On the dynamics of Jeandel-Rao tilings

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CNRS + LaBRI + Université de Bordeaux

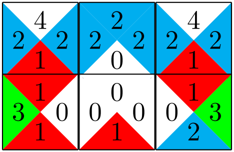
Wang tiles

A **Wang tile** is a square tile with a color on each border

Tile set T : a finite collection of such tiles.
A tiling of the plane : an assignment

$$\mathbb{Z}^2 \to T$$

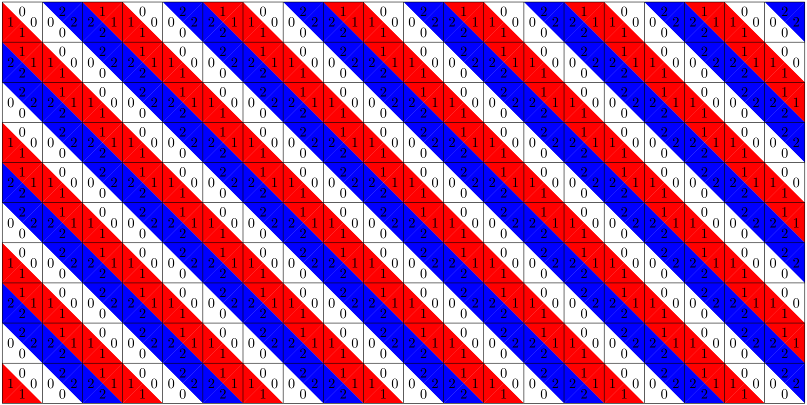
of tiles on infinite square lattice so that the contiguous edges of adjacent tiles have the same color.



Note : rotation not allowed.

Periods

A tiling is called **periodic** if it is invariant under some non-zero translation of the plane.



A Wang tile set that admits a periodic tiling also admits a **doubly periodic** tiling : a tiling with a horizontal and a vertical period.

Aperiodicity

A tile set is **finite** if there is no tiling of the plane with this set.
A tile set is **aperiodic** if it tiles the plane, but no tiling is periodic

Conjecture (Wang 1961)

Every set is either finite or periodic

False :

Theorem (Berger 1966)

There exists an aperiodic set of Wang tiles.

History

- Berger : 20426 tiles in 1966 (lowered down later to 104)
- Knuth : 92 tiles in 1968
- Robinson : 56 tiles in 1971
- Ammann : 16 tiles in 1971
- Grunbaum : 24 tiles in 1987
- Kari : 14 tiles in 1996
- Culik : (same method) 13 tiles in 1996
- Jeandel, Rao : 11 tiles in 2015
- Jeandel, Rao : every set of $\leq$ 10 tiles is finite or periodic

ELSEVIER

Discrete Mathematics 160 (1996) 259–264

DISCRETE MATHEMATICS

Note

A small aperiodic set of Wang tiles

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Received 3 January 1995

Abstract

A new aperiodic tile set containing only 14 Wang tiles is presented. The construction is based on Mealy machines that multiply Beatty sequences of real numbers by rational constants.

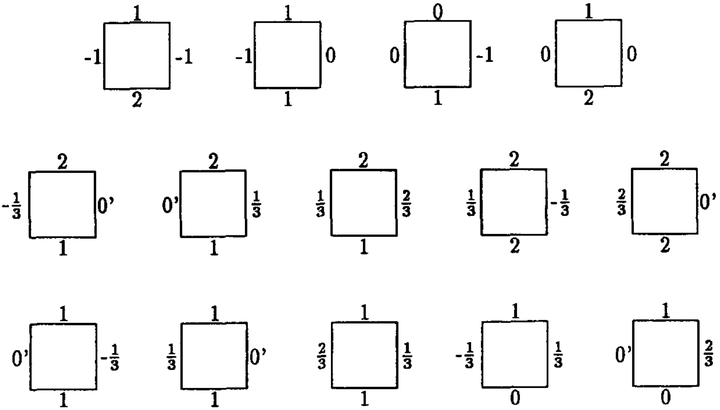
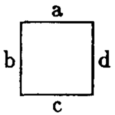


Fig. 1. Aperiodic set of 14 Wang tiles.

2. The tiles

Our set T of 14 tiles is shown in Fig. 1. The colors are rational numbers — number 0 may also be marked with a prime. Let us divide T into two disjoint sets T_2 and $T_{2/3}$, where T_2 contains the four tiles on the first row of Fig. 1 and $T_{2/3}$ the remaining ten tiles. The colors of the vertical edges are different in the two sets, so in every valid tiling all tiles on the same row must belong to the same set T_q.

We say that tile



multiplies by q if $aq + b = c + d$. In other words, the tile multiplies the number on its upper edge by q, adds the 'carry' from the left edge, and splits the result between the lower edge and the 'carry' to the right. Clearly, the tiles in T_2 multiply by 2, and the tiles in $T_{2/3}$ by $\frac{2}{3}$. (The vertical color $0'$ in $T_{2/3}$ is interpretated as 0; the prime is used to distinguish it from the color 0 used in T_2.)

Proposition 1. *The tile set T does not admit a periodic tiling.*

Proof. Assume that $f: \mathbb{Z}^2 \to T$ is a doubly periodic tiling with horizontal period a and vertical period b. For $i \in \mathbb{Z}$, let n_i denote the sum of colors on the upper edges of tiles $f(1, i), f(2, i), \ldots, f(a, i)$. Because the tiling is horizontally periodic with period a, the 'carries' on the left edge of $f(1, i)$ and the right edge of $f(a, i)$ are equal. Therefore $n_{i+1} = q_i n_i$, where $q_i = 2$ if tiles of T_2 are used on row i and $q_i = \frac{2}{3}$ if tiles of $T_{2/3}$ are used.

Because the vertical period of tiling f is b,

$$n_1 = n_{b+1} = q_1 q_2 \ldots q_b \cdot n_1,$$

and because two tiles with 0's on their upper edges cannot be next to each other, $n_1 \neq 0$. So $q_1 q_2 \ldots q_b = 1$. This contradicts the fact that no non-empty product of 2's and $\frac{2}{3}$'s can be 1. □

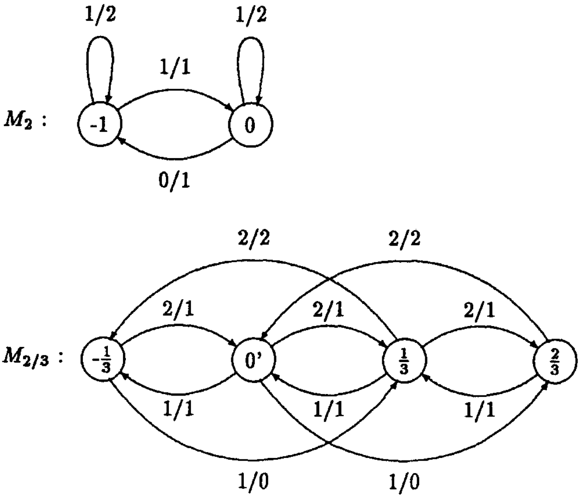


Fig. 2. Mealy machine corresponding to the aperiodic tile set.

Jeandel-Rao no aperiodic tile set of size ≤ 10

3.2.1 Easy cases

Recall that a Wang set is *not* aperiodic if

- Either there exists k so that $s(\mathcal{T}^k)$ is empty: there is no word w, w' so that $w\mathcal{T}^k w'$
- or there exists k so that $\mathcal{T}^k$ is periodic: there exists a word w so that $w\mathcal{T}^k w$

The general algorithm to test for aperiodicity is therefore clear: for each k, generate $\mathcal{T}^k$, and test if one of the two situations happen. If it does, the set is not aperiodic. Otherwise, we go to the next k. The algorithm stops when the computer program runs out of memory. In that case, the algorithm was not able to decide if the Wang set was aperiodic (it is after all an undecidable problem), and we have to examine carefully this Wang set.

This approach works quite well in practice: when launched on a computer with a reasonable amount of memory, it eliminates a very large number of tilesets. Of course, the key idea is to simplify as much as possible $\mathcal{T}^k$ before computing $\mathcal{T}^{k+1}$. Note that this should be done as fast as possible, as this will be done for *all* Wang sets. It turns out that the easy simplification that consists in deleting at each step tiles that cannot appear in a tiling of a row (i.e. vertices that are sources/terminals) is already sufficient.

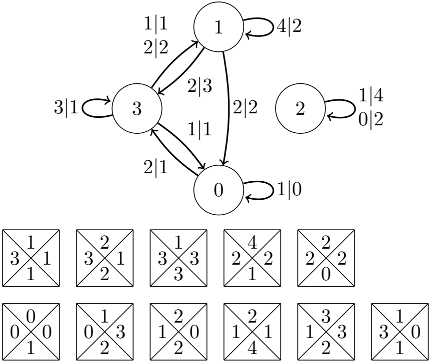
It is important to note that this approach relying on transducers (test whether the Wang set tiles k consecutive rows, and if it does so periodically) turned out in practice to be much more efficient than the naive approach using tilings of squares (test whether the Wang set tiles a square of size k, and if it does so periodically).

3.2.2 Harder cases

Once this has been done, a small number of Wang sets remain (at most 200), for which the program was not able to prove that they tile the plane periodically or that they do not tile the plane.

4 An aperiodic Wang set of 11 tiles - Proof Sketch

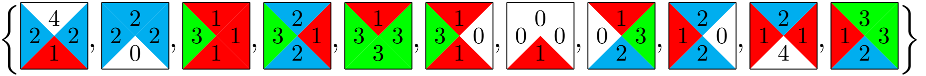
Using the same method presented in the last section, we were able to enumerate and test sets of 11 tiles, and found a few potential candidates. Of these few candidates, two of them were extremely promising and we will indeed prove that they are aperiodic sets.

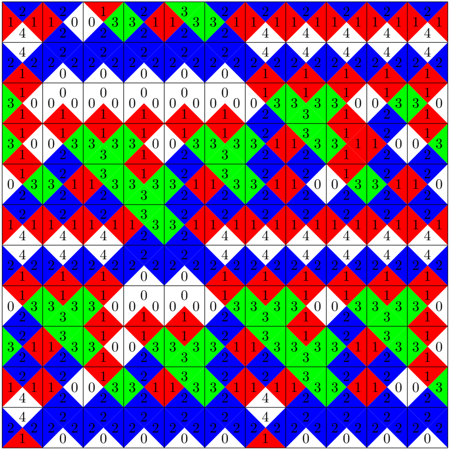


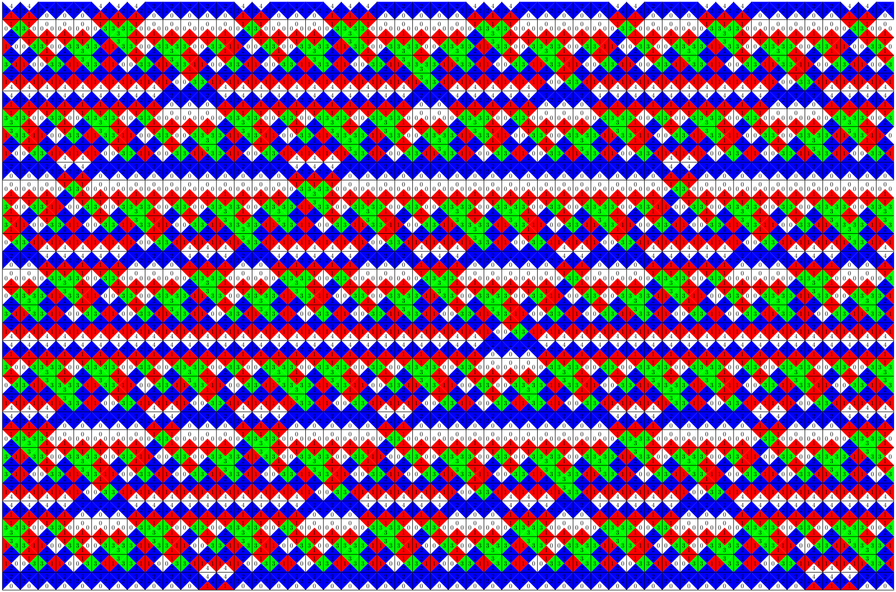
Source : Jeandel, Rao, An aperiodic set of 11 Wang tiles, `arxiv:1506.06492`.

Jeandel-Rao 11 tiles set

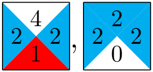
$\mathcal{T} =$







Some observations

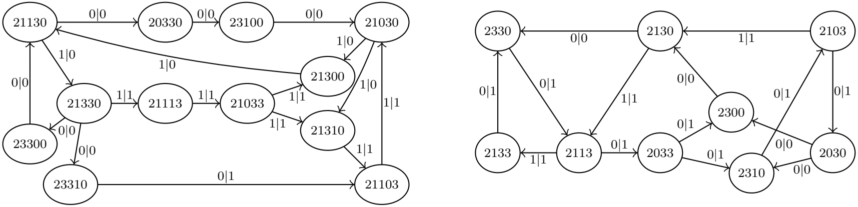
- horizontal lines are over $\mathcal{T}_1 = \{$ 

$\}$ or $\mathcal{T}_0 = \{$ 

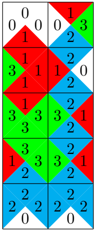
$\}$
- horizontal lines over $\mathcal{T}_1$ are often made of runs of 2 or 3 consecutive equal tiles.
- Jeandel, Rao : "It can be seen by a easy computer check that every tiling by $\mathcal{T}$ can be decomposed into a tiling by transducers $\mathcal{T}_1\mathcal{T}_0\mathcal{T}_0\mathcal{T}_0\mathcal{T}_0$ and $\mathcal{T}_1\mathcal{T}_0\mathcal{T}_0\mathcal{T}_0$."

In the proof, a transducer among others

They get a new transducer $\mathcal{T}_D$ which is the union of $\mathcal{T}_a \simeq \mathcal{T}_{10000}$ and $\mathcal{T}_b \simeq \mathcal{T}_{1000}$. The strongly connected components of $\mathcal{T}_{10000}$ (left) and $\mathcal{T}_{1000}$ (right) after deletion of unused transitions is :



Source : `arxiv:1506.06492` , Figure 7 (b) at page 17.

... in which  corresponds to two consecutive edges.

What is proved

Corollary 2. *The Wang set $\mathcal{T}_D = \mathcal{T}_a \cup \mathcal{T}_b$ is aperiodic. Furthermore, the set of words $u \in \{a,b\}^\star$ s.t. the sequence of transducers $\mathcal{T}_u$ appear in a tiling of the plane is exactly the set of factors of the Fibonacci word (*i.e. *the fixed point of the morphism $a \to ab, b \to a$), i.e. the set of factors of sturmian words of slope $1/\phi$, for ϕ the golden mean.*

The set of biinfinite words $u \in \{a,b\}^{\mathbb{Z}}$ s.t $\mathcal{T}_u$ represents a valid tiling of the plane are exactly the sturmian words of slope $1/\phi$.

See [BS02] for some references on sturmian words.

Proof. The sequence of words u_n we defined is the sequence of singular factors of the Fibonacci word (see for example [WW94]). Thus, on tilings by $\mathcal{T}_a \cup \mathcal{T}_b$, the vertical sequence on $\{a,b\}$ have the same set of factors that the Fibonacci word. □

Corollary 3. *The Wang set $\mathcal{T}$ is aperiodic. Furthermore, the set of words $u \in \{0,1\}^\star$ s.t. the sequence of transducers $\mathcal{T}_u$ appear in a tiling of the plane is exactly the set of factors of sturmian words of slope $\phi/(5\phi-1)$, for ϕ the golden mean.*

The set of biinfinite words $u \in \{0,1\}^{\mathbb{Z}}$ s.t $\mathcal{T}_u$ represents a valid tiling of the plane are exactly the sturmian words of slope $\phi/(5\phi-1)$.

Proof. Let ψ be the morphism $a \mapsto 10000, b \mapsto 1000$. The set of all words $u \in \{0,1\}^{\mathbb{Z}}$ that can appear in a tiling of the whole plane are exactly the image by ψ of the sturmian words over the alphabet $\{a,b\}$ of slope $1/\phi$.

Source : Jeandel, Rao, An aperiodic set of 11 Wang tiles, `arxiv:1506.06492`.

What is proved

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It should be $\frac{1}{\phi+2}$

$= \frac{1}{6-\phi}$

Source : Jeandel, Rao, An aperiodic set of 11 Wang tiles, `arxiv:1506.06492`.

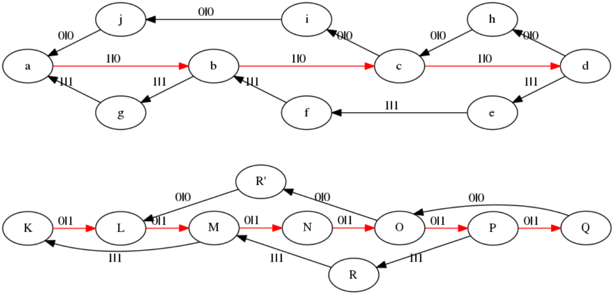
Another observation

... from the last slide of Michaël Rao's talk.

Open question 2 : "proof from the book" ?

If we look at densities of 1 on each line on an infinite tiling, one transducer add $\varphi - 1$ and the other add $\varphi - 2$.

$\rightarrow$ "additive-Kari-type" ?



The goals

- **Find the proof from the book** for the aperiodicity of Jeandel-Rao tilings : non-periodic + existence.
- **Understand the Wang shift Ω** of the Jeandel-Rao tilings. Is it minimal ? uniquely ergodic ? what is its entropy ? what is its pattern complexity ?

Wang tiles from codings of $\mathbb{Z}^2$-actions

- Let D be a **set**,
- $u, v : D \to D$ two **invertible transformations** s.t. $u \circ v = v \circ u$,
- I and J : two not necessarily disjoint **sets of colors**,
- $D = \cup_{i \in I} X_i$ and $D = \cup_{j \in J} Y_j$ be two topological **partitions** of D.

This gives the **left and bottom colors** :

$$\begin{array}{rccl} \ell : & D & \to & I \\ & \mathbf{x} & \mapsto & i \text{ if } \mathbf{x} \in X_i, \end{array} \qquad \begin{array}{rccl} b : & D & \to & J \\ & \mathbf{x} & \mapsto & j \text{ if } \mathbf{x} \in Y_j. \end{array}$$

and the **right and top colors** $r : D \to I$, $t : D \to J$ as :

$$r = \ell \circ u \quad \text{and} \quad t = b \circ v,$$

that is, the right color of an element $\mathbf{x} \in D$ is the left color of $u(\mathbf{x})$.
The **Wang tile coding** :

$$\begin{array}{rccl} c : & D & \to & I \times J \times I \times J \\ & \mathbf{x} & \mapsto & (r(\mathbf{x}), t(\mathbf{x}), \ell(\mathbf{x}), b(\mathbf{x})). \end{array}$$

Let $\mathcal{T} = c(D)$ be the associated **Wang tile set**.

Wang tilings from codings of $\mathbb{Z}^2$-actions

Lemma

Let

$$\begin{array}{rccl} f: & D & \to & \mathcal{T}^{\mathbb{Z}^2} \\ & \mathbf{x} & \mapsto & f_{\mathbf{x}} : (m,n) \mapsto c(u^m v^n \mathbf{x}). \end{array}$$

For every $\mathbf{x} \in D$, $f_{\mathbf{x}}$ is a ***Wang tiling of the plane****.*
Moreover the $\mathbb{Z}^2$-action of u and v on D is conjugate through f to $\mathbb{Z}^2$-translations of the tilings.

that is,

$$\begin{aligned} f_{u^m v^n \mathbf{x}}(m', n') &= c(u^{m'} v^{n'} u^m v^n \mathbf{x}) \\ &= c(u^{m'+m} v^{n'+n} \mathbf{x}) \\ &= f_{\mathbf{x}}(m'+m, n'+n) \\ &= f_{\mathbf{x}}\left((m', n') + (m, n)\right). \end{aligned}$$

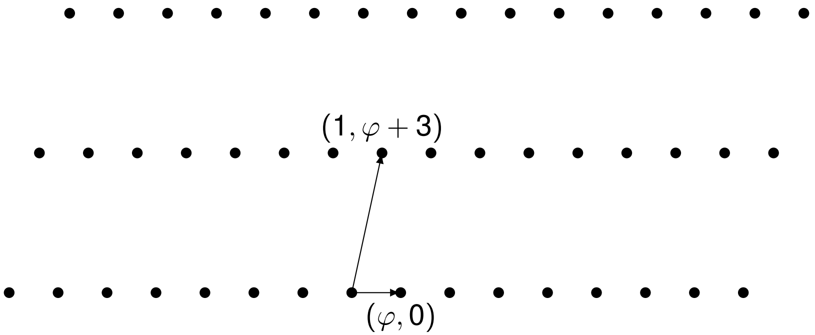
Therefore, unique ergodicity of the $\mathbb{Z}^2$-action on D will imply uniform patch frequencies in the tilings generated by f.

A first example

Sometimes, sleeping is better than takings hours to prepare a slide.
See the board...

A lattice, a torus and translations

Let $\varphi = \frac{1+\sqrt{5}}{2}$. Consider the **lattice** $\Gamma = \langle(\varphi, 0), (1, \varphi + 3)\rangle_{\mathbb{Z}}$.



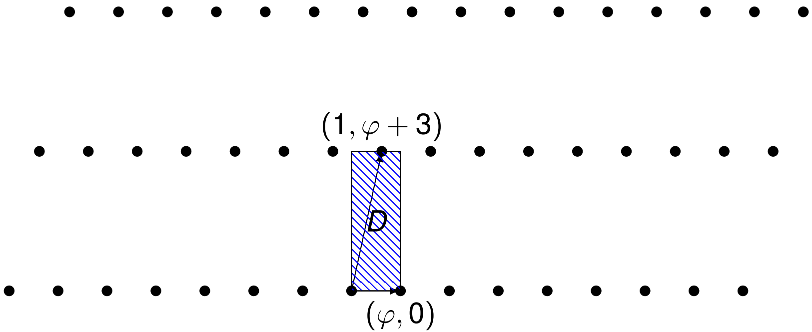
On the **torus** $\mathbb{R}^2/\Gamma$, we consider the **translations**

$$u : \begin{array}{ccc} \mathbb{R}^2/\Gamma & \to & \mathbb{R}^2/\Gamma \\ (x, y) & \mapsto & (x + 1, y) \end{array} \quad \text{and} \quad v : \begin{array}{ccc} \mathbb{R}^2/\Gamma & \to & \mathbb{R}^2/\Gamma \\ (x, y) & \mapsto & (x, y + 1). \end{array}$$

A fundamental domain

A **fundamental domain** of $\mathbb{R}^2/\Gamma$ is

$$D = [0, \varphi[\times [0, \varphi + 3[.$$

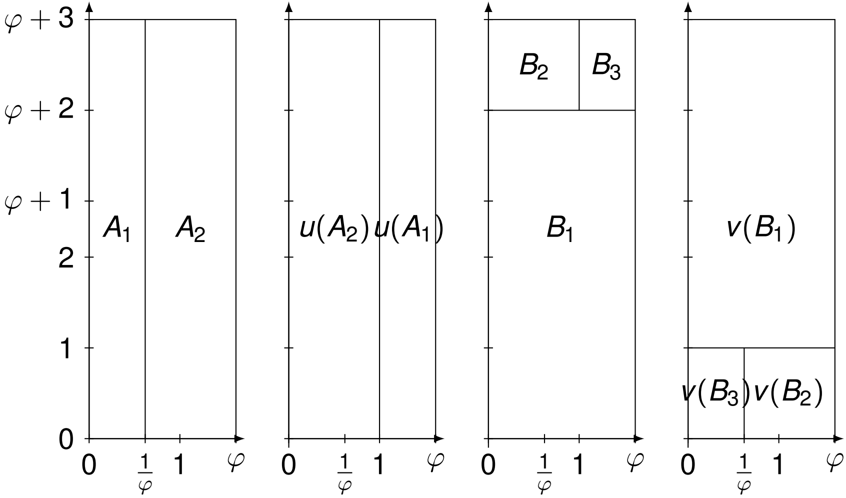


After renormalisation of transformations u and v on the torus $\mathbb{R}^2/\mathbb{Z}^2$, we observe that each translation vect. is not rationally independent :

$$\begin{pmatrix} \phi & 1 \\ 0 & \phi + 3 \end{pmatrix}^{-1} = \begin{pmatrix} \phi - 1 & -\frac{4}{11}\phi + \frac{5}{11} \\ 0 & -\frac{1}{11}\phi + \frac{4}{11} \end{pmatrix}.$$

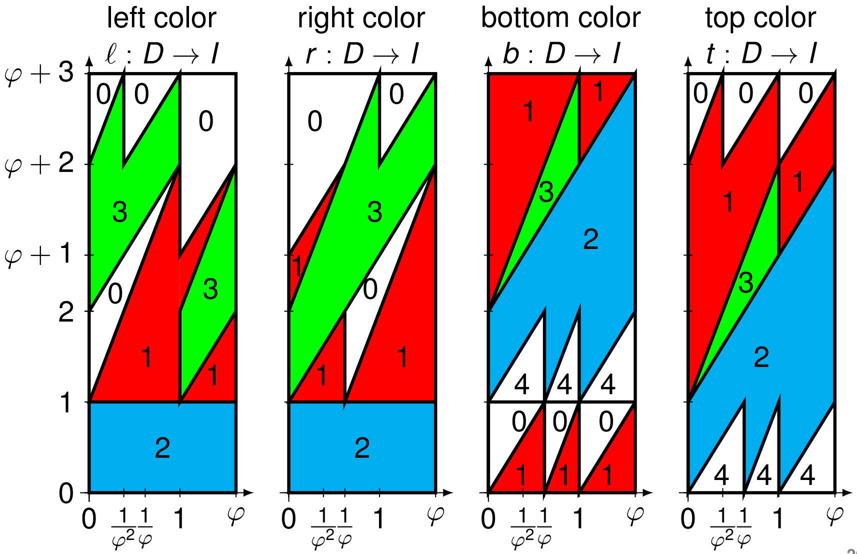
Transformations u and v on D

Transformations u and v are one-to-one **piecewise translations** of pieces on the fundamental domain D.



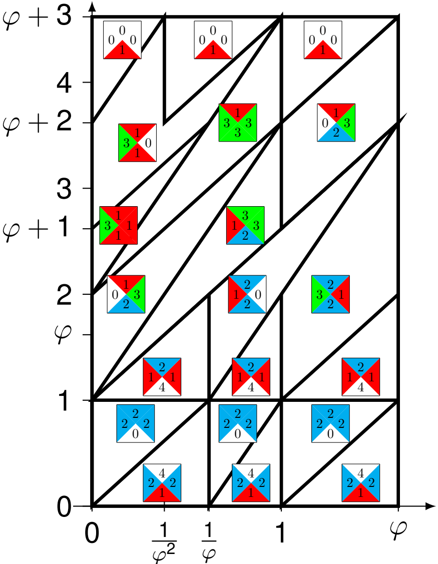
The color codings

The **left, right bottom and top color codings** satisfying $r = \ell \circ u$ and $t = b \circ v$.



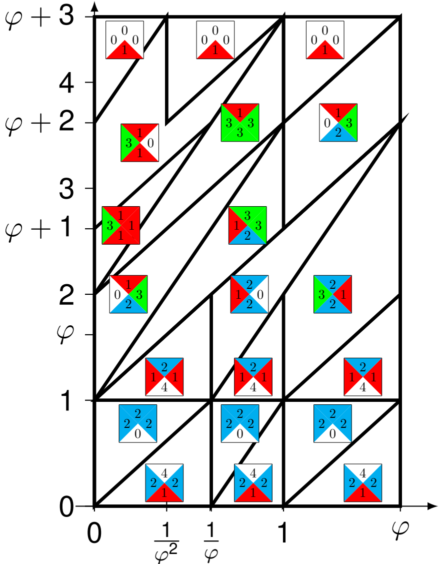
The tile coding

We deduce a **coding function** $c : D \to \mathcal{T}$



The tile coding

We deduce a **coding function** $c : D \to \mathcal{T}$ where $\mathcal{T}$ is the Jeandel-Rao tile set.



Some Jeandel-Rao tilings

Moreover we have the following result as an application of Lemma 1.

Proposition

Let $D = [0, \varphi[\times [0, \varphi + 3[$ with commuting transformations u and v, $\{X_i\}_{i \in I}$ and $\{Y_j\}_{j \in J}$ be partitions of D defined in the previous figures. For every $\mathbf{x} \in D$, the function

$$\begin{array}{rccl} f_{\mathbf{x}}: & \mathbb{Z}^2 & \to & \mathcal{T} \\ & (m,n) & \mapsto & c(u^m v^n \mathbf{x}) \end{array}$$

*is a **Wang tiling of the plane** made of the **Jeandel-Rao tile set** $\mathcal{T}$.*

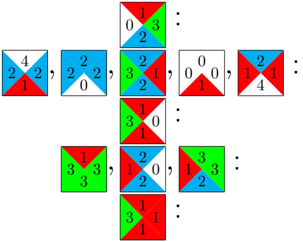
Frequency of patterns

Corollary

Since Lebesgue measure is the only invariant mesure on $\mathbb{R}^2/\Gamma$ which is invariant under both translations u and v, we have unique ergodicity of the tiling space

$$\overline{\{f_{\mathbf{x}} \mid \mathbf{x} \in D\}}$$

from which we deduce existence of pattern frequencies.



$5/(12\varphi + 14) \approx 0.1496$

$1/(2\varphi + 6) \approx 0.1083$

$1/(5\varphi + 4) \approx 0.0827$

$1/(8\varphi + 2) \approx 0.0669$

$1/(18\varphi + 10) \approx 0.0256$

Is it a complete description ?

Question

If Ω is the ***Wang shift*** *of all Wang tilings made of the Jeandel-Rao tile set $\mathcal{T}$, do we have*

$$\overline{\{f_{\mathbf{x}} \mid \mathbf{x} \in D\}} = \Omega \quad ?$$

Is it a complete description ?

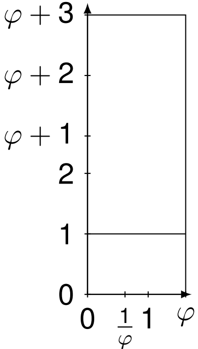
Question

If Ω is the ***Wang shift*** *of all Wang tilings made of the Jeandel-Rao tile set $\mathcal{T}$, do we have*

$$\overline{\{f_{\mathbf{x}} \mid \mathbf{x} \in D\}} = \Omega \quad ?$$

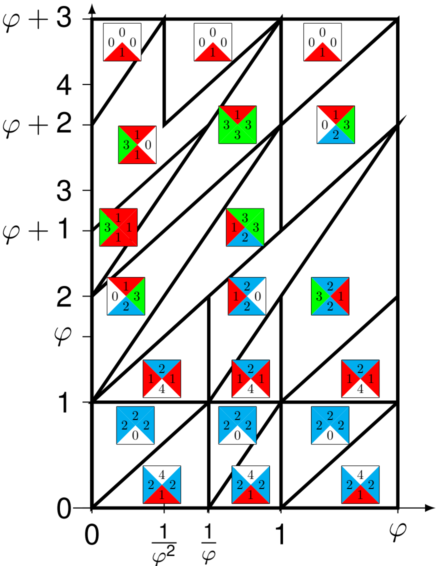
Short answer : I believe yes...

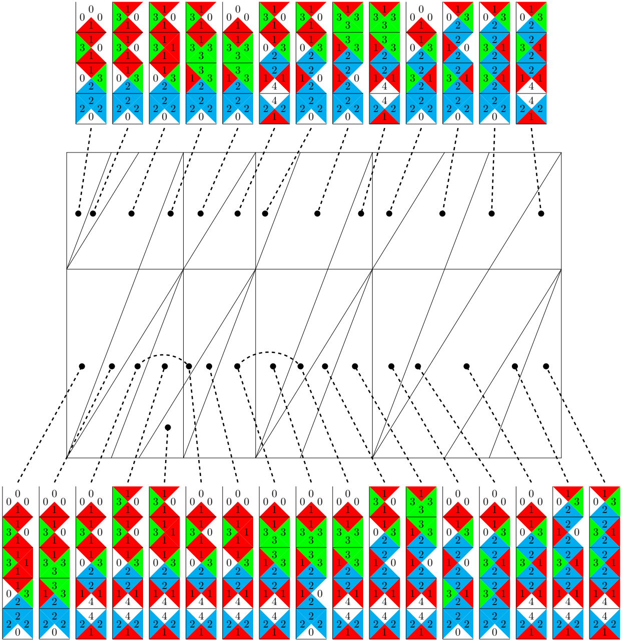
Let us do the induction of v on $[0, \varphi[\times[0, 1[$.



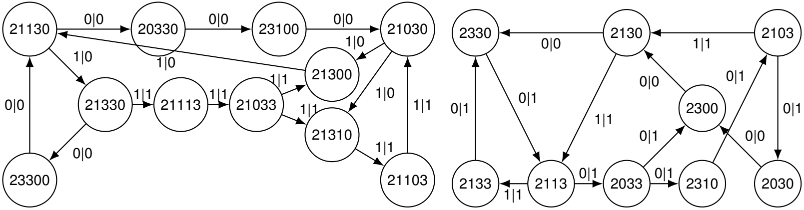
Recall the tile coding

$c : D \to \mathcal{T}$

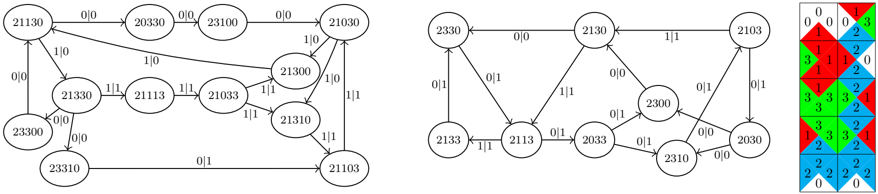




The transducer that we get from the induction of v on $[0,\varphi[\times[0,1[$ is :

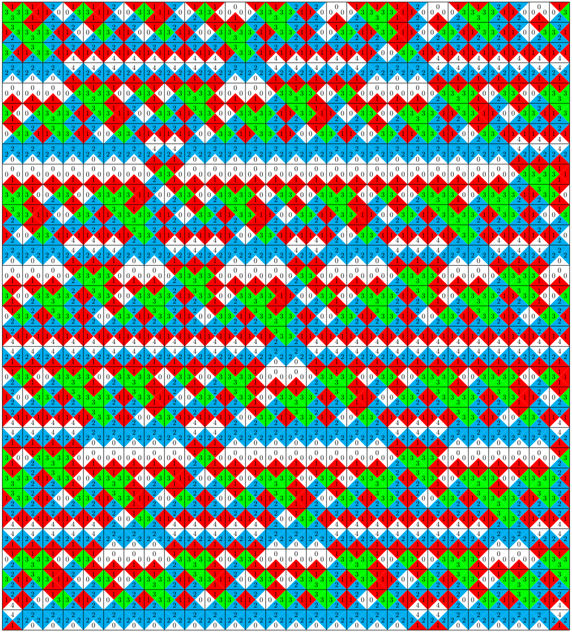


Recall Figure 7 (b) page 17 of Jeandel-Rao preprint :



In $\overline{f(D)}$, patterns 21330 $\xrightarrow{0|0}$ 23310 $\xrightarrow{0|1}$ 21103 and 2030 $\xrightarrow{0|0}$ 2310 have frequency zero... (why ?).

A surrounding of 21330 $\xrightarrow{0|0}$ 23310 $\xrightarrow{0|1}$ 21103



The questions

- **Find the proof from the book** for the aperiodicity of Jeandel-Rao tilings :
  - non-periodic
  - existence (done).
- Can we find a tiling in Ω, giving a positive frequency to patterns $21330 \xrightarrow{0|0} 23310 \xrightarrow{0|1} 21103$ and $2030 \xrightarrow{0|0} 2310$?
- **Understand the Wang shift Ω** of the Jeandel-Rao tilings. Is (Ω, shift) minimal ? uniquely ergodic ? is $\overline{f(D)} = \Omega$? what is its pattern complexity ? what is its entropy ?
- Can we generalize Jeandel-Rao tilings to **other Pisot numbers** ?
- What are the structure of the **other aperiodic tile sets** of cardinality 11 found by Jeandel-Rao ?

References : Slides of **Mickaël Rao** and **Jarkko Kari** which were very helpful to prepare mine are available at :

http://www.crm.umontreal.ca/2017/Pavages17/horaire_e.html