Questions in the topological study of self-affine tiles

Benoît Loridant
MU Leoben

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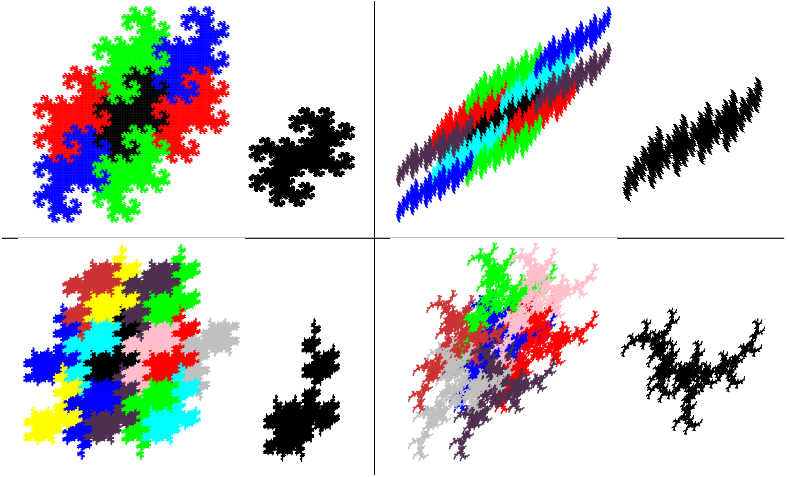
Tiles associated with numeration systems



Topological questions: topological disk, cut points, connected components of the interior, holes (fundamental group)?

Assumptions

T is a planar **integral self-affine tile** inducing a **self-similar tiling**.
Sufficient conditions: Lagarias-Wang, Gröchenig-Haas (1994).
In particular,

- T is the **attractor** of an Iterated Function System (IFS):

$$T = \bigcup_{d \in \mathcal{D}} f_d(T)$$

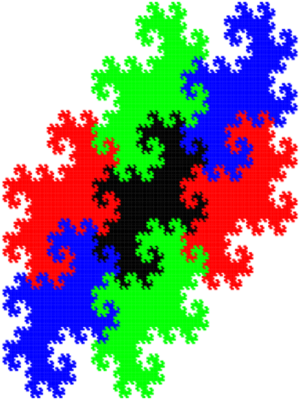
  with $f_d(x) = \mathbf{A}^{-1}(x+d)$, $\mathbf{A} \in \mathbb{Z}^{2\times 2}$ *expanding* matrix and $\mathcal{D} \subset \mathbb{Z}^2$ *digit set*.
  Note that $T = \left\{ \sum_{j \geq 1} \mathbf{A}^{-k} d_k ; d_k \in \mathcal{D} \right\} \leftrightarrow \mathcal{D}^{\mathbb{N}}$.

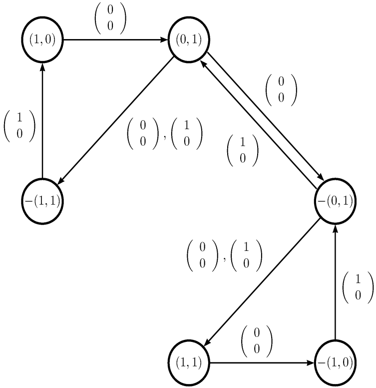
- ∂T is the **attractor** of a Graph-directed IFS (*GIFS*):

$$\begin{cases} \partial T &= \bigcup_{s \in \mathcal{S}} T \cap (T+s) \\ T \cap (T+s) &= \bigcup_{s \xrightarrow{d} s' \in \mathcal{G}} f_d(T \cap (T+s')), \end{cases}$$

and $\mathcal{G}$ is called the **boundary graph**: $\mathcal{L}(\mathcal{G}) \subset \mathcal{D}^{\mathbb{N}}$.

Example: boundary automaton of Knuth tile





$$T \cap (T + (1,0)) = \left\{ \sum_{k \geq 1} \mathbf{A}^{-k} d_k ; (1,0) \xrightarrow{d_1=(0,0)} s_1 = (0,1) \xrightarrow{d_2} s_2 \xrightarrow{d_3} \cdots \in \mathcal{G} \right\}.$$

Boundary parametrization (Shigeki-B.)

There exists a mapping $C : \begin{array}{ccc} [0,1] & \to & \partial T \\ 0._\beta a_1 a_2 \ldots & \mapsto & \sum_{k\geq 1} \mathbf{A}^{-k} d_k \end{array}$.

- Dumont-Thomas number system in $[0,1]$; β is the spectral radius of the incidence matrix of $\mathcal{G}$.
- C is onto and Hölder continuous, with exponent $1/\dim_H \partial T$ if $\mathbf{A}$ is a similarity.

To extract from C topological information on ∂T: **identifications**

$$\begin{array}{ccc} \sum_{k\geq 1} \mathbf{A}^{-k} d_k & = & \sum_{k\geq 1} \mathbf{A}^{-k} d'_k \\ \leftrightarrow \quad (d_k & , & d'_k)_{k\geq 1} \end{array}$$

recognized by a graph $\mathcal{G}_{all}$.

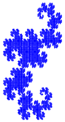
- **Trivial identifications**:

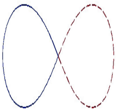
  $$\begin{array}{ccc} C(0._{\beta=10}1999\ldots) & = & C(0.200\ldots) \\ \leftrightarrow \quad (d_k & , & d'_k)_{k\geq 1} \end{array}$$

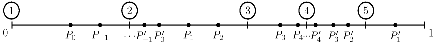
  recognized by a graph $\mathcal{G}_{trivial}$.
- **Non trivial identifications**:
  1. Compute $\mathcal{G}_{all} \setminus \mathcal{G}_{trivial}$: *complementation of Büchi automata*.
  2. Classify types of non-trivial identifications.
  3. Deduce topological properties.

About 2. Classify...

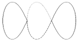
- No crossing pairs of identifications:







Case *outer identifications*.

Consider the simple arcs between two identifications: 
Then their union L is the boundary of an interior component L_0.
Description of $L, \overline{L_0}$ by a graph?

- Crossing? Inner identifications?