Geometric and Asymptotic Group Theory II

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http://www.mat.univie.ac.at/∼dosaj/GGTWien/Course.html
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Blatt 7

Residually finite groups

A group G is *residually finite* if for every its element $g \neq 1_G$ there exists a homomorphism $\varphi\colon G \to F$ into some finite group F, such that $\varphi(g) \neq 1_F$.

(1) Show that a group G is residually finite iff one of the following conditions hold.
  (a) For every element $g \neq 1_G$ in G, there exists a finite index subgroup $K \leq G$ with $g \notin K$.
  (b) For every finite set A of nontrivial elements in G, there exists a homomorphism $\varphi\colon G \to F$ into some finite group F, such that $\varphi(g) \neq 1_F$, for every $g \in A$.
  (c) The intersection of all (normal) subgroups of G of finite index is trivial.
  (d) Let $G = \pi_1(X, x_0)$. For every homotopically non-trivial loop γ in (X, x_0) there is a finite covering $p\colon \widetilde{X} \to X$ such that γ does not lift up to a loop in $\widetilde{X}$.

(2) Show that $\mathbb{Z}$ and $\mathbb{Z}^2$ are residually finite.

(3) Let T be a labeled tree of valence $k \geq 2$ (at every vertex). Let $G \leq \mathrm{Aut}(T)$ be the group generated by reflections wrt. edges. Show that G is residually finite.

(4) *Free groups are residually finite–a probabilistic approach.* Let Γ be a finite graph. Consider its double covering $p\colon \widetilde{\Gamma} \to \Gamma$. It means in particular the following. For each vertex $v \in \Gamma$ there are two vertices $\widetilde{v}_1, \widetilde{v}_2 \in \widetilde{\Gamma}$ with $p(\widetilde{v}_1) = p(\widetilde{v}_2) = v$, and if $\{\widetilde{v}, \widetilde{w}\}$ is an edge in $\widetilde{\Gamma}$ then $\{p(\widetilde{v}), p(\widetilde{w})\}$ is an edge in Γ.
  (a) Observe that $g := \mathrm{girth}(\Gamma) \leq \mathrm{girth}(\widetilde{\Gamma})$.
  (b) Let Z be a random variable counting the number of cycles (i.e. polygonal loops) of length g in a double covering of Γ. Show that EZ (the expected value of Z) equals the number of g-cycles in Γ.
  (c) Conclude that there is a double covering with fewer g-cycles.
  (d) Show that there exists a (not necessarily double) covering $\widetilde{\Gamma}$ with
$$\mathrm{girth}(\widetilde{\Gamma}) > \mathrm{girth}(\Gamma).$$
  (e) Conclude that free groups are residually finite.