z}-v\frac{\partial u}{\partial z}+\frac{2l}{z-\eta}uv\right)dz-\int_{B_2C_2}\left(u\frac{\partial v}{\partial s}-v\frac{\partial u}{\partial s}-\frac{2l}{\eta+\varepsilon-s}uv\right)ds\\
&+\int_{B_3B_2}\Big(u\frac{\partial v}{\partial z}-v\frac{\partial u}{\partial z}+\frac{4l}{z-s}uv-u\frac{\partial v}{\partial s}+v\frac{\partial u}{\partial s}\Big)dz. \qquad (2.12)
\end{aligned}
$$

Now we further evaluate some of the appearing integrals using integration by parts:

$$
\begin{aligned}
\int_{C_2A}\left(u\frac{\partial v}{\partial z}-v\frac{\partial u}{\partial z}+\frac{2l}{z}uv\right)dz&=uv\Big|_{C_2}^{A}-2\int_{C_2A}v\left(\frac{\partial u}{\partial z}-\frac{l}{z}u\right)dz\\
\int_{AP}\left(u\frac{\partial v}{\partial s}-v\frac{\partial u}{\partial s}-\frac{2l}{\xi-s}uv\right)ds&=-uv\Big|_{A}^{P}+2\int_{AP}u\left(\frac{\partial v}{\partial s}-\frac{l}{\xi-s}v\right)ds\\
\int_{PB_3}\left(u\frac{\partial v}{\partial z}-v\frac{\partial u}{\partial z}+\frac{2l}{z-\eta}uv\right)dz&=-uv\Big|_{P}^{B_3}+2\int_{PB_3}u\left(\frac{\partial v}{\partial z}+\frac{l}{z-\eta}v\right)ds.
\end{aligned}
\qquad (2.13)
$$

To proceed we introduce the Riemann-Green function v_1 to simplify some of the expressions in (2.12). v_1 shall be a solution to the following problem:

$$
\begin{aligned}
&1)\quad Mv_1=0\\
&2)\quad \frac{\partial v_1}{\partial z}+\frac{l}{z-\eta}v_1=0\quad \text{on } PB\\
&3)\quad \frac{\partial v_1}{\partial s}-\frac{l}{\xi-s}v_1=0.\quad \text{on } AP\\
&4)\quad v_1(P)=1
\end{aligned}
\qquad (2.14)
$$

In [15], [23] an explicit formula was computed:

$$
v_1(z,s;\eta,\xi)=(\eta-z)^l(s-\xi)^l(s-z)^{-2l}\,{}_2F_1\left(\begin{matrix}-l,-l\\1\end{matrix};\frac{(z-\xi)(s-\eta)}{(z-\eta)(s-\xi)}\right). \qquad (2.15)
$$

Let's give a very short sketch of proof for the previous formula. By considering symmetry groups for the Euler-Poisson-Darboux operator L, one first sees by a lengthy but rather straightforward computation, that L is invariant under the group action of $\mathrm{GL}(2,\mathbb{C})$(so especially, under $\mathrm{SL}(2,\mathbb{C})$); i.e., if $u(z,s)$ is a solution of $Lu=0$, then $\tilde{u}(z,s):=(bz+d)^l(bs+d)^l u\left(\frac{az+c}{d+bz},\frac{as+c}{d+bs}\right)$ is also a solution, if $ad-bc\neq 0$. Also motivated by symmetry group considerations, one chooses the following ansatz for the solution u: $u(z,s)=z^{\mu}u_1(\frac{s}{z})$, and it turns out, that u_1 is indeed a solution of the hypergeometric equation (A.3) with parameters $a=-\mu$, $b=-l$ and $c=1-\mu+l$. Next one sets $\mu=l$ and observes, that a solution u of $Lu=0$ can be transformed to a solution v of $Mv=0$ via $v(z,s)=(z-s)^{-2l}u(z,x)$ and thus, since u is invariant under linear transformations, we act with the matrix $\begin{pmatrix}1&1\\-\eta&-\xi\end{pmatrix}$. This indeed leads to formula (2.15) and we observe, that also the boundary conditions given in (2.14) are satisfied. For further details, especially for the connections between symmetry groups for certain PDEs and special functions, for the definition of the group action, etc..., we refer e.g. to [24]–[25]. To continue, employing (2.14) and (2.13) leads to:

$$\begin{aligned}2\iint_{APB_3B_2C_2} v_1 Ludzds = &-2\int_{C_2A} v_1\left(\frac{\partial u}{\partial z}-\frac{l}{z}u\right)dz + uv_1\Big|_{C_2}^{A} + uv_1\Big|_{A}^{P} - uv_1\Big|_{P}^{B_3}\\ &-\int_{B_2C_2}\left(u\frac{\partial v_1}{\partial s}-v_1\frac{\partial u}{\partial s}-\frac{2l}{\eta+\varepsilon-s}uv_1\right)ds\\ &+\int_{B_3B_2}\Big(u\frac{\partial v_1}{\partial z}-v_1\frac{\partial u}{\partial z}+\frac{4l}{z-s}uv_1-u\frac{\partial v_1}{\partial s}+v_1\frac{\partial u}{\partial s}\Big)dz. \end{aligned}\tag{2.16}$$

Next let's focus on the region enclosed by $0'B_1C_1$ first. Similar computations as before imply:

$$\begin{aligned}2\iint_{0'B_1C_1}\Big(vLu-uMv\Big)dzds = &\,uv\Big|_{0'}^{C_1} - 2\int_{0'C_1} v\left(\frac{\partial u}{\partial z}-\frac{l}{z}u\right)dz\\ &-\int_{C_1B_1}\left(u\frac{\partial v}{\partial s}-v\frac{\partial u}{\partial s}-\frac{2l}{\eta-\varepsilon-s}uv\right)ds\\ &+\int_{B_10'}\Big(u\frac{\partial v}{\partial z}-v\frac{\partial u}{\partial z}+\frac{4l}{z-s}uv-u\frac{\partial v}{\partial s}+v\frac{\partial u}{\partial s}\Big)dz. \end{aligned}\tag{2.17}$$

Using formula (2.15) for v_1 and the form of a second linearly independent solution to (A.3), we obtain a solution v_2, defined in $0'B_1C_1$, to the following problem:

$$\begin{aligned}&1)\quad Mv_2 = 0\\ &2)\quad v_2(z,z) = 0\\ &3)\quad v_1(z,\eta)-v_2(z,\eta) = \mathcal{O}(1) \quad \text{as} \quad z\to\eta.\end{aligned}\tag{2.18}$$

v_2 has the following explicit representation(cf. [23] for details):

$$\begin{aligned}v_2(z,s;\eta,\xi) = &(-1)^l\frac{\Gamma(1+l)}{\Gamma(-l)\Gamma(2+2l)}(z-s)(\eta-\xi)^{1+2l}(\eta-z)^{-l-1}(s-\xi)^{-l-1}\\ &\times {}_2F_1\left(\begin{matrix}1+l,1+l\\ 2+2l\end{matrix};\frac{(z-s)(\eta-\xi)}{(z-\eta)(s-\xi)}\right).\end{aligned}\tag{2.19}$$

Setting $v=v_2$ in (2.17) we end up with

$$\begin{aligned}2\iint_{0'B_1C_1} v_2Ludzds = &\,uv\Big|_{0'}^{C_1} - 2\int_{0'C_1} v_2\left(\frac{\partial u}{\partial z}-\frac{l}{z}u\right)dz\\ &-\int_{C_1B_1}\left(u\frac{\partial v_2}{\partial s}-v_2\frac{\partial u}{\partial s}-\frac{2l}{\eta-\varepsilon-s}uv_2\right)ds\\ &+\int_{B_10'}\Big(u\frac{\partial v_2}{\partial z}-v_2\frac{\partial u}{\partial z}+\frac{4l}{z-s}uv_2-u\frac{\partial v_2}{\partial s}+v_2\frac{\partial u}{\partial s}\Big)dz. \end{aligned}\tag{2.20}$$

Now if we introduce

$$v = \begin{cases} v_1, & z>\eta,\\ v_2, & z<\eta.\end{cases}$$

and combine (2.16), (2.20) with the boundary condition (2.10), this implies

$$2\iint_{0'B_1C_1+APB_3B_2C_2} vLudzds = -\frac{1}{2}\int_{0'C_1} vq(\sqrt{z})z^{l-\frac{1}{2}}dz - \frac{1}{2}\int_{C_2A} vq(\sqrt{z})z^{l-\frac{1}{2}}dz$$
$$+uv_1\Big|_{C_2}^{A} + uv_1\Big|_{A}^{P} - uv_1\Big|_{P}^{B_3} + uv\Big|_{0'}^{C_1} + \Delta_1 + \Delta_2, \tag{2.21}$$

where

$$\Delta_1 := \int_{C_2B_2}\left(u\frac{\partial v_1}{\partial s} - v_1\frac{\partial u}{\partial s} - \frac{2l}{\eta+\varepsilon-s}uv_1\right)ds$$
$$-\int_{C_1B_1}\left(u\frac{\partial v_2}{\partial s} - v_2\frac{\partial u}{\partial s} - \frac{2l}{\eta-\varepsilon-s}uv_2\right)ds, \tag{2.22}$$

$$\Delta_2 := \int_{B_3B_2}\Big(u\frac{\partial v_1}{\partial z} - v_1\frac{\partial u}{\partial z} + \frac{4l}{z-s}uv_1 - u\frac{\partial v_1}{\partial s} + v_1\frac{\partial u}{\partial s}\Big)dz$$
$$+\int_{B_10'}\Big(u\frac{\partial v_2}{\partial z} - v_2\frac{\partial u}{\partial z} + \frac{4l}{z-s}uv_2 - u\frac{\partial v_2}{\partial s} + v_2\frac{\partial u}{\partial s}\Big)dz. \tag{2.23}$$

The next lemmas show, what happens, if we perform limits in the previously introduced expressions:

Lemma 2.2. *If $\varepsilon \to 0$, we obtain*

$$\Delta_1 \to (A_2 - A_1)\left(\frac{\xi-\eta}{\eta-s}\right)^l u\Big|_{C}^{B'} \tag{2.24}$$

for some $A_1, A_2 \in \mathbb{R}$.

Proof. To start, we integrate by parts to obtain:

$$\Delta_1 = uv_1\Big|_{C_2}^{B_2} - uv_2\Big|_{C_1}^{B_1} - 2\int_{C_2B_2} v_1\left(\frac{\partial u}{\partial s} + \frac{l}{\eta+\varepsilon-s}u\right)ds$$
$$+2\int_{C_1B_1} v_2\left(\frac{\partial u}{\partial s} + \frac{l}{\eta-\varepsilon-s}u\right)ds. \tag{2.25}$$

Using that $u \in C^2(G)$, and thus $\frac{\partial u}{\partial s}(.,s)$, $u(.,s)$ being locally Lipschitz continuous, we observe the following properties:

$$\frac{\partial u}{\partial s}\Big|_{z=\eta-\varepsilon} - \frac{\partial u}{\partial s}\Big|_{z=\eta+\varepsilon} = \mathcal{O}(\varepsilon), \quad \frac{lu(\eta-\varepsilon,s)}{\eta-\varepsilon-s} - \frac{lu(\eta+\varepsilon,s)}{\eta+\varepsilon-s} = \mathcal{O}(\varepsilon). \tag{2.26}$$

Inserting (2.26) into (2.25), we obtain the following expression for Δ_1:

$$uv_1\Big|_{C_2}^{B_2} - uv_2\Big|_{C_1}^{B_1} + 2\int_0^{\eta-\varepsilon-\delta}(v_2(\eta-\varepsilon,s) - v_1(\eta+\varepsilon,s))\left(\frac{\partial u}{\partial s} + \frac{lu}{\eta-\varepsilon-s}\right)\Big|_{z=\eta-\varepsilon}ds$$
$$-2\int_{B_4B_2} v_1\left(\frac{\partial u}{\partial s} + \frac{l}{\eta+\varepsilon-s}u\right)ds + \mathcal{O}(\varepsilon).$$

Now we use (2.15), (2.19) and (A.7) to get the following asymptotic expansions:

$$v_1(\eta+\varepsilon,s) = \frac{1}{\Gamma(-l)\Gamma(l+1)}\left(\frac{\xi-\eta}{\eta-s}\right)^l \cdot \log\left(\frac{(\eta-s)(\xi-\eta)}{\varepsilon(\xi-s)}\right) + A_1\left(\frac{\xi-\eta}{\eta-s}\right)^l + \mathcal{O}(\varepsilon),$$

$$v_2(\eta-\varepsilon,s) = \frac{1}{\Gamma(-l)\Gamma(l+1)}\left(\frac{\xi-\eta}{\eta-s}\right)^l \cdot \log\left(\frac{(\eta-s)(\xi-\eta)}{\varepsilon(\xi-s)}\right) + A_2\left(\frac{\xi-\eta}{\eta-s}\right)^l + \mathcal{O}(\varepsilon).$$

Thus we end up with

$$\Delta_1 = uv_1\Big|_{C_2}^{B_2} - uv_2\Big|_{C_1}^{B_1} + 2\int_{C_1B_1}\mathcal{O}(\varepsilon)\left(\frac{\partial u}{\partial s} + \frac{lu}{\eta-\varepsilon-s}\right)ds$$
$$+2\int_{C_1B_1}(A_2-A_1)\left(\frac{\xi-\eta}{\eta-s}\right)^l\left(\frac{\partial u}{\partial s} + \frac{lu}{\eta-\varepsilon-s}\right)ds + \int_{B_4B_2}\mathcal{O}(\log(\varepsilon))ds + \mathcal{O}(\varepsilon).$$

Finally we let $\varepsilon \to 0$, so that one more integration by parts leads us to:

$$\begin{aligned}\Delta_1 =& uv_1\Big|_{C_2\to C}^{B_2\to B} + uv_2\Big|_{B_1\to B}^{C_1\to C} + 2(A_2-A_1)\left(\frac{\xi-\eta}{\eta-s}\right)^l u\Big|_C^{B'}\\ &+2\int_{CB'}(A_2-A_1)u\left(-\frac{\partial}{\partial s}\left(\frac{\xi-\eta}{\eta-s}\right)^l + \frac{l}{\eta-s}\left(\frac{\xi-\eta}{\eta-s}\right)^l\right)ds.\end{aligned}$$

Since the integral expression disappears and by observing that

$$(uv_2)(C_1) - (uv_1)(C_2) \xrightarrow{\varepsilon\to 0} (A_2-A_1)\left(\frac{\xi-\eta}{\eta-s}\right)^l u(C)$$

and

$$(uv_1)(B_2) - (uv_2)(B_1) \xrightarrow{\varepsilon\to 0} (A_1-A_2)\left(\frac{\xi-\eta}{\eta-s}\right)^l u(B'),$$

(here we again use the asymptotics for v_1 and v_2 and the fact that the log-terms cancel), the claim follows. □

Thus Lemma 2.2 leads to the following expression for (2.21), when we let $\varepsilon \to 0$:

$$\begin{aligned}2\iint_{0'B_3PA} vLudzds = -\frac{1}{2}\int_{0'A} vq(\sqrt{z})z^{l-\frac{1}{2}}dz + 2u(P) - u(0')v(0') - u(B_3)v(B_3)\\ + u(B')(A_2-A_1)\left(\frac{\xi-\eta}{\delta}\right)^l + \Delta_2.\end{aligned} \tag{2.27}$$

It remains to perform the limit $\delta \to 0$. In the next lemma this is done for Δ_2:

Lemma 2.3. *If $\delta \to 0$, we obtain $\Delta_2 \to 0$.*

Proof. Here we only sketch the proof in a way such that the main argument should be clear. Let's first divide the integral into two parts:

$$\begin{aligned}\Delta_2 = &\int_{B_3B'}\Big(u\frac{\partial v_1}{\partial z} - v_1\frac{\partial u}{\partial z} + \frac{4l}{z-s}uv_1 - u\frac{\partial v_1}{\partial s} - v_1\frac{\partial u}{\partial s}\Big)dz\\ &+\int_{B'0'}\Big(u\frac{\partial v_2}{\partial z} - v_2\frac{\partial u}{\partial z} + \frac{4l}{z-s}uv_2 - u\frac{\partial v_2}{\partial s} - v_2\frac{\partial u}{\partial s}\Big)dz =: \Delta_{2,1} + \Delta_{2,2}\end{aligned}$$

and focus on the part $\Delta_{2,1}$ first. Each summand in this expression needs to be treated separately, however, since the calculations are similar, we will only focus on

$v_1\frac{\partial u}{\partial z}$. W.l.o.g. we can also set $\varepsilon = \frac{\delta}{2}$, compute the integral along $\overline{B_3B_2}$ and then let $\delta \to 0$. An integration by parts gives:

$$\int_{B_3B_2} v_1 \frac{\partial u}{\partial z} ds = (uv_1)\big|_{B_3}^{B_2} - \int_0^{\frac{\delta}{2}} (u\frac{\partial v_1}{\partial z})(\eta + \frac{\delta}{2} + t, \eta - \frac{\delta}{2} + t) dt.$$

Next we provide an estimate for $\frac{\partial v_1}{\partial z}$ along $\overline{B_3B_2}$. A straightforward calculation using (A.2) gives:

$$\begin{aligned}\frac{\partial v_1}{\partial z} &= -l(\eta - z)^{l-1}(s-\xi)^l(s-z)^{-2l}\, {}_2F_1\left(\begin{matrix}-l,-l\\1\end{matrix}; \frac{(z-\xi)(s-\eta)}{(z-\eta)(s-\xi)}\right)\\ &\quad + 2l(\eta-z)^l(s-\xi)^l(s-z)^{-2l-1}\, {}_2F_1\left(\begin{matrix}-l,-l\\1\end{matrix}; \frac{(z-\xi)(s-\eta)}{(z-\eta)(s-\xi)}\right)\\ &\quad + (\eta-z)^l(s-\xi)^l(s-z)^{-2l}(-l)^2\frac{(s-\eta)(\xi-\eta)}{(s-\xi)(z-\eta)^2}\\ &\quad \times\, {}_2F_1\left(\begin{matrix}-l+1,-l+1\\2\end{matrix}; \frac{(z-\xi)(s-\eta)}{(z-\eta)(s-\xi)}\right),\end{aligned}$$

and thus the following upper bound follows by taking (2.15), (A.4)–(A.6) and $s = z - \delta$ into account:

$$\begin{aligned}\sup_{(z,s)\in\overline{B_3B_2}}\left|\frac{\partial v_1}{\partial z}(z,s)\right| &\le C\delta^{-l-1}\sup_{z\in(\eta+\frac{\delta}{2},\eta+\delta)}\left|{}_2F_1\left(\begin{matrix}-l,-l\\1\end{matrix}; \frac{(z-\xi)(z-\delta-\eta)}{(z-\eta)(z-\delta-\xi)}\right)\right|\\ &\le \delta^{-l-1}\begin{cases}C_l, & l > -\frac{1}{2},\\ C\log(\frac{1}{\delta}), & l = -\frac{1}{2}.\end{cases}\end{aligned}$$

Consequently, in combination with (2.10) and the fact, that the length of $\overline{B_3B_2}$ is proportional to δ, we obtain:

$$\left|\int_{B_3B_2} v_1\frac{\partial u}{\partial z} ds\right| = \begin{cases}\mathcal{O}(\delta^{\frac{1}{2p'}-\alpha}), & l > -\frac{1}{2},\\ \mathcal{O}(\delta^{\frac{1}{p'}-\alpha}\log(\frac{1}{\delta})), & l = -\frac{1}{2},\end{cases}$$

which goes to 0 as $\delta \to 0$. The reasoning for the boundary term is similar. For $\Delta_{2,2}$, we will only look at the term $\frac{4l}{z-s}uv_2$, since the others are again treated analogously(compare with the previous computations for $\Delta_{2,1}$). Thus, using (2.19) and (2.10) in combination with the observation, that the hypergeometric-function-term in (2.19) is bounded for sufficiently small $\delta > 0$, leads us to:

$$\left|\int_{B_3B'}\frac{4l}{z-s}uv_2 dz\right|$$

$$\le \begin{cases} C_l\delta^{l+\frac{1}{2p'}-\alpha}\int_\delta^{\eta-\frac{\delta}{2}}(\eta-z)^{-l-1}dz \le C_l\delta^{l+\frac{1}{2p'}-\alpha}\delta^{-l}, & l \ge 0,\\ C_l\delta^{l+\frac{1}{2p'}-\alpha}\int_\delta^{\eta-\frac{\delta}{2}}(\eta-z)^{-l-1}dz \le C_l\delta^{l+\frac{1}{2p'}-\alpha}(\eta-\frac{\delta}{2})^{-l}, & 0 > l > -\frac{1}{2},\\ C\delta^{-\frac{1}{2}+\frac{1}{p'}-\alpha}\int_\delta^{\eta-\frac{\delta}{2}}(\eta-z)^{-\frac{1}{2}}dz \le C\delta^{-\frac{1}{2}+\frac{1}{p'}-\alpha}(\eta-\frac{\delta}{2})^{\frac{1}{2}}, & l = -\frac{1}{2},\end{cases}$$

where p' and α are defined according to the different values of l in Theorem 1.1. $\square$

So finally, after δ tends to 0 and after applying Lemma 2.3, the expression (2.27) provides us the following integral equation for u:

$$u(\xi,\eta) = \frac{1}{4}\int_0^\xi vq(\sqrt{z})z^{l-\frac{1}{2}}dz + \frac{1}{4}\iint_{0BPA}(zs)^{-\frac{1}{2}}vq(\sqrt{z}+\sqrt{s})u\,dz\,ds \qquad (2.28)$$

Now we go the other way round, i.e. under our assumptions in Theorem 1.1 we want to show that this equation has indeed a solution. To do so we use the successive approximation method, which will lead to the subsequent result:

Theorem 2.4. *Under the conditions on q stated in Theorem 1.1, there is a unique continuous function $u(\xi,\eta)$ that solves (2.28) and satisfies*

$$\begin{aligned} u(\xi,\eta) \le{}& \frac{(\xi-\eta)^{l+\frac{1}{2p'}-\alpha}}{\alpha}(\sqrt{\xi}+\sqrt{\eta})^{2\alpha}\,\|q\|_{L^p((0,\sqrt{\xi}+\sqrt{\eta}])}\\ &\times\exp\left(\frac{\tilde{C}(\sqrt{\xi}+\sqrt{\eta})^{1+\frac{1}{p'}}}{\alpha}\,\|q\|_{L^p((0,\sqrt{\xi}+\sqrt{\eta}])}\right),\quad l>-\frac{1}{2},\\ u(\xi,\eta) \le{}& \frac{(\xi-\eta)^{l+\frac{1}{p'}-\alpha}}{\alpha}(\sqrt{\xi}+\sqrt{\eta})^{2\alpha}(\max(1,L))^{\frac{1}{2p'}}\,\|q\|_{L^p((0,\sqrt{\xi}+\sqrt{\eta}],z^{-\frac{p}{p'}})}\\ &\times\exp\left(\frac{\tilde{C}(\sqrt{\xi}+\sqrt{\eta})^{1+\frac{1}{p'}}(\max(1,L))^{\frac{1}{2p'}}}{\alpha}\,\|q\|_{L^p((0,\sqrt{\xi}+\sqrt{\eta}],z^{-\frac{p}{p'}})}\right),\quad l=-\frac{1}{2}. \end{aligned} \tag{2.29}$$

The constant $\tilde{C}$ only depends on l.

Indeed, we want to represent u as a series $u=u_0+\sum_{n=1}^{\infty}u_n$, where the u_n's are defined recursively as follows:

$$\begin{aligned} u_0(\xi,\eta) &:= \frac{1}{4}\int_0^{\xi} v(z,0)q(\sqrt{z})z^{l-\frac{1}{2}}dz\\ u_{n+1}(\xi,\eta) &:= \frac{1}{4}\iint_{0BPA}(zs)^{-\frac{1}{2}}v(z,s)q(\sqrt{z}+\sqrt{s})u_n(z,s)dzds. \end{aligned} \tag{2.30}$$

The crucial point here is of course to find appropriate estimates, such that this series converges. This will be done carefully in several steps and we will start with providing some estimates for v. This has basically already been done in [22], however, we will give a slightly more general version here and, for convenience, also repeat some of the arguments:

Lemma 2.5. [22, Lemma A.2] *Fix some $0\le s\le\eta$ and let $z_1(s)$, $z_2(s)$ be defined by the following equations:*

$$-1=\frac{(z_1(s)-\xi)(s-\eta)}{(z_1(s)-\eta)(s-\xi)},\quad -1=\frac{(z_2(s)-s)(\eta-\xi)}{(z_2(s)-\eta)(s-\xi)} \tag{2.31}$$

Then the functions v_1 and v_2 satisfy the following inequalities:

$$|v_1(z,s;\eta,\xi)|\le C_1(z-\eta)^l(\xi-s)^l(z-s)^{-2l},\quad z\in(z_1,\xi) \tag{2.32}$$

$$|v_1(z,s;\eta,\xi)|\le C_2(z-s)^{-2l}(\xi-z)^l(\eta-s)^l(\log\frac{(\xi-z)(\eta-s)}{(z-\eta)(\xi-s)}+1),\quad z\in(\eta,z_1) \tag{2.33}$$

$$|v_2(z,s;\eta,\xi)|\le C_3(\xi-\eta)^{1+2l}(z-s)(\xi-s)^{-l-1}(\eta-z)^{-l-1},\quad z\in(0,z_2) \tag{2.34}$$

$$|v_2(z,s;\eta,\xi)|\le C_4(\xi-\eta)^l(z-s)^{-l}(\log\frac{(z-s)(\xi-\eta)}{(\eta-z)(\xi-s)}+1),\quad z\in(z_2,\eta). \tag{2.35}$$

Proof. The proof heavily relies on estimates for the hypergeometric function, collected in Appendix A. Let's denote the argument $\frac{(z-\xi)(s-\eta)}{(z-\eta)(s-\xi)}$ of the hypergeometric function in (2.15) by σ_1. We start by proving (2.32). In this case we have that $0 \geq \sigma_1 \geq -1$ and ${}_2F_1\left({-l,-l \atop 1};\sigma_1\right)$ is bounded, since for $k > l$, the expression $|\sigma_1|^k \left(\frac{(-l)_k}{k!}\right)^2$ is monotone decreasing and converges to 0, thus the series in (A.1) is also converging, since the terms additionally alternate in sign. For (2.33), we first consider the case $l \notin \mathbb{N}_0$. We note that $\sigma_1 \leq -1$, and, employing (A.7) and (A.8), we first of all end up with the following equation:

$$
\begin{aligned}
{}_2F_1\left({-l,-l \atop 1};\sigma_1\right) = \frac{(-\sigma_1)^l}{\Gamma(-l)\Gamma(1+l)} \sum_{k=0}^{\infty} \left(\frac{(-l)_k}{k!}\right)^2 \sigma_1^{-k} \\
\times \left(\log(-\sigma_1) + 2\psi(1+k) - 2\psi(k-l) + \pi\cot\pi l\right).
\end{aligned}
$$

We already know that $\sum_{k=0}^{\infty} \left(\frac{(-l)_k}{k!}\right)^2 \sigma_1^{-k}$ is bounded. By (A.9), we also see that $\psi(1+k)-\psi(k-l) = \frac{l+1}{k-l} + \mathcal{O}(k^{-2})$ and thus $\sum_{k=0}^{\infty} \left(\frac{(-l)_k}{k!}\right)^2 \sigma_1^{-k}(2\psi(1+k)-2\psi(k-l))$ admits the absolutely convergent series $\sum_{k=0}^{\infty} \frac{l+1}{k-l}\left(\frac{(-l)_k}{k!}\right)^2$ as a uniform bound. Therefore we can deduce:

$$
\begin{aligned}
\left|{}_2F_1\left({-l,-l \atop 1};\sigma_1\right)\right| &= \frac{|\sigma_1|^l}{|\Gamma(-l)\Gamma(1+l)|} \Big|(\log(-\sigma_1) + \pi\cot\pi l) \sum_{k=0}^{\infty} \left(\frac{(-l)_k}{k!}\right)^2 \sigma_1^{-k} \\
&\quad + \sum_{k=0}^{\infty} \left(\frac{(-l)_k}{k!}\right)^2 \sigma_1^{-k}(2\psi(1+k) - 2\psi(k-l))\Big| \\
&\leq \frac{|\sigma_1|^l}{|\Gamma(-l)\Gamma(1+l)|}\left(C_1 \left|\log(-\sigma_1) + \pi\cot\pi l\right| + C_2\right) \\
&\leq C_2 |\sigma_1|^l \left(\log(-\sigma_1) + 1\right)
\end{aligned}
$$

In the case $l \in \mathbb{N}$, the hypergeometric function reduces to a polynomial and thus the proof is easy. The proof of the remaining estimates (2.34)–(2.35) is similar. $\square$

The following result now will act as a useful tool to estimate certain integral expressions:

Lemma 2.6. *Let $\gamma > 1$ and $0 \leq \tilde{z} \leq 1$. Then we have:*

$$
\|\log(z)\|_{L^\gamma((0,\tilde{z}])} \leq C\gamma, \tag{2.36}
$$

where the constant C is independent from $\overline{z}$.

Proof. We first use the transformation $u = -\log(z)$ to get

$$
\int_0^{\tilde{z}} (-\log(z))^\gamma dz = \int_{-\log(\tilde{z})}^{\infty} u^\gamma \mathrm{e}^{-u} du = \Gamma(\gamma+1, -\log(\tilde{z})),
$$

where $\Gamma(a,z)$ denotes the incomplete Gamma function, cf. [27, (8.2.2)]. Furthermore $\Gamma(a,z)$ enjoys the following asymptotics(cf. [27, (8.11.2)]):

$$\Gamma(a,z) = z^{a-1}\mathrm{e}^{-z} + \mathcal{O}(z^{-1}), \quad z \to \infty.$$

This leads to the following estimate:

$$\|\log(z)\|_{L^{\gamma}((0,\tilde{z}])} \le C(-\log(\tilde{z}))\tilde{z}^{\frac{1}{\gamma}} \le C\gamma,$$

where in the last estimate we used that the local maximum of $(-\log(\tilde{z}))\tilde{z}^{\frac{1}{\gamma}}$ on $[0,1]$ is proportional to γ. □

In the next three lemmas we investigate the iterates u_n and thus start with u_0:

Lemma 2.7. *The following estimates hold:*

$$|u_0(\xi,\eta)| \le \begin{cases} \frac{C}{\alpha}\|q\|_{L^p((0,\sqrt{\xi}+\sqrt{\eta}])}(\sqrt{\xi}+\sqrt{\eta})^{2\alpha}(\xi-\eta)^{l+\frac{1}{2p'}-\alpha}, & l > -\frac{1}{2} \\ \frac{C}{\alpha}\|q\|_{L^p((0,\sqrt{\xi}+\sqrt{\eta}],z^{-\frac{p}{p'}})}(\max(1,L))^{\frac{1}{2p'}}(\sqrt{\xi}+\sqrt{\eta})^{2\alpha} & \\ \times(\xi-\eta)^{-\frac{1}{2}+\frac{1}{p'}-\alpha}, & l = -\frac{1}{2}, \end{cases} \tag{2.37}$$

where the constant C only depends on l.

Proof. First we split the integral for u_0 into two parts:

$$\frac{1}{4}\int_0^{\xi} v(z,0)q(\sqrt{z})z^{l-\frac{1}{2}}dz = \frac{1}{4}\int_0^{\eta} v(z,0)q(\sqrt{z})z^{l-\frac{1}{2}}dz + \frac{1}{4}\int_{\eta}^{\xi} v(z,0)q(\sqrt{z})z^{l-\frac{1}{2}}dz$$

and estimate each part separately. We then estimate the integrals from η to ξ and from 0 to η respectively and use (2.32)–(2.35) to further decompose it into three more parts:

$$\begin{aligned} \frac{1}{4}\int_{\eta}^{\xi} \left|v(z,0)q(\sqrt{z})\right| z^{l-\frac{1}{2}}dz &\le C_2\eta^l \int_{\eta}^{z_1(0)} \frac{(\xi-z)^l}{z^{l+\frac{1}{2}}}\left|q(\sqrt{z})\right|dz \\ +C_1\xi^l\int_{z_1(0)}^{\xi}\frac{(z-\eta)^l}{z^{l+\frac{1}{2}}}\left|q(\sqrt{z})\right|dz &+ C_2\eta^l\int_{\eta}^{z_1(0)}\frac{(\xi-z)^l}{z^{l+\frac{1}{2}}}\log\left(\frac{(\xi-z)\eta}{(z-\eta)\xi}\right)\left|q(\sqrt{z})\right|dz \\ &=: I_1 + I_2 + I_3 \end{aligned}$$

and

$$\begin{aligned} \frac{1}{4}\int_0^{\eta}\left|v(z,0)q(\sqrt{z})\right|z^{l-\frac{1}{2}}dz &\le C_4\frac{(\xi-\eta)^{1+2l}}{\xi^{1+l}}\int_0^{z_2(0)}\frac{z^{l+\frac{1}{2}}}{(\eta-z)^{l+1}}\left|q(\sqrt{z})\right|dz \\ +C_5(\xi-\eta)^l\int_{z_2(0)}^{\eta} z^{-\frac{1}{2}}\left|q(\sqrt{z})\right|dz\eta^l &+ C_5(\xi-\eta)^l\int_{z_2(0)}^{\eta}\log\left(\frac{z(\xi-\eta)}{(\eta-z)\xi}\right)\left|q(\sqrt{z})\right|dz \\ &=: I_4 + I_5 + I_6, \end{aligned}$$

where $z_1(0)$ and $z_2(0)$ are given via (2.31). Now to bound I_1, we first note that $\frac{\xi-\eta}{2\eta} \le \frac{\xi-z}{z} \le \frac{\xi-\eta}{\eta}$, i.e. $\frac{(\xi-z)^l}{z^l} \le C_l\frac{(\xi-\eta)^l}{\eta^l}$, where $C_l = \max(1,2^{-l})$. Hence in the case $l > -\frac{1}{2}$:

$$\begin{aligned}
|I_1| &\leq 2C_l C_2(\xi-\eta)^l \int_{\eta}^{z_1(0)} z^{-\frac{1}{2}} \left|q(\sqrt{z})\right| dz = 2C_l C_2(\xi-\eta)^l 2 \int_{\sqrt{\eta}}^{\sqrt{z_1(0)}} |q(z)|\, dz \\
&\leq 2C_l C_2(\xi-\eta)^l \|q\|_{L^p((0,\sqrt{\xi}])} \left(\int_{\sqrt{\eta}}^{\sqrt{z_1(0)}} 1 dz\right)^{\frac{1}{p'}} \\
&= 2C_l C_2(\xi-\eta)^l \|q\|_{L^p((0,\sqrt{\xi}])} \left(\left(\frac{2\xi\eta}{\xi+\eta}\right)^{\frac{1}{2}} - \eta^{\frac{1}{2}}\right)^{\frac{1}{p'}} \\
&\leq 2C_l C_2(\xi-\eta)^l \left(\frac{\eta}{\xi+\eta}(\xi-\eta)\right)^{\frac{1}{2p'}} \|q\|_{L^p((0,\sqrt{\xi}])} \\
&\leq 2C_l C_2(\sqrt{\xi}+\sqrt{\eta})^{2\alpha}(\xi-\eta)^{l+\frac{1}{2p'}-\alpha} \|q\|_{L^p((0,\sqrt{\xi}+\sqrt{\eta}])},
\end{aligned}$$

where we have used Hölder's inequality in the third, the elementary estimate $\sqrt{a-b} \geq \sqrt{a} - \sqrt{b}$ in the fifth, and the fact that $\left(\frac{(\sqrt{\xi}+\sqrt{\eta})^2}{\xi-\eta}\right)^{\alpha} \geq 1$ in the last step. In the case $l = -\frac{1}{2}$ we proceed as follows:

$$\begin{aligned}
|I_1| &\leq 2C_{-\frac{1}{2}} C_2(\xi-\eta)^{-\frac{1}{2}} \int_{\sqrt{\eta}}^{\sqrt{z_1(0)}} z^{\frac{1}{p'}} z^{-\frac{1}{p'}} |q(z)|\, dz \\
&\leq 2C_{-\frac{1}{2}} C_2(\xi-\eta)^{-\frac{1}{2}} \left(\int_{\sqrt{\eta}}^{\sqrt{z_1(0)}} z dz\right)^{\frac{1}{p'}} \|q\|_{L^p((0,\sqrt{\xi}],z^{-\frac{p}{p'}})} \\
&\leq 2C_{-\frac{1}{2}} C_2(\sqrt{\xi}+\sqrt{\eta})^{2\alpha}(\xi-\eta)^{-\frac{1}{2}+\frac{1}{p'}-\alpha} \|q\|_{L^p((0,\sqrt{\xi}+\sqrt{\eta}],z^{-\frac{p}{p'}})}.
\end{aligned}$$

The calculations for I_2 are similar, we just use $\dfrac{(z-\eta)^l}{z^l} \le C_l \dfrac{(\xi-\eta)^l}{\xi^l}$ instead. Let's continue with I_3, again in the case $l > -\frac{1}{2}$ first:

$$
\begin{aligned}
&|I_3| \le C_l C_1 (\xi-\eta)^l \int_\eta^{z_1(0)} z^{-\frac{1}{2}} \log\left(\frac{\eta}{z-\eta}\right) \left|q(\sqrt{z})\right| dz \\
&= 2C_l C_1 (\xi-\eta)^l \int_{\sqrt{\eta}}^{\sqrt{z_1(0)}} \log\left(\frac{\eta}{z^2-\eta}\right) |q(z)|\, dz \\
&\le 2C_l C_1 (\xi-\eta)^l \int_{\sqrt{\eta}}^{\sqrt{z_1(0)}} \log\left(\frac{\sqrt{\eta}}{z-\sqrt{\eta}}\right) |q(z)|\, dz \\
&\le 2C_l C_1 (\xi-\eta)^l \, \|q\|_{L^p((0,\sqrt{\xi}])} \left(\int_{\sqrt{\eta}}^{\sqrt{z_1(0)}} dz\right)^{\frac{1}{p'}-2\alpha} \\
&\times \left(\int_{\sqrt{\eta}}^{\sqrt{z_1(0)}} \log^{\frac{1}{2\alpha}}\left(\frac{\sqrt{\eta}}{z-\sqrt{\eta}}\right) dz\right)^{2\alpha} \\
&\le 2C_l C_1 (\xi-\eta)^l (\xi-\eta)^{\frac{1}{2p'}-\alpha} \left(\sqrt{\eta}\int_0^{\sqrt{\frac{\xi-\eta}{\xi+\eta}}} (-\log(u))^{\frac{1}{2\alpha}}\, du\right)^{2\alpha} \|q\|_{L^p((0,\sqrt{\xi}])} \\
&\le 2C_l C_1 \frac{(\sqrt{\xi}+\sqrt{\eta})^{2\alpha}}{\alpha} (\xi-\eta)^{l+\frac{1}{2p'}-\alpha} \, \|q\|_{L^p((0,\sqrt{\xi}])},
\end{aligned}
$$

where in the fourth step we again used Hölder's inequality with indices $\frac{p'}{1-2\alpha p'}$, 2α and p, in the fifth step we did a linear transformation inside the logarithmic integral ($u = \frac{z-\sqrt{\eta}}{\sqrt{\eta}}$), and in the penultimate step we applied Lemma 2.6. In the case $l = -\frac{1}{2}$, we have to make similar changes as for I_1, namely the fourth step will read as follows:

$$
\left(\int_{\sqrt{\eta}}^{\sqrt{z_1(0)}} z^{\frac{1}{p'}\frac{p'}{1-p'\alpha}} dz\right)^{\frac{1}{p'}-\alpha} \left(\int_{\sqrt{\eta}}^{\sqrt{z_1(0)}} \log^{\frac{1}{\alpha}}\left(\frac{\sqrt{\eta}}{z-\sqrt{\eta}}\right) dz\right)^{\alpha} \|q\|_{L^p((0,\sqrt{\xi}],z^{-\frac{p}{p'}})},
$$

while the first integral can be further estimated:

$$
\left(\int_{\sqrt{\eta}}^{\sqrt{z_1(0)}} z^{\frac{1}{p'}\frac{p'}{1-p'\alpha}} dz\right)^{\frac{1}{p'}-\alpha} \le z_1(0)^{\frac{\alpha}{2}} \left(\int_{\sqrt{\eta}}^{\sqrt{z_1(0)}} z dz\right)^{\frac{1}{p'}-\alpha},
$$

and now we can proceed as for I_1 and the logarithmic integral can of course be treated the same way as in the case $l > -\frac{1}{2}$. For I_4 in the case $l > -\frac{1}{2}$ we can now deduce the following inequality:

$$
\begin{aligned}
|I_4| &\le C_3(\xi-\eta)^{1+2l}\xi^{-1-l}\int_0^{z_2(0)} z^{l+1-\frac{1}{2p'}} z^{\frac{1}{2p'}-\frac{1}{2}}(\eta-z)^{-l-1}\left|q(\sqrt{z})\right| dz \\
&\le C_3(\xi-\eta)^{1+2l}\xi^{-1-l}z_2(0)^{l+1-\frac{1}{2p'}}\left(\int_0^{z_2(0)}(\eta-z)^{p'(-l-1)}dz\right)^{\frac{1}{p'}} \\
&\times\left(\int_0^{z_2(0)} z^{p(\frac{1}{2p'}-\frac{1}{2})}\left|q(\sqrt{z})\right|^p dz\right)^{\frac{1}{p}}
\end{aligned}
$$

To further estimate this expression, we need to distinguish cases. If $(-l-1)p'+1<0$, we get:

$$
\begin{aligned}
|I_4| &\le \frac{C_3}{|(-l-1)p'+1|^{\frac{1}{p'}}}(\xi-\eta)^{1+2l}\xi^{-1-l}(\xi\eta)^{l+1-\frac{1}{2p'}} \\
&\times(2\xi-\eta)^{-l-1+\frac{1}{2p'}}\eta^{-l-1+\frac{1}{p'}}(\xi-\eta)^{-l-1+\frac{1}{p'}}(2\xi-\eta)^{l+1-\frac{1}{p'}}\left\|q\right\|_{L^p((0,\sqrt{\xi}])} \\
&= \frac{C_3}{|(-l-1)p'+1|^{\frac{1}{p'}}}(\xi-\eta)^{l+\frac{1}{2p'}}\left(\frac{\xi-\eta}{2\xi-\eta}\right)^{\frac{1}{2p'}}\left(\frac{\eta}{\xi}\right)^{\frac{1}{2p'}}\left\|q\right\|_{L^p((0,\sqrt{\xi}])} \\
&\le \frac{C_3}{|(-l-1)p'+1|^{\frac{1}{p'}}}(\sqrt{\xi}+\sqrt{\eta})^{2\alpha}(\xi-\eta)^{l+\frac{1}{2p'}-\alpha}\left\|q\right\|_{L^p((0,\sqrt{\xi}+\sqrt{\eta}])},
\end{aligned}
$$

while for $(-l-1)p'+1>0$, the following estimate is valid:

$$
\begin{aligned}
|I_4| &\le \frac{C_3}{|(-l-1)p'+1|^{\frac{1}{p'}}}(\xi-\eta)^{1+2l}\xi^{-1-l}(\xi\eta)^{l+1-\frac{1}{2p'}} \\
&\times(2\xi-\eta)^{-l-1+\frac{1}{2p'}}\eta^{-l-1+\frac{1}{p'}}\left\|q\right\|_{L^p((0,\sqrt{\xi}])} \\
&= \frac{C_3}{|(-l-1)p'+1|^{\frac{1}{p'}}}(\xi-\eta)^{l+\frac{1}{2p'}}\left(\frac{\xi-\eta}{2\xi-\eta}\right)^{l+1-\frac{1}{2p'}}\left(\frac{\eta}{\xi}\right)^{\frac{1}{2p'}}\left\|q\right\|_{L^p((0,\sqrt{\xi}])} \\
&\le \frac{C_3}{|(-l-1)p'+1|^{\frac{1}{p'}}}(\sqrt{\xi}+\sqrt{\eta})^{2\alpha}(\xi-\eta)^{l+\frac{1}{2p'}-\alpha}\left\|q\right\|_{L^p((0,\sqrt{\xi}+\sqrt{\eta}])}.
\end{aligned}
$$

Since p' only depends on l, we will also in this case end up with a constant only depending on l. One also has to treat the case $(-l-1)p'+1=0$, but we omit the details here (one of the inner integrals would result in the logarithm, but this wouldn't cause any further difficulties). In the $-\frac{1}{2}$-case, with similar changes as in $I_1 - I_3$, we get an additional factor $z(0)^{\frac{1}{2p'}}$. For the remaining part of this lemma, we will only focus on the computations in the case $l>-\frac{1}{2}$ in order to avoid writing down the same changes all the time. Next, for I_5 we get:

$$
\begin{aligned}
|I_5| \le C_4(\xi-\eta)^l\int_{\sqrt{z_2(0)}}^{\sqrt{\eta}}|q(z)|\,dz &\le C_4(\xi-\eta)^l\left\|q\right\|_{L^p((0,\sqrt{\xi}])}\left(\int_{\sqrt{z_2(0)}}^{\sqrt{\eta}}1dz\right)^{\frac{1}{p'}} \\
&\le\ C_4(\sqrt{\xi}+\sqrt{\eta})^{2\alpha}(\xi-\eta)^{l+\frac{1}{2p'}-\alpha}\left\|q\right\|_{L^p((0,\sqrt{\xi}+\sqrt{\eta}])}.
\end{aligned}
$$

It remains to look at I_6, and with similar arguments as for I_3, we obtain:

$$\begin{aligned}
|I_6| &\le C_4(\xi-\eta)^l \int_{\sqrt{z_2(0)}}^{\sqrt{\eta}} z^{-\frac{1}{2}} \log\left(\frac{\sqrt{\eta}}{\sqrt{\eta}-z}\right) |q(z)|\, dz \\
&\le C_4(\xi-\eta)^l \left(\int_{\sqrt{z_2(0)}}^{\sqrt{\eta}} dz\right)^{\frac{1}{p'}-2\alpha} \left(\int_{\sqrt{z_2(0)}}^{\sqrt{\eta}} \log^{\frac{1}{2\alpha}}\left(\frac{\sqrt{\eta}}{\sqrt{\eta}-z}\right) dz\right)^{2\alpha} \|q\|_{L^p((0,\sqrt{\xi}])} \\
&\le C_4(\xi-\eta)^l(\xi-\eta)^{\frac{1}{2p'}-\alpha} \left(\sqrt{\eta}\int_0^{\sqrt{\frac{\xi-\eta}{2\xi-\eta}}} (-\log(u))^{\frac{1}{2\alpha}}\, du\right)^{2\alpha} \|q\|_{L^p((0,\sqrt{\xi}])} \\
&\le \frac{C_4(\sqrt{\xi}+\sqrt{\eta})^{2\alpha}}{\alpha} (\xi-\eta)^{l+\frac{1}{2p'}-\alpha} \|q\|_{L^p((0,\sqrt{\xi}+\sqrt{\eta}])}\,.
\end{aligned}$$

□

In the next lemma we are concerned with proving an inequality for u_1:

Lemma 2.8. *The following estimates hold:*

$$|u_1(\xi,\eta)| \le \begin{cases} \frac{\tilde{C}}{\alpha^2}\|q\|^2_{L^p((0,\sqrt{\xi}+\sqrt{\eta}])} (\sqrt{\xi}+\sqrt{\eta})^{1+\frac{1}{p'}+2\alpha}(\xi-\eta)^{l+\frac{1}{2p'}-\alpha}, & l>-\frac{1}{2} \\ \frac{\tilde{C}}{\alpha^2}\|q\|^2_{L^p((0,\sqrt{\xi}+\sqrt{\eta}],z^{-\frac{p}{p'}})} (\max(1,L))^{\frac{2}{2p'}} & \\ \times(\sqrt{\xi}+\sqrt{\eta})^{1+\frac{2}{p'}+2\alpha}(\xi-\eta)^{-\frac{1}{2}+\frac{1}{p'}-\alpha}, & l=-\frac{1}{2}. \end{cases} \tag{2.38}$$

The constant $\tilde{C}$ only depends on l and it may differ from C in Lemma 2.7.

Proof. Similarly as in Lemma 2.38, we split the corresponding integral:

$$\begin{aligned}
&\frac{1}{4}\iint_{0BPA}(zs)^{-\frac{1}{2}}v(z,s)q(\sqrt{z}+\sqrt{s})u_0(z,s)dzds = \\
&\frac{1}{4}\int_0^\eta\int_\eta^\xi (zs)^{-\frac{1}{2}}v(z,s)q(\sqrt{z}+\sqrt{s})u_0(z,s)dzds \\
+&\frac{1}{4}\int_0^\eta\int_s^\eta (zs)^{-\frac{1}{2}}v(z,s)q(\sqrt{z}+\sqrt{s})u_0(z,s)dzds.
\end{aligned}$$

Let us denote $\tilde{C}_i := CC_i$, where C is the constant obtained in Lemma 2.38, and the C_i's again are taken from Lemma 2.5. Using the results from Lemma 2.5 and Lemma 2.7 we end up with:

$$
\begin{aligned}
&\frac{1}{4}\int_0^\eta \int_\eta^\xi (zs)^{-\frac{1}{2}} \left|v(z,s)q(\sqrt{z}+\sqrt{s})u_0(z,s)\right| dzds \\
&\leq \tilde{C}_2 \int_0^\eta s^{-\frac{1}{2}} \int_\eta^{z_1(s)} z^{-\frac{1}{2}} \left|q(\sqrt{z}+\sqrt{s})\right| (z-s)^{l+\frac{1}{2p'}-\alpha} \\
&\times (\xi-z)^l(\eta-s)^l(z-s)^{-2l} dzds \\
&+ \tilde{C}_1 \int_0^\eta s^{-\frac{1}{2}} \int_{z_1(s)}^{\xi} z^{-\frac{1}{2}} \left|q(\sqrt{z}+\sqrt{s})\right| (z-s)^{l+\frac{1}{2p'}-\alpha} \\
&\times (z-\eta)^l(\xi-s)^l(z-s)^{-2l} dzds \\
&+ \tilde{C}_2 \int_0^\eta s^{-\frac{1}{2}} \int_\eta^{z_1(s)} z^{-\frac{1}{2}} \left|q(\sqrt{z}+\sqrt{s})\right| (z-s)^{l+\frac{1}{2p'}-\alpha} (\xi-z)^l(\eta-s)^l(z-s)^{-2l} \\
&\times \log\frac{(\xi-z)(\eta-s)}{(z-\eta)(\xi-s)} dzds =: J_1+J_2+J_3,
\end{aligned}
$$

and similarly:

$$
\begin{aligned}
&\frac{1}{4}\int_0^\eta \int_s^\eta (zs)^{-\frac{1}{2}} \left|v(z,s)q(\sqrt{z}+\sqrt{s})u_0(z,s)\right| dzds \\
&\leq \tilde{C}_3 \int_0^\eta s^{-\frac{1}{2}} \int_s^{z_2(s)} z^{-\frac{1}{2}} \left|q(\sqrt{z}+\sqrt{s})\right| (z-s)^{l+1+\frac{1}{2p'}-\alpha} \\
&\times (\xi-\eta)^{1+2l}(\eta-z)^{-l-1}(\xi-s)^{-l-1} dzds \\
&+ \tilde{C}_4 \int_0^\eta s^{-\frac{1}{2}} \int_{z_2(s)}^{\eta} z^{-\frac{1}{2}} \left|q(\sqrt{z}+\sqrt{s})\right| (z-s)^{l+\frac{1}{2p'}-\alpha} (\xi-\eta)^l(z-s)^{-l} dzds \\
&+ \tilde{C}_4 \int_0^\eta s^{-\frac{1}{2}} \int_\eta^{z_1(s)} z^{-\frac{1}{2}} \left|q(\sqrt{z}+\sqrt{s})\right| (z-s)^{l+\frac{1}{2p'}-\alpha} (\xi-\eta)^l(z-s)^{-l} \\
&\times \log\frac{(z-s)(\xi-\eta)}{(\eta-z)(\xi-s)} dzds =: J_4+J_5+J_6,
\end{aligned}
$$

where $z_1(s)$ and $z_2(s)$ are again given via (2.31). So let's consider J_1. A similar reasoning as in the Lemma 2.38 gives $\left(\frac{\xi-z}{z-s}\right)^l \leq C_l \left(\frac{\xi-\eta}{\eta-s}\right)^l$ and thus:

$$
\begin{aligned}
|J_1| \leq \tilde{C}_1(\xi-\eta)^l \int_0^\eta s^{-\frac{1}{2}} \int_\eta^{z_1(s)} z^{-\frac{1}{2}} \|q\|_{L^p((0,\sqrt{z}+\sqrt{s}])} \left|q(\sqrt{z}+\sqrt{s})\right| \\
\times (\sqrt{z}+\sqrt{s})^{2\alpha}(z-s)^{\frac{1}{2p'}-\alpha} dzds
\end{aligned}
$$

Next, by Hölder's inequality, we obtain:

$$\begin{aligned}
|J_1| &\le \tilde{C}_1(\xi-\eta)^l \int_0^\eta (\sqrt{\xi}+\sqrt{s})^{\frac{1}{p'}} s^{-\frac{1}{2}} \int_{\sqrt{\eta}}^{\sqrt{z_1(s)}} \|q\|_{L^p((0,z+\sqrt{s}])} \left|q(z+\sqrt{s})\right| dz ds \\
&\le \tilde{C}_1(\xi-\eta)^l \int_0^\eta (\sqrt{\xi}+\sqrt{s})^{\frac{1}{p'}} s^{-\frac{1}{2}} \left(\int_{\sqrt{\eta}}^{\sqrt{z_1(s)}} dz\right)^{\frac{1}{p'}} \\
&\times \left(\int_{\sqrt{\eta}}^{\sqrt{z_1(s)}} \|q\|^p_{L^p((0,z+\sqrt{s}])} \left|q(z+\sqrt{s})\right|^p dz\right)^{\frac{1}{p}} ds \\
&\le \tilde{C}_1(\xi-\eta)^{l+\frac{1}{2p'}-\alpha}(\sqrt{\xi}+\sqrt{\eta})^{2\alpha} \frac{\|q\|^2_{L^p((0,\sqrt{\xi}+\sqrt{\eta}])}}{2^{\frac{1}{p}}} \int_0^{\sqrt{\eta}} (\sqrt{\xi}+\sqrt{s})^{\frac{1}{p'}} ds \\
&\le \frac{\tilde{C}_1}{1+\frac{1}{p'}}(\xi-\eta)^{l+\frac{1}{2p'}-\alpha}(\sqrt{\xi}+\sqrt{\eta})^{1+\frac{1}{p'}+2\alpha} \|q\|^2_{L^p((0,\sqrt{\xi}+\sqrt{\eta}])}
\end{aligned}$$

In the $l=-\frac{1}{2}$-case, we also only remark, that in the end one gets a factor $(\sqrt{\xi}+\sqrt{\eta})^{1+\frac{2}{p'}+2\alpha}$ instead of $(\sqrt{\xi}+\sqrt{\eta})^{1+\frac{1}{p'}+2\alpha}$, with similar changes as in the previous Lemma 2.7. From now on we restrict ourselves to provide details only for the case $l>-\frac{1}{2}$. The tiny modifications in the $l=-\frac{1}{2}$-case will always be similar to Lemma 2.7. Concerning J_2, we use $\left(\frac{z-\eta}{z-s}\right)^l \le C_l\left(\frac{\xi-\eta}{\xi-s}\right)^l$ instead, the remaining procedure is the same as for J_1. Now we come to J_3. As for I_3, we use Hölder's inequality with indices $\frac{p'}{1-2\alpha p'}$, 2α and p, the inequality $z_1(s)\le z_1(0)$, and Lemma 2.6 to arrive at:

$$\begin{aligned}
|J_3| \le& \tilde{C}_3(\xi-\eta)^l \int_0^\eta (\sqrt{\xi}+\sqrt{s})^{\frac{1}{p'}} s^{-\frac{1}{2}} \left(\int_{\sqrt{\eta}}^{\sqrt{z_1(s)}} dz\right)^{\frac{1}{p'}-2\alpha} \\
&\times \left(\int_{\sqrt{\eta}}^{\sqrt{z_1(0)}} \log^{\frac{1}{2\alpha}}\left(\frac{\sqrt{\eta}}{z-\sqrt{\eta}}\right) dz\right)^{2\alpha} \\
&\times \left(\int_{\sqrt{\eta}}^{\sqrt{z_1(s)}} \|q\|^p_{L^p((0,z+\sqrt{s}])} \left|q(z+\sqrt{s})\right|^p dz\right)^{\frac{1}{p}} ds \\
&\le \frac{\tilde{C}_3}{(1+\frac{1}{p'})\alpha}(\xi-\eta)^{l+\frac{1}{2p'}-\alpha}(\sqrt{\xi}+\sqrt{\eta})^{1+\frac{1}{p'}+2\alpha} \|q\|^2_{L^p((0,\sqrt{\xi}+\sqrt{\eta}])}.
\end{aligned}$$

We continue with J_4, and here, for brevity, only consider the case $(-l-1)p'+1<0$(cf. the calculations for I_4 for the other cases). First we use $z_2(s)-s=\frac{(\xi-s)(\eta-s)}{2\xi-\eta-s}$

and Hölder's inequality to obtain:

$$|J_4| \leq \tilde{C}_4(\xi-\eta)^{1+2l}\int_0^\eta (\sqrt{\xi}+s)^{2\alpha}s^{-\frac{1}{2}}(\xi-s)^{-l-1}(\xi-s)^{l+1-\alpha}(\eta-s)^{l+1-\alpha}$$
$$\times (2\xi-\eta-s)^{-l-1+\alpha}\left(\int_0^{z_2(s)}(\eta-z)^{(-l-1)p'}dz\right)^{\frac{1}{p'}}$$
$$\times\left(\int_0^{z_2(s)} z^{p(\frac{1}{2p'}-\frac{1}{2})}\left|q(\sqrt{z}+\sqrt{s})\right|^p \|q\|^p_{L^p((0,\sqrt{z}+\sqrt{s}])}\,dz\right)^{\frac{1}{p}} ds.$$

We further estimate this expression by first calculating the inner integrals. After that, we use $\eta - z_2(s) = \frac{(\xi-\eta)(\eta-s)}{2\xi-\eta-s}$ and we group the remaining terms in an appropriate way:

$$|J_4| \leq \frac{\tilde{C}_4}{|(-l-1)p'+1|^{\frac{1}{p'}}}(\sqrt{\xi}+\sqrt{\eta})^{2\alpha}(\xi-\eta)^{1+2l}\int_0^\eta s^{-\frac{1}{2}}(\xi-s)^{-\alpha}(\eta-s)^{l+1-\alpha}$$
$$\times (2\xi-\eta-s)^{-l-1+\alpha}(\eta-s)^{-l-1+\frac{1}{p'}}(\xi-\eta)^{-l-1+\frac{1}{p'}}$$
$$\times (2\xi-\eta-s)^{l+1-\frac{1}{p'}}\|q\|^2_{L^p((0,\sqrt{\xi}+\sqrt{\eta}])}\,ds$$
$$\leq \frac{\tilde{C}_4}{|(-l-1)p'+1|^{\frac{1}{p'}}}(\xi-\eta)^{l+\frac{1}{2p'}-\alpha}(\sqrt{\xi}+\sqrt{\eta})^{2\alpha}\|q\|^2_{L^p((0,\sqrt{\xi}+\sqrt{\eta}])}$$
$$\times\int_0^\eta(\sqrt{\xi}+\sqrt{s})^{\frac{1}{p'}}s^{-\frac{1}{2}}\left(\frac{\eta-s}{(\sqrt{\xi}+\sqrt{s})^2}\right)^{\frac{1}{2p'}}\left(\frac{\eta-s}{2\xi-\eta-s}\right)^{\frac{1}{2p'}-\alpha}\left(\frac{\xi-\eta}{2\xi-\eta-s}\right)^{\frac{1}{2p'}}ds.$$

The last expression immediately leads to:

$$|J_4| \leq \frac{\tilde{C}_3}{(1+\frac{1}{p'})\alpha\,|(-l-1)p'+1|^{\frac{1}{p'}}}(\xi-\eta)^{l+\frac{1}{2p'}-\alpha}(\sqrt{\xi}+\sqrt{\eta})^{1+\frac{1}{p'}+2\alpha}\|q\|^2_{L^p((0,\sqrt{\xi}+\sqrt{\eta}])}.$$

We omit the details for J_5 and J_6. Concerning J_6, we only remark that we follow the same procedure as for J_3, at one point though we have to use the estimate $z_2(s) \geq s$ in order to get s as the lower bound of the inner integral. $\square$

The next lemma treats the n-th iterate u_n:

Lemma 2.9. *The following estimates hold:*

$$|u_n(\xi,\eta)|$$
$$\leq \begin{cases} \frac{\tilde{C}^n}{\alpha^{n+1}n!}\|q\|^{n+1}_{L^p((0,\sqrt{\xi}+\sqrt{\eta}])}(\sqrt{\xi}+\sqrt{\eta})^{n(1+\frac{1}{p'})+2\alpha}(\xi-\eta)^{l+\frac{1}{2p'}-\alpha}, & l>-\frac{1}{2}\\ \frac{\tilde{C}^n}{\alpha^{n+1}n!}\|q\|^{n+1}_{L^p((0,\sqrt{\xi}+\sqrt{\eta}],z^{-\frac{p}{p'}})}(\max(1,L))^{\frac{n+1}{2p'}} & \\ \times(\sqrt{\xi}+\sqrt{\eta})^{n(1+\frac{1}{p'})}(\xi-\eta)^{-\frac{1}{2}+\frac{1}{p'}-\alpha}, & l=-\frac{1}{2}.\end{cases}$$

The constant is identical to the one obtained in Lemma 2.8.

Proof. We do a similar integral splitting as before, and, as an example, only provide details for the inequality for J_3^n. The other expressions can be treated in a similar way. We will proceed inductively:

$$\begin{aligned}
\left|J_3^{n+1}\right| &\leq \frac{\tilde{C}^n C_3(\xi-\eta)^l}{\alpha^{n+1} n!} \int_0^\eta s^{-\frac{1}{2}} \int_{\sqrt{\eta}}^{\sqrt{z_1(s)}} \log\left(\frac{\sqrt{\eta}}{z-\sqrt{\eta}}\right) \\
&\quad \times \left|q(z+\sqrt{s})\right| \|q\|_{L^p((0,z+\sqrt{s}])}^{n+1} (\sqrt{z}+\sqrt{s})^{n(1+\frac{1}{p'})} (z-s)^{\frac{1}{2p'}} \, dz ds \\
&\leq \frac{\tilde{C}^n C_3(\xi-\eta)^l}{\alpha^{n+1} n!} \int_0^\eta (\sqrt{\xi}+\sqrt{s})^{n(1+\frac{1}{p'})+\frac{1}{p'}} s^{-\frac{1}{2}} \left(\int_{\sqrt{\eta}}^{\sqrt{z_1(s)}} dz\right)^{\frac{1}{p'}-2\alpha} \\
&\quad \times \left(\int_{\sqrt{\eta}}^{\sqrt{z_1(0)}} \log^{\frac{1}{2\alpha}}\left(\frac{\sqrt{\eta}}{z-\sqrt{\eta}}\right) dz\right)^{2\alpha} \\
&\quad \times \left(\int_{\sqrt{\eta}}^{\sqrt{z_1(s)}} \|q\|_{L^p((0,z+\sqrt{s}])}^{p(n+1)} \left|q(z+\sqrt{s})\right|^p dz\right)^{\frac{1}{p}} ds.
\end{aligned}$$

With an analogical reasoning as for J_3, we obtain the expected inequality:

$$\left|J_3^{n+1}\right| \leq \frac{\tilde{C}^{n+1}(\xi-\eta)^{l+\frac{1}{2p'}-\alpha}}{\alpha^{n+2}(n+1)!} (\sqrt{\xi}+\sqrt{\eta})^{(n+1)(\frac{1}{p'}+1)+2\alpha} \|q\|_{L^p((0,\sqrt{\xi}+\sqrt{\eta}])}^{n+2}.$$

□

We are now in the position to finish the proof of Theorem 2.4:

Proof of Theorem 2.4. Everything now follows from the Lemmas 2.5–2.9, since $\sum_{n=0}^\infty |u_n|$ converges uniformly on compact sets. □

We continue now with some remarks, which aim at relating previously obtained results to our work:

Remark 2.10. *It has already been mentioned in* [21, Appendix, Page 21]*, that the estimates for u in* [30] *contain an error. Indeed, if they were true, we would have the inequality $|B(x,x)| \leq Cx^{2-2\rho}$ for any $0 \leq \rho < 1$, which is impossible for $\rho < \frac{1}{2}$ due to $\frac{\partial B(x,x)}{\partial x} = \frac{q(x)}{2}$ ((2.7)), because not even a constant potential $q(x) = C$ would satisfy the condition. In* [21, Appendix]*, the authors tried to give valid estimates for u, but it seems there is also a small inconsistency in the estimate for I_4. That's the main reason, why we have been very careful in the proof of the previous Theorem and also provided many details regarding the technical estimates. Moreover, in the case $l > -\frac{1}{2}$, our computations also allow to generalize to more general potentials lying in some L^p-space, while in* [21] *and* [30] *only continuous potentials were considered.*

Remark 2.11. *In the $-\frac{1}{2}$ case, however, it seems, that not even continuous potentials suffice, and we imposed some extra decay condition near 0, mentioned in Theorem 1.1. It was not clear to the author, why, e.g. the asymptotics given in* [21, Theorem 3.1.]*, i.e. $u(z,s) = \mathcal{O}((z-s)^{-\alpha})$, are good enough for the proof of Lemmas 2.2–2.3.*

Remark 2.12. *In* [19] *and* [12] *it was conjectured, that Theorem 1.1 continues to hold for any $q \in L^1_{\text{loc}}([0,\infty))$. This would of course be very convenient, but it seems that to treat the logarithmic singularities e.g. in I_3 and I_6, one has to work with Hölder's inequality, which of course isn't available for locally integrable potentials.*

If we are now able to prove that, in addition to Theorem 2.4, u is also a C_2 function, then we can indeed conclude that it satisfies all the equations from Lemma 2.1. This will be discussed next:

Lemma 2.13. *Let $q \in C^1([0,L])$. Then $B(x,\cdot) \in C^2([0,x])$.*

Proof. This proof closely follows the arguments from [28](c.f. end of page 6). Let the corresponding kernel $B(x,y)$ be given by (2.8). We start by establishing an integral equation for B. We thus introduce the new coordinates $\tilde{z} := \sqrt{z} = \frac{x+y}{2}$, $\quad \tilde{s} := \sqrt{s} = \frac{x-y}{2}$, and the function $\tilde{u}(\tilde{z},\tilde{s}) := B(x,y) = B(\tilde{z}+\tilde{s}, \tilde{z}-\tilde{s})$, so that (2.6) transforms to

$$\frac{\partial^2 \tilde{u}}{\partial \tilde{z} \partial \tilde{s}} + \frac{4l(l+1)\tilde{z}\tilde{s}}{(\tilde{z}^2 - \tilde{s}^2)^2}\tilde{u} = -q(\tilde{z}+\tilde{s})\tilde{u}.$$

Now we integrate with respect to $\tilde{z}$ and $\tilde{s}$ and transform back to x and y coordinates($\tilde{x} = \tilde{z}+\tilde{s}$, $\quad \tilde{y} = \tilde{z}-\tilde{s}$) and obtain the following equation for B(the first term origins from (2.7), the upper bounds of the inner integrals come from the inequalities $\frac{\tilde{x}+\tilde{y}}{2} \leq \frac{x+y}{2}$ and $\tilde{y} \leq \tilde{x}$):

$$\begin{aligned} B(x,y) = \int_0^{\frac{x+y}{2}} q(\tilde{x})d\tilde{x} + \frac{1}{2}\left(\int_0^{\frac{x+y}{2}} d\tilde{x} \int_0^{\tilde{x}} + \int_{\frac{x+y}{2}}^{x} d\tilde{x} \int_0^{x+y-\tilde{x}}\right) \\ \times \left[q(\tilde{x}) + l(l+1)\left(\frac{1}{\tilde{x}^2} - \frac{1}{\tilde{y}^2}\right)\right] B(\tilde{x},\tilde{y})d\tilde{y}, \quad 0 < y \leq x. \end{aligned}$$

This immediately shows that B obtains second derivatives, if q is differentiable. $\square$

So far we have shown that for a smooth potential q the transformation operators exist. Now we suppose the assumptions on q from Theorem 1.1 and proceed as follows: Approximate the function q by a sequence of smooth functions q_n, such that for any $x \in (0,L]$, q_n converges to q in the $L^p((0,x])$-norm(or $L^p(z^{-\frac{1}{p'}}, (0,x])$-norm in the $-\frac{1}{2}$-case). Let $B_n(x,y)$, $B(x,y)$ be the kernels obtained from the potentials q_n, q reps. via Theorem 1.1. Then from (1.5) we can conclude that B_n converges to B uniformly on $[0,L]^2$. This proves Theorem 1.1.

3. Transformation Operators near ∞

Completely analogous computations as in the beginning of the previous section lead to the following set of equations for the transformation operator K:

$$\left(\frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial y^2} + \frac{l(l+1)}{y^2} - \frac{l(l+1)}{x^2} - q(x)\right) K(x,y) = 0, \quad 0 < x < y \tag{3.1}$$

$$\frac{\partial K(x,x)}{\partial x} = -\frac{q(x)}{2}, \quad \lim_{y\to\infty} K(x,y) = 0 = \lim_{y\to\infty} \frac{\partial K(x,y)}{\partial y}. \tag{3.2}$$

The next step is to put the problem into integral form using the same transformation as in Lemma 2.13, i.e. $\xi := \frac{x+y}{2}, \eta := \frac{y-x}{2}$, $\quad w(\xi,\eta) := K(x,y) = K(\xi-\eta, \xi+\eta)$, so that (3.1) transforms to

$$\frac{\partial^2 w}{\partial \xi \partial \eta} + \frac{4l(l+1)\xi\eta}{(\xi^2-\eta^2)^2} w = -q(\xi-\eta)w \tag{3.3}$$

$$w(\xi,0) = \frac{1}{2}\int_x^\infty q(z)dz, \quad \lim_{\xi\to\infty} w(\xi,\eta) = 0, \quad \eta > 0. \tag{3.4}$$

Again, as in the previous section, for the time being we assume q to be differentiable. We furthermore introduce the Riemann function v_3 as a solution to the problem

$$\begin{aligned}
&\frac{\partial^2 v_3}{\partial z \partial s} + \frac{4l(l+1)zs}{(z^2-s^2)^2} v_3 = 0, \quad 0 < s < \eta < \xi < z < \infty \\
&v_3(z,s;\xi,\eta)\Big|_{z=\xi} = 1 \quad s \in [0,\eta], \\
&v_3(z,s;\xi,\eta)\Big|_{s=\eta} = 1 \quad z \in [\xi,\infty).
\end{aligned} \tag{3.5}$$

Using the transformation $\tilde{z} = z^2$, $\tilde{s} = s^2$ and defining $\tilde{v_3} := (\tilde{z} - \tilde{s})^l v_3$, we see that $\tilde{v_3}$ satisfies the equation $L\tilde{v_3} = 0$, where L is the Euler-Poisson-Darboux operator defined in the previous section. Similar considerations as for v_1 finally lead to the following explicit formula for v_3:

$$v_3(z,s;\eta,\xi) = \left(\frac{z^2-\eta^2}{z^2-s^2} \cdot \frac{\xi^2-s^2}{\xi^2-\eta^2}\right)^l {}_2F_1\left(\begin{matrix}-l,-l\\1\end{matrix}; \frac{z^2-\xi^2}{z^2-\eta^2} \cdot \frac{\eta^2-s^2}{\xi^2-s^2}\right). \tag{3.6}$$

If we apply Riemann's method to (3.3), we end up with the following integral equation for w:

$$w(\xi,\eta) = \frac{1}{2}\int_\xi^\infty v_3(z,0;\xi,\eta)q(z)dz + \int_\xi^\infty \int_0^\eta q(z-s)v_3(z,s;\xi,\eta)w(z,s)dsdz. \tag{3.7}$$

In the case $l > 0$, instead of w, we will use a little trick and consider an integral equation for the function $\tilde{w} := \left(\frac{\xi^2}{\xi^2-\eta^2}\right)^{-l} w(\xi,\eta)$ instead. Thus in the sequel we are concerned with the following expression:

$$\begin{aligned}
\tilde{w}(\xi,\eta) = &\frac{1}{2}\left(\frac{\xi^2}{\xi^2-\eta^2}\right)^{-l} \int_\xi^\infty v_3(z,0;\xi,\eta)q(z)dz \\
&+ \int_\xi^\infty \int_0^\eta q(z-s)v_3(z,s;\xi,\eta)\tilde{w}(z,s)dsdz.
\end{aligned}$$

Again we intend to solve this equation via successive approximation and set $\tilde{w} = \sum_{n=0}^\infty \tilde{w}_n$, where the $\tilde{w}_n$'s are defined recursively as follows:

$$\begin{aligned}
\tilde{w}_0(\xi,\eta) &:= \left(\frac{\xi^2}{\xi^2-\eta^2}\right)^{-l} \frac{1}{2}\int_\xi^\infty v_3(z,0;\xi,\eta)q(z)dz \\
\tilde{w}_{n+1}(\xi,\eta) &:= \left(\frac{\xi^2}{\xi^2-\eta^2}\right)^{-l} \int_\xi^\infty \int_0^\eta q(z-s)v_3(z,s;\xi,\eta)\tilde{w}_n(z,s)dsdz.
\end{aligned} \tag{3.8}$$

The iterates w_n in the case $l \leq 0$ are obviously defined in the same way, just without the factor $\left(\frac{\xi^2}{\xi^2-\eta^2}\right)^{-l}$. Finally we will end up with the following theorem:

Theorem 3.1. *Under the conditions on q stated in Theorem 1.2, there is a unique continuous function $w(\xi,\eta)$ that solves* (3.7) *and satisfies*

$$|w(\xi,\eta)| \leq \begin{cases} \frac{C_l}{2}\left(\frac{\xi^2}{\xi^2-\eta^2}\right)^l \tilde{\sigma}_0(\xi)\mathrm{e}^{C_l[\tilde{\sigma}_1(\xi-\eta)-\tilde{\sigma}_1(\xi)]} & l > 0, \\ \frac{C_l}{2}\tilde{\sigma}_0(\xi)\mathrm{e}^{C_l[\tilde{\sigma}_1(\xi-\eta)-\tilde{\sigma}_1(\xi)]}, & -\frac{1}{2} \leq l \leq 0. \end{cases} \tag{3.9}$$

To establish this result we start with proving estimates for v_3:

Lemma 3.2. *The Riemann function v_3 satisfies the following estimate:*

$$|v_3(z,s;\xi,\eta)| \leq \begin{cases} C_l \left(\frac{\xi^2}{\xi^2-\eta^2}\right)^l, & l>0 \\ C_l, & -\frac{1}{2} \leq l \leq 0, \end{cases} \tag{3.10}$$

where $0<s<\eta<\xi<z<\infty$.

Proof. Set $t := \frac{z^2-\xi^2}{z^2-\eta^2} \cdot \frac{\eta^2-s^2}{\xi^2-s^2}$. Clearly $0<t<1$. A short calculation also shows $1-t = \frac{z^2-s^2}{z^2-\eta^2} \cdot \frac{\xi^2-\eta^2}{\xi^2-s^2}$, thus we also have $0<1-t<1$. It remains to look at asymptotics for $(1-t)^{-\alpha}\, {}_2F_1\left({-l,-l \atop 1};t\right)$, where $0 \leq t \leq 1$. In the case $l \neq -\frac{1}{2}$, by employing (A.4) we obtain

$$\left|(1-t)^{-\alpha}\, {}_2F_1\left({-l,-l \atop 1};t\right)\right| \leq \begin{cases} C_l(1-t)^{-l}, & l>0 \\ C_l, & -\frac{1}{2}<l\leq 0, \end{cases}$$

which immediately leads to (3.10). Moreover, in the case $l=-\frac{1}{2}$, using (A.6), we get

$$(1-t)^{\frac{1}{2}}\, {}_2F_1\left({\frac{1}{2},\frac{1}{2} \atop 1};t\right) = \mathcal{O}\left((1-t)^{\frac{1}{2}} \log\left(\frac{1}{1-t}\right)\right), \quad t \to 0,$$

and thus again $|v_3(z,s;\xi,\eta)| \leq C$, since $(1-t)^{\frac{1}{2}}\log\left(\frac{1}{1-t}\right)$ is bounded on $[0,1]$. □

The next step is to find suitable estimates for the iterates w_n, which will immediately lead to a complete proof of Theorem 2.4:

Lemma 3.3. *We have the following estimates for our iterates w_n defined in* (3.8)*:*

$$|\tilde{w_0}(\xi,\eta)| \leq \frac{C_l}{2}\tilde{\sigma}_0(\xi), \quad l>0$$
$$|w_0(\xi,\eta)| \leq \frac{C_l}{2}\tilde{\sigma}_0(\xi), \quad -\frac{1}{2} \leq l \leq 0,$$

$$|\tilde{w_1}(\xi,\eta)| \leq \frac{C_l}{2}\tilde{\sigma}_0(\xi)C_l[\tilde{\sigma}_1(\xi-\eta)-\tilde{\sigma}_1(\xi)], \quad l>0$$
$$|w_1(\xi,\eta)| \leq \frac{C_l}{2}\tilde{\sigma}_0(\xi)C_l[\tilde{\sigma}_1(\xi-\eta)-\tilde{\sigma}_1(\xi)], \quad -\frac{1}{2} \leq l \leq 0,$$

and finally

$$|\tilde{w_n}(\xi,\eta)| \leq \frac{C_l}{2}\tilde{\sigma}_0(\xi)\frac{(C_l[\tilde{\sigma}_1(\xi-\eta)-\tilde{\sigma}_1(\xi)])^n}{n!}, \quad l>0$$
$$|w_n(\xi,\eta)| \leq \frac{C_l}{2}\tilde{\sigma}_0(\xi)\frac{(C_l[\tilde{\sigma}_1(\xi-\eta)-\tilde{\sigma}_1(\xi)])^n}{n!}, \quad -\frac{1}{2} \leq l \leq 0.$$

Proof. The estimate for $\tilde{w_0}$(or w_0 resp.) follows immediately from (3.8) and (3.10). Let's proceed with $\tilde{w_1}$. We only consider the case $l>0$, since the other one is similar(just without the factors $\left(\frac{\xi^2}{\xi^2-\eta^2}\right)^{-l}$):

$$\begin{aligned}
|\tilde{w_1}(\xi,\eta)| &\le \left(\frac{\xi^2}{\xi^2-\eta^2}\right)^{-l}\int_\xi^\infty\int_0^\eta |q(z-s)v_3(z,s;\xi,\eta)w_0(z,s)|\,dsdz\\
&\le \left(\frac{\xi^2}{\xi^2-\eta^2}\right)^{-l}\int_\xi^\infty\int_0^\eta \frac{C_l^2}{2}\left(\frac{\xi^2}{\xi^2-\eta^2}\right)^{l}\tilde{\sigma_0}(z)\,|q(z-s)|\,dsdz\\
&\le \frac{C_l^2}{2}\tilde{\sigma_0}(\xi)\int_\xi^\eta(\tilde{\sigma_0}(z-\eta)-\tilde{\sigma_0}(z))dz\\
&= \frac{C_l^2}{2}\tilde{\sigma_0}(\xi)(\tilde{\sigma_1}(\xi-\eta)-\tilde{\sigma_1}(\xi)).
\end{aligned}$$

Now we come to the estimate for $\tilde{w_n}$, which is done inductively:

$$\begin{aligned}
|\tilde{w_n}(\xi,\eta)| &\le \left(\frac{\xi^2}{\xi^2-\eta^2}\right)^{-l}\int_\xi^\infty\int_0^\eta |q(z-s)v_3(z,s;\xi,\eta)\tilde{w_{n-1}}(z,s)|\,dsdz\\
&\le \left(\frac{\xi^2}{\xi^2-\eta^2}\right)^{-l}\int_\xi^\infty\int_0^\eta \frac{C_l^2}{2}\left(\frac{\xi^2}{\xi^2-\eta^2}\right)^{l}\tilde{\sigma_0}(z)\\
&\times|q(z-s)|\,\frac{(C_l(\tilde{\sigma_1}(z-s)-\tilde{\sigma_1}(z)))^{n-1}}{(n-1)!}dsdz\\
&\le \frac{C_l^2}{2}\tilde{\sigma_0}(\xi)\int_\xi^\infty\frac{(C_l(\tilde{\sigma_1}(z-\eta)-\tilde{\sigma_1}(z)))^{n-1}}{(n-1)!}(\tilde{\sigma_1}(z-\eta)-\tilde{\sigma_1}(z))dz\\
&= \frac{C_l^2}{2}\tilde{\sigma_0}(\xi)\frac{(C_l(\tilde{\sigma_1}(\xi-\eta)-\tilde{\sigma_1}(\xi)))^{n}}{n!}.
\end{aligned}$$

□

Proof of Theorem 3.1. Everything now follows from the Lemma 3.3 again by uniform convergence of the corresponding series $\tilde{w}=\sum_{n=0}^\infty\tilde{w_n}$(or $w=\sum_{n=0}^\infty w_n$ resp.). □

Also here we want to state some remarks about existing results in the literature:

Remark 3.4. *The content of this section is strongly influenced by* [28]*, however, in* [28] *only the case $l\ge 0$ was considered. We extended this work to $l\ge-\frac{1}{2}$ by providing a more detailed analysis of the Riemann function v_3 in Lemma 3.2, but also provide the remaining details for the reader's convenience.*

The next result is the analogous version of Lemma 2.13

Lemma 3.5. *Let q satisfy the assumptions of Theorem 1.2 and additionally $q\in C^1([x,\infty))$. Then $K(x,.)\in C^2([x,\infty))$.*

Proof. Similar ideas as in the previous section lead to the following integral equation for $K(x,y)$:

$$\begin{aligned}
K(x,y) = \int_{\frac{x+y}{2}}^\infty q(\tilde{x})d\tilde{x} &+ \frac{1}{2}\left(\int_x^{\frac{x+y}{2}}d\tilde{x}\int_{-\tilde{x}+x+y}^{\tilde{x}+y-x}+\int_{\frac{x+y}{2}}^\infty d\tilde{x}\int_{\tilde{x}}^{\tilde{x}+y-x}\right)\\
&\times\left[q(\tilde{x})+l(l+1)\left(\frac{1}{\tilde{x}^2}-\frac{1}{\tilde{y}^2}\right)\right]K(\tilde{x},\tilde{y})d\tilde{y},\quad 0<x\le y,
\end{aligned}$$

which gives the desired claim. □

The arguments for the approximation procedure, that conclude the proof of Theorem 1.2, are now exactly the same as in the previous section.

Appendix A. The Gauss Hypergeometric function

Here we collect basic formulas and information on the Gauss hypergeometric function (see, e.g., [1],[27]). First of all by Γ is denoted the classical gamma function [27, (5.2.1)]. For $x \in \mathbb{C}$ and $n \in \mathbb{N}_0$

$$(x)_n := x(x+1)\cdots(x+n-1) \quad (n>0), \quad (x)_0 := 1; \quad \binom{n+x}{n} := \frac{(x+1)_n}{n!}$$

denote the Pochhammer symbol [27, (5.2.4)] and the binomial coefficient, respectively. Notice that for $-x \notin \mathbb{N}_0$

$$(x)_n = \frac{\Gamma(x+n)}{\Gamma(x)}, \qquad \binom{n+x}{n} = \frac{\Gamma(x+n+1)}{\Gamma(x+1)\Gamma(n+1)}.$$

Moreover, the above formulas allow to define the Pochhammer symbol and the binomial coefficient for noninteger x, $n>0$ as well. For $-c \notin \mathbb{N}_0$ the Gauss hypergeometric function [27, (15.2.1)] is defined by

$${}_2F_1\left(\begin{matrix} a,b \\ c \end{matrix}; z\right) := \sum_{k=0}^{\infty} \frac{(a)_k(b)_k}{(c)_k k!} z^k \quad (|z|<1 \text{ or else } -a \text{ or } -b \in \mathbb{N}_0). \tag{A.1}$$

The branch cut is chosen along the positive real axis. By analytic continuation this definition may also be extended to other values of z. Thus the derivative is also easy to compute and given by the following formula(see [27, (15.5.1)]):

$$\frac{\partial}{\partial z}\, {}_2F_1\left(\begin{matrix} a,b \\ c \end{matrix}; z\right) = \frac{ab}{c}\, {}_2F_1\left(\begin{matrix} a+1,b+1 \\ c+1 \end{matrix}; z\right). \tag{A.2}$$

Functions of the form (A.1) are closely related to the Hypergeometric equation

$$x(1-x)\frac{d^2f}{dx^2} + (c-(a+b+1)x)\frac{df}{dx} - abx = 0. \tag{A.3}$$

Indeed, the hypergeometric functions appear in explicit formulas for solutions of (A.3), one has to be careful with certain values of the parameters a, b and c though. For a summary of the types of solutions that may occur, see [27, (15.10)]. Next, we also need the asymptotic behavior near the possible singular points 1 and ∞ for ${}_2F_1\left(\begin{smallmatrix} a,b \\ c \end{smallmatrix}; z\right)$ for specific values of a, b and c(see [27, (15.4.20), (15.4.21), (15.8.8)]):

$${}_2F_1\left(\begin{matrix} a,b \\ c \end{matrix}; 1\right) = \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}, \quad \mathrm{Re}(c-a-b)>0 \tag{A.4}$$

$$\lim_{z\to 1-} \frac{{}_2F_1\left(\begin{smallmatrix} a,b \\ c \end{smallmatrix}; z\right)}{(1-z)^{c-a-b}} = \frac{\Gamma(c)\Gamma(a+b-c)}{\Gamma(a)\Gamma(b)}, \quad \mathrm{Re}(c-a-b)<0 \tag{A.5}$$

$$\lim_{z\to 1-} \frac{{}_2F_1\left(\begin{smallmatrix} a,b \\ a+b \end{smallmatrix}; z\right)}{-\log(1-z)} = \frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)} \tag{A.6}$$

and

$$\begin{aligned} {}_2F_1\left(\begin{matrix} a,a \\ c \end{matrix}; z\right) = \frac{\Gamma(c)(-z)^{-a}}{\Gamma(a)} \sum_{k=0}^{\infty} &\frac{(a)_n}{(k!)^2\Gamma(c-a-k)}(-1)^k z^{-k} \\ &\times(\log(-z)+2\psi(k+1)-\psi(a+k)-\psi(c-a-k)), \end{aligned} \tag{A.7}$$

if $|z| > 1$. Here ψ denotes the digamma function [27, (5.2.2)]. It satisfies the reflection formula (c.f. [27, (5.5.2)])

$$\psi(z+1) = \psi(z) + \frac{1}{z} \tag{A.8}$$

and we also briefly mention an estimate near ∞ (c.f. [27, (5.11.2)]):

$$\psi(z) = \log z - \frac{1}{2z} + \mathcal{O}(z^{-2}), \quad z \to \infty, \tag{A.9}$$

which will be useful in order to show absolute convergence of the series in (A.7). To conclude, we also need to mention that for integer values of a, the hypergeometric function reduces to a polynomial.

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References

[1] G. Andrews, R. Askey, and R. Roy, *Special Functions*, Cambridge University Press, Cambridge, 1999.
[2] A. A. Bolibruch, Yu. S. Osipov and Ya. G. Sinai, *Mathematical events of the twentieth century*, Springer-Verlag, Berlin; PHASIS, Moscow, 2006.
[3] R. Castillo-Pèrez, V. V. Kravchenko and S. M. Torba, *Spectral parameter power series for perturbed Bessel equations*, Appl. Math. Comput. **220** (2013), 676-694 (2013).
[4] R. Castillo-Pèrez, V. V. Kravchenko and S. M. Torba, *Analysis of graded-index optical fibers by the spectral parameter power series method*, J. Optics **17** , 025607 (9pp) (2015).
[5] H. Chèbli, A. Fitouhi and M. M. Hamza, *Expansion in series of Bessel functions and transmutations for perturbed Bessel operators*, J. Math. Anal. Appl. **181** , no. 3, 789–802 (1994).
[6] R. Courant, D. Hilbert, *Methods of Mathematical Physics: Part 2*, Springer, Berlin, 1937.
[7] K. Chadan and P. C. Sabatier, *Inverse Problems in Quantum Scattering Theory*, 2nd ed., Springer-Verlag, 1989.
[8] E. Copson, *Partial Differential Equations*, Cambridge Univ. Press, 1975.
[9] M. Coz and C. Coudray, *The Riemann solution and the inverse quantum mechanical problem*, J. Math. Phys. **17**, no. 6, 888–893 (1976).
[10] L. Faddeev, *The inverse problem in quantum scattering theory*, J. Math. Phys. **4**, 72–104 (1963).
[11] M. Holzleitner, A. Kostenko, and G. Teschl, *Dispersion estimates for spherical Schrödinger equations: The effect of boundary conditions*, Opuscula Math. **36**, no. 6, 769–786 (2016).
[12] M. Holzleitner, A. Kostenko, and G. Teschl, *Dispersion estimates for spherical Schrdinger equations with critical angular momentum*, in Partial Differential Equations, Mathematical Physics, and Stochastic Analysis, F. Gesztesy et al. (eds), EMS Congress Reports (to appear).
[13] R. Hryniv and Ya. Mykytyuk, *Transformation Operators for Sturm-Liouville Operators with Singular Potentials* Math. Phys. Anal. Geom. **7**, 119–149 (2004)
[14] I. S. Kac, *On the behavior of spectral functions of second-order differential systems*, Dokl. Akad. Nauk SSSR **106**, 183–186 (1956). [in Russian]
[15] N. S. Koshlyakov, M. M. Smirnov and E. B. Gliner *Differential Equations of Mathematical Physics*, North-Holland publishing company, Amsterdam, 1964.
[16] A. Kostenko, A. Sakhnovich, and G. Teschl, *Weyl–Titchmarsh theory for Schrödinger operators with strongly singular potentials*, Int. Math. Res. Not. **2012**, 1699–1747 (2012).
[17] A. Kostenko and G. Teschl, *On the singular Weyl–Titchmarsh function of perturbed spherical Schrödinger operators*, J. Differential Equations **250**, 3701–3739 (2011).
[18] A. Kostenko and G. Teschl, *Spectral asymptotics for perturbed spherical Schrödinger operators and applications to quantum scattering*, Comm. Math. Phys. **322**, 255–275 (2013).
[19] A. Kostenko, G. Teschl and J. H. Toloza, *Dispersion estimates for spherical Schrödinger equations*, Ann. Henri Poincaré **17**, no. 11, 3147–3176 (2016).

[20] H. Kovařík and F. Truc, *Schrödinger operators on a half-line with inverse square potentials*, Math. Model. Nat. Phenom. **9**, no. 5, 170–176 (2014).

[21] V. V. Kravchenko, S. M. Torba, R. Castillo-Pèrez, *A Neumann series of Bessel functions representation for solutions of perturbed Bessel equations.*, Appl. Anal. , 1-28, (2017).

[22] V. V. Kravchenko, S. M. Torba, J. Yu, *Generalized wave polynomials and transmutations related to perturbed Bessel equations*, ArXiv: 1606.07850.

[23] B. M. Levitan, *Expansion in Fourier series and integrals with Bessel functions*, Uspekhi Mat. Nauk (N.S.) **6**, 102–143 (1951)

[24] W. Miller Jr., *Lie Theory and Special Functions.*, New York: Academic Press, 1968.

[25] W. Miller Jr., *Symmetries of differential equations. The hypergeometric and Euler - Darboux equations.*, SIAM J. Math. Anal. **4**, no. 2, 314-328, (1973).

[26] K. Okamoto, *Fundamentals of optical waveguides*, San Diego: Academic Press, 2000.

[27] F. W. J. Olver et al., *NIST Handbook of Mathematical Functions*, Cambridge University Press, Cambridge, 2010.

[28] A. S. Sohin, *On a class of transformation operators*, Trudy Fiz.-Teh. Inst. Nizkih Temp. AN USSR, Mat. Fiz., Funkts. Analiz, no. 1, 117–125 (1969) (in Russian); *English transl.* in Sel. Math. Sov. **3**, no. 3, 301–308 (1983).

[29] A. S. Sohin, *The inverse scattering problem for an equation with a singularity*, Trudy Fiz.-Teh. Inst. Nizkih Temp. AN USSR, Mat. Fiz., Funkts. Analiz, no.**2**, 182–235 (1971) (in Russian).

[30] V. Ya. Volk, *On inversion formulas for a differential equation with a singularity at $x = 0$*, Uspehi Matem. Nauk (N.S.) **8**, 141–151 (1953).

[31] J. Weidmann, *Spectral Theory of Ordinary Differential Operators*, Lecture Notes in Mathematics **1258**, Springer, Berlin, 1987.

E-mail address: `markus.holzleitner1@gmail.com`