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Relative Oscillation Theory for Jacobi Operators

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Abstract

Classical oscillation theory for Jacobi matrices connects the number of eigenvalues below a given value with the number of nodes (sign flips) of certain solutions of the underlying difference equation. The aim of this thesis is to develop a novel relative oscillation theory for Jacobi matrices which, rather than counting the number of eigenvalues of one single matrix, counts the difference between the number of eigenvalues of two different matrices. This is done by replacing nodes of solutions associated with one matrix by weighted nodes of Wronskians of solutions of two different matrices.

Zusammenfassung

Klassische Oszillationstheorie für Jacobi-Matrizen verknüpft die Anzahl der Eigenwerte unter einem vorgegebenen Wert mit der Anzahl der Knoten (Vorzeichenwechsel) gewisser Lösungen der zugrundeliegenden Differenzengleichung. Ziel dieser Diplomarbeit ist es eine neuartige relative Oszillationstheorie für Jacobi-Matrizen zu entwickeln, welche — anstatt die Eigenwerte einer einzigen Matrix zu zählen — die Differenz der Anzahl der Eigenwerte zweier verschiedener Matrizen zählt. Dazu ersetzen wir die Knoten einer zu einer Matrix gehörenden Lösung durch gewichtete Knoten der Wronski-Determinante von Lösungen zweier verschiedener Matrizen.

CONTENTS

CHAPTER 0

Introduction

Jacobi operators can be regarded as a discrete analogue to Sturm-Liouville operators, given by

$$\tau = \frac{1}{r}\left(-\frac{d}{dx}p\frac{d}{dx} + q\right),$$

which arise in quantum mechanics as a generalisation of Schrödinger operators in one dimension. Jacobi operators – as well as Sturm-Liouville operators – have a variety of possible applications in physics, for example the infinite harmonic crystal in one dimension, a model from solid state physics.
Of central importance in this respect is the investigation of the spectrum of these operators and one of the main tools is oscillation theory. Classical oscillation theory investigates the number of nodes (sign flips) of solutions of the underlying difference equation which can in turn be related to the number of eigenvalues below a certain value. We refere for example to the recent monograph on discrete oscillation theory by Agarwal et al. [1]. The aim of this thesis is to develop an extended oscillation theory which allows us to measure the difference between the spectra of two Jacobi operators. That is, we compare the Jacobi matrix $H^0_{0,n}$ and a perturbation $H^1_{0,n}$, given by

$$H^j_{0,n} = \begin{pmatrix} b_j(1) & a(1) & 0 & 0 & 0 \\ a(1) & b_j(2) & \ddots & 0 & 0 \\ 0 & \ddots & \ddots & \ddots & 0 \\ 0 & 0 & a(n-1) & b_j(n-2) & a(n-2) \\ 0 & 0 & 0 & a(n-2) & b_j(n-1) \end{pmatrix}$$

where $j \in \{0, 1\}$. We will show how the number of eigenvalues inside a given interval can be counted by counting the nodes of the Wronskian of two appropriate solutions. Since $H^0_{0,n} - H^1_{0,n}$ is not of one sign, we will weight the nodes of the Wronskian according to the sign of $H^0_{0,n} - H^1_{0,n}$.
Similar findings for Sturm-Liouville operators were presented recently by Helge Krüger and Gerald Teschl in [10], [9], [8]. Preliminary work was done in 1996 by Fritz Gesztesy, Barry Simon and Gerald Teschl in [5] and [18]. They were able

to show that the spectrum of Sturm Liouville operators and Jacobi operators are connected closely with the number of nodes of the Wronskian. I recommend Chapter 4 of [14] for further information and references. The main findings of this thesis will appear in [2].

Chapter 1 gives a short overview on difference expressions, i.e. endomorphisms of complex-valued sequences, and introduces the second order symmetric difference expression

$$\begin{aligned} \tau :\ & \ell(\mathbb{Z}) \to \ell(\mathbb{Z}) \\ & f(n) \mapsto a(n)f(n+1) + a(n-1)f(n-1) + b(n)f(n) \end{aligned}$$

which leads us to the Jacobi difference equation $\tau u = \lambda u$, $\lambda \in \mathbb{C}$. We will regard some basic facts about the corresponding solutions and their Wronskians. Last but not least we will give attention to the Jacobi operator

$$\begin{aligned} H : \ell^2(\mathbb{Z}) &\to \ell^2(\mathbb{Z}) \\ f &\mapsto \tau f \end{aligned}$$

and its finite restrictions. The content of this chapter is described in [14] in a more comprehensive way. This chapter also recapitulates some well-known facts from functional analysis as given in [20], [19] or [15].

In **Chapter 2** we introduce Prüfer variables and establish a connection between the number of sign changes, called nodes, of the solutions of the Jacobi difference equation and their Prüfer angles. As a main reference the reader is referred to [14], Chapter 4 and [18].

In **Chapter 3** we compare the solutions of two Jacobi operators with different b. Moreover, we infer a few properties of their Wronskians, which will be helpful for our further investigations and which provides us with the fact that in a proper setting the Prüfer angle is monotonically increasing, resp. decreasing, which is a key ingredient for our proofs in chapter 5.

Chapter 4 gives a proof of Sturm's Separation Theorem, cf. [6], Section 6.2.

Chapter 5 proves the fact that the difference of the number of eigenvalues below λ of two finite Jacobi operators differing in b equals the number of weighted nodes of the Wronskian of suitable solutions u of the Jacobi difference equation $\tau u = \lambda u$. Therefore, we first show how the sign changes of the Wronskian and the difference of the corresponding Prüfer angles are connected in detail. By weighting the nodes of the Wronskian according to the sign of $b_0 - b_1$, we will see that it is possible to count them using Prüfer angles. The connection between the spectrum of a Jacobi operator and the nodes of a suitable solution goes back to the work of Gerald Teschl, cf. [18].

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CHAPTER 1

A Brief Summary of Jacobi Operators

1.1 The Jacobi Difference Equation

In this chapter we will review some basic terms and definitions on Jacobi operators. Most of these facts can be found more comprehensive in [14]. For this purpose especially Chapters 1, 2, and 4 are relevant.

Definition 1.1. *Let I be a subset of $\mathbb{Z}$ and $M = \mathbb{R}$ or $M = \mathbb{C}$. By $\ell(I, M)$ we denote the set of M-valued sequences $(f(n))_{n\in I}$ and $\ell(I) = \ell(I, \mathbb{C})$. Furthermore we define*

$$\begin{aligned} \ell(n_1, n_2) &= \ell(\{n \in \mathbb{Z} \,|\, n_1 < n < n_2\}), \\ \ell(n_1, \infty) &= \ell(\{n \in \mathbb{Z} \,|\, n_1 < n\}), \\ \ell(\infty, n_2) &= \ell(\{n \in \mathbb{Z} \,|\, n < n_2\}) \end{aligned} \tag{1.1}$$

for all n_1, $n_2 \in \mathbb{Z}$. Moreover,

$$\delta_n(m) = \delta_{n,m} = \begin{cases} 0 & \text{if} \quad n \neq m \\ 1 & \text{if} \quad n = m \end{cases} \tag{1.2}$$

denotes the canonical basis of $\ell(\mathbb{Z})$.

Definition 1.2. *A* difference expression *is an endomorphism of $\ell(\mathbb{Z})$, given by*

$$\begin{aligned} R:\ \ell(\mathbb{Z}) &\to \ell(\mathbb{Z}) \\ u &\mapsto Ru. \end{aligned} \tag{1.3}$$

Each difference expression is uniquely determined by its corresponding matrix representation, given by

$$(R(m,n))_{m,n\in\mathbb{Z}} = (R\delta_n)(m) = \langle \delta_m, R\delta_n \rangle. \tag{1.4}$$

Definition 1.3. *We call a difference expression R* symmetric *if*

$$R(m,n) = R(n,m) \quad \text{for all } m,n \in \mathbb{Z}. \tag{1.5}$$

The order of a difference expression R is given by $N \in \mathbb{N}$ if N is the smallest non-negative integer which satisfies $R(m,n) = 0$ for all $n-m > N_+$ and $m-n > N_-$ with $N = N_+ + N_-$. If no such number exists we call R of infinite order.

Definition 1.4. *Let $a, b \in \ell(\mathbb{Z})$ be real-valued sequences which satisfy $a(n) \in \mathbb{R}\setminus\{0\}$, $b(n) \in \mathbb{R}$, $n \in \mathbb{Z}$. Then τ denotes the second order symmetric difference expression given by*

$$\begin{aligned} \tau :\ & \ell(\mathbb{Z}) \to \ell(\mathbb{Z}) \\ & f(n) \mapsto a(n)f(n+1) + a(n-1)f(n-1) + b(n)f(n) \end{aligned} \tag{1.6}$$

associated with the tridiagonal matrix

$$\begin{pmatrix} \ddots & \ddots & 0 & 0 & 0 \\ a(n-2) & b(n-1) & a(n-1) & 0 & 0 \\ 0 & a(n-1) & b(n) & a(n) & 0 \\ 0 & 0 & a(n) & b(n+1) & a(n+1) \\ 0 & 0 & 0 & \ddots & \ddots \end{pmatrix}. \tag{1.7}$$

Moreover, the Jacobi difference equation *is defined as*

$$\tau u = \lambda u, \tag{1.8}$$

where $u \in \ell(\mathbb{Z})$ and $\lambda \in \mathbb{C}$.

A short calculation shows that

$$\begin{aligned} (\tau f)(n) &= -(\partial^* a \partial f)(n) + \big(a(n-1) + a(n) + b(n)\big) f(n) \\ &= \partial(a(n-1)\partial f(n-1)) + \big(a(n-1) + a(n) + b(n)\big) f(n) \end{aligned} \tag{1.9}$$

where

$$\begin{aligned} (\partial f)(n) &= f(n+1) - f(n), \\ (\partial^* f)(n) &= f(n-1) - f(n) \end{aligned} \tag{1.10}$$

and we find that τ is a discrete analogue to the well-known Sturm-Liouville operator, given by

$$\tau = \frac{1}{r}\left(-\frac{d}{dx}p\frac{d}{dx} + q\right). \tag{1.11}$$

Lemma 1.5 (Summation by Parts)**.**

$$\sum_{i=m}^{n} g(i)(\partial f)(i) = g(n)f(n+1) - g(m-1)f(m) + \sum_{i=m}^{n} (\partial^* g)(i) f(i) \tag{1.12}$$

Definition 1.6. *For all f, $g \in \ell(\mathbb{Z})$ we define the* (modified) Wronskian

$$W_n(f,g) \ = a(n)\big(f(n)g(n+1) - f(n+1)g(n)\big) \tag{1.13}$$

and

$$W_{\pm\infty}(f,g) \ = \ \lim_{n\to\pm\infty} W_n(f,g) \tag{1.14}$$

provided the limit exists.

The Wronskian is given by

$$W_n(f,g) = \begin{vmatrix} f(n) & g(n) \\ f(n+1) & g(n+1) \end{vmatrix} = \begin{vmatrix} f(n) & g(n) \\ (\partial f)(n) & (\partial g)(n) \end{vmatrix} \tag{1.15}$$

and satisfies

$$\begin{aligned} W_n(f,f) &= 0, \\ W_n(f,g) &= -W_n(g,f), \\ W_n(cf,g) &= W_n(f,cg) = cW_n(f,g) \quad \text{for all } c \in \mathbb{C}, \\ W_n(f+\tilde{f},g) &= W_n(f,g) + W_n(\tilde{f},g), \\ W_n(f,g+\tilde{g}) &= W_n(f,g) + W_n(f,\tilde{g}), \end{aligned} \tag{1.16}$$

and if f, g are real-valued we have

$$W_n(f,g) = 0 \quad \Rightarrow \quad \exists\, c \in \mathbb{R}\text{: } f(n) = cg(n) \text{ and } f(n+1) = cg(n+1). \tag{1.17}$$

Lemma 1.7 (Green's Formula)**.** *Let f, $g \in \ell(\mathbb{Z})$, then*

$$\sum_{j=m}^{n} \big(f(\tau g) - (\tau f)g\big)(j) = W_n(f,g) - W_{m-1}(f,g). \tag{1.18}$$

Proof. We have

$$\begin{aligned} &\sum_{j=m}^{n} \big(f(\tau g) - (\tau f)g\big)(j) \\ &= \sum_{j=m}^{n} f(j)\big(a(j)g(j+1) + a(j-1)g(j-1) + b(j)g(j)\big) \\ &- \sum_{j=m}^{n} \big(a(j)f(j+1) + a(j-1)f(j-1) + b(j)f(j)\big)g(j) \\ &= \sum_{j=m}^{n} \big(W_j(f,g) - W_{j-1}(f,g)\big) \\ &= W_n(f,g) - W_{m-1}(f,g). \end{aligned} \tag{1.19}$$

□

Lemma 1.8. *Let f, $g \in \ell(\mathbb{Z})$, then the Wronskian $W(f,g)$ is nonzero if and only if f and g are linearly independent. If $\tau f = \lambda f$ and $\tau g = \lambda g$ for some $\lambda \in \mathbb{C}$, then the Wronskian $W_n(f,g)$ is constant.*

Proof. We say f and g are linearly dependent if $f(n) = cg(n)$ for some $c \in \mathbb{C}$ and for all $n \in \mathbb{Z}$. Thus,

$$\begin{aligned} & f,\ g \text{ are linearly dependent} \\ \Leftrightarrow\ & f(n) = cg(n) \\ \Leftrightarrow\ & f(n)cg(n+1) = cg(n)f(n+1) \\ \Leftrightarrow\ & W_n(f,g) = 0, \end{aligned} \tag{1.20}$$

for all $n \in \mathbb{N}$, which proofs the lemma. If f, g solve (1.8) with the same $\lambda \in \mathbb{C}$ we have

$$W_n(f,g) - W_{m-1}(f,g) = \sum_{i=m}^{n} \big(f(\tau g) - (\tau f)g\big)(i) = 0. \tag{1.21}$$

□

Suppose that $a(n) \neq 0$ for all $n \in \mathbb{Z}$, then for any arbitrary given values in two consecutive points $u(n_0)$ and $u(n_0+1)$ there is a unique solution of $\tau u = \lambda u$ in $\ell(\mathbb{Z})$. Furthermore, for all λ in $\mathbb{C}$ there are two linearly independent solutions of the Jacobi difference equation. Moreover, two linearly independent solutions cannot have a common zero, since the Wronskian has no zeros.

1.2 The Fundamental Solutions

The solution space of the Jacobi difference equation (1.8) is two-dimensional and hence it is possible to choose two linearly independent solutions of (1.8), namely c, s, the fundamental solutions.

Definition 1.9. *For any $n_0 \in \mathbb{Z}$, $\lambda \in \mathbb{C}$ we define the* fundamental solutions *c, $s \in \ell(\mathbb{Z})$ as*

$$\tau c(.,n_0) = \lambda c(.,n_0) \quad \textit{where} \quad c(n_0,n_0) = 1,\ c(n_0+1,n_0) = 0 \tag{1.22}$$

and

$$\tau s(.,n_0) = \lambda s(.,n_0) \quad \textit{where} \quad s(n_0,n_0) = 0,\ s(n_0+1,n_0) = 1. \tag{1.23}$$

We will omit n_0 whenever it is zero, that is,

$$c(n) := c(n,0) \quad \textit{and} \quad s(n) := s(n,0). \tag{1.24}$$

Any solution u of (1.8) is a linear combination of the fundamental solutions, such that

$$u(n) = \frac{W(u,s)}{W(c,s)}c(n) - \frac{W(u,c)}{W(c,s)}s(n). \tag{1.25}$$

Using induction it is straightforward to show that, for any $k \geq 0$, $c(n_0+k,n_0)$ and $s(n_0+k,n_0)$ are polynomials of order at most k with respect to λ.

Definition 1.10. *For all n_1, $n_2 \in \mathbb{Z}$ where $n_1 > n_2 + 1$ we define the* Jacobi matrix

$$J_{n_1,n_2} = \begin{pmatrix} b(n_1+1) & a(n_1+1) & 0 & 0 & 0 \\ a(n_1+1) & b(n_1+2) & \ddots & 0 & 0 \\ 0 & \ddots & \ddots & \ddots & 0 \\ 0 & 0 & a(n_2-1) & b(n_2-2) & a(n_2-2) \\ 0 & 0 & 0 & a(n_2-2) & b(n_2-1) \end{pmatrix}. \tag{1.26}$$

Lemma 1.11. *Let n_1, $n_2 \in \mathbb{Z}$, $n_1 > n_2+1$ and let s be the fundamental solution of $\tau s = \lambda s$, then we have*

$$s(\lambda, n_1, n_2) = \frac{\det(\lambda - J_{n_1,n_2})}{\prod_{i=n_1+1}^{n_2-1} a(i)}. \tag{1.27}$$

Proof. cf. [14], (1.65). □

Since the Wronskian of $c(.,n_0)$ and $s(.,n_0)$ is constant, we can evaluate it at the point n_0 and thus we have

$$W_n(c(.,n_0),s(.,n_0)) = a(n_0). \tag{1.28}$$

Hence, for arbitrary solutions u, v of the Jacobi difference equation (1.25) implies

$$u(n) = u(n_0)c(n,n_0) + u(n_0+1)s(n,n_0) \tag{1.29}$$

and

$$W_n(u(.,n_0),v(.,n_0)) = a(n_0)(u(n_0)v(n_0+1) - u(n_0+1)v(n_0)). \tag{1.30}$$

1.3 The Jacobi Difference Operator

In this section we will have a look at operators acting on the Hilbert space $\ell^2(\mathbb{Z})$ associated with the Jacobi difference equation. From now on we assume that a, $b \in \ell^\infty(\mathbb{Z})$ are bounded and as already noted before we will furthermore assume that $a(n) \in \mathbb{R}\setminus\{0\}$, $b(n) \in \mathbb{R}$, $n \in \mathbb{Z}$.

Definition 1.12. *For $1 \le p < \infty$ we define the Banach spaces*

$$\ell^p(I,M) = \Big\{ f \in \ell(I,M) \mid \sum_{n\in I} |f(n)|^p < \infty \Big\} \tag{1.31}$$

and

$$\ell^\infty(I,M) = \Big\{ f \in \ell(I,M) \mid \sup_{n\in I} |f(n)| < \infty \Big\} \tag{1.32}$$

with the corresponding norms

$$\begin{aligned} \|f\|_p &= \Big(\sum_{n\in I}|f(n)|^p\Big)^{\frac{1}{p}} \quad \text{for all } f \in \ell^p(I,M), \\ \|f\|_\infty &= \sup_{n\in I}|f(n)| \quad \text{for all } f \in \ell^\infty(I,M). \end{aligned} \tag{1.33}$$

Definition 1.13. *We define $\ell^p_\pm(\mathbb{Z},M)$ as the set of sequences in $\ell(\mathbb{Z},M)$ which are ℓ^p near $\pm\infty$, i.e. all sequences in $\ell(\mathbb{Z},M)$ whose restriction to $\ell(\pm\mathbb{N},M)$ belongs to $\ell^p(\pm\mathbb{N},M)$.*

The vector space $\ell^2(\mathbb{Z})$ is a Hilbert space with scalar product

$$\langle f,g\rangle = \sum_{n\in\mathbb{Z}} f(n)^* g(n) \quad \text{and} \quad \|f\| = \sqrt{\sum_{n\in\mathbb{Z}} |f(n)|^2} \tag{1.34}$$

where $f(n)^*$ denotes the complex conjugation of $f(n)$.

Lemma 1.14. *For all f, g, τf, $\tau g \in \ell^2_\pm(\mathbb{Z})$ we have*

$$W_{\pm\infty}(f,g) = 0. \tag{1.35}$$

Proof. We infer

$$W_{\pm\infty}(f,g) = \lim_{n\to\pm\infty} a(n)(f(n)g(n+1) - f(n+1)g(n)). \tag{1.36}$$

Since $a \in \ell^\infty$ the last term is in $\ell^2_\pm$, thus the limit is zero. □

Definition 1.15. *Let a and b be in $\ell^\infty(\mathbb{Z},\mathbb{R})$ and $a(n) \neq 0$. Then we call*

$$\begin{aligned} H : \ell^2(\mathbb{Z}) &\to \ell^2(\mathbb{Z}) \\ f &\mapsto \tau f \end{aligned} \tag{1.37}$$

the Jacobi operator associated with *a, b.*

The norm of H is given by

$$\|H\| = \sup_{f\,:\,\|f\|=1} \|Hf\|_2 \tag{1.38}$$

and since $a(n)$ and $b(n)$ are real we have $(Hf)^* = Hf^*$.

Definition 1.16. *Let $\mathfrak{H}$ be a Hilbert space and let $A : \mathfrak{D}(A) \to \mathfrak{H}$, $\mathfrak{D}(A)$ dense. The* adjoint operator *A^* is defined by*

$$\begin{aligned} &\mathfrak{D}(A^*) = \{f \in \mathfrak{H} \,|\, \exists \tilde{f} \in \mathfrak{H} : \langle f, Ag\rangle = \langle \tilde{f}, g\rangle \forall\ g \in \mathfrak{D}(A)\} \\ &A^* f = \tilde{f} \end{aligned} \tag{1.39}$$

A is called self-adjoint *if $A = A^*$.*

Remark 1.17. *We have*

$$\begin{aligned} \|H\| &= \sup_{f\,:\,\|f\|=1} \|(a(n)f(n+1) + a(n-1)f(n-1) + b(n)f(n))_{n\in\mathbb{Z}}\| \\ &\leq \sup_{f\,:\,\|f\|=1} \|a\|_\infty \|f^-\| + \|a\|_\infty \|f^+\| + \|b\|_\infty \|f\| \\ &\leq 2\|a\|_\infty + \|b\|_\infty, \end{aligned} \tag{1.40}$$

where $f^-(n) = f(n+1)$ and $f^+(n) = f(n-1)$ for all $n \in \mathbb{Z}$.

Theorem 1.18. *The Jacobi operator H is bounded and self-adjoint.*

Proof. Green's formula and Lemma 1.14 imply

$$\begin{aligned} &\langle f, Hf\rangle - \langle Hf, f\rangle \\ &= \sum_{n\in\mathbb{Z}} (f^*(n)(Hf)(n) - (Hf)^*(n)f(n)) \\ &= \sum_{n\in\mathbb{Z}} (f^*(n)(\tau f)(n) - (\tau f^*)(n)f(n)) \\ &= W_\infty(f^*, f) - W_{-\infty}(f^*, f) = 0. \end{aligned} \tag{1.41}$$

Since H is bounded, symmetric and defined on the entire Hilbert space $\ell^2(\mathbb{Z})$, H is self-adjoint. (Hellinger-Toeplitz [cf. e.g. [20] Th. V.5.5]) □

Definition 1.19. *Let a, b_0 and b_1 be in $\ell^\infty(\mathbb{Z},\mathbb{R})$ and $a(n) \neq 0$, then we define*

$$\begin{aligned} H_t : \ell^2(\mathbb{Z}) &\to \ell^2(\mathbb{Z}) \\ f &\mapsto \tau_t f \end{aligned} \tag{1.42}$$

where

$$(\tau_t f)(n) = a(n)f(n+1) + a(n-1)f(n-1) + ((1-t)b_0(n) + tb_1(n))f(n) \tag{1.43}$$

for all $t \in [0,1]$.

Remark 1.20. *We have*

$$H_t = (1-t)H_0 + tH_1 = H_0 + t(b_1 - b_0). \tag{1.44}$$

Definition 1.21. *For any $n_0 \in \mathbb{Z}$ we define the restriction H_{+,n_0} of H to the subspace $\ell^2(n_0,\infty)$ as*

$$H_{+,n_0}f(n) = \begin{cases} a(n_0+1)f(n_0+2) + b(n_0+1)f(n_0+1) & \text{if} \quad n = n_0+1 \\ (\tau f)(n) & \text{if} \quad n > n_0+1 \end{cases} \tag{1.45}$$

and the restriction H_{-,n_0} of H to the subspace $\ell^2(-\infty,n_0)$ as

$$H_{-,n_0}f(n) = \begin{cases} a(n_0-2)f(n_0-2) + b(n_0-1)f(n_0-1) & \text{if} \quad n = n_0-1 \\ (\tau f)(n) & \text{if} \quad n < n_0-1 \end{cases} \tag{1.46}$$

and the restriction H_{n_1,n_2} of H to the subspace $\ell^2(n_1,n_2)$ as

$$H_{n_1,n_2}f(n) = \begin{cases} a(n_1+1)f(n_1+2) + b(n_1+1)f(n_1+1) & \text{if} \quad n = n_1+1 \\ (\tau f)(n) \quad \text{if} \quad n_1+1 < n < n_2-1 \\ a(n_2-2)f(n_2-2) + b(n_2-1)f(n_2-1) & \text{if} \quad n = n_2-1 \end{cases} \tag{1.47}$$

These operators are again bounded and self-adjoint. Moreover, H_{n_1,n_2} is associated with the Jacobi matrix J_{n_1,n_2}.

Definition 1.22. *Moreover, we set $H^{\infty,\infty}_{n_1,n_2} = H_{n_1,n_2}$, $H^{0,\beta_2}_{n_1,n_2} = H^{\infty,\beta_2}_{n_1+1,n_2}$, and*

$$\begin{aligned} H^{\beta_1,\beta_2}_{n_1,n_2} &= H^{\infty,\beta_2}_{n_1,n_2} - a(n_1)\beta_1^{-1}\langle\delta_{n_1+1},.\rangle\delta_{n_1+1}, \quad \beta_1 \neq 0, \\ H^{\beta_1,\beta_2}_{n_1,n_2} &= H^{\beta_1,\infty}_{n_1,n_2+1} - a(n_2)\beta_2\langle\delta_{n_2},.\rangle\delta_{n_2}, \quad \beta_2 \neq \infty. \end{aligned} \tag{1.48}$$

Remark 1.23. *Note that H^β_{+,n_0} can be associated with the following domain*

$$\mathfrak{D}(H^\beta_{+,n_0}) = \{f \in \ell^2(n_0,\infty) | \cos(\alpha)f(n_0) + \sin(\alpha)f(n_0+1) = 0\}, \tag{1.49}$$

$\beta = \cot(\alpha) \neq 0$, if one agrees that only points with $n > n_0$ are of significance and that the last point is only added as a dummy variable so that one does not have to specify an extra expression for $(\tau f)(n_0+1)$. In particular, the case $\beta = \infty$ (i.e., corresponding to the boundary condition $f(n_0) = 0$) is known as Dirichlet boundary condition at n_0. Analogously for H^β_{-,n_0} and $H^{\beta_1,\beta_2}_{n_1,n_2}$.

CHAPTER 2

Oscillation Theory

2.1 Prüfer Variables

We will first recall a few basic results on the solutions of the Jacobi difference equation given in (1.8) before we subsequently will have a look at the Wronskian of two solutions solving (1.8) with different $b \in \ell(\mathbb{Z})$.

Remark 2.1. *(cf. [18], Remark 2.2) Introduce $H_\varepsilon = U_\varepsilon H U_\varepsilon^{-1}$ where $U_\varepsilon = U_\varepsilon^{-1}$ is a unitary operator defined via $(U_\varepsilon f)(n) = \tilde{\varepsilon}(n) f(n)$ with $\tilde{\varepsilon}(n) \in \{+1, -1\}$ and $\tilde{\varepsilon}(n)\tilde{\varepsilon}(n+1) = \varepsilon(n)$. Then H_ε is associated with the sequences $a_\varepsilon(n) = \varepsilon(n)a(n)$, $b_\varepsilon(n) = b(n)$, $n \in \mathbb{Z}$ and the case $a(n) \neq 0$ can be easily reduced to the case $a(n) < 0$.*

Thus, for $a(n) \in \mathbb{R} \setminus \{0\}$ it is no restriction to assume that $a(n) < 0$ for all $n \in \mathbb{Z}$, what we will do from now on. Furthermore, we assume that $b(n) \in \mathbb{R}$ for all $n \in \mathbb{Z}$ and a solution of (1.8) will always mean a real valued, nonzero solution.

Lemma 2.2. *Let u be a solution of (1.8) and $u(n) = 0$ for some $n \in \mathbb{N}$, then we have*

$$u(n-1)u(n+1) < 0. \tag{2.1}$$

Proof. Suppose that $u(n) = 0$, then (1.8) implies

$$a(n)u(n+1) = -a(n-1)u(n-1) \tag{2.2}$$

and thus we have

$$-\frac{u(n-1)}{u(n+1)} = \frac{a(n)}{a(n-1)} > 0. \tag{2.3}$$

Note that a solution (not vanishing identically!) cannot be zero at two consecutive points and that we made the assumption $a(n) < 0$ for all $n \in \mathbb{Z}$. □

In some sense this lemma says that nontrivial solutions of (1.8) can only have "simple" zeros. Moreover, every (nontrivial) solution fulfills

$$(u(n), u(n+1)) \neq (0, 0) \quad \text{for all } n \in \mathbb{Z}, \tag{2.4}$$

which allows us to introduce Prüfer variables. For all $n \in \mathbb{Z}$ we set

$$\begin{aligned} u(n) &= \rho_u(n) \sin \theta_u(n) \quad \text{and} \\ u(n+1) &= \rho_u(n) \cos \theta_u(n). \end{aligned} \tag{2.5}$$

That is,

$$\rho_u(n) \sin \theta_u(n) = \rho_u(n-1) \cos \theta_u(n-1) \tag{2.6}$$

and

$$\theta_u(n) = \begin{cases} \operatorname{arccot}\left(\frac{u(n+1)}{u(n)}\right), & \text{if } u(n) \neq 0, \\ k\pi \text{ for some } k \in \mathbb{Z}, & \text{otherwise.} \end{cases} \tag{2.7}$$

Note that $\rho_u(n) > 0$ for all $n \in \mathbb{Z}$ and $\theta_u(n)$ is only defined up to an additive integer multiple of 2π, depending on n. For our further investigations it is essential to gain unique values for the Prüfer angle and therefore we fix $\theta_u(0)$ and require

$$\lceil \theta_u(n)/\pi \rceil \leq \lceil \theta_u(n+1)/\pi \rceil \leq \lceil \theta_u(n)/\pi \rceil + 1, \tag{2.8}$$

where $x \mapsto \lceil x \rceil$ denotes the ceiling function given by

$$x \mapsto \lceil x \rceil = \min\{n \in \mathbb{Z} \,|\, n \geq x\}. \tag{2.9}$$

The function $x \mapsto \lceil x \rceil - 1$ is a left-continuous analogue to the well-known floor function defined as

$$x \mapsto \lfloor x \rfloor = \max\{n \in \mathbb{Z} \,|\, n \leq x\} \tag{2.10}$$

which itself is a right-continuous step function.

2.2 Nodes of Solutions

Whereas in the continuous case one considers zeros of solutions of Sturm-Liouville differential equations, in the discrete case we are interested in zeros and sign changes of a solution of (1.8), denoted as *nodes*.

Definition 2.3. *We call $n \in \mathbb{Z}$ a* node *of a solution u of (1.8) if either*

$$u(n) = 0 \quad \text{or} \quad u(n)u(n+1) < 0. \tag{2.11}$$

We say that a node n_0 of u lies between *m* and *n if either*

$$m < n_0 < n \quad \text{or} \quad n_0 = m \text{ but } u(m) \neq 0. \tag{2.12}$$

$\#(u)$ denotes the number of nodes of u and $\#_{(m,n)}(u)$ denotes the number of nodes of u between m and n.

Lemma 2.4. *Suppose that (2.8) holds for a solution u of (1.8) and for an arbitrary $n_0 \in \mathbb{Z}$ we choose some $k \in \mathbb{Z}$ such that $\theta_u(n_0) = k\pi + \gamma$ with $\gamma \in (0, \pi]$ and $\theta_u(n_0 + 1) = k\pi + \Gamma$, then we have*

$$\gamma \in \begin{cases} (0, \frac{\pi}{2}] & \text{iff} \quad n_0 \text{ is not a node} \\ (\frac{\pi}{2}, \pi] & \text{iff} \quad n_0 \text{ is a node} \end{cases} \tag{2.13}$$

and

$$\Gamma \in \begin{cases} (0,\pi] & \textit{iff} \quad n_0 \textit{ is not a node} \\ (\pi, 2\pi) & \textit{iff} \quad n_0 \textit{ is a node} \end{cases} \tag{2.14}$$

Moreover,

$$\theta_u(n) = k\pi + \frac{\pi}{2} \quad \Leftrightarrow \quad \theta_u(n+1) = (k+1)\pi. \tag{2.15}$$

Proof. Condition (2.8) implies that $\Gamma \in (0, 2\pi]$.
Suppose that there is no node at n, then $\gamma \in (0,\pi)$. Suppose that $\pm u(n) > 0$, then $\pm\sin(k\pi+\gamma) > 0$. Furthermore we have $\pm u(n+1) \geq 0$ and thus $\pm\cos(k\pi+\gamma) \geq 0$. Therefore $\gamma \in (0, \frac{\pi}{2})$ and $k \equiv {0 \atop 1} \mod 2$. Moreover,

$$\pm \sin\theta_u(n+1) = \pm(-1)^k \sin(\Gamma) \geq 0 \tag{2.16}$$

implies $\Gamma \in (0,\pi]$.
On the other hand suppose that n is a node, then there are two cases:
Either we have $u(n) = 0$, so $\gamma = \pi$ and $\theta_u(n) = (k+1)\pi$ where $k \equiv {0 \atop 1} \mod 2$. Thus,

$$\pm \rho_u(n)\cos((k+1)\pi) = \pm u(n+1) < 0. \tag{2.17}$$

And hence

$$\pm \sin\theta_u(n+1) = \pm(-1)^k \sin(\Gamma) < 0 \tag{2.18}$$

implies $\Gamma \in (\pi, 2\pi)$. Or we have $\pm u(n) > 0$ and $\pm u(n+1) < 0$, thus we infer

$$\pm \sin\theta_u(n) = \pm(-1)^k \sin(\gamma) > 0 \tag{2.19}$$

and

$$\pm \sin\theta_u(n+1) = \pm(-1)^k \sin(\Gamma) < 0. \tag{2.20}$$

Now, $\gamma \in (0,\pi)$ implies $\Gamma \in (\pi, 2\pi)$. Moreover, by

$$\pm u(n+1) = \pm\rho_u(n)\cos\theta_u(n) < 0 \tag{2.21}$$

we conclude that $\sin\theta_u(n)$ and $\cos\theta_u(n)$ are of different sign and thus $\gamma \in (\frac{\pi}{2}, \pi)$.
Suppose that $\theta_u(n) = k\pi + \frac{\pi}{2}$, then we have $u(n+1) = \rho_u(n)\cos\theta_u(n) = 0$ and thus Γ is an integer multiply of π and $u(n)u(n+1) < 0$. Hence,

$$\sin(k\pi + \frac{\pi}{2})\cos(k\pi + \Gamma) = (-1)^k \sin(\frac{\pi}{2})(-1)^k\cos(\Gamma) < 0 \tag{2.22}$$

implies $\Gamma = \pi$.
Conversely, suppose that $\theta_u(n+1) = (k+1)\pi$, then we have $\cos\theta_u(n) = 0$ and thus $\theta_u(n) = l\pi + \frac{\pi}{2}$ where $l \in \mathbb{Z}$. Furthermore Lemma 2.2 implies $u(n)u(n+2) < 0$ and we have

$$\sin(l\pi + \frac{\pi}{2})\cos((k+1)\pi) = (-1)^l \sin(\frac{\pi}{2})(-1)^{k+1} < 0. \tag{2.23}$$

Hence, $l \equiv k \mod 2$ and (2.8) implies $l = k$. □

Corollary 2.5. *Suppose that (2.8) holds for a solution u of (1.8), then*

$$\lceil \frac{\theta_u(n+1)}{\pi} \rceil = \begin{cases} \lceil \frac{\theta_u(n)}{\pi} \rceil + 1 & \textit{if } n \textit{ is a node,} \\ \lceil \frac{\theta_u(n)}{\pi} \rceil & \textit{otherwise.} \end{cases} \tag{2.24}$$

By this means it is possible to count nodes of solutions of the Jacobi difference equation using Prüfer variables and the number of nodes in an interval (m, n), possibly infinite, is given by

Theorem 2.6. *Let $m < n$. Suppose that (2.8) holds for a solution u of (1.8), then*

$$\#_{(m,n)}(u) = \lceil \frac{\theta_u(n)}{\pi} \rceil - \lfloor \frac{\theta_u(m)}{\pi} \rfloor - 1 \tag{2.25}$$

and

$$\#(u) = \lim_{n\to\infty} \Big(\lceil \frac{\theta_u(n)}{\pi} \rceil - \lfloor \frac{\theta_u(-n)}{\pi} \rfloor - 1 \Big). \tag{2.26}$$

Proof. We proof the theorem by induction.
Let $n = m + 1$.
If $u(m) = 0$, then $u(n) \neq 0$ and according to the definition the node m of u doesn't lie in (m, n). For any k in $\mathbb{Z}$ we have $\theta_u(m) = (k+1)\pi$ and $\theta_u(n) = k\pi+\Gamma$ with $\Gamma \in (\pi, 2\pi]$. Hence,

$$\lceil \frac{\theta_u(n)}{\pi} \rceil - \lfloor \frac{\theta_u(m)}{\pi} \rfloor - 1 = 0 = \#_{(m,n)}(u). \tag{2.27}$$

If $u(m) \neq 0$, we have

$$\lfloor \frac{\theta_u(m)}{\pi} \rfloor = \lceil \frac{\theta_u(m)}{\pi} \rceil - 1 = \begin{cases} \lceil \frac{\theta_u(n)}{\pi} \rceil - 2 & \text{if } m \text{ is a node,} \\ \lceil \frac{\theta_u(n)}{\pi} \rceil - 1 & \text{otherwise.} \end{cases} \tag{2.28}$$

Hence,

$$\lceil \frac{\theta_u(n)}{\pi} \rceil - \lfloor \frac{\theta_u(m)}{\pi} \rfloor - 1 = \#_{(m,n)}(u). \tag{2.29}$$

We assume that the theorem already holds for some $n \geq m + 1$ and if n is a node we have

$$\begin{aligned} \#_{(m,n+1)}(u) &= \#_{(m,n)}(u) + 1 \\ &= \lceil \frac{\theta_u(n)}{\pi} \rceil - \lfloor \frac{\theta_u(m)}{\pi} \rfloor \\ &= \lceil \frac{\theta_u(n+1)}{\pi} \rceil - \lfloor \frac{\theta_u(m)}{\pi} \rfloor - 1. \end{aligned} \tag{2.30}$$

The same holds if n is no node of u. □

Since the fundamental solution s of the Jacobi difference equation has boundary condition $s(0) = 0$ and we normalize θ_s such that $\theta_s(0) = 0$ we have

Corollary 2.7. *Let $n > 0$. Suppose that (2.8) holds for the fundamental solution s of (1.8), then*

$$\#_{(0,n)}(s) = \lceil \frac{\theta_s(n)}{\pi} \rceil - 1. \tag{2.31}$$

2.3 The Riccati Equation

Lemma 2.8 (Riccati Equation)**.** *If* $\sin\theta_u(n) \neq 0$ *and* $\cos\theta_u(n) \neq 0$ *for all* $n \in \mathbb{N}$*, then*

$$\tau u = \lambda u \;\Leftrightarrow\; a(n)\cot\theta_u(n) + a(n-1)\tan\theta_u(n-1) = \lambda - b(n). \tag{2.32}$$

Proof. We have

$$\begin{aligned}
& (\tau - \lambda)u = 0 \\
\Leftrightarrow\quad & a(n)u(n+1) + a(n-1)u(n-1) + (b(n) - \lambda)u(n) = 0 \\
\Leftrightarrow\quad & a(n)\rho_u(n)\cos\theta_u(n) + a(n-1)\rho_u(n-1)\sin\theta_u(n-1) \\
& + (b(n) - \lambda)\rho_u(n)\sin\theta_u(n) = 0 \qquad (2.33) \\
\Leftrightarrow\quad & a(n)\cot\theta_u(n) + a(n-1)\frac{\rho_u(n-1)\sin\theta_u(n-1)}{\rho_u(n)\sin\theta_u(n)} + b(n) - \lambda = 0 \\
\Leftrightarrow\quad & a(n)\cot\theta_u(n) + a(n-1)\tan\theta_u(n-1) = \lambda - b(n).
\end{aligned}$$

□

CHAPTER 3

The Modified Jacobi Difference Equation

Definition 3.1. *Let a, b_0, $b_1 \in \ell(\mathbb{Z})$ be real-valued sequences which satisfy $a(n) < 0$. We define*

$$\begin{aligned}\tau_0 :\ & \ell(\mathbb{Z}) \to \ell(\mathbb{Z})\\ & u(n) \mapsto a(n)u(n+1) + a(n-1)u(n-1) + b_0(n)u(n),\\ \tau_1 :\ & \ell(\mathbb{Z}) \to \ell(\mathbb{Z})\\ & u(n) \mapsto a(n)u(n+1) + a(n-1)u(n-1) + b_1(n)u(n)\end{aligned} \tag{3.1}$$

and $\tau_t = \tau_0 + t(b_1 - b_0)$ for all $t \in [0,1]$.

Lemma 3.2. *For all $t \in [0,1]$ let u_t be solutions of $\tau_t u_t = \lambda u_t$ where $u_t(n_0) = (1-t)u_0(n_0) + tu_1(n_0)$ and $u_t(n_0+1) = (1-t)u_0(n_0+1) + tu_1(n_0+1)$, $u_0(n_0)$, $u_1(n_0)$, $u_0(n_0+1)$, $u_1(n_0+1) \in \mathbb{R}$, $n_0 \in \mathbb{Z}$, then for any $m \in \mathbb{Z}$, $t \mapsto u_t(m)$ is a polynomial in t of order at most*

$$\begin{cases} m - n_0 & \textit{if} \quad m > n_0, \\ |m - n_0| + 1 & \textit{if} \quad m \le n_0. \end{cases} \tag{3.2}$$

Proof. We will proof the lemma by induction. By assumption $u_t(n_0)$ and $u_t(n_0+1)$ are polynomials of order 1 with respect to t. Now, suppose that $u_t(n)$ is a polynomial of order at most $m - n_0$ for all $n_0 + 1 < n \le m$. Since u_t is a solution of $\tau_t u_t = \lambda u_t$ we have

$$a(m)u_t(m+1) + a(m-1)u_t(m-1) + b_t(m)u_t(m) = \lambda u_t(m), \tag{3.3}$$

where $b_t(m) = b_0(m) + t(b_1(m) - b_0(m))$. By

$$a(m)u_t(m+1) = (\lambda - b_0(m))u_t(m) - a(m-1)u_t(m-1) + (b_0(m) - b_1(m))tu_t(m) \tag{3.4}$$

we infer that $u_t(m+1)$ is a polynomial of order at most $m + 1 - n_0$.
The same holds for $m \le n_0$. $\square$

Lemma 3.3. *For all $t \in [0,1]$, $n_0 \in \mathbb{Z}$, let u_t be solutions of $\tau_t u_t = \lambda u_t$ where $u_t(n_0) = (1-t)u_0(n_0) + tu_1(n_0)$ and $u_t(n_0+1) = (1-t)u_0(n_0+1) + tu_1(n_0+1)$ with $u_0(n_0)$, $u_1(n_0)$, $u_0(n_0+1)$, $u_1(n_0+1) \in \mathbb{R}$. Suppose that (2.8) holds for θ_{u_0}, then (2.8) holds for all θ_{u_t} where $t \in [0,1]$.*

Proof. Assume that there is a $t \in (0,1]$, such that (2.8) doesn't hold for θ_{u_t}, then there is an n such that

$$\lceil \frac{\theta_{u_t}(n)}{\pi} \rceil > \lceil \frac{\theta_{u_t}(n+1)}{\pi} \rceil \quad \text{or} \quad \lceil \frac{\theta_{u_t}(n)}{\pi} \rceil + 1 < \lceil \frac{\theta_{u_t}(n+1)}{\pi} \rceil. \tag{3.5}$$

Since θ_{u_t} depends continuously on t and (2.8) holds for θ_{u_0} there exists a $t_0 \in [0,t]$ such that for any $k \in \mathbb{Z}$ one of the following cases holds:

$$\begin{array}{llll} (1) & \theta_{u_{t_0}}(n) = k\pi & \text{and} & \theta_{u_{t_0}}(n+1) \in ((k+1)\pi, (k+2)\pi] \\ (2) & \theta_{u_{t_0}}(n) = (k+1)\pi & \text{and} & \theta_{u_{t_0}}(n+1) \in (k\pi, (k+1)\pi] \\ (3) & \theta_{u_{t_0}}(n) \in (k\pi, (k+1)\pi] & \text{and} & \theta_{u_{t_0}}(n+1) = (k+2)\pi \\ (4) & \theta_{u_{t_0}}(n) \in (k\pi, (k+1)\pi] & \text{and} & \theta_{u_{t_0}}(n+1) = k\pi \end{array}$$

Suppose that (1) or (2) holds, then $u_{t_0}(n+1)^2 > 0$ and thus

$$0 < \sin\theta_{u_{t_0}}(n+1)\cos\theta_{u_{t_0}}(n) = (-1)^{k+1}(-1)^k \sin\Gamma \tag{3.6}$$

where Γ lies in $(0,\pi]$, which leads to a contradiction.
Suppose that (3) or (4) holds, then $u_{t_0}(n+1) = 0$ implies $u_{t_0}(n)u_{t_0}(n+2) < 0$ and thus

$$0 > \sin\theta_{u_{t_0}}(n)\cos\theta_{u_{t_0}}(n+1) = (-1)^k \sin(\gamma)(-1)^k \cos\Gamma \tag{3.7}$$

with $\gamma \in (0,\pi]$ and resp. $\Gamma = 2\pi$ in the case of (3), resp. $\Gamma = 0$ in the case of (4), which leads to a contradiction.

□

The previous lemma ensures that it is always possible to fix $\theta_{u_0}(0)$ and require

$$\lceil \theta_{u_0}(n)/\pi \rceil \le \lceil \theta_{u_0}(n+1)/\pi \rceil \le \lceil \theta_{u_0}(n)/\pi \rceil + 1 \tag{3.8}$$

to use continuity to gain unique values for the Prüfer angle for all $t \in [0,1]$ such that

$$\lceil \theta_{u_t}(n)/\pi \rceil \le \lceil \theta_{u_t}(n+1)/\pi \rceil \le \lceil \theta_{u_t}(n)/\pi \rceil + 1. \tag{3.9}$$

3.1 A Few Properties of the Wronskian

A couple of facts about the Wronskian of two solutions of (1.8) such as Green's formula given in Lemma 1.7 in a slightly modified way still hold in our context, i.e. for the Wronskian of two solutions of Jacobi difference equations with a modified b.

Lemma 3.4 (Green's Formula). *Let $f, g \in \ell(\mathbb{Z})$ and $m \le n$, then*

$$\begin{aligned} &\sum_{j=m}^{n} \big(f(\tau_1 g) - (\tau_0 f)g\big)(j) \\ &= W_n(f,g) - W_{m-1}(f,g) + \sum_{j=m}^{n} (b_1(j) - b_0(j))f(j)g(j). \end{aligned} \tag{3.10}$$

Proof. We have

$$\begin{aligned}
&\sum_{j=m}^{n} \big(f(\tau_1 g) - (\tau_0 f)g\big)(j) \\
=\ &\sum_{j=m}^{n} \big(f(j)\big(a(j)g(j+1) + a(j-1)g(j-1) + b_1(j)g(j)\big) \\
&- \big(a(j)f(j+1) + a(j-1)f(j-1) + b_0(j)f(j)\big)g(j)\big) \qquad (3.11) \\
=\ &\sum_{j=m}^{n} \big(W_j(f,g) - W_{j-1}(f,g) + (b_1(j) - b_0(j))g(j)f(j)\big) \\
=\ &W_n(f,g) - W_{m-1}(f,g) + \sum_{j=m}^{n} (b_1(j) - b_0(j))g(j)f(j).
\end{aligned}$$

□

Now we fix some $\lambda \in \mathbb{R}$ and have a look at the solutions of the corresponding Jacobi difference equations, especially the properties of their Wronskian.

Corollary 3.5. *Let $\lambda \in \mathbb{R}$, let $u_0, u_1 \in \ell(\mathbb{Z})$ be solutions of $\tau_{0,1}u_{0,1} = \lambda u_{0,1}$ and $m \leq n$, then*

$$W_n(u_0,u_1) - W_{m-1}(u_0,u_1) = \sum_{j=m}^{n} (b_0(j) - b_1(j))u_0(j)u_1(j). \qquad (3.12)$$

In particular,

$$W_{n+1}(u_0,u_1) - W_n(u_0,u_1) = (b_0(n+1) - b_1(n+1))u_0(n+1)u_1(n+1). \qquad (3.13)$$

Since $u_t(n)$ is a polynomial in t for all $n \in \mathbb{Z}$ it is differentiable and we let the dot denote the derivative with respect to t, given by

$$\dot{u}_{t_0}(n) = \lim_{t \to t_0} \frac{u_t(n) - u_{t_0}(n)}{t - t_0}. \qquad (3.14)$$

To shorten notation we will use the following abbreviation for sums

$$\sum_{i=m_0}^{m-1}{}^{*} f(i) = \begin{cases} \sum_{i=m_0}^{m-1} f(i) & \text{if} \quad m > m_0 \\ 0 & \text{if} \quad m = m_0 \\ -\sum_{i=m}^{m_0-1} f(i) & \text{if} \quad m < m_0 \end{cases}. \qquad (3.15)$$

Lemma 3.6. *Let $\lambda \in \mathbb{R}$, $n_0 \in \mathbb{Z}$ and for all $t \in [0,1]$ let u_t be a solution of $\tau_t u_t = \lambda u_t$ with boundary conditions $u_t(n_0) = u_0(n_0) + t(u_1(n_0) - u_0(n_0))$ and $u_t(n_0+1) = u_0(n_0+1) + t(u_1(n_0+1) - u_0(n_0+1))$ where $u_0(n_0)$, $u_0(n_0+1)$, $u_1(n_0)$, $u_1(n_0+1) \in \mathbb{R}$, then we have*

$$W_n(u_t, \dot{u}_t) = W_{n_0}(u_0,u_1) + \sum_{i=n_0+1}^{n}{}^{*} (b_0(i) - b_1(i))u_t(i)^2. \qquad (3.16)$$

Proof. For all t, $\tilde{t} \in [0,1]$ we have

$$\begin{aligned} W_{n_0}(u_t, \dot{u}_t) &= a(n_0)(u_t(n_0)\dot{u}_t(n_0+1) - u_t(n_0+1)\dot{u}_t(n_0)) \\ &= W_{n_0}(u_0, u_1) \end{aligned} \tag{3.17}$$

and

$$\begin{aligned} W_{n_0}(u_t, u_{\tilde{t}}) &= a(n_0)(u_t(n_0)u_{\tilde{t}}(n_0+1) - u_t(n_0+1)u_{\tilde{t}}(n_0)) \\ &= (\tilde{t} - t)W_{n_0}(u_0, u_1) \end{aligned} \tag{3.18}$$

Moreover,

$$\begin{aligned} W_n(u_t, \dot{u}_t) &= \lim_{\tilde{t}\to t} \frac{W_n(u_t, u_{\tilde{t}}) - W_n(u_t, u_t)}{\tilde{t} - t} \\ &= \lim_{\tilde{t}\to t} \frac{W_n(u_t, u_{\tilde{t}})}{\tilde{t} - t}. \end{aligned} \tag{3.19}$$

If $n > n_0$ by Corollary 3.5 and $b_t(i) = b_0(i) + t(b_1(i) - b_0(i))$ we further conclude that

$$\begin{aligned} W_n(u_t, \dot{u}_t) - W_{n_0}(u_0, u_1) &= \lim_{\tilde{t}\to t} \frac{W_n(u_t, u_{\tilde{t}}) - W_{n_0}(u_t, u_{\tilde{t}})}{\tilde{t} - t} \\ &= \lim_{\tilde{t}\to t} \frac{1}{\tilde{t} - t} \sum_{i=n_0+1}^{n} (b_t(i) - b_{\tilde{t}}(i))u_t(i)u_{\tilde{t}}(i) \\ &= \lim_{\tilde{t}\to t} \frac{\tilde{t} - t}{\tilde{t} - t} \sum_{i=n_0+1}^{n} (b_0(i) - b_1(i))u_t(i)u_{\tilde{t}}(i) \\ &= \sum_{i=n_0+1}^{n} (b_0(i) - b_1(i))u_t(i)^2. \end{aligned} \tag{3.20}$$

Similarly, if $n < n_0$ we have

$$W_n(u_t, \dot{u}_t) - W_{n_0}(u_0, u_1) = -\sum_{i=n+1}^{n_0} (b_0(i) - b_1(i))u_t(i)^2. \tag{3.21}$$

□

It is further possible to investigate sign changes near one or more consecutive zeros of the Wronskian.

Lemma 3.7. *Let $\lambda \in \mathbb{R}$, let $u_{0,1}$ be solutions of $\tau_{0,1}u_{0,1} = \lambda u_{0,1}$ and*

(1) *let $\prod_{i=n}^{n+1}(b_0(i) - b_1(i)) > 0$, $W_{n-1}(u_0, u_1)W_{n+1}(u_0, u_1) \neq 0$ and $W_n(u_0, u_1) = 0$, then*

$$W_{n-1}(u_0, u_1)W_{n+1}(u_0, u_1) < 0. \tag{3.22}$$

(2) *let $b_1(n+1) \neq b_0(n+1)$ and $W_n(u_0, u_1) = W_{n+1}(u_0, u_1) = 0$, then*

$$u_0(n+1) = u_1(n+1) = 0. \tag{3.23}$$

(3) *let* $\pm b_0(n) > b_1(n)$, $\pm b_0(n+2) > b_1(n+2)$ *and* $W_n(u_0, u_1) = W_{n+1}(u_0, u_1) = 0$, *then*

$$W_{n-1}(u_0, u_1)W_{n+2}(u_0, u_1) < 0. \tag{3.24}$$

Proof. **(1)** We have

$$\begin{aligned}
&W_{n-1}(u_0, u_1)W_{n+1}(u_0, u_1) \\
&= -(W_n(u_0, u_1) - W_{n-1}(u_0, u_1))(W_{n+1}(u_0, u_1) - W_n(u_0, u_1)) \\
&= -(b_0(n) - b_1(n))u_0(n)u_1(n)(b_0(n+1) - b_1(n+1))u_0(n+1)u_1(n+1) \\
&= -(b_0(n) - b_1(n))(b_0(n+1) - b_1(n+1))c^2 u_0(n)^2 u_0(n+1)^2 < 0.
\end{aligned} \tag{3.25}$$

In the last step we used (1.17) since u_0 and u_1 are real-valued.

(2) By (3.13) and $W_n(u_0, u_1) - W_{n+1}(u_0, u_1) = 0$ either $u_0(n+1)$ or $u_1(n+1)$ must be zero. Since (1.17) and $W_n(u_0, u_1) = 0$ both are 0.

(3) By (2) there is a node at $n+1$ and hence (3.13) and (2.2) imply

$$\begin{aligned}
&W_{n-1}(u_0, u_1)W_{n+2}(u_0, u_1) \\
&= (W_{n-1}(u_0, u_1) - W_n(u_0, u_1))(W_{n+2}(u_0, u_1) - W_{n+1}(u_0, u_1)) \\
&= -(b_0(n) - b_1(n))u_1(n)u_0(n)(b_0(n+2) - b_1(n+2))u_1(n+2)u_0(n+2) \\
&< 0.
\end{aligned} \tag{3.26}$$

□

3.2 The Derivative of the Prüfer Angle

Lemma 3.8. *Let* $\lambda \in \mathbb{R}$, $n_0 \in \mathbb{Z}$ *and for all* $t \in [0,1]$ *let* u_t *be a solution of* $\tau_t u_t = \lambda u_t$ *with boundary conditions* $u_t(n_0) = u_0(n_0) + t(u_1(n_0) - u_0(n_0))$ *and* $u_t(n_0+1) = u_0(n_0+1) + t(u_1(n_0+1) - u_0(n_0+1))$ *where* $u_0(n_0)$, $u_1(n_0)$, $u_0(n_0+1)$, $u_1(n_0+1) \in \mathbb{R}$, *then we have*

$$\dot\theta_{u_t}(n) = \frac{W_{n_0}(u_0, u_1)}{-a(n)\rho_{u_t}(n)^2} + \sum_{i=n_0+1}^{n}{}^{*} \frac{(b_0(i) - b_1(i))u_t(i)^2}{-a(n)\rho_{u_t}(n)^2}. \tag{3.27}$$

Proof. We have

$$\begin{aligned}
&W_n(u_t, \dot u_t) \\
&= a(n)(u_t(n)\dot u_t(n+1) - \dot u_t(n)u_t(n+1)) \\
&= a(n)(\rho_{u_t}(n)\sin\theta_{u_t}(n)(\dot\rho_{u_t}(n)\cos\theta_{u_t}(n) - \rho_{u_t}(n)\sin\theta_{u_t}(n)\dot\theta_{u_t}(n)) \\
&\quad - (\dot\rho_{u_t}(n)\sin\theta_{u_t}(n) + \rho_{u_t}(n)\cos\theta_{u_t}(n)\dot\theta_{u_t}(n))\rho_{u_t}(n)\cos\theta_{u_t}(n)) \\
&= a(n)\rho_{u_t}(n)(\sin\theta_{u_t}(n)\dot\rho_{u_t}(n)\cos\theta_{u_t}(n) - \sin\theta_{u_t}(n)\rho_{u_t}(n)\sin\theta_{u_t}(n)\dot\theta_{u_t}(n)) \\
&\quad - (\cos\theta_{u_t}(n)\dot\rho_{u_t}(n)\sin\theta_{u_t}(n) + \cos\theta_{u_t}(n)\rho_{u_t}(n)\cos\theta_{u_t}(n)\dot\theta_{u_t}(n))) \\
&= a(n)\rho_{u_t}(n)^2\dot\theta_{u_t}(n)(-\sin^2\theta_{u_t}(n) - \cos^2\theta_{u_t}(n)).
\end{aligned} \tag{3.28}$$

To finish the proof apply Lemma 3.6. □

Lemma 3.9. *Let $\lambda \in \mathbb{R}$ and let n_0, $n \in \mathbb{Z}$, b_0, $b_1 \in \ell(\mathbb{Z})$ and for all $t \in [0,1]$ let u_t be a solution of $\tau_t u_t = \lambda u_t$ with boundary conditions $u_t(n_0) = u_0(n_0) + t(u_1(n_0) - u_0(n_0))$ and $u_t(n_0+1) = u_0(n_0+1) + t(u_1(n_0+1) - u_0(n_0+1))$ where $u_0(n_0)$, $u_0(n_0+1)$, $u_1(n_0)$, $u_1(n_0+1) \in \mathbb{R}$. Let $b_0(n+1) \geq b_1(n+1)$, and let $\dot{\theta}_t(n) > 0$, then $\dot{\theta}_t(n+1) > 0$.*

Proof. Suppose $n = n_0$ and $\dot{\theta}_{u_t}(n) = \frac{W_{n_0}(u_0,u_1)}{-a(n)\rho_{u_t}(n)^2} > 0$, then

$$\dot{\theta}_{u_t}(n+1) = \frac{W_{n_0}(u_0,u_1) + (b_0(n_0+1) - b_1(n_0+1))u_t(n_0+1)^2}{-a(n+1)\rho_{u_t}(n+1)^2} > 0. \quad (3.29)$$

Suppose $n+1 = n_0$ and

$$\dot{\theta}_{u_t}(n) = \frac{W_{n_0}(u_0,u_1)}{-a(n)\rho_{u_t}(n)^2} - \frac{(b_0(n_0) - b_1(n_0))u_t(n_0)^2}{-a(n)\rho_{u_t}(n)^2} > 0, \quad (3.30)$$

then

$$\dot{\theta}_{u_t}(n+1) = \frac{W_{n_0}(u_0,u_1)}{-a(n+1)\rho_{u_t}(n+1)^2} > 0. \quad (3.31)$$

Otherwise suppose $n \neq n_0$, $n+1 \neq n_0$ and $\dot{\theta}_{u_t}(n) > 0$ then

$$\dot{\theta}_{u_t}(n+1) = \dot{\theta}_{u_t}(n)\frac{-a(n)\rho_t(n)^2 + (b_0(n+1) - b_1(n+1))u_t(n+1)^2}{-a(n+1)\rho_t(n+1)^2} > 0. \quad (3.32)$$

□

Theorem 3.10. *Let $\lambda \in \mathbb{R}$, $n_0 \in \mathbb{Z}$ and for all $t \in [0,1]$ let u_t be a solution of $\tau_t u_t = \lambda u_t$ with boundary conditions $u_t(n_0) = c_0$ and $u_t(n_0+1) = c_1$ where c_0, $c_1 \in \mathbb{R}$ and let $b_0(n) \geq b_1(n)$ (resp. $b_0(n) \leq b_1(n)$) for all $n \in \mathbb{Z}$, then we have*

$$\dot{\theta}_{u_t}(n_0) \begin{cases} \geq 0 \ (\textit{resp.} \leq 0) & \textit{if} \quad n > n_0, \\ = 0 & \textit{if} \quad n = n_0, \\ \leq 0 \ (\textit{resp.} \geq 0) & \textit{if} \quad n < n_0. \end{cases} \quad (3.33)$$

Proof. Due to Lemma 3.8 we have

$$\dot{\theta}_{u_t}(n) = \sum_{i=n_0+1}^{n} \frac{(b_0(i) - b_1(i))u_t(i)^2}{-a(n)\rho_{u_t}(n)^2} \quad (3.34)$$

for all $n > n_0$ and

$$\dot{\theta}_{u_t}(n) = \sum_{i=n+1}^{n_0} \frac{(b_0(i) - b_1(i))u_t(i)^2}{a(n)\rho_{u_t}(n)^2} \quad (3.35)$$

for all $n < n_0$. □

3.3 More on a Fundamental Solution

Lemma 3.11. *Let $\lambda \in \mathbb{R}$ and for any $n_0 > 0$ let s be the fundamental solution of $\tau s = \lambda s$ associated with $b(n)$ sufficiently large for all $0 < n < n_0$ where $s(0) = 0$ and $s(1) = 1$, then*

$$s(n) > 0 \quad \textit{for all} \quad 0 < n \leq n_0. \quad (3.36)$$

For any $n_0 < 0$ let s be the fundamental solution of $\tau s = \lambda s$ associated with $b(n)$ sufficiently large for all $n_0 < n < 0$ where $s(0) = 0$ and $s(1) = 1$, then

$$s(n) < 0 \quad \textit{for all} \ \ n_0 \leq n < 0. \tag{3.37}$$

Proof. We will proof the lemma by induction. We have $s(1) = 1$. Suppose that $s(n) > 0$ for all $0 < n \leq n_0$, thus by

$$s(n_0 + 1) = a(n_0)^{-1}((\lambda - b(n_0))s(n_0) - a(n_0 - 1)s(n_0 - 1)) \tag{3.38}$$

we infer that $s(n_0 + 1) > 0$ if and only if

$$b(n_0) > \lambda - a(n_0 - 1)s(n_0 - 1)s(n_0)^{-1}. \tag{3.39}$$

In an analogous manner we prove the case $n < n_0$. □

Remark 3.12. *Let $\lambda \in \mathbb{R}$ and let s_t be the fundamental solution of $\tau_t s_t = \lambda s_t$ where $s_t(0) = 0$ and $s_t(1) = 1$. For all $t \in [0, 1]$ we normalize θ_{s_t} such that*

$$\theta_{s_t}(0) = 0 \quad \textit{and} \quad \theta_{s_t}(-1) = -\frac{\pi}{2}. \tag{3.40}$$

Let $b_t(n)$ be sufficiently large for all $n > 0$, then we have $\sin\theta_{s_t}(n) > 0$ and $\cos\theta_{s_t}(n) > 0$ and hence

$$\theta_{s_t}(n) \in (0, \frac{\pi}{2}). \tag{3.41}$$

Let $b_t(n)$ be sufficiently large for all $n < 0$, then we have $\sin\theta_{s_t}(n) < 0$ and $\cos\theta_{s_t}(n) < 0$ and hence

$$\theta_{s_t}(n) \in (-\pi, -\frac{\pi}{2}). \tag{3.42}$$

Lemma 3.13. *Let $\lambda \in \mathbb{R}$ and let s be a solution of $\tau_b s = \lambda s$ where $s(0) = 0$ and $s(1) = 1$, then*

$$\begin{array}{lll} \lim_{b(n)\to\infty} \frac{\cot\theta_s(n)}{b(n)} = -\frac{1}{a(n)} & \textit{if} & n > 0, \\ \lim_{b(n)\to\infty} \frac{\tan\theta_s(n-1)}{b(n)} = -\frac{1}{a(n-1)} & \textit{if} & n < 0. \end{array} \tag{3.43}$$

Proof. Since $b(n) \to \infty$, s has no nodes beside 0, thus $\sin\theta_s(n) \neq 0$ and $\cos\theta_s(n) \neq 0$ for all $n \neq 0$, $n \neq -1$. Lemma 2.8 implies

$$\lim_{b(n)\to\infty} \frac{a(n)\cot\theta_s(n)}{b(n)} + \lim_{b(n)\to\infty} \frac{a(n-1)\tan\theta_s(n-1)}{b(n)} = \lim_{b(n)\to\infty} \frac{\lambda}{b(n)} - 1. \tag{3.44}$$

Suppose that $n > 0$, then as $b(n) \to \infty$ the Prüfer angle $\theta_s(n-1)$ is in $(0, \frac{\pi}{2})$ (Remark 3.12) and decreases. Thus, $\exists\ M > 0$ such that $\tan\theta_s(n-1) \leq M$ as $b(n) \to \infty$ and thus we have

$$\lim_{b(n)\to\infty} \frac{a(n-1)\tan\theta_s(n-1)}{b(n)} = 0 \quad \Rightarrow \quad \lim_{b(n)\to\infty} \frac{\cot\theta_s(n)}{b(n)} = -\frac{1}{a(n)}. \tag{3.45}$$

Suppose that $n < 0$, then as $b(n) \to \infty$ the Prüfer angle $\theta_s(n)$ is in $(-\pi, -\frac{\pi}{2})$ (Remark 3.12) and increases. Thus, $\exists\ M > 0$ such that $\cot\theta_s(n) \leq M$ as $b(n) \to \infty$ and thus we have

$$\lim_{b(n)\to\infty} \frac{a(n-1)\cot\theta_s(n)}{b(n)} = 0 \quad \Rightarrow \quad \lim_{b(n)\to\infty} \frac{\tan\theta_s(n-1)}{b(n)} = -\frac{1}{a(n-1)}. \tag{3.46}$$

□

CHAPTER 4

Sturm's Separation Theorem

We prove a discrete analogue of the well-known Sturm separation theorem for differential equations. Unlike the corresponding result in the continuous case the Sturm separation theorem for difference equations is not valid for all second order homogenous difference equations. It is believed that Sturm actually proved the theorem for difference equations before he proved the corresponding result for differential equations (cf. [6] Theorem 6.5, [3]).

Theorem 4.1 (Sturm Separation Theorem)**.** *Let $\lambda \in \mathbb{R}$ and let $u_{0,1}$ be solutions of $\tau_{0,1} u_{0,1} = \lambda u_{0,1}$ corresponding to $b_0(j) \geq b_1(j)$ for all $j \in \mathbb{Z}$, let $m < n$. Suppose that either m is a node of u_0 or $W_m(u_0, u_1) = 0$ and either n is a node of u_0 or $W_n(u_0, u_1) = 0$ ($m = -\infty$ or $n = +\infty$ are allowed if u_0 and u_1 are both in $\ell^2_\pm(\mathbb{Z})$ and $W_{\pm\infty}(u_0, u_1) = 0$). If $W_.(u_0, u_1)$ is not vanishing identically on $[m, n]$, then u_1 has at least one node between m and $n + 1$.*

Proof. W.l.o.g. u_0 has no node between n and m. Moreover, let $u_0(m + 1) > 0$ and $u_1(m + 1) > 0$, thus we have

$$\left.\begin{array}{c} u_0(m) \leq 0 \\ \text{or} \\ W_m(u_0, u_1) = 0 \end{array}\right\}, u_0(m+1) > 0, \dots, u_1(n) \geq 0, \left\{\begin{array}{c} u_0(n+1) < 0 \\ \text{or} \\ W_{n+1}(u_0, u_1) = 0 \end{array}\right. . \tag{4.1}$$

Suppose u_1 has no node between m and $n + 1$, then

$$u_1(m) \geq 0, u_1(m+1) > 0, \dots, u_1(n) > 0, u_1(n+1) \geq 0 \tag{4.2}$$

holds. Observe

$$W_n(u_0, u_1) - W_m(u_0, u_1) = \sum_{i=m+1}^{n} (b_0(i) - b_1(i)) u_0(i) u_1(i) \geq 0 \tag{4.3}$$

with equality if and only if $b_0(i) = b_1(i)$ for all $i = m + 1, \dots, n - 1$ and either $b_0(n) = b_1(n)$ or $u_0(n) = 0$. Suppose u_0 has a node at m, then

$$W_m(u_0, u_1) = a(m)(u_0(m) u_1(m+1) - u_0(m+1) u_1(m)) \geq 0. \tag{4.4}$$

Of course this also holds trivially if $W_m(u_0, u_1) = 0$. Similarly, if u_0 has a node at n, then

$$W_n(u_0, u_1) = a(n)(u_0(n)u_1(n+1) - u_0(n+1)u_1(n)) \leq 0. \tag{4.5}$$

This contradicts (4.3) unless both Wronskians on the left and the sum on the right are zero, which is the case if and only if the Wronskian vanishes identically on $[m, n]$. □

This leads to the question whether solutions are oscillatory or not.

Definition 4.2. *τ is said to be* oscillatory *if there is a solution of $\tau u = 0$ with infinitely many nodes. τ is said to be* oscillatory near $\pm\infty$ *if there is a solution u of $\tau u = 0$ with infinitely many nodes near $\pm\infty$.*

Remark 4.3. *If one solution of τ has infinitely many nodes, then each such solution has infinitely many nodes. Due to Sturm's Theorem 4.1 we have: if $\tau_0 - \lambda$ is oscillatory, $\tau_1 - \lambda$ also is oscillatory whenever $b_0(n) \geq b_1(n)$ for all $n \in \mathbb{Z}$ and thus, if $\tau_1 - \lambda$ is non-oscillatory then $\tau_0 - \lambda$ is non-oscillatory.*

CHAPTER 5

Relative Oscillation Theory for Jacobi Matrices

In this chapter we investigate sign changes, i.e. *weighted nodes*, of the Wronskian of two solutions of a Jacobi difference equation modified in b as defined in 3.1. Subsequently we will investigate how the difference of the number of eigenvalues of two Jacobi operators is correlated to the number of weighted nodes of the Wronskian of their solutions.

5.1 Comparing Prüfer Angles

First we will have a closer look at the difference of the Prüfer angles of two solutions.

Definition 5.1. *Let u_0, u_1 be solutions of $\tau_{0,1}u_{0,1} = \lambda u_{0,1}$, $\lambda \in \mathbb{R}$, then we define*

$$\Delta_{u_0,u_1}(n) = \theta_{u_1}(n) - \theta_{u_0}(n). \tag{5.1}$$

Let u_t denote solutions of $\tau_t u_t = \lambda u_t$, for some $\lambda \in \mathbb{R}$ and for all $t \in [0,1]$, where τ_t corresponds to $b_t = (1-t)b_0 + tb_1$ and where $u_t(n_0) = (1-t)u_0(n_0) + tu_1(n_0)$ and $u_t(n_0+1) = (1-t)u_0(n_0+1) + u_1(n_0+1)$ with $u_0(n_0)$, $u_1(n_0)$, $u_0(n_0+1)$, $u_1(n_0+1) \in \mathbb{R}$, $n_0 \in \mathbb{Z}$. To shorten notation we will denote $\Delta = \Delta_{t,\tilde{t}} = \Delta_{u_t,u_{\tilde{t}}}$, $\theta_t = \theta_{u_t}$ and $\rho_t = \rho_{u_t}$. We will furthermore assume that (2.8) holds for the solution u_0 and thus (2.8) holds for all solutions u_t as shown in Lemma 3.3. Hence, it is possible to use the following notation:

$$\begin{aligned} \theta_t(n) &= k_t\pi + \gamma_t && \text{for some} \quad \gamma_t \in (0,\pi], \\ \theta_t(n+1) &= k_t\pi + \Gamma_t && \text{for some} \quad \Gamma_t \in (0,2\pi] \end{aligned} \tag{5.2}$$

and for some $k_t \in \mathbb{Z}$.

Lemma 5.2. *For some $n_0 \in \mathbb{Z}$ choose some k_0, $k_1 \in \mathbb{Z}$ such that $\theta_{0,1}(n_0) = k_{0,1}\pi + \gamma_{0,1}$ with $\gamma_{0,1} \in (0,\pi]$ and $\theta_{0,1}(n_0+1) = k_{0,1}\pi + \Gamma_{0,1}$ with $\Gamma_{0,1} \in (0,2\pi]$, then we have*

$$\Delta(n_0) = (k_1 - k_0)\pi + \gamma_1 - \gamma_0 \quad \text{and} \quad \Delta(n_0+1) = (k_1 - k_0)\pi + \Gamma_1 - \Gamma_0 \tag{5.3}$$

where

(1) *either u_0 and u_1 have a node at n_0 or both do not have a node at n_0, then*

$$\gamma_1 - \gamma_0 \in (-\frac{\pi}{2}, \frac{\pi}{2}) \quad \textit{and} \quad \Gamma_1 - \Gamma_0 \in (-\pi, \pi). \tag{5.4}$$

(2) *u_1 has no node at n_0, but u_0 has a node at n_0, then*

$$\gamma_1 - \gamma_0 \in (-\pi, 0) \quad \textit{and} \quad \Gamma_1 - \Gamma_0 \in (-2\pi, 0). \tag{5.5}$$

(3) *u_1 has a node at n_0, but u_0 has no node at n_0, then*

$$\gamma_1 - \gamma_0 \in (0, \pi) \quad \textit{and} \quad \Gamma_1 - \Gamma_0 \in (0, 2\pi). \tag{5.6}$$

Proof. Use Lemma 2.4. □

Lemma 5.3. *For all t, $\tilde{t} \in [0,1]$ and all $n \in \mathbb{Z}$ we have*

$$\lceil \Delta_{u_t, u_{\tilde{t}}}(n)/\pi \rceil - 1 \le \lceil \Delta_{u_t, u_{\tilde{t}}}(n+1)/\pi \rceil \le \lceil \Delta_{u_t, u_{\tilde{t}}}(n)/\pi \rceil + 1. \tag{5.7}$$

Proof. For all $n \in \mathbb{Z}$ using the notation from (5.2) where $k = k_{\tilde{t}} - k_t$ by Lemma 5.2 we have either

$$\begin{array}{lll} \Delta_{t,\tilde{t}}(n) \in (k\pi - \frac{\pi}{2}, k\pi + \frac{\pi}{2}) & \text{and} & \Delta_{t,\tilde{t}}(n+1) \in (k\pi - \pi, k\pi + \pi), \\ \Delta_{t,\tilde{t}}(n) \in (k\pi - \pi, k\pi) & \text{and} & \Delta_{t,\tilde{t}}(n+1) \in (k\pi - 2\pi, k\pi) \text{ or} \\ \Delta_{t,\tilde{t}}(n) \in (k\pi, k\pi + \pi) & \text{and} & \Delta_{t,\tilde{t}}(n+1) \in (k\pi, k\pi + 2\pi). \end{array} \tag{5.8}$$

In each case the lemma holds. □

Thus, we can use the following notation:

$$\begin{array}{lll} \Delta_{t,\tilde{t}}(n) = k\pi + \gamma & \text{for some} & \gamma \in (0, \pi], \\ \Delta_{t,\tilde{t}}(n+1) = k\pi + \Gamma & \text{for some} & \Gamma \in (-\pi, 2\pi]. \end{array} \tag{5.9}$$

Lemma 5.4. *For all $n \in \mathbb{Z}$*

$$W_n(u_0, u_1) = -a(n)\rho_{u_0}(n)\rho_{u_1}(n) \sin \Delta_{u_0,u_1}(n) \tag{5.10}$$

holds.

Proof. We have

$$\begin{aligned} & W_n(u_0, u_1) \\ &= a(n)(u_0(n)u_1(n+1) - u_1(n)u_0(n+1)) \\ &= a(n)\rho_{u_0}(n)\rho_{u_1}(n)(\sin\theta_{u_0}(n)\cos\theta_{u_1}(n) - \sin\theta_{u_1}(n)\cos\theta_{u_0}(n)) \\ &= a(n)\rho_{u_0}(n)\rho_{u_1}(n)\sin(\theta_{u_0}(n) - \theta_{u_1}(n)). \end{aligned} \tag{5.11}$$

□

We will now show that the same relation as the one given in (2.8) which allows us to count nodes of solutions still holds for the difference of their Prüfer angles, which is responsible for the sign changes of the Wronskian as stated in the previous lemma.

Lemma 5.5. *Fix some n. Then, if $b_0(n+1) \geq b_1(n+1)$, we have*

$$\lceil \Delta_{u_0,u_1}(n)/\pi \rceil \leq \lceil \Delta_{u_0,u_1}(n+1)/\pi \rceil \leq \lceil \Delta_{u_0,u_1}(n)/\pi \rceil + 1 \tag{5.12}$$

and if $b_0(n+1) \leq b_1(n+1)$, we have

$$\lceil \Delta_{u_0,u_1}(n)/\pi \rceil - 1 \leq \lceil \Delta_{u_0,u_1}(n+1)/\pi \rceil \leq \lceil \Delta_{u_0,u_1}(n)/\pi \rceil. \tag{5.13}$$

Proof. We will use the notation from Lemma 5.2 where we assume $k_0 = k_1 = 0$ without loss of generality. In particular, Lemma 5.2 implies

$$\lceil \Delta_{u_0,u_1}(n)/\pi \rceil - 1 \leq \lceil \Delta_{u_0,u_1}(n+1)/\pi \rceil \leq \lceil \Delta_{u_0,u_1}(n)/\pi \rceil + 1.$$

Hence, to show (5.12) there are two cases to exclude. Namely, (i) $\Delta_{u_0,u_1}(n) \in (0, \frac{\pi}{2})$, $\Delta_{u_0,u_1}(n+1) \in (-\pi, 0]$ (from case (1)) and (ii) $\Delta_{u_0,u_1}(n) \in (-\pi, 0)$, $\Delta_{u_0,u_1}(n+1) \in (-2\pi, -\pi]$ (from case (2)). But in case (i) we obtain a contradiction from (3.13):

$$\underbrace{W_{n+1}(u_0,u_1)}_{\leq 0} = \underbrace{W_n(u_0,u_1)}_{>0} + \underbrace{(b_0(n+1) - b_1(n+1))}_{\geq 0} \underbrace{u_0(n+1)u_1(n+1)}_{\geq 0}.$$

Similarly, in case (ii) equation (3.13) implies

$$\underbrace{W_{n+1}(u_0,u_1)}_{\geq 0} = \underbrace{W_n(u_0,u_1)}_{<0} + \underbrace{(b_0(n+1) - b_1(n+1))}_{\geq 0} \underbrace{u_0(n+1)u_1(n+1)}_{\leq 0}.$$

Equation (5.13) can be established in a similar fashion. □

5.2 Nodes of the Wronskian

Now we have all the necessary tools to establish a connection between the sign changes of the Wronskian of two solutions and the difference of their Prüfer angles. The next Lemma shows how they exactly relate to each other.

Lemma 5.6. *Let $n \in \mathbb{Z}$, then*

(1) $W_n(u_0,u_1) = W_{n+1}(u_0,u_1) = 0$ *or* $W_n(u_0,u_1)W_{n+1}(u_0,u_1) > 0$ *implies*

$$\lceil \frac{\Delta_{u_0,u_1}(n+1)}{\pi} \rceil = \lceil \frac{\Delta_{u_0,u_1}(n)}{\pi} \rceil. \tag{5.14}$$

(2) $W_n(u_0,u_1)W_{n+1}(u_0,u_1) < 0$ *implies*

$$\lceil \frac{\Delta_{u_0,u_1}(n+1)}{\pi} \rceil = \begin{cases} \lceil \frac{\Delta_{u_0,u_1}(n)}{\pi} \rceil + 1, & \text{if } b_0(n+1) > b_1(n+1), \\ \lceil \frac{\Delta_{u_0,u_1}(n)}{\pi} \rceil - 1, & \text{if } b_0(n+1) < b_1(n+1). \end{cases} \tag{5.15}$$

(3) $W_n(u_0,u_1) = 0$ *and* $W_{n+1}(u_0,u_1) \neq 0$ *implies*

$$\lceil \frac{\Delta_{u_0,u_1}(n+1)}{\pi} \rceil = \begin{cases} \lceil \frac{\Delta_{u_0,u_1}(n)}{\pi} \rceil + 1, & \text{if } b_0(n+1) > b_1(n+1), \\ \lceil \frac{\Delta_{u_0,u_1}(n)}{\pi} \rceil, & \text{if } b_0(n+1) < b_1(n+1). \end{cases} \tag{5.16}$$

(4) $W_n(u_0, u_1) \neq 0$ *and* $W_{n+1}(u_0, u_1) = 0$ *implies*

$$\lceil \frac{\Delta_{u_0,u_1}(n+1)}{\pi} \rceil = \begin{cases} \lceil \frac{\Delta_{u_0,u_1}(n)}{\pi} \rceil, & \textit{if } b_0(n+1) > b_1(n+1), \\ \lceil \frac{\Delta_{u_0,u_1}(n)}{\pi} \rceil - 1, & \textit{if } b_0(n+1) < b_1(n+1). \end{cases} \tag{5.17}$$

Note that in the cases (2)–(4) we necessarily have $b_0(n+1) \neq b_1(n+1)$.

Proof. We will use the notation from Lemma 5.2 where we assume $k_0 = k_1 = 0$ without loss of generality. Moreover, interchanging u_0 and u_1 using $\Delta_{u_1,u_0} = -\Delta_{u_0,u_1}(n)$ and

$$\lceil -x \rceil = \begin{cases} -\lceil x \rceil & \text{if } x \in \mathbb{Z}, \\ -\lceil x \rceil + 1 & \text{otherwise}, \end{cases}$$

we see that it suffices to show one case $b_0(n+1) \geq b_1(n+1)$ or $b_0(n+1) \leq b_1(n+1)$.
Suppose $W_n(u_0, u_1) = W_{n+1}(u_0, u_1) = 0$ and $W_n(u_0, u_1) W_{n+1}(u_0, u_1) > 0$ do not hold, then by (3.13) we have

$$W_{n+1}(u_0, u_1) - W_n(u_0, u_1) = (b_0(n+1) - b_1(n+1)) u_0(n+1) u_1(n+1) \neq 0$$

and hence $b_0(n+1) \neq b_1(n+1)$.
(1) and (2). Suppose $W_n(u_0, u_1) = W_{n+1}(u_0, u_1) = 0$, then by (5.4) we infer

$$\sin(\Delta_{u_0,u_1}(n)) = \sin(\gamma_1 - \gamma_0) = 0, \quad \sin(\Delta_{u_0,u_1}(n+1)) = \sin(\Gamma_1 - \Gamma_0) = 0,$$

where γ_0, $\gamma_1 \in (0, \pi]$. Thus $\gamma_0 = \gamma_1$ and we have case (1) of Lemma 5.2 which implies $\Gamma_1 - \Gamma_0 \in (-\pi, \pi)$ and we conclude $\Gamma_1 - \Gamma_0 = 0$. In summary, $\Delta_{u_0,u_1}(n) = \Delta_{u_0,u_1}(n+1) = 0$ as claimed.
Next suppose $W_n(u_0, u_1) W_{n+1}(u_0, u_1) \neq 0$, then by (5.4) the sign of the Wronskian at n equals the sign of $\sin(\Delta_{u_0,u_1}(n))$ and hence (5.12) respectively (5.13) finish the proof of case (1) and (2).
(3). By (5.4) we conclude $\Delta_{u_0,u_1}(n) = \gamma_1 - \gamma_0 \equiv 0 \mod \pi$, where γ_0, $\gamma_1 \in (0, \pi]$ and thus $\gamma_1 - \gamma_0 = 0$. So we have case (1) of Lemma 5.2 and hence $\Delta_{u_0,u_1}(n+1) = \Gamma_1 - \Gamma_0 \in (-\pi, \pi)$. That is,

$$\lceil \Delta_{u_0,u_1}(n)/\pi \rceil \leq \lceil \Delta_{u_0,u_1}(n+1)/\pi \rceil \leq \lceil \Delta_{u_0,u_1}(n)/\pi \rceil + 1$$

and (5.13) finishes the proof of case (3) for $b_0(n+1) < b_1(n+1)$.
(4). By (5.4) we have $\Delta_{u_0,u_1}(n+1) = \Gamma_1 - \Gamma_0 \equiv 0 \mod \pi$ and Lemma 5.2 leaves us with the following possibilities

$$\begin{array}{llll} \text{(a)} & \Delta_{u_0,u_1}(n) \in (-\frac{\pi}{2}, \frac{\pi}{2}) & \text{and} & \Delta_{u_0,u_1}(n+1) = 0, \\ \text{(b)} & \Delta_{u_0,u_1}(n) \in (-\pi, 0) & \text{and} & \Delta_{u_0,u_1}(n+1) = -\pi, \\ \text{(c)} & \Delta_{u_0,u_1}(n) \in (0, \pi) & \text{and} & \Delta_{u_0,u_1}(n+1) = \pi. \end{array}$$

and (5.12) shows (4) if $b_0(n+1) > b_1(n+1)$. □

Definition 5.7. *Set*

$$\#_n W(u_0,u_1) = \begin{cases} 1, & \begin{array}{l} \text{if } b_0(n+1)-b_1(n+1)>0 \text{ and} \\ \text{either } W_n(u_0,u_1)W_{n+1}(u_0,u_1)<0 \\ \text{or } W_n(u_0,u_1)=0 \text{ and } W_{n+1}(u_0,u_1)\neq 0, \end{array} \\ -1, & \begin{array}{l} \text{if } b_0(n+1)-b_1(n+1)<0 \text{ and} \\ \text{either } W_n(u_0,u_1)W_{n+1}(u_0,u_1)<0 \\ \text{or } W_n(u_0,u_1)\neq 0 \text{ and } W_{n+1}(u_0,u_1)=0, \end{array} \\ 0, & \text{otherwise.} \end{cases} \tag{5.18}$$

We say the Wronskian has a weighted node *at n if $\#_n W(u_0,u_1)\neq 0$. The weighted number of nodes of the Wonskian between n_1 and n_2 is denoted as*

$$\#_{(n_1,n_2)} W(u_0,u_1) = \sum_{j=n_1}^{n_2-1} \#_j W(u_0,u_1) - \left\{ \begin{array}{lll} 0 & \text{if} & W_{n_1}(u_0,u_1)\neq 0 \\ 1 & \text{if} & W_{n_1}(u_0,u_1)=0 \end{array} \right. . \tag{5.19}$$

If the Wronskian is non-zero at n_1 and n_2, we have

$$\#_{(n_1,n_2)} W(u_0,u_1) = -\#_{(n_1,n_2)} W(u_1,u_0). \tag{5.20}$$

Corollary 5.8. *We have*

$$\lceil \frac{\Delta_{u_0,u_1}(n+1)}{\pi} \rceil = \lceil \frac{\Delta_{u_0,u_1}(n)}{\pi} \rceil + \#_n W(u_0,u_1). \tag{5.21}$$

Thus, it is possible to count weighted nodes of the Wronskian using the difference of the corresponding Prüfer angles. The weighted number of nodes at an interval (n_1,n_2), possibly infinite, is given by

Lemma 5.9. *Let $n_1<n_2$, then*

$$\#_{(n_1,n_2)} W(u_0,u_1) = \lceil \Delta_{u_0,u_1}(n_2)/\pi \rceil - \lfloor \Delta_{u_0,u_1}(n_1)/\pi \rfloor - 1. \tag{5.22}$$

$$\#_{(-\infty,\infty)} W(u_0,u_1) = \lim_{n\to\infty} (\lceil \Delta_{u_0,u_1}(n)/\pi \rceil - \lfloor \Delta_{u_0,u_1}(-n)/\pi \rfloor - 1). \tag{5.23}$$

Proof. Let $n_2=n_1+1$. If $W_{n_1}(u_0,u_1)\neq 0$, then by Corollary 5.8

$$\begin{aligned} &\#_{(n_1,n_2)} W(u_0,u_1) = \#_{n_1} W(u_0,u_1) \\ &= \lceil \frac{\Delta_{u_0,u_1}(n_2)}{\pi} \rceil - \lceil \frac{\Delta_{u_0,u_1}(n_1)}{\pi} \rceil = \lceil \frac{\Delta_{u_0,u_1}(n_2)}{\pi} \rceil - \lfloor \frac{\Delta_{u_0,u_1}(n_1)}{\pi} \rfloor - 1 \end{aligned} \tag{5.24}$$

holds. If $W_{n_1}(u_0,u_1)=0$, then by Corollary 5.8

$$\begin{aligned} &\#_{(n_1,n_2)} W(u_0,u_1) = \#_{n_1} W(u_0,u_1) - 1 \\ &= \lceil \frac{\Delta_{u_0,u_1}(n_2)}{\pi} \rceil - \lceil \frac{\Delta_{u_0,u_1}(n_1)}{\pi} \rceil - 1 = \lceil \frac{\Delta_{u_0,u_1}(n_2)}{\pi} \rceil - \lfloor \frac{\Delta_{u_0,u_1}(n_1)}{\pi} \rfloor - 1 \end{aligned} \tag{5.25}$$

holds. Now, we assume that the claim already holds for an interval $[n_1,n_2]$ where $n_2\geq n_1+1$ and we will show that it holds for the interval $[n_1,n_2+1]$ as well. We have

$$\begin{aligned} \#_{(n_1,n_2+1)} W(u_0,u_1) &= \#_{(n_1,n_2)} W(u_0,u_1) + W_{n_2}(u_0,u_1) \\ &= \lceil \Delta(n_2)/\pi \rceil + W_{n_2}(u_0,u_1) - \lfloor \Delta(n_1)/\pi \rfloor - 1 \\ &= \lceil \Delta(n_2+1)/\pi \rceil - \lfloor \Delta(n_1)/\pi \rfloor - 1 \end{aligned} \tag{5.26}$$

where we used Corollary 5.8 again in the last step. □

Remark 5.10. *Let a, $b \in \mathbb{R}$, then*

$$\lceil \frac{a-b}{\pi} \rceil = \begin{cases} \lceil \frac{a}{\pi} \rceil - \lceil \frac{b}{\pi} \rceil + 1 & \text{iff } b < a \mod \pi \\ \lceil \frac{a}{\pi} \rceil - \lceil \frac{b}{\pi} \rceil & \text{iff } b \geq a \mod \pi \end{cases} \tag{5.27}$$

and

$$\lfloor \frac{a-b}{\pi} \rfloor = \begin{cases} \lfloor \frac{a}{\pi} \rfloor - \lfloor \frac{b}{\pi} \rfloor & \text{iff } b \leq a \mod \pi \\ \lfloor \frac{a}{\pi} \rfloor - \lfloor \frac{b}{\pi} \rfloor - 1 & \text{iff } b > a \mod \pi \end{cases} \tag{5.28}$$

Proof. We choose

$$\begin{aligned} a &= k_a \pi + \gamma_a, \\ b &= k_b \pi + \gamma_b, \end{aligned} \tag{5.29}$$

where γ_a, $\gamma_b \in (0, \pi]$ for some k_a, $k_b \in \mathbb{Z}$. This implies

$$a - b = (k_a - k_b)\pi + \gamma_a - \gamma_b. \tag{5.30}$$

Suppose that $\gamma_a > \gamma_b$, thus we have $\gamma_a - \gamma_b \in (0, \pi)$ and hence

$$\lceil \frac{a-b}{\pi} \rceil = k_a - k_b + 1. \tag{5.31}$$

Suppose that $\gamma_a \leq \gamma_b$, thus we have $\gamma_a - \gamma_b \in (-\pi, 0]$ and hence

$$\lceil \frac{a-b}{\pi} \rceil = k_a - k_b. \tag{5.32}$$

Use $\lceil x \rceil = -\lfloor -x \rfloor$, for all $x \in \mathbb{R}$, to gain the second claim. □

Theorem 5.11. *Let $n_1 < n_2$, then*

$$|\#_{(n_1,n_2)} W(u_0, u_1) - (\#_{(n_1,n_2)}(u_1) - \#_{(n_1,n_2)}(u_0))| \leq 2. \tag{5.33}$$

Proof. We have

$$\begin{aligned} &|\#_{(n_1,n_2)} W(u_0, u_1) - (\#_{(n_1,n_2)}(u_1) - \#_{(n_1,n_2)}(u_0))| \\ &= |\lceil \frac{\Delta_{u_0,u_1}(n_2)}{\pi} \rceil - \lfloor \frac{\Delta_{u_0,u_1}(n_1)}{\pi} \rfloor - 1 \\ &\quad - \lceil \frac{\theta_{u_1}(n_2)}{\pi} \rceil + \lfloor \frac{\theta_{u_1}(n_1)}{\pi} \rfloor + \lceil \frac{\theta_{u_0}(n_2)}{\pi} \rceil - \lfloor \frac{\theta_{u_0}(n_1)}{\pi} \rfloor| \\ &\leq |\lceil \frac{\theta_{u_1}(n_2) - \theta_{u_0}(n_2)}{\pi} \rceil - (\lceil \frac{\theta_{u_1}(n_2)}{\pi} \rceil - \lceil \frac{\theta_{u_0}(n_2)}{\pi} \rceil + 1)| \\ &\quad + |(\lfloor \frac{\theta_{u_1}(n_1) - \theta_{u_0}(n_1)}{\pi} \rfloor - (\lfloor \frac{\theta_{u_1}(n_1)}{\pi} \rfloor - \lfloor \frac{\theta_{u_0}(n_1)}{\pi} \rfloor))| \\ &\leq 1 + 1, \end{aligned} \tag{5.34}$$

where we used Theorem 5.9 and Theorem 2.6 in the first step and Remark 5.10 in the last step. □

Theorem 5.12 (Triangle Inequality for Wronskians)**.** *Let u_j be solutions of $\tau_j u_j = \lambda u_j$, $j \in \{0, 1, 2\}$. Then,*

$$\#_{(n_1,n_2)} W(u_0, u_1) + \#_{(n_1,n_2)} W(u_1, u_2) - 1 \leq \#_{(n_1,n_2)} W(u_0, u_2) \tag{5.35}$$

and

$$\#_{(n_1,n_2)} W(u_0, u_2) \leq \#_{(n_1,n_2)} W(u_0, u_1) + \#_{(n_1,n_2)} W(u_1, u_2) + 1 \tag{5.36}$$

holds.

Proof. We have $\Delta_{u_0,u_1} + \Delta_{u_1,u_2} = \Delta_{u_0,u_2}$ and

$$\begin{aligned}
&\#_{(n_1,n_2)} W(u_0,u_1) + \#_{(n_1,n_2)} W(u_1,u_2) \\
&= \lceil \frac{\Delta_{u_0,u_1}(n_2)}{\pi} \rceil - \lfloor \frac{\Delta_{u_0,u_1}(n_1)}{\pi} \rfloor - 1 + \lceil \frac{\Delta_{u_1,u_2}(n_2)}{\pi} \rceil - \lfloor \frac{\Delta_{u_1,u_2}(n_1)}{\pi} \rfloor - 1 \\
&\leq \lceil \frac{\Delta_{u_0,u_2}}{\pi} \rceil - \lfloor \frac{\Delta_{u_0,u_2}(n_1)}{\pi} \rfloor = \#_{(n_1,n_2)} W(u_0,u_2) + 1
\end{aligned} \tag{5.37}$$

where we used $\lceil x \rceil + \lceil y \rceil - 1 \leq \lceil x + y \rceil$ and $\lfloor x \rfloor + \lfloor y \rfloor + 1 \geq \lfloor x + y \rfloor$ for all $x, y \in \mathbb{R}$. Furthermore,

$$\#_{(n_1,n_2)} W(u_0,u_2) \leq \#_{(n_1,n_2)} W(u_0,u_1) + \#_{(n_1,n_2)} W(u_1,u_2) + 1 \tag{5.38}$$

holds by $\lceil x \rceil + \lceil y \rceil \geq \lceil x + y \rceil$ and $\lfloor x \rfloor + \lfloor y \rfloor \leq \lfloor x + y \rfloor$ for all $x, y \in \mathbb{R}$. $\square$

Theorem 5.13 (Comparison Theorem for Wronskians)**.** *Let u_j be solutions of $\tau_j u_j = \lambda u_j$, $j \in \{0,1,2\}$ where $b_0(j) \geq b_1(j) \geq b_2(j)$ for all $j \in \mathbb{Z}$. Let n_1 and n_2 be weighted nodes of $W(u_0,u_1)$, then $\#_{(n_1,n_2+1)} W(u_0,u_2) \geq 1$.*

Proof. W.l.o.g. $W(u_0,u_1)$ has no weighted node between n_1 and n_2.
First, suppose $\#_{(n_1,n_2+1)} W(u_1,u_2) = -1$, i.e. $W(u_1,u_2)$ is vanishing identically on $[n_1, n_2+1]$. For all $j \in [n_1, n_2+1]$ and some $c \in \mathbb{R}$ we have $W_j(u_0,u_2) = W_j(u_0,cu_1) = cW_j(u_0,u_1)$ and thus both Wronskians have weighted nodes in n_1 and n_2. Otherwise, the claim follows obviously from (5.35) except if $\#_{(n_1,n_2+1)} W(u_0,u_0) = 1$ and $\#_{(n_1,n_2+1)} W(u_1,u_2) = 0$. If so, for some k, $\tilde{k} \in \mathbb{Z}$ we have

$$\Delta_{0,1}(n_1) = k\pi, \qquad \Delta_{0,1}(n_2+1) \in ((k+1)\pi, (k+2)\pi), \tag{5.39}$$

$$\Delta_{1,2}(n_1) \in [\tilde{k}\pi, (\tilde{k}+1)\pi), \text{ and } \qquad \Delta_{1,2}(n_2+1) \in (\tilde{k}\pi, (\tilde{k}+1)\pi]. \tag{5.40}$$

Hence, by $\Delta_{0,2} = \Delta_{0,1} + \Delta_{1,2}$ we have $\#_{(n_1,n_2+1)} W(u_0,u_2) \geq 1$. $\square$

5.3 Finite Jacobi Operators

From now on $s_t(.,n_0)$ will denote the solutions of $\tau_t s_t = \lambda s_t$ with boundary conditions $s_t(n_0,n_0) = 0$ and $s_t(n_0+1,n_0) = 1$ for all $t \in [0,1]$ and some $n_0 \in \mathbb{Z}$.

Definition 5.14. *Let a, b_0, $b_1 \in \ell(\mathbb{Z})$ and let $n_0 > 1$, $n_0 \in \mathbb{Z}$. For all $t \in [0,1]$ let H^t_{0,n_0} denote the finite Jacobi operator*

$$\begin{aligned}
H^t_{0,n_0} &: \ell^2(0,n_0) \mapsto \ell^2(0,n_0) \\
H^t_{0,n_0} f &= J^t_{0,n_0} f
\end{aligned} \tag{5.41}$$

where J^t_{0,n_0} is the Jacobi matrix given by

$$J^t_{0,n_0} = \begin{pmatrix}
b_t(1) & a(1) & 0 & 0 & 0 \\
a(1) & b_t(2) & \ddots & 0 & 0 \\
0 & \ddots & \ddots & \ddots & 0 \\
0 & 0 & \ddots & \ddots & a(n_0-2) \\
0 & 0 & 0 & a(n_0-2) & b_t(n_0-1)
\end{pmatrix} \tag{5.42}$$

and $b_t(n) = b_0(n) + t(b_1(n) - b_0(n))$ *for all* $n \in \mathbb{Z}$.

Lemma 5.15. *Let* τ *be a Jacobi difference expression associated with* $a, b \in \ell(\mathbb{Z})$ *and let* J_{0,n_0} *be the corresponding finite Jacobi operator where* $1 < n_0 \in \mathbb{Z}$. *Furthermore, let* u *be a solution of* $\tau u = \lambda u$, *then*

$$u(0) = 0 \quad \text{and} \quad u(n_0) = 0 \quad \Leftrightarrow \quad J_{0,n_0} \begin{pmatrix} u(1) \\ u(2) \\ \vdots \\ u(n_0-1) \end{pmatrix} = \lambda \begin{pmatrix} u(1) \\ u(2) \\ \vdots \\ u(n_0-1) \end{pmatrix}. \tag{5.43}$$

Proof. Straightforward. $\square$

The lemma states that λ is an eigenvalue of H^t_{0,n_0} if and only if the fundamental solution $s_t(.,0)$ of $\tau_t s_t = \lambda s_t$ has a zero at n_0, resp. $s_t(.,n_0)$ has a zero at 0. Let P_Ω denote the family of spectral projections for H_{0,n_0}.

Theorem 5.16. *Let* $\lambda \in \mathbb{R}$ *and* H_{0,n_0} *be a finite restriction of the Jacobi operator* H, *then*

$$\dim \operatorname{Ran} P_{(-\infty,\lambda)}(H_{0,n_0}) = \#_{(0,n_0)}(s(.,0)) = \lceil \frac{\theta_{s(.,0)}(n_0)}{\pi} \rceil - 1. \tag{5.44}$$

Proof. Confer [18] (3.8). $\square$

Lemma 5.17. *Let* $\lambda \in \mathbb{R}$, $b_0(j) \geq b_1(j)$ *(resp.* $b_0(j) \leq b_1(j)$*) for all* $j \in \mathbb{Z}$ *and let* $1 < n_0 \in \mathbb{Z}$. *Then,*

$$\begin{aligned} &f : [0,1] \mapsto \mathbb{Z} \\ &f(t) = \dim \operatorname{Ran} P_{(-\infty,\lambda)}(H^t_{0,n_0}) - \dim \operatorname{Ran} P_{(-\infty,\lambda]}(H^0_{0,n_0}) \end{aligned} \tag{5.45}$$

is a monotonically increasing (decreasing) step function, which is continuous from below (above) and jumps by 1 *whenever* $s_t(n_0, 0) = 0$. *Moreover,*

$$f(0) = \begin{cases} -1 & \textit{iff} \quad \lambda \in \sigma(H^0_{0,n_0}) \\ 0 & \textit{iff} \quad \lambda \notin \sigma(H^0_{0,n_0}) \end{cases}. \tag{5.46}$$

Proof. By Theorem 5.16 we have

$$f(t) = \lceil \frac{\theta_{s_t(.,0)}(n_0)}{\pi} \rceil - \lceil \frac{\theta_{s_0(.,0)}(n_0)}{\pi} \rceil - \begin{cases} 1 & \text{iff} \quad \lambda \in \sigma(H^0_{0,n_0}), \\ 0 & \text{iff} \quad \lambda \notin \sigma(H^0_{0,n_0}). \end{cases} \tag{5.47}$$

As $t \in [0,1]$ increases, due to Theorem 3.10 $\theta_{s_t(.,0)}(n_0)$ is non-decreasing (non-increasing) and thus $f(t)$ monotonically increases (decreases) as well. In particular $f(t)$ increases (decreases) by 1 at the end (beginning) of each interval $[t_i, \tilde{t}_i]$, possibly $t_i = \tilde{t}_i$, where the Prüfer angle $\theta_{s_t(.,0)}(n_0) \equiv 0 \mod \pi$ for all $t \in [t_i, \tilde{t}_i]$, resp. $s_t(n_0, 0) = 0$ such that $f(t)$ is continuous from below (above). $\square$

Lemma 5.18. *Let* $\lambda \in \mathbb{R}$, $b_0(j) \geq b_1(j)$ *(resp.* $b_0(j) \leq b_1(j)$*) for all* $j \in \mathbb{Z}$ *and let* $1 < n_0 \in \mathbb{Z}$. *Then,*

$$\begin{aligned} &\dim \operatorname{Ran} P_{(-\infty,\lambda)}(H^1_{0,n_0}) - \dim \operatorname{Ran} P_{(-\infty,\lambda]}(H^0_{0,n_0}) \\ &= \lceil \frac{\Delta_{s_0(.,n_0),s_1(.,0)}(n_0)}{\pi} \rceil - \lfloor \frac{\Delta_{s_0(.,n_0),s_1(.,0)}(0)}{\pi} \rfloor - 1 \\ &= \lceil \frac{\Delta_{s_0(.,0),s_1(.,n_0)}(n_0)}{\pi} \rceil - \lfloor \frac{\Delta_{s_0(.,0),s_1(.,n_0)}(0)}{\pi} \rfloor - 1. \end{aligned} \tag{5.48}$$

Proof. Let

$$
\begin{aligned}
&g : [0,1] \mapsto \mathbb{Z} \\
&g(t) \;=\; \lceil \frac{\Delta_{s_0(.,n_0),s_t(.,0)}(n_0)}{\pi} \rceil - \lfloor \frac{\Delta_{s_0(.,n_0),s_t(.,0)}(0)}{\pi} \rfloor - 1 \\
&\quad\;\; = \lceil \frac{\theta_{s_t(.,0)}(n_0) - \theta_{s_0(.,n_0)}(n_0)}{\pi} \rceil - \lfloor \frac{\theta_{s_t(.,0)}(0) - \theta_{s_0(.,n_0)}(0)}{\pi} \rfloor - 1.
\end{aligned}
\tag{5.49}
$$

If $t = 0$ we have $\tau_t = \tau_0$. Thus, the solutions $s_0(.,n_0)$ and $s_t(.,0)$ solve the same Jacobi difference equation and by Lemma 1.8 the Wronskian $W(s_0(.,n_0), s_t(.,0))$ is constant. If λ is an eigenvalue of H^0_{0,n_0} we have $s_t(n_0,0) = 0$ and thus $W_{n_0}(s_0(.,n_0), s_t(.,0)) = 0$. This implies that the Wronskian is vanishing and by Lemma 5.6 we have $\Delta_{s_0(.,n_0),s_t(.,0)}(j) = k\pi$ for some $k \in \mathbb{Z}$ and for all $j \in \mathbb{Z}$, hence $g(0) = -1$. If λ is not an eigenvalue of H^0_{0,n_0} we have $s_t(n_0,0) \neq 0$ and $s_0(0,n_0) \neq 0$, but $0 = s_t(0,0) = s_0(n_0,n_0)$. Thus, $s_0(.,n_0)$ and $s_t(.,0)$ are linearly independent and by Lemma 1.8 the Wronskian is not vanishing. By Lemma 5.6

$$
\lceil \frac{\Delta_{s_0(.,n_0),s_t(.,0)}(n_0)}{\pi} \rceil = \lceil \frac{\Delta_{s_0(.,n_0),s_t(.,0)}(0)}{\pi} \rceil \tag{5.50}
$$

holds and by $\lceil x \rceil - 1 = \lfloor x \rfloor$ for all $x \notin \mathbb{Z}$ we have $g(0) = 0$. That is,

$$
g(0) = \begin{cases} -1 & \text{iff} \quad \lambda \in \sigma(H^0_{0,n_0}) \\ 0 & \text{iff} \quad \lambda \notin \sigma(H^0_{0,n_0}) \end{cases} \tag{5.51}
$$

and therefore (5.46) proofs the claim for $t = 0$.
As t increases by Theorem 3.10 we have $\dot\theta_{s_t(.,0)}(n_0) \geq 0$ (resp. $\dot\theta_{s_t(.,0)}(n_0) \leq 0$). Hence $g(t)$ is a monotonically increasing (decreasing) step function, which is continuous from below (above) and jumps by 1 at the end (beginning) of each interval $[t_i, \tilde t_i]$, possibly $t_i = \tilde t_i$, where $\theta_{s_t(.,0)}(n_0) \equiv \theta_{s_0(.,n_0)}(n_0) \mod \pi$ for all $t \in [t_i, \tilde t_i]$, i.e. where $s_t(n_0,0) = 0$ (by $\theta_{s_0(.,n_0)}(n_0) = 0$). To finish the proof apply Lemma 5.17.
The same holds for

$$
\begin{aligned}
&g : [0,1] \mapsto \mathbb{Z} \\
&g(t) \;=\; \lceil \frac{\Delta_{s_0(.,0),s_1(.,n_0)}(n_0)}{\pi} \rceil - \lfloor \frac{\Delta_{s_0(.,0),s_1(.,n_0)}(0)}{\pi} \rfloor - 1.
\end{aligned}
\tag{5.52}
$$

□

Corollary 5.19. *Let $\lambda \in \mathbb{R}$, $b_0(j) \geq b_1(j)$ (resp. $b_0(j) \leq b_1(j)$) for all $j \in \mathbb{Z}$ and let $1 < n_0 \in \mathbb{Z}$. Then,*

$$
\begin{aligned}
&\dim \operatorname{Ran} P_{(-\infty,\lambda)}(H^1_{0,n_0}) - \dim \operatorname{Ran} P_{(-\infty,\lambda)}(H^0_{0,n_0}) \\
&= \lceil \frac{\Delta_{s_0(.,n_0),s_1(.,0)}(n_0)}{\pi} \rceil - \lceil \frac{\Delta_{s_0(.,n_0),s_1(.,0)}(0)}{\pi} \rceil \\
&= \lceil \frac{\Delta_{s_0(.,0),s_1(.,n_0)}(n_0)}{\pi} \rceil - \lceil \frac{\Delta_{s_0(.,0),s_1(.,n_0)}(0)}{\pi} \rceil.
\end{aligned}
\tag{5.53}
$$

Theorem 5.20. *Let $\lambda \in \mathbb{R}$ and let $1 < n_0 \in \mathbb{Z}$. Then,*

$$
\begin{aligned}
&\dim \operatorname{Ran} P_{(-\infty,\lambda)}(H^1_{0,n_0}) - \dim \operatorname{Ran} P_{(-\infty,\lambda]}(H^0_{0,n_0}) \\
&= \#_{(0,n_0)} W(s_0(.,n_0), s_1(.,0)) \\
&= \#_{(0,n_0)} W(s_0(.,0), s_1(.,n_0)).
\end{aligned}
\tag{5.54}
$$

Proof. Let

$$\tilde{b}(j) = \min\{b_0(j), b_1(j)\} \tag{5.55}$$

for all $j \in \mathbb{Z}$, then we have $b_0(j) \geq \tilde{b}(j)$ and $b_1(j) \geq \tilde{b}(j)$ for all $j \in \mathbb{Z}$. Thus we apply Corollary 5.19 and Lemma 5.18 and we infer

$$\begin{aligned}
&\dim \operatorname{Ran} P_{(-\infty,\lambda)}(H^1_{0,n_0}) - \dim \operatorname{Ran} P_{(-\infty,\lambda]}(H^0_{0,n_0})\\
&= \lceil \frac{\Delta_{\tilde{s}(.,n_0),s_1(.,0)}(n_0)}{\pi} \rceil - \lceil \frac{\Delta_{\tilde{s}(.,n_0),s_1(.,0)}(0)}{\pi} \rceil\\
&+ \lceil \frac{\Delta_{s_0(.,n_0),\tilde{s}(.,0)}(n_0)}{\pi} \rceil - \lfloor \frac{\Delta_{s_0(.,n_0),\tilde{s}(.,0)}(0)}{\pi} \rfloor - 1\\
&= \lceil \frac{\theta_{s_1(.,0)}(n_0)}{\pi} \rceil - \lceil \frac{-\theta_{\tilde{s}(.,n_0)}(0)}{\pi} \rceil + \lceil \frac{\theta_{\tilde{s}(.,0)}(n_0)}{\pi} \rceil - \lfloor \frac{-\theta_{s_0(.,n_0)}(0)}{\pi} \rfloor - 1\\
&= \lceil \frac{\theta_{s_1(.,0)}(n_0) - \theta_{s_0(.,n_0)}(n_0)}{\pi} \rceil - \lfloor \frac{\theta_{s_1(.,0)}(0) - \theta_{s_0(.,n_0)}(0)}{\pi} \rfloor - 1\\
&= \lceil \frac{\Delta_{s_0(.,n_0),s_1(.,0)}(n_0)}{\pi} \rceil - \lfloor \frac{\Delta_{s_0(.,n_0),s_1(.,0)}(0)}{\pi} \rfloor - 1
\end{aligned} \tag{5.56}$$

where we used

$$\begin{aligned}
&\lceil \frac{\theta_{\tilde{s}(.,0)}(n_0)}{\pi} \rceil - \lceil \frac{-\theta_{\tilde{s}(.,n_0)}(0)}{\pi} \rceil\\
&= \lceil \frac{\Delta_{\tilde{s}(.,n_0),\tilde{s}(.,0)}(n_0)}{\pi} \rceil - \lceil \frac{\Delta_{\tilde{s}(.,n_0),\tilde{s}(.,0)}(0)}{\pi} \rceil = 0.
\end{aligned} \tag{5.57}$$

Similarly,

$$\begin{aligned}
&\dim \operatorname{Ran} P_{(-\infty,\lambda)}(H^1_{0,n_0}) - \dim \operatorname{Ran} P_{(-\infty,\lambda]}(H^0_{0,n_0})\\
&= \lceil \frac{\Delta_{s_0(.,0),s_1(.,n_0)}(n_0)}{\pi} \rceil - \lfloor \frac{\Delta_{s_0(.,0),s_1(.,n_0)}(0)}{\pi} \rfloor - 1
\end{aligned} \tag{5.58}$$

holds. Apply Lemma 5.9. $\square$

APPENDIX A

Notation

$a(n)$	sequence in $\ell(\mathbb{Z})$, first upper and lower secondary diagonal of J
$b(n)$	sequence in $\ell(\mathbb{Z})$, diagonal of J
$b_t(n)$	sequence in $\ell(\mathbb{Z})$, diagonal of J^t
$\mathbb{C}$	set of complex numbers
c	fundamental solution of $\tau s = \lambda s$ vanishing in 1
dim	dimension of a linear space
det	determinant
$\dot{f}$	derivative of f with respect to t
H	Jacobi operator
H_{0,n_0}	finite Jacobi operator associated with J_{0,n_0}
H^t_{0,n_0}	finite Jacobi operator associated with J^t_{0,n_0}
J	Jacobi matrix associated with τ
J^t	Jacobi matrix associated with τ_t
J_{0,n_0}	finite Jacobi matrix associated with τ
J^t_{0,n_0}	finite Jacobi matrix associated with τ_t
λ	a real number
$\ell(\mathbb{Z})$	set of all complex valued sequences $(f(n))_{n\in\mathbb{Z}}$
$\ell^p(\mathbb{Z})$	space of all p-power summable sequences $(f(n))_{n\in\mathbb{Z}}$
$\ell^\infty(\mathbb{Z})$	space of all bounded sequences $(f(n))_{n\in\mathbb{Z}}$
P_Ω	family of spectral projections
$\rho(n)$	Prüfer variable of a solution u at n
$\rho_t(n)$	Prüfer variable of a solution u_t at n
$\mathbb{R}$	set of real numbers
Ran	range of an operator
$\sigma_{ess}(A)$	essential spectrum of A
s	fundamental solution of $\tau s = \lambda s$ vanishing at 0
s_t	fundamental solution of $\tau_t s_t = \lambda s_t$ vanishing at 0
$s(.,n_0)$	fundamental solution of $\tau s = \lambda s$ vanishing at n_0
$s_t(.,n_0)$	fundamental solution of $\tau_t s_t = \lambda s_t$ vanishing at n_0
τ	second order, symmetric difference expression
τ_t	modified second order, symmetric difference expression
t	real number in $[0,1]$

$\theta_u(n)$	the Prüfer angle of a solution u at n
$\theta_0(n)$	the Prüfer angle of a solution u_0 at n
$u(n)$	solution of $\tau u = \lambda u$
$u_t(n)$	solution of $\tau_t u_t = \lambda u_t$
$\#(u)$	number of nodes of u
$\#_{(m,n)}(u)$	number of nodes of u between m and n
$W(f,g)$	Wronskian
$\#_{(m,n)}W(u_0,u_1)$	number of nodes of $W(u_0,u_1)$ between m and n
$\lceil x \rceil$	$\sup\{n \in \mathbb{Z} \mid n \geq x\}$, ceiling function
$\lfloor x \rfloor$	$\sup\{n \in \mathbb{Z} \mid n \leq x\}$, floor function
$\mathbb{Z}$	set of Integers
z^*	complex conjugate of z

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Curriculum Vitae

I was born on October 21, 1977, in Hohenems, Vorarlberg. After receiving my A-levels (Matura) in June 1996 I started to study mathematics at the University of Vienna. During my studies I was working as a network administrator and as a software engineer where I was – among other activities – involved in the project BASES at the University of Innsbruck funded by the Federal Ministry for Education, Science and Culture (bm:bwk). From 2001 to 2004 I was working as a referee for the Austrian National Association of Students (ÖH) at my university. During 2003 and 2004 I was the chairman of the Students' Association at the Faculty of Natural and Theoretical Sciences at the University of Vienna. At that time I started to study Software and Information Engineering at the TU Vienna. From 2004 to 2007 I was a teaching assistant at the University of Vienna. Since 2004 I am working as an IT professional at the Faculty of Mathematics at the University of Vienna. Since 2008 I teach the Introductory Seminar Mathematics for Computer Science Ed. 1 at the University of Vienna.