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Spectral Deformations and Singular Weyl–Titchmarsh–Kodaira Theory for Dirac Operators

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Abstract

The aim of the present thesis is to investigate the connection between singular Weyl–Titchmarsh–Kodaira theory and the double commutation method for one-dimensional Dirac operators. In particular, we compute the singular Weyl function of the commuted operator in terms of the data from the original operator. The results obtained are then applied to radial Dirac operators in order to show that the singular Weyl function of such an operator is a generalized Nevanlinna function.

Zusammenfassung

Das Ziel der vorliegenden Arbeit ist es den Zusammenhang zwischen singulärer Weyl–Titchmarsh–Kodaira Theorie und der Doppelkommutatormethode für eindimensionale Dirac Operatoren zu untersuchen. Insbesondere berechnen wir die singuläre Weyl-Funktion des kommutierten Operators anhand der Daten des ursprünglichen Operators. Die gewonnenen Resultate wenden wir anschließend auf radiale Dirac Operatoren an, um zu zeigen, dass die singuläre Weyl-Funktion eines solchen Operators eine verallgemeinerte Nevanlinna-Funktion ist.

Contents

Introduction

The main object of this thesis will be the one-dimensional Dirac differential expression

$$\tau = \frac{1}{i}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\frac{d}{dx} + Q(x), \qquad x \in (a,b),$$

where Q denotes some locally integrable symmetric potential. Associating to τ a suitable domain in the complex Hilbert space $\mathfrak{H} = L^2((a,b),\mathbb{C}^2)$, one obtains a self-adjoint Dirac operator H. The Dirac operator is the relativistic counterpart to the non-relativistic Schrödinger operator and plays a central role in relativistic quantum mechanics. For its physical relevance we refer to any textbook covering relativistic quantum mechanics. For its mathematical properties we refer e.g. to the monographs [4], [25], [28], [32], [34], or [35]. Particularly relevant for the one-dimensional case considered here is Chapter 15 of [35] which we recommend for further background.

One approach to analyze the spectrum of such an operator is to assign an analytic function to H, the so-called Weyl m-function, such that this function contains all the spectral information of H. The corresponding classical Weyl–Titchmarsh–Kodaira theory usually assumes that at least on of the endpoints is regular (cf. Chapter 15 of [35]). However, it was shown only recently by Brunnhuber et al. [6] — based on previous work for the one-dimensional Schrödinger equation [11], [12], [13], [14], [7], [15], [16], [18], [19], [20], [21], [22], [24] — that similar results still prevail at a singular endpoint under certain assumptions.

Following [6], the key ingredient for defining such a function is a linearly independent system of real entire, with respect to the spectral parameter z, solutions $\Phi(z,x)$, $\Theta(z,x)$ of the underlying differential equation $\tau u = zu$ such that Φ is in the domain of H near the singular endpoint a (i.e. Φ is square integrable near a and satisfies the boundary conditions of H if any). Moreover, the Wronskian of Φ and Θ is has to be normalized. If such a system of solutions exists, one defines the singular Weyl function M by the requirement that the solution

$$\Psi(z,x) = \Theta(z,x) + M(z)\Phi(z,x)$$

is in the domain of H near b.

Basic properties of $M(z)$ and some key elements of Weyl–Titchmarsh–Kodaira theory for Dirac operators were established in [5], [6], [10]. The objective of the present thesis is to connect singular Weyl–Titchmarsh–Kodaira theory with the double commutation method which is used to remove or insert any finite number of prescribed eigenvalues into spectral gaps for a given operator. One obtains a deformed operator H_γ which will be unitarily equivalent to H when restricted to the complement of the subspace spanned by the newly inserted/removed eigenvalues. In the case of Schrödinger operators this method is classical and the connection with singular Weyl–Titchmarsh–Kodaira theory was investigated in [20]. For one-dimensional Dirac operators an analogous method was found in [29] and it was already used in the present singular context in a special case in [2]. The aim here is to give a systematic treatment. To this end we are going to derive a system of real entire solutions Φ_γ, Θ_γ of the deformed operator in terms of Φ and Θ. As a consequence we will be able to explicitly compute the singular Weyl function M_γ of the deformed operator in terms of M.

Finally we will apply our results to radial Dirac operators, which are given by differential expressions of the form

$$\tau = \frac{1}{i}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\frac{d}{dx} + \frac{\kappa}{x}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} + Q(x), \quad \kappa \geq 0, \quad x \in (0,b),$$

and arise during separation of variables for multi-dimensional Dirac operates with radially symmetric potentials. One of the open questions in this context is if the corresponding M function is a generalized Nevanlinna function. To answer this question we will use the fact the radial Dirac operator is limit circle at $x = 0$ for $\kappa \in [0, \frac{1}{2})$ in which case the M function is known to be a Nevanlinna function. Starting with such an operator in the limit point case and making use of the double commutation method, we will be able to decrease the angular momentum κ until a limit circle situation for the deformed operator is obtained. In this way we are able to answer the above question in the affirmative as one of the main novel results in this thesis.

Section 1 starts with preliminary facts and definitions about the Dirac differential expression. The Dirac operator is introduced in a suitable functional analytic setting. The definitions and results in this section have been collected from [5], [6], [30], [31], [33] and [35].

Section 2 introduces singular Weyl–Titchmarsh–Kodaira theory for one-dimensional Dirac operators following [6]. Hypothesis 2.1 guarantees the existence of the desired solutions Φ and Θ in order to define the singular Weyl function. The section is concluded with results about the singular Weyl function in the limit circle case, which will be used later on in Section 6.

In **Section 3** the supersymmetric Dirac operator is discussed. In this case the Dirac operator is related to so-called generalized Schrödinger operators. The given results are used in **Section 4** in order to calculate a system of solutions Φ, Θ together with the singular Weyl function for the unperturbed radial Dirac operator. The results are collected from [6], [8], [9] and [18].

Our main **Section 5** is about the connection between the double commutation method and singular Weyl–Titchmarsh–Kodaira theory for one-dimensional Dirac operators. We start this section by introducing the transformation operator which links the original operator H unitarily to the deformed operator H_γ when restricted to the complement of the subspace spanned by the newly inserted eigenvalue. Our main results are stated in Theorem 5.6 and Theorem 5.8, where we calculate the singular Weyl function of the deformed operator in terms of the original operator. Our results are strongly motivated by [20] for the case of one-dimensional Schrödinger operators. The construction of the double commutation method follows [29].

Finally in **Section 6** we apply our results from the previous section to radial Dirac operators. Lemma 6.2 shows how the angular momentum κ can be reduced until the deformed operator is limit circle at $x = 0$. Using the formulas for M and M_γ from Theorem 5.8, we are able to state Theorem 6.4 and Theorem 6.5, which tell us that the singular Weyl function of the original operator is a generalized Nevanlinna function. Lemma 6.2 and its proof rely on the observations from Section 3 of [2].

Appendix A is about Hardy-type inequalities which are used in **Appendix B**. Appendix B itself deals with properties of the regular and singular solution. The collected results are required in the proof of Lemma 6.2. The results in Appendix

A can be found in the appendix of [19]. In Appendix B we follow closely the presentation of Appendix B in [2].

Appendix C is a collection of definitions and some results about Nevanlinna functions from Appendix B in [19].

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1. Preliminaries

In this thesis we study the one-dimensional Dirac differential expression

$$\tau = \frac{1}{i}\sigma_2\frac{d}{dx} + Q(x), \qquad x \in I. \tag{1.1}$$

Here $I = (a, b)$, $-\infty \leq a < b \leq \infty$, denotes an arbitrary interval and $Q(x) \in \mathbb{R}^{2\times 2}$ is a symmetric potential matrix,

$$Q(x) = q_{el}(x)\mathbb{1} + q_{am}(x)\sigma_1 + (m + q_{sc}(x))\sigma_3, \tag{1.2}$$

together with the Pauli matrices

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{1.3}$$

Moreover, $m \in [0, \infty), q_{el}, q_{am}, q_{sc} \in L^1_{loc}(I, \mathbb{R})$ describe the mass, electrostatic, anomalous magnetic moment and scalar potential, respectively. For more details, we refer to Chapter 4 of [32]. A multi-dimensional Dirac operator with differential expression (1.1) and radial symmetric potential can be decomposed into the direct sum of radial Dirac operators, whose differential expressions are of the form

$$\tau_\kappa = \frac{1}{i}\sigma_2\frac{d}{dx} + \frac{\kappa}{x}\sigma_1 + Q(x), \quad \kappa \geq 0, \quad x \in (0, \infty), \tag{1.4}$$

where κ is called the angular momentum (see [32, Section 4.6], [35, Section 20.3]).

Under the assumptions $a > -\infty$, the potential Q entrywise integrable near a, that is, $Q_{ij}(.) \in L^1((a, c), \mathbb{R})$ $(c \in (a, b), 1 \leq i, j \leq 2)$, the Dirac differential expression is called **regular** at a. If it is not regular at a, then it is said to be **singular** at a. The same notation applies to the right endpoint b. If τ is regular at both a and b, then it is called **regular**, and **singular** in the other case. Note that in the case of the radial Dirac operator, its differential expression is singular.

Since we want to work in a functional analytic framework, we introduce the Hilbert space $\mathfrak{H} = L^2(I, \mathbb{C}^2) = \{f : I \to \mathbb{C}^2 \mid \|f\|_2 < \infty\}$ together with the inner product

$$\langle f, g\rangle = \int_a^b f(x)^* g(x)\, dx, \qquad \|f\|_2 = \sqrt{\langle f, f\rangle}, \tag{1.5}$$

where $f^* = \overline{f}^\top$ denotes conjugate transpose of the vector $f(x) \in \mathbb{C}^2$. The maximal domain on which τ is defined is given by

$$\mathfrak{D}(\tau) = \{f \in \mathfrak{H} \mid f \in AC(I, \mathbb{C}^2), \tau f \in \mathfrak{H}\}, \tag{1.6}$$

where $f \in AC(I, \mathbb{C}^2)$ if every component of the function belongs to the set of locally absolutely continuous functions

$$AC(I, \mathbb{C}) = \left\{ f \in C(I, \mathbb{C}) | f(x) = f(c) + \int_c^x g(t)\, dt,\ g \in L^1_{loc}(I, \mathbb{C}),\ c \in I \right\}. \tag{1.7}$$

For $f,g \in AC(I)$ one obtains the **Lagrange identity** $(a<c<d<b)$

$$\begin{aligned}\int_c^d (\tau f)^* g\,dx &= \int_c^d \left(\frac{1}{i}\sigma_2 f' + Qf\right)^* g\,dx = \int_c^d \left[-(f^*)'\frac{1}{i}\sigma_2 g + f^* Q g\right] dx\\ &= -f^*\frac{1}{i}\sigma_2 g\Big|_c^d + \int_c^d f^*\left(\frac{1}{i}\sigma_2 g' + Qg\right) dx\\ &= W_d(\overline{f},g) - W_c(\overline{f},g) + \int_c^d f^*\tau g\,dx, \end{aligned} \tag{1.8}$$

where

$$\begin{aligned} W_x(\overline{f},g) &= -f(x)^*\frac{1}{i}\sigma_2 g(x) = \left(\frac{1}{i}\sigma_2 f(x)\right)^* g(x)\\ &= \det\begin{pmatrix} \overline{f_1(x)} & g_1(x)\\ \overline{f_2(x)} & g_2(x)\end{pmatrix} = \overline{f_1(x)}g_2(x) - \overline{f_2(x)}g_1(x)\end{aligned} \tag{1.9}$$

is called the **Wronskian**. Two solutions of (1.12) are linearly independent if their Wronskian does not vanish. Note that the Wronskian of two solutions u,v of the homogeneous equation (1.12) does not depend on x,

$$\begin{aligned}\frac{d}{dx}W_x(u,v) &= u_1'v_2 - u_2'v_1 + u_1v_2' - u_2v_1' = \left(\frac{1}{i}\sigma_2 u'\right)^\top v - u^\top\left(\frac{1}{i}\sigma_2 v'\right)\\ &= (zu-Qu)^\top v - u^\top(zv-Qv) = 0.\end{aligned} \tag{1.10}$$

We will therefore write $W(u,v) = W_x(u,v)$ in this case.

If $f,g \in \mathfrak{D}(\tau)$, the limits $\lim_{x\to a,b} W(\overline{f},g) = W_{a,b}(\overline{f},g)$ exist, cf. [35, Theorem 15.2], and one obtains from the Lagrange identity

$$\langle \tau f, g\rangle = W_b(\overline{f},g) - W_a(\overline{f},g) + \langle f,\tau g\rangle. \tag{1.11}$$

To proceed further, we need to investigate the homogeneous Dirac equation

$$\tau u = zu, \qquad z\in\mathbb{C}. \tag{1.12}$$

We collect further results which can be found in [35].

Theorem 1.1 ([35, Corollary 15.2])**.** *Let $g \in L^1_{loc}(I,\mathbb{C}^2)$. Then for arbitrary $c\in I$, $(\alpha,\beta)^\top \in \mathbb{C}^2$ there exists a unique solution of the initial value problem*

$$(\tau - z)f = g, \quad f(c) = \begin{pmatrix}\alpha\\ \beta\end{pmatrix}, \quad z\in\mathbb{C}. \tag{1.13}$$

Moreover, for fixed $x\in I$, the solution $f(z,x)$ is an entire function with respect to the energy parameter z.

Lemma 1.2. *Suppose $g \in L^1_{loc}(I)$, u_1 and u_2 solutions of $(\tau - z)u = 0$ with $W(u_1,u_2) = 1$. Then any solution of $(\tau - z)u = g$ can be written as*

$$f(x) = u_1(x)\Big(\alpha + \int_c^x u_2^\top g\,dy\Big) + u_2(x)\Big(\beta - \int_c^x u_1^\top g\,dy\Big), \tag{1.14}$$

with $\alpha,\beta\in\mathbb{C}$, $c\in I$.

Proof.

$$\begin{aligned}
\frac{1}{i}\sigma_2 f' &= \frac{1}{i}\sigma_2 u_1'\Big(\alpha + \int u_2^\top g\,dy\Big) + \frac{1}{i}\sigma_2 u_2'\Big(\beta - \int u_1^\top g\,dy\Big)\\
&\quad + \frac{1}{i}\sigma_2(u_2^\top g u_1 - u_1^\top g u_2)\\
&= (z-Q)u_1\Big(\alpha + \int u_2^\top g\,dy\Big) + (z-Q)u_2\Big(\beta - \int u_1^\top g\,dy\Big)\\
&\quad + \frac{1}{i}\sigma_2\begin{pmatrix} g_2 \\ -g_1 \end{pmatrix}\\
&= (z-Q)f + g.
\end{aligned}$$

□

Note that the constants α, β coincide with those from Theorem 1.1 if

$$u_1(c) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad u_2(c) = \begin{pmatrix} 0 \\ 1 \end{pmatrix}. \tag{1.15}$$

Clearly, one wants to associate a self-adjoint operator to the differential expression τ. Thus, for our operator to be symmetric, we will need that the Wronskians at the endpoints vanish. In order find a proper domain for a symmetric operator, one needs to study the behavior of solutions of the homogeneous equation $(\tau - z)u = 0$ near the endpoints a, b. Similarly as in the definition of regular or singular endpoints, one calls a measurable function $f : (a,b) \to \mathbb{C}^2$ square integrable near a if $f \in L^2((a,c),\mathbb{C}^2)$ $(c \in (a,b))$ and similarly for square integrable near b.

Theorem 1.3 ([35, Theorem 15.14])**.** *Let τ be a Dirac differential expression on (a,b). If for one $z_0 \in \mathbb{C}$ all solutions of $(\tau - z_0)u = 0$ are square integrable near a, then this holds for any $z \in \mathbb{C}$. The same argument holds for the right endpoint b.*

Theorem 1.4 (Weyl alternative [35, Theorem 15.15])**.** *Let τ be a Dirac differential expression on (a,b). Then one of the following holds:*

(i) *For any $z \in \mathbb{C}$, every solution of $(\tau - z)u = 0$ is square integrable near a.*
(ii) *For any $z \in \mathbb{C}$, there exists at least one solution of $(\tau - z)u = 0$ which is not square integrable near a. In this case, there exists for any $z \in \mathbb{C}\backslash\mathbb{R}$ exactly one (up to a scalar multiple) solution of $(\tau - z)u = 0$ which is square integrable near a.*

Similarly, this holds for b.

The Dirac differential expression τ is called **limit circle** at a if case (i) occurs. It is called **limit point** at a if (ii) holds. The same definition applies for the endpoint b. Another characterization of limit circle/point can be given by the following

Theorem 1.5 ([30, Theorem 9.9])**.** *The operator τ is limit point at a if and only there exists $v \in \mathfrak{D}(\tau)$ with $W_a(\overline{v}, v) = 0$ such that $W_a(v, f) \neq 0$ for at least one $f \in \mathfrak{D}(\tau)$. The same result holds for the right endpoint b.*

The following observation explains how to define boundary conditions at a, b, respectively, in order to obtain a symmetric operator:

Lemma 1.6 ([30, Lemma 9.5])**.** *Suppose $v \in \mathfrak{D}(\tau)$ with $W_a(\overline{v}, v) = 0$ and suppose there is a $\hat{f} \in \mathfrak{D}(\tau)$ with $W_a(\overline{v}, \hat{f}) \neq 0$. Then for $f, g \in \mathfrak{D}(\tau)$ we have*

$$W_a(v, f) = 0 \quad \Leftrightarrow \quad W_a(v, \overline{f}) = 0 \tag{1.16}$$

and

$$W_a(v,f) = W_a(v,g) = 0 \quad \Rightarrow \quad W_a(\overline{g}, f) = 0. \tag{1.17}$$

Similarly this holds for the endpoint b.

Hence if the Dirac differential expression τ is limit circle at a, choose $v \in \mathfrak{D}(\tau)$ with the properties from the lemma above and set a boundary condition at a by $W_a(v,.) = 0$. Then for any $f, g \in \mathfrak{D}(\tau)$ with $W_a(v,f) = W_a(v,g) = 0$ it follows that $W_a(\overline{g}, f) = W_a(\overline{f}, g) = 0$. Note that τ will be limit point at a if and only if $W_a(f,g) = 0$ for any $f, g \in \mathfrak{D}(\tau)$. In particular, the operator will be symmetric since the Wronskians at the boundaries vanish (cf. (1.11)) if boundary conditions are specified in the limit circle case at a, b, respectively.

Moreover, if τ is limit point at both a and b, then τ gives rise to a unique self-adjoint operator when defined maximal (cf. e.g. [25],[34],[35]). Otherwise, if τ is limit circle at a and/or b, the Dirac operator

$$\begin{array}{rlcl} H: & \mathfrak{D}(H) & \to & L^2(I, \mathbb{C}^2) \\ & f & \mapsto & \tau f \end{array} \tag{1.18}$$

with

$$\begin{aligned} \mathfrak{D}(H) = \{f \in \mathfrak{D}(\tau) \,|\, & W_a(v,f) = 0 \textit{ if l.c. at } a \\ & W_b(w,f) = 0 \textit{ if l.c. at } b\} \end{aligned} \tag{1.19}$$

is self-adjoint. Here $v \in \mathfrak{D}(\tau)$ is chosen with $W_a(\overline{v}, v) = 0$ and $W_a(v,f) \neq 0$ for some $f \in \mathfrak{D}(\tau)$ if τ is limit circle at a and w in the case if τ is limit circle at b.

Theorem 1.7 ([33, Theorem A.4])**.** *The Dirac differential expression τ is*

(i) *limit circle at regular endpoints,*
(ii) *limit point at infinite endpoints.*

We briefly recall some notation about resolvents and spectra. For a densely defined closed operator $H : \mathfrak{D}(H) \to \mathfrak{H}$, its **resolvent set** is defined as

$$\rho(H) = \{z \in \mathbb{C} | (H - z) : \mathfrak{D}(H) \to \mathfrak{H} \text{ is bijective}\}. \tag{1.20}$$

The function

$$\begin{array}{rlcl} R_H: & \rho(H) & \to & \mathfrak{L}(\mathfrak{H}) \\ & z & \mapsto & (H-z)^{-1} \end{array} \tag{1.21}$$

is the **resolvent** of H. The complement of the resolvent set is called the **spectrum** of H,

$$\sigma(H) = \mathbb{C} \backslash \rho(H). \tag{1.22}$$

A point $z \in \sigma(H)$ is called an eigenvalue if $(H - z)$ has non-trivial kernel. An element $0 \neq \psi \in \mathrm{Ker}(H - z)$ is an eigenvector corresponding to the eigenvalue z. The **discrete spectrum** $\sigma_d(H)$ is the set of all eigenvalues which are isolated points of the spectrum and whose corresponding eigenspace is finite dimensional. The complement of the discrete spectrum is called the **essential spectrum** $\sigma_{ess}(H) = \sigma(H) \backslash \sigma_d(H)$.

Non-trivial solutions $u_\pm(z,x) \in AC_{loc}(I, \mathbb{C}^2)$ of $(\tau - z)u = 0$, $z \in \mathbb{C}$, are called **Weyl solutions** (whenever they exist) if they are in the domain of H near the endpoints, i.e. $u_+(z,.)$ is square integrable near b and satisfies the boundary condition at b if any and similarly $u_-(z,.)$ for the left endpoint a.

Supposing the Weyl solutions exist, the resolvent can be written in terms of $u_\pm(z,.)$,

$$(H-z)^{-1}f(x) = \int_a^b G(z,x,y)f(y)\,dy, \tag{1.23}$$

where

$$G(z,x,y) = \frac{1}{W(u_+(z),u_-(z))}\begin{cases} u_+(z,x)u_-(z,y)^\top, & y<x, \\ u_-(z,x)u_+(z,y)^\top, & y>x, \end{cases} \tag{1.24}$$

is the Green's function of H.

Now fix a point $c \in I$ and denote by $H^D_{(a,c)}$ and $H^D_{(c,b)}$ the self-adjoint restrictions of H to the intervals (a,c) and (c,b) together with a Dirichlet condition at c, $f_1(c)=0$. Let $c(z,x)$ and $s(z,x)$ be solutions of the homogeneous equation together with the initial conditions $c_1(z,c)=1$, $c_2(z,c)=0$ and $s_1(z,c)=0$, $s_2(z,c)=1$. Note that such solutions to the initial value problem exist due to Theorem 1.1. The **Weyl m-functions** $m_\pm(z)$ (corresponding to the base point c) are defined by

$$G_{c,\pm}(z,c,c) = \begin{pmatrix} 0 & \pm\frac{1}{2} \\ \pm\frac{1}{2} & m_\pm(z) \end{pmatrix}, \tag{1.25}$$

where $G_{c,\pm}(z,.,..)$ is the Green's function of $H^D_{(c,b)}$, respectively $H^D_{(a,c)}$, where $G_{c,\pm}(z,x,x) := \lim_{\varepsilon\to 0}(G_{c,\pm}(z,x+\varepsilon,x)+G_{c,\pm}(z,x-\varepsilon,x))/2$. The Weyl solutions exist if we are away from the essential spectrum of $H^D_{(a,c)}$, $H^D_{(c,b)}$, respectively. Moreover, they can be assumed to be real analytic if the whole spectrum is excluded (Lemma 1.1 of [31]). The Weyl solutions can be defined by

$$u_-(z,x) = c(z,x) - m_-(z)s(z,x), \qquad z \in \mathbb{C}\setminus\sigma(H^D_{(a,c)}), \tag{1.26}$$

$$u_+(z,x) = c(z,x) + m_+(z)s(z,x), \qquad z \in \mathbb{C}\setminus\sigma(H^D_{(c,b)}). \tag{1.27}$$

Note that the functions $m_\pm(z)$ are Herglotz–Nevanlinna functions, see Appendix C.

2. Singular Weyl-Titchmarsh-Kodaira Theory

To define an analogous singular Weyl function at the, in general, singular endpoint a, an analogous system of real entire solutions of the underlying differential equation is required. To be more precise, one needs a system of real entire solutions $\Phi(z,x)$, $\Theta(z,x)$ of the underlying homogeneous equation $(\tau - z)u = 0$, $z \in \mathbb{C}$, such that $\Phi(z,x)$ lies in the domain of H near a and $W(\Theta(z), \Phi(z)) = 1$. Needless to say, the question arises if such a suitable fundamental system of solutions exists at all. This question is answered positively if one assumes the following hypothesis.

Hypothesis 2.1. *Suppose that the spectrum of $H^D_{(a,c)}$ is purely discrete for one (and hence for all) $c \in (a,b)$.*

The following results were shown by Kostenko, Sakhnovich and Teschl in [19] in the case of Schrödinger operators and were adapted for Dirac operators by Brunnhuber, Eckhardt, Kostenko and Teschl in [5], [6]. One should also mention the work of Gesztesy and Zinchenko [14]. For an overview of the development of the singular Weyl-Titchmarsh theory, we refer to [19] and the references therein.

Lemma 2.2 ([6, Lemma 2.2])**.** *The following are equivalent:*

(i) *The spectrum of $H^D_{(a,c)}$ is purely discrete.*
(ii) *There is a real entire solution $\Phi(z,x)$ that is non-trivial and lies in the domain of H near a for each $z \in \mathbb{C}$.*
(iii) *There are real entire solutions $\Theta(z,x), \Phi(z,x)$ with $W(\Theta(z), \Phi(z)) = 1$, such that $\Phi(z,x)$ is nontrivial and lies in the domain of H near a for each $z \in \mathbb{C}$.*

Suppose Hypothesis 2.1 holds, the **singular Weyl function**

$$M(z) = -\frac{W(\Theta(z), u_+(z))}{W(\Phi(z), u_+(z))} \tag{2.1}$$

is defined such that the solution which is in the domain of H near b is given by

$$u_+(z,x) = a(z)(\Theta(z,x) + M(z)\Phi(z,x)), \tag{2.2}$$

where $a(z) = -W(\Phi(z), u_+(z))$.

Lemma 2.3 ([5],[19])**.** *The singular Weyl function is analytic in $\mathbb{C}\backslash\mathbb{R}$ and satisfies the symmetry condition $M(z) = \overline{M(\overline{z})}$.*

The constant $a(z)$ in (2.2) is of no importance and will be set to one. Hence, instead of $u_+(z,x)$, we will work with

$$\Psi(z,x) = \Theta(z,x) + M(z)\Phi(z,x). \tag{2.3}$$

We stress the fact that there is no natural choice of a fundamental system Φ and Θ. The regular solution can be multiplied by a real entire function that must not vanish, say $\tilde{\Phi}(z,x) = \mathrm{e}^{g(z)}\Phi(z,x)$ for some real entire function g. Due to the requirement that the Wronskian has to be normalized, the singular function needs to be of the form $\tilde{\Theta}(z,x) = \mathrm{e}^{-g(z)}\Theta(z,x) - f(z)\Phi(z,x)$, where f is some real entire function. This will change the Weyl function to

$$\tilde{M}(z) = \mathrm{e}^{-2g(z)}M(z) + \mathrm{e}^{g(z)}f(z). \tag{2.4}$$

Hence one can control the growth of the singular Weyl function by choosing a certain fundamental system. This is an important point about on whole theory, namely that one should always keep in mind the chosen fundamental system when speaking of *the* singular Weyl function.

The following results including detailed proofs can be found in Chapter 4 of [5].

One infers that associated with $M(z)$ is a corresponding spectral measure ρ given by the Stieltjes–Livšić inversion formula

$$\frac{1}{2}\left(\rho\big((\lambda_0,\lambda_1)\big)+\rho\big([\lambda_0,\lambda_1]\big)\right)=\lim_{\varepsilon\downarrow 0}\frac{1}{\pi}\int_{\lambda_0}^{\lambda_1}\mathrm{Im}\big(M(\lambda+\mathrm{i}\varepsilon)\big)\,d\lambda. \tag{2.5}$$

Then there exists a spectral transformation which maps the Dirac operator H in $L^2(I,\mathbb{C}^2)$ to the multiplication operator on $L^2(\mathbb{R},d\rho)$:

Theorem 2.4 ([5, Theorem 4.3])**.** *Suppose Hypothesis 2.1 and let the spectral measure ρ be given by* (2.5)*. The mapping*

$$U: L^2(I,\mathbb{C}^2)\to L^2(\mathbb{R},d\rho), \qquad f\mapsto \hat{f} \tag{2.6}$$

where $\hat{f}$ is defined by

$$\hat{f}(\lambda)=\lim_{c\uparrow b}\int_a^c \Phi_1(\lambda,x)f_1(x)+\Phi_2(\lambda,x)f_2(x)\;dx \tag{2.7}$$

is unitary and its inverse

$$U^{-1}: L^2(\mathbb{R},d\rho)\to L^2(I,\mathbb{C}^2), \qquad \hat{f}\mapsto f \tag{2.8}$$

is given by

$$f(x)=\lim_{r\to\infty}\int_{-r}^{r}\Phi(\lambda,x)\hat{f}(\lambda)\,d\rho(\lambda)=\lim_{r\to\infty}\begin{pmatrix}\int_{-r}^{r}\Phi_1(\lambda,x)\hat{f}(\lambda)\,d\rho(\lambda)\\ \int_{-r}^{r}\Phi_2(\lambda,x)\hat{f}(\lambda)\,d\rho(\lambda)\end{pmatrix}. \tag{2.9}$$

Moreover, U maps H to multiplication by λ. Note that the right-hand sides of (2.7) *and* (2.9) *are to be understood as limits in $L^2(\mathbb{R},d\rho)$ and $L^2(I,\mathbb{C}^2)$, respectively.*

Hence the spectral types of H can be read off from the boundary behavior of the singular Weyl function:

Corollary 2.5 ([5, Corollary 4.4])**.** *Consider the following sets*

$$\Sigma_{ac}=\{\lambda\,|\,0<\limsup_{\varepsilon\downarrow 0}\mathrm{Im}(M(\lambda+\mathrm{i}\varepsilon))<\infty\}, \tag{2.10}$$

$$\Sigma_{s}=\{\lambda\,|\,\limsup_{\varepsilon\downarrow 0}\mathrm{Im}(M(\lambda+\mathrm{i}\varepsilon))=\infty\}, \tag{2.11}$$

$$\Sigma_{p}=\{\lambda\,|\,\lim_{\varepsilon\downarrow 0}\varepsilon\mathrm{Im}(M(\lambda+\mathrm{i}\varepsilon))>0\}, \tag{2.12}$$

$$\Sigma=\Sigma_{ac}\cup\Sigma_s=\{\lambda\,|\,0<\limsup_{\varepsilon\downarrow 0}\mathrm{Im}(M(\lambda+\mathrm{i}\varepsilon))\}. \tag{2.13}$$

The spectrum of H is given by the closure of Σ,

$$\sigma(H)=\overline{\Sigma}, \tag{2.14}$$

the point spectrum (the set of eigenvalues) is given by Σ_p,

$$\sigma_p(H)=\Sigma_p, \tag{2.15}$$

and the absolutely continuous spectrum is given by the essential closure of Σ_{ac},

$$\sigma_{ac}(H)=\overline{\Sigma}_{ac}^{ess}. \tag{2.16}$$

Recall that $\overline{\Omega}^{ess}=\{\lambda\in\mathbb{R}\,|\,|(\lambda-\varepsilon,\lambda+\varepsilon)\cap\Omega|>0$ for all $\varepsilon>0\}$ where $|\Omega|$ denotes the Lebesgue measure of a Borel set Ω.

The singular Weyl function gives rise to a spectral measure ρ via the Stieltjes–Livšić inversion formula. On the other hand, $M(z)$ can be reconstructed from ρ up to an entire function.

Theorem 2.6 ([19, Theorem 4.1])**.** *Let $M(z)$ be a singular Weyl function and ρ its associated spectral measure. Then there exists an entire function $g(z)$ such that $g(\lambda) \geq 0$ for $\lambda \in \mathbb{R}$ and $\mathrm{e}^{-g(\lambda)} \in L^2(\mathbb{R}, d\rho)$.*

Moreover, for any entire function $\hat{g}(z)$ such that $\hat{g}(\lambda) > 0$ for $\lambda \in \mathbb{R}$ and $(1+\lambda^2)^{-1}\hat{g}(\lambda)^{-1} \in L^1(\mathbb{R}, d\rho)$ (e.g. $\hat{g}(z) = \mathrm{e}^{2g(z)}$) we have the integral representation

$$M(z) = E(z) + \hat{g}(z) \int_{\mathbb{R}} \left(\frac{1}{\lambda - z} - \frac{\lambda}{1+\lambda^2} \right) \frac{d\rho(\lambda)}{\hat{g}(\lambda)}, \qquad z \in \mathbb{C}\backslash\sigma(H), \tag{2.17}$$

where $E(z)$ is a real entire function.

The following theorem gives a criterion when the singular Weyl function belongs to the class $\mathcal{N}_\kappa^\infty$ of generalized Nevanlinna functions with no non-real poles and the only generalized pole of nonpositive type at ∞, see Appendix C.

Theorem 2.7 ([19, Theorem 4.3])**.** *Fix the solution $\Phi(z,x)$. Then there is a corresponding solution $\Theta(z,x)$ such that $M(z) \in \mathcal{N}_\kappa^\infty$ for some $\kappa \leq k$ if and only if $(1+\lambda^2)^{-k-1} \in L^1(\mathbb{R}, d\rho)$. Moreover, $\kappa = k$ if $k = 0$ or $(1+\lambda^2)^{-k} \notin L^1(\mathbb{R}, d\rho)$.*

When the operator H happens to be limit circle at a singular endpoint a, then the Weyl function is even a Herglotz–Nevanlinna function. This result is of particular importance for our applications to radial Dirac operators later on in Section 6.

Theorem 2.8 ([6, Theorem 5.7])**.** *Let H be some self-adjoint Dirac operator associated with τ limit circle at a. Then the singular Weyl function defined in* (2.3) *is a Herglotz–Nevanlinna function and satisfies*

$$\mathrm{Im}(M(z)) = \mathrm{Im}(z) \int_a^b |\Psi(z,x)|^2 dx. \tag{2.18}$$

Corollary 2.9 ([6, Corollary 5.9])**.** *Suppose the same assumptions as in Theorem 2.8. Then we have*

$$M(z) = \mathrm{Re}(M(\mathrm{i})) + \int_{\mathbb{R}} \left(\frac{1}{\lambda - z} - \frac{\lambda}{1+\lambda^2} \right) d\rho(\lambda), \tag{2.19}$$

where ρ satisfies $\int_{\mathbb{R}} d\rho = \infty$ and $\int_{\mathbb{R}} \frac{d\rho(\lambda)}{1+\lambda^2} < \infty$.

3. The Supersymmetric Case

Following Section 3 of [6], we first discuss the supersymmetric Dirac operator where everything can be computed explicitly. In this case $q_{el} = q_{sc} = 0$ and the differential expression reads

$$\tau = \begin{pmatrix} m & \tau_- \\ \tau_+ & -m \end{pmatrix}, \tag{3.1}$$

where $\tau_\pm = \pm\frac{d}{dx} + q(x)$ for some real-valued $q \in L^1_{loc}(I)$. Now consider the operators

$$\begin{aligned} &\mathfrak{D}(A_\pm) = \{f \in L^2(I,\mathbb{C}) \,|\, f \in AC_{loc}(I,\mathbb{C}), \tau_\pm f \in L^2(I)\}, \\ &A_\pm f = \tau_\pm f, \end{aligned} \tag{3.2}$$

and $\mathfrak{D}(A_{0,\pm}) = \mathfrak{D}(A_\pm) \cap AC_c(I,\mathbb{C})$, $A_{0,\pm} f = \tau_\pm f$, where $AC_c(I,\mathbb{C})$ denotes the set of absolutely continuous functions with compact support in I. Then one obtains, by following the arguments in [30, Lemma 9.4], that $A^*_{0,\pm} = A_\mp$ and

$$\mathfrak{D}(\overline{A_{0,\pm}}) = \{f \in \mathfrak{D}(A_\pm) \,|\, \lim_{x\to a,b} f(x)g(x) = 0 \,\forall\, g \in \mathfrak{D}(A_\mp)\}. \tag{3.3}$$

Moreover, A_- is closed and the Dirac operator

$$H = \begin{pmatrix} m & A_- \\ A_-^* & -m \end{pmatrix}, \qquad \mathfrak{D}(H) = \mathfrak{D}(\overline{A_{0,+}}) \oplus \mathfrak{D}(A_-), \tag{3.4}$$

is self-adjoint because it is a bounded perturbation of the self-adjoint operator (3.4) with $m = 0$. Next we want to look for solutions of the homogeneous equation $(\tau - z)u = 0$, $z \in \mathbb{C}$. This system of equations is equivalent to the systems

$$\begin{aligned} \begin{cases} u_1 = (z-m)^{-1}\tau_- u_2 \\ u_2 = (z+m)^{-1}\tau_+ u_1 \end{cases} &\Leftrightarrow \begin{cases} \tau_+\tau_- u_2 = (z^2-m^2)u_2 \\ u_1 = (z-m)^{-1}\tau_- u_2 \end{cases} \\ &\Leftrightarrow \begin{cases} \tau_-\tau_+ u_1 = (z^2-m^2)u_1 \\ u_2 = (z+m)^{-1}\tau_+ u_1 \end{cases}. \end{aligned} \tag{3.5}$$

In order to find a system of entire solutions for the Dirac operator H, we make a detour over H^2,

$$H^2 = \begin{pmatrix} A_-A_-^* + m^2 & 0 \\ 0 & A_-^*A_- + m^2 \end{pmatrix}. \tag{3.6}$$

The self-adjoint operators $A_-A_-^*$ and $A_-^*A_-$ are so-called generalized Schrödinger operators, cf. [8], [9]. For the mapping $x \mapsto f(x) = x^2$ one infers by spectral mapping $\sigma(f(H)) = f(\sigma(H))$. Hence the Hypothesis 2.1 will hold if and only if it holds for $A_-A_-^*$ or $A_-^*A_-$. By Theorem 8.4 of [9], a fundamental system of entire solutions $\phi(\zeta,x), \theta(\zeta,x)$ of the equation $(\tau_-\tau_+ - \zeta)u = 0$ exists such that $\phi(\zeta,x)$ is in the domain of $A_-A_-^*$ near a and $W(\theta(z),\phi(z)) = 1$. Hence one obtains a system for H by virtue of (3.5),

$$\Phi(z,x) = \begin{pmatrix} (z+m)\phi(z^2-m^2,x) \\ \tau_+\phi(z^2-m^2,x) \end{pmatrix}, \quad \Theta(z,x) = \begin{pmatrix} \theta(z^2-m^2,x) \\ \frac{1}{z+m}\tau_+\theta(z^2-m^2,x) \end{pmatrix}. \tag{3.7}$$

Note that the solutions satisfy $W(\Theta(z),\Phi(z)) = 1$.

Theorem 3.1 ([6, Theorem 3.1])**.** *Let the Dirac operator H be given by* (3.4)*. Suppose the restriction of the Schrödinger-type operator $A_-A_-^*$ to (a,c), together with a Dirichlet boundary condition at c, has purely discrete spectrum. Then H*

satisfies Hypothesis 2.1 and the singular Weyl function with the fundamental system (3.7) *is given by*

$$M(z) = \frac{m_q(z^2 - m^2)}{z + m}, \tag{3.8}$$

where $m_q(\zeta)$ *is the singular Weyl function of* $A_- A_-^*$.

4. The Unperturbed Radial Dirac Operator

We now turn to the analysis of the unperturbed radial Dirac operator, whose differential expression is given by

$$\tau_\kappa = \frac{1}{i}\sigma_2 \frac{d}{dx} + \frac{\kappa}{x}\sigma_1 + m\sigma_3 = \begin{pmatrix} m & \tau_- \\ \tau_+ & -m \end{pmatrix}, \quad \kappa \geq 0, \quad x \in (0,\infty), \tag{4.1}$$

where $\tau_\pm = \pm\frac{d}{dx} + \frac{\kappa}{x}$. Since

$$\tau_-\tau_+ u = \left(-\frac{d}{dx} + \frac{\kappa}{x}\right)\left(\frac{d}{dx} + \frac{\kappa}{x}\right) u = -u'' + \frac{\kappa(\kappa+1)}{x^2} u, \tag{4.2}$$

and by virtue of Theorem 3.1, the analysis of a fundamental system of real entire solutions for the unperturbed radial Dirac operator boils down to the investigation of the Bessel equation

$$-u'' + \frac{l(l+1)}{x^2} u = \zeta u, \tag{4.3}$$

also known an the unperturbed radial Schrödinger operator. This operator is limit circle at $x = 0$ for $l \in [-1/2, 1/2)$ and limit point for $l \geq 1/2$. We just state the final results, following Section 4 of [6].

Two suitable real entire solutions of (4.3) with normalized Wronskian are given by

$$\phi_l(\zeta, x) = \zeta^{-\frac{2l+1}{4}} \sqrt{\frac{\pi x}{2}} J_{l+\frac{1}{2}}(\sqrt{\zeta} x), \tag{4.4}$$

$$\theta_l(\zeta, x) = -\zeta^{\frac{2l+1}{4}} \sqrt{\frac{\pi x}{2}} \begin{cases} \frac{-1}{\sin((l+\frac{1}{2})\pi)} J_{-l-\frac{1}{2}}(\sqrt{\zeta} x), & l + \frac{1}{2} \in \mathbb{R}_+ \setminus \mathbb{N}_0, \\ Y_{l+\frac{1}{2}}(\sqrt{\zeta} x) - \frac{1}{\pi}\log(\zeta) J_{l+\frac{1}{2}}(\sqrt{\zeta} x), & l + \frac{1}{2} \in \mathbb{N}_0, \end{cases} \tag{4.5}$$

where J_ν and Y_ν are the usual Bessel and Neumann functions [27].

The singular Weyl function is

$$m_l(\zeta) = \begin{cases} \frac{-1}{\sin((l+\frac{1}{2})\pi)} (-\zeta)^{l+\frac{1}{2}}, & l + \frac{1}{2} \in \mathbb{R}_+ \setminus \mathbb{N}_0, \\ \frac{-1}{\pi} z^{l+\frac{1}{2}} \log(-\zeta), & l + \frac{1}{2} \in \mathbb{N}_0. \end{cases} \tag{4.6}$$

Inserting

$$\tau_+\phi_\kappa(\zeta, x) = \zeta^{-\frac{2\kappa-1}{4}} \sqrt{\frac{\pi x}{2}} J_{\kappa-\frac{1}{2}}(\sqrt{\zeta} x) = \begin{cases} \phi_{\kappa-1}(\zeta, x), & \kappa \geq \frac{1}{2}, \\ \cos(\pi\kappa)\theta_{-\kappa}(\zeta, x), & \kappa \in [0, \frac{1}{2}), \end{cases} \tag{4.7}$$

as well as

$$\tau_+\theta_\kappa(\zeta, x) = \begin{cases} \zeta\theta_{\kappa-1}(\zeta, x), & \kappa \geq \frac{1}{2}, \\ \frac{\zeta}{\cos(\pi\kappa)} \phi_{-\kappa}(\zeta, x), & \kappa \in [0, \frac{1}{2}), \end{cases} \tag{4.8}$$

into (3.7), $\zeta = z^2 - m^2$, one obtains a system of solutions $\Phi_\kappa(z, x)$ and $\Theta_\kappa(z, x)$, which are real entire with respect to z. Finally, the singular Weyl function, defined by

$$\Psi_\kappa(z, x) = \Theta_\kappa(z, x) + M_\kappa(z)\Phi_\kappa(z, x) \in L^2((1,\infty), \mathbb{C}^2), \tag{4.9}$$

is given by

$$M_\kappa(z) = \frac{1}{z+m} m_\kappa(z^2 - m^2), \quad z \in \mathbb{C} \setminus (-\infty, -m] \cup [m, \infty). \tag{4.10}$$

The associated spectral measure is given by

$$d\rho_\kappa(\lambda) = \chi_{(-\infty,-m]\cup[m,\infty)}(\lambda)\frac{|\lambda^2 - m^2|^{\kappa+1/2}}{|\lambda| + m}\frac{d\lambda}{\pi}. \tag{4.11}$$

By Theorem 2.7, one infers that $M_\kappa(z)$ is in the generalized Nevanlinna class $\mathcal{N}_{\kappa_0}^\infty$ with $\kappa_0 = \lfloor \kappa + 1/2 \rfloor$.

5. The Double Commutation Method

We are going to give a description on the double commutation method for one-dimensional Dirac operators. This method is used to remove or insert any finite number of prescribed eigenvalues into spectral gaps for a given operator. We follow the notation in [29] and point out the references therein. We will begin with the construction of a transformation operator which will link the original and deformed operator up to unitarily equivalence when restricted to the complement of the subspace spanned by the newly inserted eigenvalue.

Let us begin with some preliminary definitions. Let $\mathfrak{H}_1, \mathfrak{H}_2$ be two Hilbert spaces. A bounded linear operator $U : \mathfrak{H}_1 \to \mathfrak{H}_2$ is called **unitary** if it is onto and preserves the inner product. By the polarization identity, this is equivalent to preserving the norms: $\|U\psi\|_2 = \|\psi\|_1$ for all $\psi \in \mathfrak{H}_1$. Note that a unitary operator is automatically bijective. If there exists such a unitary operator, then the two Hilbert spaces are said to be **unitarily equivalent**. Two linear operators

$$
\begin{array}{rlcl} A_1 : & \mathfrak{H}_1 \supseteq \mathfrak{D}(A_1) & \to & \mathfrak{H}_1 \\ A_2 : & \mathfrak{H}_2 \supseteq \mathfrak{D}(A_2) & \to & \mathfrak{H}_2 \end{array} \tag{5.1}
$$

are called **unitarily equivalent** if there exists a unitary operator $U : \mathfrak{H}_1 \to \mathfrak{H}_2$ such that

$$
UA_1 = A_2 U \quad \text{and} \quad U\mathfrak{D}(A_1) = \mathfrak{D}(A_2). \tag{5.2}
$$

For a general setting, let $n \in \mathbb{N}$, $I = (a, b)$ with $-\infty \le a < b \le \infty$ and let $K \in L^1_{loc}(I, \mathbb{R}^{n\times n})$ be a positive definite n by n matrix. Choose the Hilbert space $\mathfrak{H} = L^2(I, \mathbb{C}^n; K\,dx)$ together with the inner product

$$
\langle f, g\rangle = \int_a^b f(t)^* K(t) g(t)\,dt, \quad \|f\|^2 = \langle f, f\rangle. \tag{5.3}
$$

Denote by $\mathfrak{H}_-$ the set of locally square-integrable functions which are in $\mathfrak{H}$ near a,

$$
\mathfrak{H}_- = \{f \in L^2_{loc}(I, \mathbb{C}^n; K\,dx) \,|\, f \in L^2((a, c), \mathbb{C}^n; K\,dx)\}, \quad c \in I. \tag{5.4}
$$

Similarly, $\mathfrak{H}_+$ is defined as the set of locally square-integrable functions in $\mathfrak{H}$ near b. Now consider a fixed element $u \in \mathfrak{H}_-$, choose $\gamma \in [-\|u\|^{-2}, \infty) \cup \{\infty\}$ and define the real-valued functions on I:

$$
\begin{array}{rcl} c_\gamma(x) & = & \frac{1}{\gamma} + \langle u, u\rangle_a^x, \quad \gamma \neq 0, \\ c_\infty(x) & = & \langle u, u\rangle_a^x. \end{array} \tag{5.5}
$$

Here we have abbreviated

$$
\langle f, g\rangle_y^x = \int_y^x f(t)^* K(t) g(t)\,dt. \tag{5.6}
$$

Set

$$
u_\gamma(x) = \frac{u(x)}{c_\gamma(x)}, \quad u_0 = 0, \tag{5.7}
$$

and define the linear transformation

$$
\begin{array}{llcl} U_\gamma : & \mathfrak{H} & \to & L^2_{loc}(I, \mathbb{C}^n; K\,dx) \\ & f(x) & \mapsto & f(x) - u_\gamma(x)\langle u, f\rangle_a^x, \\ U_0 = \mathbb{1}. & & & \end{array} \tag{5.8}
$$

Note that due to

$$u_\gamma(x)^* K(x) u_\gamma(x) = -\frac{d}{dx}\frac{1}{c_\gamma(x)}, \tag{5.9}$$

one gets

$$\|u_\gamma\|^2 = \begin{cases} \gamma, & u \notin \mathfrak{H}, \\ \frac{\gamma^2\|u\|^2}{1+\gamma\|u\|^2}, & u \in \mathfrak{H}. \end{cases} \tag{5.10}$$

The inverse transformation reads

$$\begin{array}{rccl} U_\gamma^{-1}: & \mathfrak{H} & \to & L^2_{loc}(I,\mathbb{C}^n; K\,dx) \\ & g(x) & \mapsto & \begin{cases} g(x) + u(x)\langle u_\gamma, g\rangle_a^x, & \gamma \in \mathbb{R} \\ g(x) - u(x)\langle u_\infty, g\rangle_x^b, & \gamma = \infty \end{cases} \end{array} \tag{5.11}$$

We point out that U_γ can also be defined on $\mathfrak{H}_-$. This extension will again be denoted by U_γ.

Denote by P, P_γ the orthogonal projections onto the one-dimensional subspaces of $\mathfrak{H}$ spanned by u, u_γ (set $P, P_\gamma = 0$ if $u, u_\gamma \notin \mathfrak{H}$). The results in Section 2 of [29] can be summarized in the following

Lemma 5.1 ([29, Lemma 2.1])**.** *The element u_γ fulfills*

$$u_\gamma \in \mathfrak{H} \Leftrightarrow -\|u\|^{-2} < \gamma < \infty, \tag{5.12}$$

$$\begin{aligned} \gamma = -\|u\|^{-2} \quad &\Rightarrow \quad u_\gamma \in \mathfrak{H}_-, \\ \gamma = \infty \quad &\Rightarrow \quad u_\gamma \in \mathfrak{H}_+. \end{aligned} \tag{5.13}$$

The operator U_γ is unitary from $(\mathbb{1} - P)\mathfrak{H}$ onto $(\mathbb{1} - P_\gamma)\mathfrak{H}$ with inverse U_γ^{-1}. If $P, P_\gamma \neq 0$, then U_γ can be extended to a unitary transformation $\tilde{U}_\gamma$ on $\mathfrak{H}$ by

$$\tilde{U}_\gamma = U_\gamma(\mathbb{1} - P_\gamma) + \sqrt{1+\gamma\|u\|^2}\, U_\gamma P_\gamma. \tag{5.14}$$

Moreover we point out the following identity,

$$\langle f_\gamma, g_\gamma\rangle_a^x = \langle f, g\rangle_a^x - \frac{\langle f, u\rangle_a^x \langle u, g\rangle_a^x}{c_\gamma(x)}, \tag{5.15}$$

where $f_\gamma = U_\gamma f$, $g_\gamma = U_\gamma g$.

We will now turn to the deformed Dirac operator which will be linked to the original operator by the unitary transformation introduced above. In order to establish a connection between the unitary transformation and the Dirac operator, we are going to assume the following

Hypothesis 5.2. *Suppose $(\lambda, \gamma) \in \mathbb{R}^2$ satisfies the following conditions:*

(i) *The Weyl solution $u_-(\lambda, x)$ exists.*
(ii) $\gamma \in [-\|u_-(\lambda)\|^{-2}, \infty) \cup \{\infty\}$.
(iii) *If $u_-(\lambda) \in \mathfrak{H}$, then $\lambda \in \sigma_p(H)$.*

Recall that $u_-(\lambda, x)$ exists for $\lambda \notin \sigma(H^D_{(a,c)})$ (Lemma 1.1 of [31]). The unitary transformation reads as follows:

$$\begin{array}{rccl} U_\gamma: & (\mathbb{1} - P(\lambda))\mathfrak{H} & \to & (\mathbb{1} - P_\gamma(\lambda))\mathfrak{H} \\ & f(x) & \mapsto & f(x) - u_{\gamma,-}(\lambda, x)\langle u_-(\lambda), f\rangle_a^x, \end{array} \tag{5.16}$$

$$u_{\gamma,-}(\lambda, x) = \frac{u_-(\lambda, x)}{c_\gamma(\lambda, x)}, \qquad c_\gamma(\lambda, x) = \frac{1}{\gamma} + \langle u_-(\lambda), u_-(\lambda)\rangle_a^x. \tag{5.17}$$

Subject of the following investigation will be the deformed Dirac operator,

$$H_\gamma f = \tau_\gamma f := (\tau + Q_\gamma) f, \quad \mathfrak{D}(H_\gamma) = \{ \begin{array}{l} f \in \mathfrak{H} | \, f \in AC_{loc}(I, \mathbb{C}^2); \tau_\gamma f \in \mathfrak{H}; \\ W_a(u_{\gamma,-}(\lambda), f) = W_b(u_{\gamma,-}(\lambda), f) = 0\}, \end{array} \tag{5.18}$$

$$\begin{aligned} Q_\gamma(x) &= \frac{1}{c_\gamma(\lambda,x)} \left[\frac{1}{i}\sigma_2 u_-(\lambda,x) u_-(\lambda,x)^\top - u_-(\lambda,x) u_-(\lambda,x)^\top \frac{1}{i}\sigma_2 \right] \\ &= \frac{u_{-,1}(\lambda,x)^2 - u_{-,2}(\lambda,x)^2}{c_\gamma(\lambda,x)} \sigma_1 - 2 \frac{u_{-,1}(\lambda,x) u_{-,2}(\lambda,x)}{c_\gamma(\lambda,x)} \sigma_3. \end{aligned} \tag{5.19}$$

Note that the electrostatic potential $q_{\rm el}$ does not change. A Dirac operator with $q_{\rm el} = 0$ in (1.2) is said to be in Ablowitz-Kaup-Newell-Segur (AKNS) normal form [1]. Hence the deformed operator will be in AKNS normal form if the original operator is.

Combining

$$\begin{aligned} \tau u_{\gamma,-}(\lambda) &= \tau \frac{u_-(\lambda)}{c_\gamma(\lambda)} \\ &= \lambda u_{\gamma,-}(\lambda) + \frac{1}{i}\sigma_2 \Big(\frac{1}{c_\gamma(\lambda)}\Big)' u_-(\lambda) \\ &= \lambda u_{\gamma,-}(\lambda) - \frac{1}{i}\sigma_2 \frac{u_-(\lambda)^\top u_-(\lambda)}{c_\gamma(\lambda)^2} u_-(\lambda) \\ &= \lambda u_{\gamma,-}(\lambda) - \frac{1}{c_\gamma(\lambda)} \frac{1}{i}\sigma_2 u_-(\lambda) u_-(\lambda)^\top u_{\gamma,-}(\lambda), \end{aligned} \tag{5.20}$$

together with the identity

$$u_-(\lambda)^\top \frac{1}{i}\sigma_2 u_-(\lambda) = -W(u_-(\lambda), u_-(\lambda)) = 0, \tag{5.21}$$

yields

$$\tau_\gamma u_{\gamma,-} = \lambda u_{\gamma,-}(\lambda). \tag{5.22}$$

By (5.12) it follows that the differential expression τ_γ will be limit point at a, b if $\gamma = \infty$, $\gamma = -\|u_-(\lambda)\|^{-2}$, respectively.

We are now going to shed some light on what we have pointed out in the beginning of this section, namely the unitary correspondence between the original Dirac operator H and its deformed counterpart H_γ. Let $P_\gamma(\lambda)$ be the orthogonal projection on the one-dimensional subspace spanned by $u_{\gamma,-}(\lambda)$. Then the original and deformed operator are unitarily equivalent when restricted to $(\mathbb{1} - P_\gamma(\lambda))\mathfrak{H}$ (Theorem 3.2 of [29]). In order to show this, one needs to check that the domains $(\mathbb{1} - P_\gamma(\lambda)) U_\gamma \mathfrak{D}(H)$ and $(\mathbb{1} - P_\gamma(\lambda))\mathfrak{D}(H_\gamma)$ coincide and

$$H_\gamma(\mathbb{1} - P_\gamma(\lambda)) = U_\gamma H U_\gamma^{-1}(\mathbb{1} - P_\gamma(\lambda)). \tag{5.23}$$

Let us turn first to the differential expressions. For $f \in \mathfrak{D}(H)$ calculate

$$\begin{aligned}\tau_\gamma(U_\gamma f)(x) &= \tau_\gamma(f(x) - u_{\gamma,-}(\lambda,x)\langle u_-(\lambda), f\rangle_a^x)\\ &= \tau f(x) + Q_\gamma(x) f(x) - \frac{1}{i}\sigma_2 u_-(\lambda,x)^\top f(x) u_{\gamma,-}(\lambda,x)\\ &\quad - \lambda u_{\gamma,-}(\lambda,x)\langle u_-(\lambda), f\rangle_a^x\\ &= \tau f(x) - \frac{1}{c_\gamma(\lambda,x)} u_-(\lambda,x) u_-(\lambda,x)^\top \frac{1}{i}\sigma_2 f(x)\\ &\quad - u_{\gamma,-}(\lambda,x)\langle \tau u_-(\lambda), f\rangle_a^x\\ &= \tau f(x) - u_{\gamma,-}(\lambda,x)\langle u_-(\lambda), \tau f\rangle_a^x\\ &= U_\gamma(\tau f)(x). \end{aligned} \tag{5.24}$$

Here the Lagrangian identity has been used together with $f \in \mathfrak{D}(H)$,

$$\begin{aligned}\langle \tau u_-(\lambda), f\rangle_a^x &= W_x(u_-(\lambda), f) - W_a(u_-(\lambda), f) + \langle u_-(\lambda), \tau f\rangle_a^x\\ &= -u_-(\lambda,x)^\top \frac{1}{i}\sigma_2 f(x) + \langle u_-(\lambda), \tau f\rangle_a^x. \end{aligned} \tag{5.25}$$

Notice that for this calculation we only needed that f satisfies the boundary condition of H at the left endpoint a. This shows that if $u_-(z) \in \mathfrak{H}_-$ is a solution of $\tau u = zu$ which satisfies the boundary condition at a (if τ is limit circle at a), then $\tau_\gamma U_\gamma u_-(z) = z U_\gamma u_-(z)$.

We have seen that the differential expressions transform properly under U_γ. Now let us turn to the domains, that is, we need to check the boundary conditions at both endpoints. Let $f \in \mathfrak{D}(H)$ and note that

$$W_x(u_{\gamma,-}(\lambda), U_\gamma f) = W_x(u_{\gamma,-}(\lambda), f) = \frac{W_x(u_\gamma(\lambda), f)}{c_\gamma(\lambda,x)} \tag{5.26}$$

holds for all $\gamma \in [-\|u_-(\lambda)\|^{-2}, \infty) \cup \{\infty\}$, $x \in (a,b)$. One can assume $\gamma \neq \infty$ since we have already seen that τ_γ will be limit point at a in that case. Henceforth the denominator does not vanish at $x = a$ and thus $W_a(u_{\gamma,-}(\lambda), U_\gamma f) = 0$. For the right endpoint b one can assume $u \neq -\|u_-(\lambda)\|^{-2}$ since τ_γ will be limit point at b. Distinguish two cases: If $u_-(\lambda) \in \mathfrak{H}$ then λ is an eigenvalue by assumption and therefore $u_-(\lambda)$ satisfies the boundary condition of H at b if H is limit circle at b. Again the denominator does not vanish at $x = b$ and one obtains $W_b(u_{\gamma,-}(\lambda), U_\gamma f) = 0$. If $u_-(\lambda) \notin \mathfrak{H}$ use

$$|W_x(u_{\gamma,-}(\lambda), U_\gamma f)|^2 = \frac{|\langle u_-(\lambda), (\tau - \lambda) f\rangle_a^x|^2}{c_\gamma(\lambda,x)^2}, \tag{5.27}$$

which tends to zero for $f \in \mathfrak{D}(H)$ as $x \to b$. These considerations reveal

$$(\mathbb{1} - P_\gamma(\lambda)) U_\gamma \mathfrak{D}(H) \subseteq (\mathbb{1} - P_\gamma(\lambda))\mathfrak{D}(H_\gamma). \tag{5.28}$$

By the property of self-adjoint operators being maximal, $(\mathbb{1} - P_\gamma(\lambda)) U_\gamma \mathfrak{D}(H)$ cannot be properly contained in $(\mathbb{1} - P_\gamma(\lambda))\mathfrak{D}(H_\gamma)$. Thus we have shown (5.23).

We point out that this result explains the procedure on how to remove or insert a single eigenvalue:

Corollary 5.3 ([29, Corollary 3.3])**.** *Suppose $u_-(\lambda) \notin \mathfrak{H}$.*

(i) *If $\gamma > 0$, then H and $(\mathbb{1} - P_\gamma(\lambda))H_\gamma$ are unitarily equivalent. Moreover, H_γ has the additional eigenvalue λ with eigenfunction $u_{\gamma,-}(\lambda)$.*

(ii) *If $\gamma = \infty$, then H and H_γ are unitarily equivalent.*

Suppose $u_-(\lambda) \in \mathfrak{H}$ (i.e., λ is an eigenvalue of H).

(i) *If $\gamma \in (-\|u_-(\lambda)\|^{-2}, \infty)$, then H and H_γ are unitarily equivalent (using $\tilde{U}_\gamma$).*

(ii) *If $\gamma = -\|u_-(\lambda)\|^{-2}$ or ∞, then $(\mathbb{1} - P(\lambda))H$, H_γ are unitarily equivalent, that is, the eigenvalue λ is removed.*

Remark 5.4. *All the previous considerations still hold if one commutes the operator at the right endpoint b instead of the endpoint a. One just needs to interchange the following roles:*

$$\begin{array}{c} \langle ., .. \rangle_a^x \\ u_-(\lambda) \in \mathfrak{H}_- \\ u_{\gamma,-}(\lambda, x) \\ Q_\gamma \end{array} \quad \Longleftrightarrow \quad \begin{array}{c} \langle ., .. \rangle_x^b \\ u_+(\lambda) \in \mathfrak{H}_+ \\ u_{\gamma,+}(\lambda, x) \\ -Q_\gamma \end{array}, \tag{5.29}$$

Also Hypothesis 5.2 needs to be adjusted properly. In addition, note that commuting from the right side ensues τ_γ is limit point at a, b if $\gamma = -\|u\|^{-2}, \infty$, respectively.

Note that depending on the choice of γ, τ_γ might not preserve limit point, limit circle conditions at the endpoints already noticed above. In particular, one has, cf. [29, Theorem 3.7]:

- For $\gamma < \infty$, τ_γ is limit circle at a if and only if τ is.
- For $\gamma > 0$, τ_γ is limit point at b if and only if τ is.

We are now almost ready to connect the double commutation method with singular Weyl-Titchmarsh-Kodaira theory, i.e., we are going to compute the singular Weyl function of the deformed operator in terms of the original operator. Before we proceed, we need some preliminary identities.

Lemma 5.5 ([29, Lemma 3.4])**.** *Let $u \in AC_{loc}(I, \mathbb{C}^2)$ fulfill $\tau u = zu$, $z \in \mathbb{C}\backslash\{\lambda\}$, and let*

$$v_\pm(z, x) = u(z, x) \pm \frac{u_{\gamma,\pm}(\lambda, x)}{\lambda - z} W_x(u_\pm(\lambda), u(z)). \tag{5.30}$$

Then $v_\pm \in AC_{loc}(I, \mathbb{C}^2)$ and $v_\pm$ fulfills $\tau_\gamma v_\pm = z v_\pm$. If u is square integrable near b, a and fulfills the boundary condition at b, a, respectively, (if any) we have $v_\pm = U_\gamma u$. We also note

$$|v_\pm(z, x)|^2 = |u(z, x)|^2 \pm \frac{1}{|\lambda - z|^2} \frac{d}{dx} \left(\frac{|W_x(u_\pm(\lambda), u(z))|^2}{c_\gamma(\lambda, x)} \right), \tag{5.31}$$

and if $\hat{u}$, $\hat{v}$ are constructed analogously, then

$$\begin{aligned} W_x(v_\pm(z), \hat{v}_\pm(\hat{z})) = W_x(u(z), \hat{u}(\hat{z})) \pm \frac{1}{c_\gamma(\lambda, x)} \times \\ \frac{z - \hat{z}}{(\lambda - z)(\lambda - \hat{z})} W_x(u_\pm(\lambda), u(z)) W_x(u_\pm(\lambda), \hat{u}(\hat{z})). \end{aligned} \tag{5.32}$$

In addition, the solutions

$$u_{\pm,\gamma,-}(z, x) = u_\pm(z, x) - \frac{u_{\gamma,-}(\lambda, x)}{\lambda - z} W_x(u_-(\lambda), u_\pm(z)), \tag{5.33}$$

$$u_{\pm,\gamma,+}(z, x) = u_\pm(z, x) + \frac{u_{\gamma,+}(\lambda, x)}{\lambda - z} W_x(u_+(\lambda), u_\pm(z)), \tag{5.34}$$

are square integrable near a, b and satisfy the boundary condition of H_γ at a, b, respectively.

Proof. Only the formulas corresponding to $u_-(\lambda) \in \mathfrak{H}_-$ are considered. The case $u_+(\lambda) \in \mathfrak{H}_+$ is similar. Note the different sign in the formulas above is a result of the interchanged integral $\langle ., .. \rangle_x^b$ when commuting from the right endpoint b.

The first claim follows from

$$\begin{aligned}\tau_\gamma v(z) &= zu(z) + Q_\gamma u(z) \\ &\quad - \frac{1}{\lambda - z}\left[\frac{1}{i}\sigma_2 u_{\gamma,-}(\lambda) W_x(u_-(\lambda), u(z))' + \tau_\gamma u_{\gamma,-}(\lambda) W_x(u_-(\lambda), u(z))\right] \\ &= zu(z) + Q_\gamma u(z) - \frac{1}{i}\sigma_2 u_{\gamma,-}(\lambda) u_-(\lambda)^\top u(z) \\ &\quad - \frac{\lambda}{\lambda - z} u_{\gamma,-}(\lambda) W_x(u_-(\lambda), u(z)) \\ &= zu(z) - u_{\gamma,-}(\lambda) u_-(\lambda)^\top \frac{1}{i}\sigma_2 u(z) - \frac{\lambda}{\lambda - z} u_{\gamma,-}(\lambda) W_x(u_-(\lambda), u(z)) \\ &= zv(z),\end{aligned}$$

where we have used $-u_{\gamma,-}(\lambda) u_-(\lambda)^\top \frac{1}{i}\sigma_2 u(z) = u_{\gamma,-}(\lambda) W_x(u_-(\lambda), u(z))$ in the last step. If u additionally satisfies the boundary condition at a, i.e., u is in the domain of H near a, then $v = U_\gamma u$ follows simply from the Lagrange identity

$$(\lambda - z)\int_a^x u_-(\lambda, t)^\top u(z, t)\, dt = W_x(u_-(\lambda), u(z)).$$

Equations (5.31) and (5.32) are shown by straightforward calculation. Using (5.31) together with the Cauchy-Schwarz inequality plus

$$\begin{aligned}W_x(u_-(\lambda), u_+(z)) &= W_c(u_-(\lambda), u_+(z)) + (\lambda - z)\int_c^x u_-(\lambda, y)^\top u_+(z, y)\, dy, \\ W_x(u_-(\lambda), u_-(z)) &= W_c(u_-(\lambda), u_-(z)) - (\lambda - z)\int_x^c u_-(\lambda, y)^\top u_-(z, y)\, dy,\end{aligned}$$

respectively, reveals that the solutions $u_{\pm,\gamma,-}$ are square integrable near b, a, respectively. That $u_{\pm,\gamma,-}(z)$ satisfy the boundary conditions at b, respectively a, follows from

$$W_x(u_{\gamma,-}(\lambda), u_{\pm,\gamma,-}(z)) = \frac{W_x(u_-(\lambda), u_\pm(z))}{c_\gamma(\lambda, x)}$$

and the same discussion as after equation (5.26). □

Theorem 5.6. *Assume Hypothesis 2.1 and let $u_+(\lambda) = \Phi(\lambda) \in \mathfrak{H}_+$ (i.e., λ is an eigenvalue), $\gamma \in (-\|\Phi(\lambda)\|^{-2}, \infty) \cup \{\infty\}$, and set $\widetilde{\Phi}_\gamma(\lambda, x) = \Phi(\lambda, x)/c_\gamma(\lambda, x)$.*

The operator H_γ has a system of real entire solutions

(5.35)

$$\Phi_\gamma(z, x) = \Phi(z, x) + \frac{1}{\lambda - z}\widetilde{\Phi}_\gamma(\lambda, x) W_x(\Phi(\lambda), \Phi(z))$$

(5.36)

$$\Theta_\gamma(z, x) = \Theta(z, x) + \frac{1}{\lambda - z}\Big(\widetilde{\Phi}_\gamma(\lambda, x) W_x(\Phi(\lambda), \Theta(z)) + \frac{1}{\gamma^{-1} + \|\Phi(\lambda)\|^2}\Phi_\gamma(z, x)\Big)$$

with $W(\Theta_\gamma(z), \Phi_\gamma(z)) = 1$*. In fact, for* $z = \lambda$*, we have*

$$\Phi_\gamma(\lambda, x) = \left(\frac{1}{\gamma} + \|\Phi(\lambda)\|^2\right) \widetilde{\Phi}_\gamma(\lambda, x) \tag{5.37}$$

$$\Theta_\gamma(\lambda, x) = \Theta_\gamma(\lambda, x) - \widetilde{\Phi}_\gamma(\lambda, x) W_x(\Phi(\lambda), \dot{\Theta}(\lambda)) - \frac{1}{\gamma^{-1} + \|\Phi(\lambda)\|^2} \dot{\Phi}_\gamma(\lambda, x). \tag{5.38}$$

Here the dot denotes the derivative with respect to the spectral parameter z*. In particular,* H_γ *satisfies again Hypothesis 2.1.*

The Weyl solutions of H_γ *are given by*

$$\begin{aligned} \Phi_\gamma(z, x), \quad \Psi_\gamma(z, x) &= \Psi(z, x) + \frac{1}{\lambda - z} \widetilde{\Phi}_\gamma(\lambda, x) W_x(\Phi(\lambda), \Psi(z)) \\ &= \Theta_\gamma(z, x) + M_\gamma(z) \Phi_\gamma(z, x), \end{aligned} \tag{5.39}$$

where

$$M_\gamma(z) = M(z) - \frac{1}{\gamma^{-1} + \|\Phi(\lambda)\|^2} \frac{1}{\lambda - z} \tag{5.40}$$

is the singular Weyl function of H_γ*.*

Proof. First note that $\Phi_\gamma(z)$ and $\Theta_\gamma(z)$ are solutions of the equation $\tau u = zu$ by the first part of Lemma 5.5. That $\Phi_\gamma(z)$ is entire in z is obvious. For the singular solution note that the pole at $z = \lambda$ has been removed since

$$\widetilde{\Phi}_\gamma(\lambda, x) W(\Phi(\lambda), \Theta(\lambda)) + \frac{\gamma}{1 + \gamma \|\Phi(\lambda)\|^2} \Phi_\gamma(\lambda, x) = -\widetilde{\Phi}_\gamma(\lambda, x) + \widetilde{\Phi}_\gamma(\lambda, x) = 0$$

by (5.37). Thus we have a system of real entire solutions. Their Wronskian satisfies $W(\Theta_\gamma(z), \Phi_\gamma(z)) = 1$ by (5.32). Hence by Lemma 2.2, H_γ will satisfy again Hypothesis 2.1.

Using (5.34), we infer that the solutions $\Phi_\gamma(z)$, $\Psi_\gamma(z)$ are in the domain of H_γ near a, b, respectively. The claim about M_γ follows easily by inspection,

$$\begin{aligned} \Psi_\gamma(z, x) &= \Psi(z, x) + \frac{1}{\lambda - z} \widetilde{\Phi}_\gamma(\lambda, x) W_x(\Phi(\lambda), \Psi(z)) \\ &= \Theta(z, x) + \frac{1}{\lambda - z} \widetilde{\Phi}_\gamma(\lambda, x) W_x(\Phi(\lambda), \Theta(z)) \\ &\quad + M(z)\Big(\Phi(z, x) + \frac{1}{\lambda - z} \widetilde{\Phi}_\gamma(\lambda, x) W_x(\Phi(\lambda), \Phi(z))\Big) \\ &= \Theta_\gamma(z, x) - \frac{1}{\gamma^{-1} + \|\Phi(\lambda)\|^2} \frac{1}{\lambda - z} \Phi_\gamma(z, x) + M(z) \Phi_\gamma(z, x) \\ &= \Theta_\gamma(z, x) + M_\gamma(z) \Phi_\gamma(z, x). \end{aligned}$$

□

Remark 5.7. *Clearly, one can also commute at the left endpoint with the regular solution. This has been done in Theorem 4.2 and Theorem 4.4 of* [20] *in the case of one-dimensional Schrödinger operators. The results can be identically transferred to the case of one-dimensional Dirac operators.*

The following observation shows us how the singular Weyl function transforms if we insert the singular solution $\Theta(\lambda)$ into the transformation operator (5.16). This result is essential for the application to the perturbed radial Dirac operator in the next section.

Theorem 5.8. *Assume Hypothesis 2.1 and suppose* $\lambda \in \mathbb{R}$ *exists such that* $u_+(\lambda) = \Theta(\lambda) \in \mathfrak{H}_+$. *Let* $\gamma \in (0,\infty) \cup \{\infty\}$ *and set* $\widetilde{\Theta}_\gamma(\lambda, x) = \Theta(\lambda, x)/c_\gamma(\lambda, x)$.

The operator H_γ *has a system of real entire solutions*

$$\Phi_\gamma(z,x) = (\lambda - z)\Big(\Phi(z,x) + \frac{1}{\lambda - z}\widetilde{\Theta}_\gamma(\lambda,x)W_x(\Theta(\lambda),\Phi(z))\Big), \tag{5.41}$$

$$\begin{aligned}\Theta_\gamma(z,x) = \frac{1}{\lambda - z}\Big[&\Theta(z,x) + \frac{1}{\lambda - z}\widetilde{\Theta}_\gamma(\lambda,x)W_x(\Theta(\lambda),\Theta(z))\\ &- \Big(\frac{1}{\gamma} - W_b(\Theta(\lambda),\dot{\Theta}(\lambda))\Big)\Phi_\gamma(z,x)\Big],\end{aligned} \tag{5.42}$$

with $W(\Theta_\gamma(z), \Phi_\gamma(z)) = 1$. *Moreover,*

$$\Phi_\gamma(\lambda,x) = \widetilde{\Theta}_\gamma(\lambda,x). \tag{5.43}$$

In particular, H_γ *satisfies again Hypothesis 2.1.*

The Weyl solutions of H_γ *are given by*

$$\begin{aligned}\Phi_\gamma(z,x), \quad \Psi_\gamma(z,x) &= \Psi(z,x) + \frac{1}{\lambda - z}\widetilde{\Theta}_\gamma(\lambda,x)W_x(\Theta(\lambda),\Psi(z))\\ &= \Theta_\gamma(z,x) + M_\gamma(z)\Phi_\gamma(z,x),\end{aligned} \tag{5.44}$$

where

$$M_\gamma(z) = \frac{M(z) - W_b(\Theta(\lambda),\dot{\Theta}(\lambda))(\lambda - z)}{(\lambda - z)^2} + \frac{\gamma^{-1}}{\lambda - z} \tag{5.45}$$

is the singular Weyl function of H_γ.

Proof. That Φ_γ is entire is obvious. For Θ_γ use l'Hôpital's rule,

$$\begin{aligned}\lim_{z\to\lambda}\Big(\Theta(z,x)+&\frac{1}{\lambda - z}\widetilde{\Theta}_\gamma(\lambda,x)W_x(\Theta(\lambda),\Theta(z))\Big)\\ &= \Theta(\lambda,x) - \widetilde{\Theta}_\gamma(\lambda,x)W_x(\Theta(\lambda),\dot{\Theta}(\lambda))\\ &= \Big(\frac{1}{\gamma} - W_b(\Theta(\lambda),\dot{\Theta}(\lambda))\Big)\widetilde{\Theta}_\gamma(\lambda,x).\end{aligned}$$

Here we used the identity

$$W_x(\Theta(\lambda),\dot{\Theta}(\lambda)) = W_b(\Theta(\lambda),\dot{\Theta}(\lambda)) + \langle\Theta(\lambda),\Theta(\lambda)\rangle_x^b,$$

which is obtained by differentiating the Lagrange identity

$$(\lambda - z)\int_x^b \Theta(\lambda,y)^\top\Theta(z,y)\,dy = W_b(\Theta(\lambda),\Theta(z)) - W_x(\Theta(\lambda),\Theta(z))$$

with respect to z and evaluating at $z = \lambda$. Hence the pole of $\Theta_\gamma(z)$ at $z = \lambda$ is removed and the solution is entire. The claim about the Wronskian follows by the same arguments as in the proof of the Theorem above. It remains to calculate

$M_\gamma(z)$. Abbreviating $\alpha := \frac{1}{\gamma} - W_b(\Theta(\lambda), \dot{\Theta}(\lambda))$, we compute

$$\begin{aligned}
\Psi_\gamma(z,x) &= \Psi(z,x) + \frac{1}{\lambda - z}\widetilde{\Theta}_\gamma(\lambda,x) W_x(\Theta(\lambda), \Psi(z)) \\
&= \Theta(z,x) + \frac{1}{\lambda - z}\widetilde{\Theta}_\gamma(\lambda,x) W_x(\Theta(\lambda), \Theta(z)) \\
&\quad + M(z)\left(\Phi(z,x) + \frac{1}{\lambda - z}\widetilde{\Theta}_\gamma(\lambda,x) W_x(\Theta(\lambda), \Phi(z))\right) \\
&= (\lambda - z)\Theta_\gamma(z,x) + \alpha\Phi_\gamma(z,x) + \frac{1}{\lambda - z} M(z)\Phi_\gamma(z,x) \\
&= (\lambda - z)\left[\Theta_\gamma(z,x) + \left(\frac{M(z)}{(\lambda - z)^2} + \frac{\alpha}{\lambda - z}\right)\Phi_\gamma(z,x)\right]. \qquad \square
\end{aligned}$$

Note that the first summand in (5.45) has no residue at $z = \lambda$ since the residue of $M_\gamma(z)$ must be given by $-\|\Phi_\gamma(\lambda)\|^{-2} = -\|\widetilde{\Theta}_\gamma(\lambda)\|^{-2} = -\gamma^{-1}$. Furthermore, if H is limit circle at b and $\gamma < \infty$ then H_γ will be again limit circle at b by [29, Thm. 3.7] (clearly H_∞ is always limit point at b). In the limit circle case the boundary condition of H_γ will be generated by $\Phi_\gamma(\lambda,x) = \widetilde{\Theta}_\gamma(\lambda,x) \in \mathfrak{H}$ and hence to repeat this procedure at every zero of

$$\begin{aligned}
z \mapsto W_b(\widetilde{\Theta}_\gamma(\lambda), \Theta_\gamma(z)) = &\frac{1}{c_\gamma(\lambda,b)}\Big[\frac{1}{\lambda - z} W_b(\Theta(\lambda), \Theta(z)) \\
&- \Big(\frac{1}{\gamma} - W_b(\Theta(\lambda), \dot{\Theta}(\lambda))\Big) W_b(\Theta(\lambda), \Phi(z))\Big].
\end{aligned}$$

Since we have $u_{\gamma,+}(z,b) = C(z)\widetilde{\Theta}_\gamma(\lambda,b)$ with a nonzero entire function $C(z)$ this implies $W(u_{\gamma,+}(z), \Theta_\gamma(z)) = C(z) W_b(\widetilde{\Theta}_\gamma(\lambda), \Theta_\gamma(z))$. Now equation (2.1) implies that the zeros of this Wronskian coincide with the zeros of $M_\gamma(z)$. But the residues of $M_\gamma(z)$ are always negative and hence there must be an odd number of zeros between two consecutive poles of $M_\gamma(z)$. In particular, we see that the above Wronskian has an infinite number of zeros and we can iterate this procedure which will be important later on. We also mention that if the function $E_\gamma(z)$ in the representation (2.17) is zero, then the derivative at every zero of $M_\gamma(z)$ is positive and there will be precisely one zero between each pole.

6. Applications to Radial Dirac Operators

In this section we are going to apply the double commutation method to perturbed radial Dirac operators. The strategy will be as follows: Starting with such an operator H in the limit point case at $x = 0$ for some certain κ and commuting the operator at the right endpoint $x = b$ with the singular solution plus applying a simple Gauge transform to the commuted operator H_γ results in a radial Dirac operator of the same form, but with decreased angular momentum $\kappa - 1$. Applying this procedure finitely many times, we end up in the limit circe case $\kappa \in [0, \frac{1}{2})$ where the singular Weyl function is a Herglotz–Nevanlinna function (Theorem 2.8). Using the results from the previous section, we will be able to show that the singular Weyl function of the perturbed radial Dirac operator will be a generalized Nevanlinna function.

Throughout this section, assume the following

Hypothesis 6.1. *Let b be a regular endpoint and suppose that the spectrum of $H^D_{(a,c)}$ is purely discrete for one (and hence for all) $c \in (0, b)$. Moreover, assume that $\lambda \in \mathbb{R}$ exists such that $\Theta(\lambda, x) = u_+(\lambda, x)$ is a Weyl solution for the right endpoint b.*

The hypothesis is essential in order to apply our findings from the last section. By [6, Lemma 8.2], one can find a regular solution Φ which has the same asymptotics as Φ_κ from Section 4 near $x = 0$. Hence Hypothesis 2.1 is satisfied by Lemma 2.2. Moreover, the last part of the hypothesis is always satisfied if the operator is limit circle at b by choosing a boundary condition. For a chosen boundary condition there exist infinitely many real λ which have the desired property (see the discussion after Theorem 5.8). That b is assumed to be regular will ease the proof in the following lemma since we can apply the results from Appendix B on the (finite) interval $(0, b)$. Moreover, since regular endpoints are limit circle (Lemma 1.7), the commuted operator H_γ inherits this property if $\gamma \in (0, \infty)$.

Lemma 6.2. *Assume Hypothesis 6.1. Let $\kappa \geq \frac{1}{2}$, $\gamma \in (0, \infty)$ and suppose*

$$H = \frac{1}{i}\sigma_2 \frac{d}{dx} + \frac{\kappa}{x}\sigma_1 + Q(x), \quad Q \in L^1(0, b), \quad x \in (0, b). \tag{6.1}$$

Then

$$H_\gamma = \frac{1}{i}\sigma_2 \frac{d}{dx} - \frac{\kappa - 1}{x}\sigma_1 + \widetilde{Q}(x), \quad \widetilde{Q} \in L^1(0, b). \tag{6.2}$$

Proof. By formula (5.19) and Remark 5.4, the commuted operator is of the form $H + Q_\gamma$, where

$$\begin{aligned} Q_\gamma(\lambda, x) &= -\frac{\Theta_1(\lambda, x)^2 - \Theta_2(\lambda, x)^2}{c_\gamma(\lambda, x)}\sigma_1 + 2\frac{\Theta_1(\lambda, x)\Theta_2(\lambda, x)}{c_\gamma(\lambda, x)}\sigma_3 \\ &= \frac{c'_\gamma(\lambda, x)}{c_\gamma(\lambda, x)}\sigma_1 + \frac{2\Theta_2(\lambda, x)^2}{c_\gamma(\lambda, x)}\sigma_1 + 2\frac{\Theta_1(\lambda, x)\Theta_2(\lambda, x)}{c_\gamma(\lambda, x)}\sigma_3. \end{aligned}$$

By Appendix B, the denominator is of the form

$$c_\gamma(\lambda, x) = \frac{1}{\gamma} + \int_x^b |\Theta(\lambda, y)|^2\, dy = x^{-2\kappa+1}\Big(\frac{1}{\gamma}x^{2\kappa-1} + w(x)\Big),$$

with $w \in W^{1,1}(0, b) \cap C([0, b])$ and $w > 0$ on $[0, b]$. Note that since $\kappa \geq \frac{1}{2}$, the mapping $x \mapsto x^{2\kappa-1}$ lies in $W^{1,1}(0, b)$ too and therefore $c_\gamma(\lambda, x) = x^{-2\kappa+1}\widetilde{w}(x)$, where $\widetilde{w}$ shares the same properties as w. Hence

$$\frac{c'_\gamma(\lambda,x)}{c_\gamma(\lambda,x)} = \frac{d}{dx}\log(c_\gamma(\lambda,x)) = \frac{-2\kappa+1}{x} + \frac{\widetilde{w}'(x)}{\widetilde{w}(x)},$$

with $\widetilde{w}'(x)/\widetilde{w}(x) \in L^1(0,b)$. Using the properties of the singular solution derived in Appendix B, one infers that $\Theta_2(\lambda,.)^2/c_\gamma(\lambda,.)$ and $(\Theta_1(\lambda,.)\Theta_2(\lambda,.))/c_\gamma(\lambda,.)$ lie in $L^1(0,b)$ too and the claim follows. □

Remark 6.3. *The commuted operator has a negative angular momentum if* $\kappa > 1$. *In this case we employ the Gauge transform* $-\sigma_1 H_\gamma \sigma_1$ *resulting in a positive angular momentum. The system of fundamental solutions has to be replaced by* $\sigma_1\Phi_\gamma(z,x)$ *and* $-\sigma_1\Theta_\gamma(z,x)$. *Note that* $W(-\sigma_1\Theta_\gamma(z),\sigma_1\Phi_\gamma(z)) = W(\Theta_\gamma(z),\Phi_\gamma(z)) = 1$ *and the formula* (5.45) *stays the same.*

Now let $\kappa \geq 0$, $\kappa + \frac{1}{2} \notin \mathbb{N}$ and the Dirac operator be given by (6.1), together with the usual boundary condition $\lim_{x\to 0} x^\kappa f_1(x) = 0$ at $x = 0$ if we are in the limit circle case $\kappa \in [0,\frac{1}{2})$. Writing $\kappa = \lfloor\kappa\rfloor + \alpha$, with $\alpha \in [0,1)\backslash\{\frac{1}{2}\}$, and applying the algorithm (lemma plus remark above) $\lfloor\kappa\rfloor$ many times, we end up with the operator

$$\widehat{H} = \frac{1}{i}\sigma_2\frac{d}{dx} + \frac{\alpha}{x}\sigma_1 + \widehat{Q}(x), \quad \widehat{Q} \in L^1(0,b). \tag{6.3}$$

If $\alpha \in [0,\frac{1}{2})$, the operator $\widehat{H}$ is limit circle at the singular endpoint $x = 0$. Otherwise, if $\alpha \in (\frac{1}{2},1)$ we need to employ the Lemma 6.2 one more time to obtain a limit circle situation.

By Theorem 2.8, the singular Weyl function in the limit circle case will be a Herglotz function. Combining these results with Theorem 5.8, one obtains by induction:

Theorem 6.4. *Assume Hypothesis 6.1 and let* $\kappa \geq 0$. *Suppose*

$$H = \frac{1}{i}\sigma_2\frac{d}{dx} + \frac{\kappa}{x}\sigma_1 + Q(x), \quad Q \in L^1, \tag{6.4}$$

where $\kappa + \frac{1}{2} \notin \mathbb{N}$. *Then there is a singular Weyl function of the form*

$$M(z) = P_{\lfloor\kappa+\frac{1}{2}\rfloor}(z)^2 M_0(z) - \sum_{n=0}^{\lfloor\kappa+\frac{1}{2}\rfloor-1} c_n P_n(z)^2(\lambda_n - z), \tag{6.5}$$

where $M_0(z)$ *is a Herglotz-Nevanlinna function and*

$$c_n = \gamma_n^{-1} - W_b(\Theta_n(\lambda_n),\dot{\Theta}_n(\lambda_n)),$$
$$P_n(z) = \prod_{j=0}^{n-1}(z-\lambda_j), \quad P_0(z) = 1,$$

depends on the choice of λ_n *and* γ_n *in every step of Lemma 6.2. The corresponding spectral measure is given by*

$$d\rho(t) = P_{\lfloor\kappa+\frac{1}{2}\rfloor}(t)d\rho_0(t), \tag{6.6}$$

where the measure ρ_0 *satisfies* $\int_{\mathbb{R}} d\rho_0(t) = \infty$ *and* $\int_{\mathbb{R}} \frac{d\rho_0(t)}{1+t^2} < \infty$.

Proof. If $\lfloor \kappa + \frac{1}{2} \rfloor = 0$, then $\kappa \in [0, \frac{1}{2})$ and we are in the limit circle case where the singular Weyl function is a Herglotz–Nevanlinna function.

If $\lfloor \kappa + \frac{1}{2} \rfloor = 1$, then $\kappa \in (\frac{1}{2}, \frac{3}{2})$ and we need to apply the lemma (plus remark) above once to end up in the limit circle case. Choose λ, γ according to Lemma 6.2. The Weyl function M_γ of the commuted operator will be a Herglotz–Nevanlinna function and by formula (5.45), we have

$$M(z) = (\lambda - z)^2 M_\gamma(z) - [\gamma^{-1} - W_b(\Theta(\lambda), \dot{\Theta}(\lambda))].$$

In the case $\lfloor \kappa + \frac{1}{2} \rfloor > 1$, the commuted operator H_γ will have angular momentum $\kappa - 1$. Therefore we can use (6.5) with $\lfloor \kappa - 1 + \frac{1}{2} \rfloor = \lfloor \kappa + \frac{1}{2} \rfloor - 1$ for M_γ together with (5.45) and the claim for M follows. Formula (6.6) for the spectral measure follows from the Stieltjes–Livšić inversion formula. □

As another consequence we obtain

Theorem 6.5. *Assume Hypothesis 6.1 and let* $\kappa \geq 0$. *Suppose*

$$H = \frac{1}{i}\sigma_2 \frac{d}{dx} + \frac{\kappa}{x}\sigma_1 + Q(x), \quad Q \in L^1, \tag{6.7}$$

where $\kappa + \frac{1}{2} \notin \mathbb{N}$. *Then there is a corresponding solution* $\Theta(z, x)$ *such that* $M(z) \in \mathcal{N}_{\kappa_0}^{\infty}$ *with some* $\kappa_0 \leq \lfloor \kappa + \frac{1}{2} \rfloor$.

Proof. Combining (6.6) with $\int (1 + t^2)^{-1} d\rho_0(t) < \infty$ the claim follows by applying Theorem 2.7 with $k = \lfloor \kappa + \frac{1}{2} \rfloor$. □

We remark that in the last theorem the assumption that b is regular is superfluous. In fact, Lemma 7.1 from [5] shows that the asymptotics of $M(z)$ as $\mathrm{Im}(z) \to \infty$ depend only on the behavior of the potential near $a = 0$. Furthermore, Lemma C.1 shows that the required integrability properties of the spectral measure $d\rho$ depend only on the asymptotics of $M(z)$ and hence also depend only on the behavior of the potential near $a = 0$.

Appendix A. Hardy Inequalities

In this Appendix we are going to provide Hardy type inequalities which are needed in Appendix B. We will follow closely [21].

For $l > -1$ and $(x,y) \in (0,\infty) \times (0,\infty)$ define the kernel

$$K_l(x,y) := \begin{cases} x^{-(l+1)} y^l, & y \le x, \\ 0, & y > x, \end{cases} \tag{A.1}$$

and introduce the following integral operators,

$$(\mathcal{K}_l f)(x) := \int_0^\infty K_l(x,y) f(y) dy = \frac{1}{x^{l+1}} \int_0^x y^l f(y) dy, \tag{A.2}$$

$$(\hat{\mathcal{K}}_l g)(y) := \int_0^\infty K_l(x,y) g(x) dx = y^l \int_y^\infty x^{-(l+1)} g(x) dx. \tag{A.3}$$

Theorem 319 of [17] implies continuity of these operators,

$$\|\mathcal{K}_l f\|_p \le \frac{p}{p(l+1)-1} \|f\|_p, \quad f \in L^p(0,\infty), \tag{A.4}$$

$$\|\hat{\mathcal{K}}_l g\|_q \le \frac{q}{ql+1} \|g\|_q, \quad g \in L^q(0,\infty), \tag{A.5}$$

whenever $p \in (1,\infty)$, $\frac{1}{p} + \frac{1}{q} = 1$ and $(l+1)p > 1$, $lq > -1$, respectively. In fact, one can even consider bounded intervals:

Lemma A.1 ([21, Lemma A.1])**.** *Let $a > 0$ and $l > -1$. The operator $\mathcal{K}_l$ is a bounded operator in $L^p(0,a)$ satisfying*

$$\|\mathcal{K}_l f\|_p \le \frac{p}{p(l+1)-1} \|f\|_p, \quad f \in L^p(0,a), \tag{A.6}$$

for any $p \in (\frac{1}{l+1}, \infty]$ if $-1 < l \le 0$ and any $p \in [1,\infty]$ if $l > 0$. Moreover, if $f \in C[0,a]$, then $\mathcal{K}_l(f) \in C[0,a]$ with

$$\lim_{x \to 0} \mathcal{K}_l(f)(x) = \frac{f(0)}{l+1}. \tag{A.7}$$

Similarly, the operator $\hat{\mathcal{K}}_l$ is a bounded operator in $L^p(0,a)$ satisfying

$$\|\hat{\mathcal{K}}_l f\|_p \le \frac{p}{pl+1} \|f\|_p, \quad f \in L^p(0,a), \tag{A.8}$$

for any $p \in [1, \frac{1}{-l})$ if $-1 < l \le 0$ and any $p \in [1,\infty]$ if $l > 0$. Moreover, if $l > 0$ and $f \in C[0,a]$, then $\hat{\mathcal{K}}_l(f) \in C[0,a]$ with

$$\lim_{x \to 0} \hat{\mathcal{K}}_l(f)(x) = \frac{f(0)}{l}. \tag{A.9}$$

In addition, we need boundedness on the Sobolev spaces $W^{1,p}(0,a)$. The Sobolev space $W^{1,p}(0,a)$ consists of absolutely continuous functions in $L^p(0,a)$ whose derivative belongs again to $L^p(0,a)$. For $f \in W^{1,p}(0,1)$, its norm is defined by $\|f\|_{W^{1,p}} = \|f\|_p + \|f'\|_p$.

Lemma A.2 ([21, Lemma A.2])**.** *Let $a > 0$ and $l > -1$. The operator $\mathcal{K}_l$ is a bounded operator in $W^{1,p}(0,a)$ viz.*

$$\|\mathcal{K}_l f\|_{W^{1,p}} \le C_l \|f\|_{W^{1,p}}, \quad f \in W^{1,p}(0,a), \tag{A.10}$$

for any $p \in [1, \infty]$. *Moreover,*

$$\lim_{x \to 0} x(\mathcal{K}_l f)'(x) = \frac{1}{l+1} \lim_{x \to 0} x f'(x) \tag{A.11}$$

whenever the limit on the right-hand side exists.

Similarly, the operator $\hat{\mathcal{K}}_l$ *is bounded in* $W^{1,p}(0,a)$ *viz.*

$$\|\hat{\mathcal{K}}_l f\|_{W^{1,p}} \le \hat{C}_l \|f\|_{W^{1,p}}, \quad f \in W^{1,p}(0,a), \tag{A.12}$$

for any $p \in [1, \frac{1}{1-l})$ *if* $0 < l \le 1$ *and any* $p \in [1, \infty]$ *if* $l > 1$. *Moreover,*

$$\lim_{x \to 0} x(\hat{\mathcal{K}}_l f)'(x) = \frac{1}{l} \lim_{x \to 0} x f'(x) \tag{A.13}$$

whenever the limit on the right-hand side exists.

Appendix B. Properties of Solutions

The double commutation method requires some analysis of solutions of $\tau_\kappa u = \lambda u$ (where τ_κ is given by (1.4)) since they appear in the potential of the commuted operator. We are going to derive required properties of such if we consider the unperturbed case first. A corresponding fundamental system of solutions Φ_κ and Θ_κ is given by virtue of the usual Bessel and Neumann functions, cf. (4.4) and (4.5). An important essence in the following calculations will be the representation of those solutions near $x = 0$ due to the asymptotic behavior of the Bessel and Neumann functions. In particular, they can be written as

$$\Phi_\kappa(\lambda, x) = \begin{pmatrix} x^{\kappa+1}\widetilde{\Phi}_1(\lambda, x) \\ x^{\kappa}\widetilde{\Phi}_2(\lambda, x) \end{pmatrix}, \quad \Theta_\kappa(\lambda, x) = \begin{pmatrix} x^{-\kappa}\widetilde{\Theta}_1(\lambda, x) \\ x^{-\kappa+1}\widetilde{\Theta}_2(\lambda, x) \end{pmatrix}, \tag{B.1}$$

with $\widetilde{\Phi}_{1,2}(\lambda, .), \widetilde{\Theta}_{1,2}(\lambda, .) \in C([0,1], \mathbb{R})$ which do not vanish at $x = 0$ if $\mathbb{R} \ni \lambda \neq 0$.

Let us shed some light first on the regular solution $\Phi(\lambda, x)$ of the equation $\tau_\kappa u = \lambda u$ for some arbitrary $Q \in L^p(0,1)$, $p \in [1, \infty)$ and $\lambda \in \mathbb{C}$. Assume the case $\kappa > 0$ in the following. To obtain a desired solution set $\hat{\Phi}_\kappa(\lambda, x) := x^{-\kappa}\Phi_\kappa(\lambda, x)$, $\hat{\Theta}_\kappa(\lambda, x) := x^{\kappa}\Theta_\kappa(\lambda, x)$ and define the kernel

$$\widehat{G}(\lambda, x, y) := \hat{\Phi}_\kappa(\lambda, x)\hat{\Theta}_\kappa(\lambda, y)^\top - \left(\frac{y}{x}\right)^{2\kappa}\hat{\Theta}_\kappa(\lambda, x)\hat{\Phi}_\kappa(\lambda, y)^\top. \tag{B.2}$$

Note that $\widehat{G}(\lambda, ., ..)$ is continuous on the triangle $\{(x, y) | 0 \leq y \leq x \leq 1\}$ and thus one obtains a Volterra operator,

$$\begin{array}{rccl} \widehat{\mathcal{G}}: & C([0,1], \mathbb{C}^2) & \to & C([0,1], \mathbb{C}^2) \\ & f(x) & \mapsto & \int_0^x \widehat{G}(\lambda, x, y)Q(y)f(y)\, dy. \end{array} \tag{B.3}$$

The Volterra integral equation of second kind,

$$f(x) = \hat{\Phi}_\kappa(\lambda, x) + \widehat{\mathcal{G}}f(x), \tag{B.4}$$

possesses a unique solution $\hat{\Phi}(\lambda, .) \in C([0,1], \mathbb{C}^2)$ by means of a Neumann series. Multiplying the solution of the above integral equation by x^κ, one infers that

$$\begin{aligned} \Phi(\lambda, x) := x^\kappa\hat{\Phi}(\lambda, x) &= \Phi_\kappa(\lambda, x) + x^\kappa \int_0^x \widehat{G}(\lambda, x, y)Q(y)\Phi(\lambda, y)\, dy \\ &= \Phi_\kappa(\lambda, x) + \Phi_\kappa(\lambda, x) \int_0^x \Theta_\kappa(\lambda, y)^\top Q(y)\Phi(\lambda, y)\, dy \\ &\quad - \Theta_\kappa(\lambda, x) \int_0^x \Phi_\kappa(\lambda, y)^\top Q(y)\Phi(\lambda, y)\, dy \end{aligned} \tag{B.5}$$

solves $\tau_\kappa u = \lambda u$ by formula (1.14). Since the regular solutions Φ_κ and Φ are of the form $x^\kappa f(x)$ with $f \in C([0,1], \mathbb{C}^2)$, the last integrand can be written as $\Phi_\kappa(\lambda, y)^\top Q(y)\Phi(\lambda, y) = y^{2\kappa}v(\lambda, y)$ with $v(\lambda, .) \in L^p(0,1)$. Making use of Lemma A.1, one gets

$$\int_0^x \Phi_\kappa(\lambda, y)^\top Q(y)\Phi(\lambda, y)\, dy = x^{2\kappa+1}\tilde{v}(\lambda, x), \quad \tilde{v}(\lambda, .) \in L^p(0,1), \quad p \in [1, \infty). \tag{B.6}$$

Note that the case $p = 1$ has to be excluded if one admits $\kappa = 0$. Summing up, the regular solution can be expressed as

$$\Phi(\lambda, x) = \Phi_\kappa(\lambda, x)(1 + o(1)) + \Theta_\kappa(\lambda, x)o(x^{2\kappa}), \tag{B.7}$$

as $x \to 0$. The results can be summarized as follows ([2, Lemma A.1]):

- $x^{-\kappa}\Phi(\lambda, x)$ is continuous on $[0,1]$,
- $\lim_{x\to 0} x^{-\kappa}\Phi(\lambda, x) = \lim_{x\to 0} x^{-\kappa}\Phi_\kappa(\lambda, x) \neq 0$ and
- regarding the first component of the regular solution, $x^{-(\kappa+1)}\Phi_1(\lambda, x) \in L^p(0,1)$.

In order to understand how the double commutation method shifts the angular momentum κ, we will need a more comprehensible representation of the integral $\int_0^x |\Phi(\lambda, y)|^2\, dy$. To obtain a better understanding if this integral, consider the function $g(x) := x^{-2\kappa}|\Phi(\lambda, x)|^2$. Note that by the above observations, g is continuous on $[0,1]$ with $g(0) > 0$. Calculating its derivative,

$$g'(x) = -2\kappa \frac{|\Phi(\lambda, x)|^2}{x^{2\kappa+1}} + \frac{2}{x^{2\kappa}}(\Phi_1(\lambda, x)\Phi_1'(\lambda, x) + \Phi_2(\lambda, x)\Phi_2'(\lambda, x)), \tag{B.8}$$

combined with

$$\tau_\kappa \Phi = \lambda\Phi \ \Leftrightarrow \left\{ \begin{array}{l} \Phi_1' = (\lambda - Q_{22})\Phi_2 - Q_{21}\Phi_1 - \frac{\kappa}{x}\Phi_1 \\ \Phi_2' = (Q_{11} - \lambda)\Phi_1 + Q_{12}\Phi_2 + \frac{\kappa}{x}\Phi_2 \end{array} \right. \tag{B.9}$$

yields

$$g'(x) = \frac{-4\kappa\Phi_1(\lambda, x)^2}{x^{2\kappa+1}} + 2\sum_{1\le i\le j\le 2} \widetilde{Q}_{ij} \frac{\Phi_i(\lambda, x)}{x^\kappa}\frac{\Phi_j(\lambda, x)}{x^\kappa}, \tag{B.10}$$

for some $\widetilde{Q}_{ij} \in L^p(0,1)$. Again note that $x^{-\kappa}\Phi(\lambda, x)$ is continuous on $[0,1]$, so the right summand in the equation above lies in $L^p(0,1)$. In addition, since $x^{-(\kappa+1)}\Phi_1(\lambda, x) \in L^p(0,1)$, one concludes that $g \in W^{1,p}(0,1)$. By Lemma A.2, $\mathcal{K}_{2\kappa} g \in W^{1,p}(0,1) \cap C([0,1],\mathbb{C})$ with $\mathcal{K}_{2\kappa} g(0) = g(0)/(2\kappa+1) > 0$. Summarizing, ([2, Lemma A.2]),

$$\int_0^x |\Phi(\lambda, y)|^2\, dy = x^{2\kappa+1}\widetilde{w}(x), \quad \widetilde{w} \in W^{1,p}(0,1) \cap C([0,1],\mathbb{R}), \quad \widetilde{w} > 0. \tag{B.11}$$

To investigate the singular solution $\Theta(\lambda, x)$ of $\tau_\kappa u = \lambda u$, we will follow the approach from above but with the slight difference that we only allow $\kappa > \frac{1}{2}$. The choice of the admissible angular momentum will become clear later on when we make use of the integral operators from Appendix A.

It is enough to consider the singular solution in a vicinity of $x = 0$ since it will be in the Sobolev space $W^{1,p}(\varepsilon, 1)$ away from the endpoint $x = 0$. Let again $\hat{\Phi}_\kappa(\lambda, x)$ and $\hat{\Theta}_\kappa(\lambda, x)$ be defined as above. The kernel

$$\widetilde{G}(\lambda, x, y) := \left(\frac{x}{y}\right)^{2\kappa} \hat{\Phi}_\kappa(\lambda, x)\hat{\Theta}_\kappa(\lambda, y)^\top - \hat{\Theta}_\kappa(\lambda, x)\hat{\Phi}_\kappa(\lambda, y)^\top \tag{B.12}$$

is continuous on the triangle $\{(x,y) | 0 \le x \le y \le \varepsilon\}$ and defines a Volterra operator,

$$\begin{array}{llll} \widetilde{\mathcal{G}}: & C([0,\varepsilon],\mathbb{C}^2) & \to & C([0,\varepsilon],\mathbb{C}^2) \\ & f(x) & \mapsto & \int_x^\varepsilon \widetilde{G}(\lambda, x, y)Q(y)f(y)\, dy. \end{array} \tag{B.13}$$

Hence there exists a unique solution $\hat{\Theta}_\varepsilon(\lambda, .) \in C([0,\varepsilon],\mathbb{C}^2)$ of the Volterra integral equation of second kind,

$$f(x) = \hat{\Theta}_\kappa(\lambda, x) + \widetilde{\mathcal{G}} f(x). \tag{B.14}$$

Similarly as for the regular solution, $\Theta_\varepsilon(\lambda, x) := x^{-\kappa}\hat{\Theta}(\lambda, x)$ will solve $\tau_\kappa u = \lambda u$.

Next we want to show that $\lim_{x\to 0} x^\kappa \Theta_\varepsilon(\lambda, x) \neq 0$. Observe that the norm of $\widetilde{\mathcal{G}}$ is bounded by

$$C_\varepsilon := \max_{0\le x\le y\le\varepsilon} |\widetilde{G}(\lambda, x, y)| \int_0^\varepsilon |Q(y)|\, dy. \tag{B.15}$$

Hence, if one chooses ε small enough such that $C_\varepsilon \le \frac{1}{2}$, we can estimate equation (B.14) and observe that the solution $\hat{\Theta}_\varepsilon$ is bounded by $\hat{\Theta}_\kappa$:

$$|\hat{\Theta}_\varepsilon(\lambda, x)| \le 2 \max_{0\le x\le\varepsilon} |\hat{\Theta}_\kappa(\lambda, x)|. \tag{B.16}$$

Thus by dominated convergence,

$$\lim_{x\to 0} \int_x^\varepsilon \left(\frac{x}{y}\right)^{2\kappa} \hat{\Theta}_\kappa(\lambda, y)^\top Q(y) \hat{\Theta}_\varepsilon(\lambda, y)\, dy = 0, \tag{B.17}$$

and therefore,

$$\begin{aligned} \lim_{x\to 0} \hat{\Theta}_\varepsilon(\lambda, x) = \lim_{x\to 0} \Big[\hat{\Theta}_\kappa(\lambda, x) \left(1 + \int_x^\varepsilon \hat{\Phi}_\kappa(\lambda, y)^\top Q(y) \hat{\Theta}_\varepsilon(\lambda, y)\, dy\right) \\ - \hat{\Phi}_\kappa(\lambda, x) \int_x^\varepsilon \left(\frac{x}{y}\right)^{2\kappa} \hat{\Theta}_\kappa(\lambda, y)^\top Q(y) \hat{\Theta}_\varepsilon(\lambda, y)\, dy \Big] \end{aligned} \tag{B.18}$$

$$= \hat{\Theta}_\kappa(\lambda, 0) \left(1 + \int_0^\varepsilon \hat{\Phi}_\kappa(\lambda, y)^\top Q(y) \hat{\Theta}_\varepsilon(\lambda, y)\, dy\right) \neq 0 \tag{B.19}$$

for ε small enough. Note that $\hat{\Theta}_\kappa(\lambda, .)^\top Q(.) \hat{\Theta}_\varepsilon(\lambda, .) \in L^p(0, \varepsilon)$ since $\hat{\Theta}_\kappa$ and $\hat{\Theta}_\varepsilon$ are continuous on $[0, \varepsilon]$. Making again use of Lemma A.1, one obtains

$$\int_x^\varepsilon \Theta_\kappa(\lambda, y)^\top Q(y) \Theta_\varepsilon(\lambda, y)\, dy = x^{-2\kappa+1} \tilde{v}(\lambda, x), \quad \tilde{v}(\lambda, .) \in L^p(0, \varepsilon), \quad p \in [1, \infty). \tag{B.20}$$

Summing up, the singular solution $\Theta(\lambda, x)$ has the following properties ([2, Lemma A.1]):

- $x^\kappa \Theta(\lambda, x)$ is continuous on $[0, 1]$,
- $\lim_{x\to 0} x^\kappa \Theta_2(\lambda, x) \neq 0$ and
- regarding the second component of the singular solution, $x^{\kappa-1}\Theta_2(\lambda, x) \in L^p(0, 1)$.

To get a better picture of the integral $\int_x^1 |\Theta(\lambda, y)|^2\, dy$. Set $g(x) := x^{2\kappa} |\Theta(\lambda, x)|^2$ and observe that g is continuous on $[0, 1]$ with $g(0) > 0$ and following the exact same calculations as for the regular solution above, one concludes that $g \in W^{1,p}(0, 1)$. Then by Lemma A.2, $\hat{\mathcal{K}}_{2\kappa-1} g \in W^{1,p}(0, 1) \cap C([0, 1], \mathbb{C})$ with $\hat{\mathcal{K}}_{2\kappa-1} g(0) = g(0)/(2\kappa - 1) > 0$. In total ([2, Lemma A.2]),

$$\int_x^1 |\Theta(\lambda, y)|^2\, dy = x^{-2\kappa+1} \widetilde{w}(x), \quad \widetilde{w} \in W^{1,p}(0, 1) \cap C([0, 1], \mathbb{R}), \quad \widetilde{w} > 0. \tag{B.21}$$

Appendix C. Nevanlinna Functions

In this Appendix some relevant results about Nevanlinna functions are stated. We begin with some preliminary facts about Herglotz-Nevanlinna functions which are a subclass of generalized Nevanlinna functions. The proofs of the stated results for Herglotz functions can be found in Section 3.4 of [30], the ones for generalized Nevanlinna functions in Appendix B of [21] and the references therein. For an overview of generalized Nevanlinna functions we refer to [26].

A holomorphic mapping $H : \mathbb{C}_+ \to \mathbb{C}_+$ is called a **Herglotz–Nevanlinna** or **Herglotz function**. It can be defined on the lower half plane $\mathbb{C}_-$ by $H(\overline{z}) = \overline{H(z)}$. One of its most important properties is that it has an integral representation,

$$H(z) = \alpha + \beta z + \int_{\mathbb{R}} \Big(\frac{1}{\lambda - z} - \frac{\lambda}{1+\lambda^2}\Big) d\mu(\lambda) \tag{C.1}$$

$$= \alpha + \beta z + \int_{\mathbb{R}} \frac{1 + z\lambda}{\lambda - z} \frac{d\mu(\lambda)}{1+\lambda^2}, \tag{C.2}$$

with $\alpha \in \mathbb{R}$, $\beta \geq 0$ and a measure μ satisfying $(1+\lambda^2)^{-1} \in L^1(\mathbb{R}, d\mu)$.

Given an arbitrary measure μ on $\mathbb{R}$, its spectrum is denoted as the set of all growth points $\sigma(\mu) = \{\lambda \in \mathbb{R} | \mu((\lambda - \varepsilon, \lambda + \varepsilon)) > 0 \,\text{for all}\, \varepsilon > 0\}$. The spectrum of μ is a support of the measure, i.e., $\mu(\mathbb{R} \backslash \sigma(\mu)) = 0$. Starting with a finite Borel measure μ on $\mathbb{R}$, its **Borel transform** is defined by the mapping

$$z \mapsto F(z) = \int_{\mathbb{R}} \frac{1}{\lambda - z} d\mu(\lambda). \tag{C.3}$$

It turns out that the corresponding Borel transform is a Herglotz function and satisfies a certain growth estimate,

$$|F(z)| \leq \frac{\mu(\mathbb{R})}{\mathrm{Im}(z)}, \quad z \in \mathbb{C}_+. \tag{C.4}$$

Moreover, F will be holomorphic on $\mathbb{C} \backslash \sigma(\mu)$.

Conversely, if one starts with a Herglotz function F satisfying the growth estimate $|F(z)| \leq M/\Im(z)$, $z \in \mathbb{C}_+$, then there is a finite Borel measure μ, with $\mu(\mathbb{R}) \leq M$, such that F is the Borel transform of μ. The measure is unique and can be reconstructed by the Stieltjes–Livšić inversion formula

$$\frac{1}{2}\left(\mu\big((\lambda_0, \lambda_1)\big) + \mu\big([\lambda_0, \lambda_1]\big)\right) = \lim_{\varepsilon \downarrow 0} \frac{1}{\pi} \int_{\lambda_0}^{\lambda_1} \mathrm{Im}\big(M(\lambda + \mathrm{i}\varepsilon)\big) d\lambda, \quad \lambda_0 < \lambda_1. \tag{C.5}$$

By $\mathcal{N}_\kappa$, $\kappa \in \mathbb{N}_0$, we denote the classes of **generalized Nevanlinna functions** [23]. A complex-valued functions $M \in \mathcal{N}_\kappa$ satisfies the following properties: M is meromorphic in $\mathbb{C}_+ \cup \mathbb{C}_-$, i.e., M is holomorphic on $\mathcal{D}_M := (\mathbb{C}_+ \cup \mathbb{C}_-) \backslash P$, where P is an isolated subset of $\mathbb{C}_+ \cup \mathbb{C}_-$ consisting of the poles of M. It satisfies the symmetry condition $M(\overline{z}) = \overline{M(z)}$ and the so-called Nevanlinna kernel,

$$N_M(z, \zeta) = \frac{M(z) - \overline{M(\zeta)}}{z - \overline{\zeta}}, \quad z, \zeta \in \mathcal{D}_M, \quad z \neq \overline{\zeta}, \tag{C.6}$$

has κ negative squares. This means that for any choice of finitely many points $\{z_j\}_{j=1}^n \subset \mathcal{D}_M$, the matrix

$$\{N_M(z_j, z_k)\}_{1\le j,k\le n} \tag{C.7}$$

has at most κ negative eigenvalues and at least one such matrix has exactly κ many. In the case $\kappa = 0$, $\mathcal{N}_0$ consists precisely of the set of Herglotz functions.

For $\kappa \ge 1$, $M \in \mathcal{N}_\kappa$, a point $\lambda_0 \in \mathbb{R}$ is called a generalized pole of non-positive type of M if either

$$\limsup_{\varepsilon\downarrow 0} \varepsilon|M(\lambda_0 + \mathrm{i}\varepsilon)| = \infty \tag{C.8}$$

or the limit

$$\lim_{\varepsilon\downarrow 0}(-\mathrm{i}\varepsilon)M(\lambda_0 + \mathrm{i}\varepsilon) \tag{C.9}$$

exists and is finite and negative. The point $\lambda_0 = \infty$ is said to be a generalized pole of non-positive type of M if either

$$\limsup_{y\uparrow\infty} \frac{|M(\mathrm{i}y)|}{y} = \infty$$

or

$$\lim_{y\uparrow\infty} \frac{M(\mathrm{i}y)}{\mathrm{i}y}$$

exists and is finite and negative. All limits can be replaced by non-tangential limits.

The set of generalized Nevanlinna functions in $\mathcal{N}_\kappa$ with no non-real poles where the only generalized pole of non-positive type is at ∞ will be denoted by $\mathcal{N}_\kappa^\infty$. A function $M \in \mathcal{N}_\kappa^\infty$ admits the integral representation

$$M(z) = (1+z^2)^k \int_{\mathbb{R}} \left(\frac{1}{\lambda - z} - \frac{\lambda}{1+\lambda^2}\right) \frac{d\rho(\lambda)}{(1+\lambda^2)^k} + \sum_{j=0}^{l} a_j z^j, \tag{C.10}$$

where $k \le \kappa$, $l \le 2\kappa + 1$,

$$a_j \in \mathbb{R}, \quad \text{and} \quad \int_{\mathbb{R}} (1+\lambda^2)^{-k-1} d\rho(\lambda) < \infty. \tag{C.11}$$

The measure ρ is given by the Stieltjes–Livšić inversion formula (C.5). The representation (C.10) is called irreducible if k is chosen minimal, that is, either $k = 0$ or $\int_{\mathbb{R}} (1+\lambda^2)^{-k} d\rho(\lambda) = \infty$.

Conversely, if (C.11) holds, then $M(z)$ defined via (C.10) is in $\mathcal{N}_\kappa^\infty$ for some κ. If k is minimal, κ is given by:

$$\kappa = \begin{cases} k, & l \le 2k, \\ \lfloor \frac{l}{2} \rfloor, & l \ge 2k+1, \ l \text{ even, or, } l \text{ odd and } a_l > 0, \\ \lfloor \frac{l}{2} \rfloor + 1, & l \ge 2k+1, \ l \text{ odd and, } a_l < 0, \end{cases} \tag{C.12}$$

where $\lfloor x \rfloor = \max\{n \in \mathbb{Z} | n \le x\}$ is the floor function.

For $M \in \mathcal{N}_\kappa^\infty$, the index $\kappa \in \mathbb{N}_0$ is given by the multiplicity of the generalized pole at ∞ which is determined by the facts that the following limits exist and take values as indicated:

$$\lim_{y\uparrow\infty} -\frac{M(\mathrm{i}y)}{(\mathrm{i}y)^{2\kappa-1}} \in (0, \infty], \qquad \lim_{y\uparrow\infty} \frac{M(\mathrm{i}y)}{(\mathrm{i}y)^{2\kappa+1}} \in [0, \infty). \tag{C.13}$$

Again the limits can be replaced by non-tangential ones. To this end note that if $M(z) \in \mathcal{N}_\kappa$, then $-M(z)^{-1}$, $-M(1/z)$, and $1/M(1/z)$ also belong to $\mathcal{N}_\kappa$.

Lemma C.1 ([19, Lemma C.2])**.** *Let $M(z)$ be a generalized Nevanlinna function given by* (C.10)–(C.11) *with $l < 2k+1$. Then, for every $0 < \gamma < 2$, we have*

$$\int_{\mathbb{R}} \frac{d\rho(\lambda)}{1+|\lambda|^{2k+\gamma}} < \infty \quad \Longleftrightarrow \quad \int_1^\infty \frac{(-1)^k \mathrm{Im}(M(\mathrm{i}y))}{y^{2k+\gamma}} dy < \infty. \tag{C.14}$$

Concerning the case $\gamma = 0$,we have

$$\int_{\mathbb{R}} \frac{d\rho(\lambda)}{(1+\lambda^2)^k} = \lim_{y\to\infty} \frac{(-1)^k \mathrm{Im}(M(\mathrm{i}y))}{y^{2k-1}}, \tag{C.15}$$

where the two sides are either both finite and equal or both infinite.

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