Dimension groups, orbit equivalence and eigenvalues

Fabien Durand

Université de Picardie Jules Verne

Workshop on substitutions and tiling spaces-Vienna

September, 26-30, 2017

Dye's theorem

Any two non atomic ergodic (invertible, bi-measurable) dynamical systems $(X_1, \mathcal{B}_1, \mu_1, T_1)$ and $(X_2, \mathcal{B}_2, \mu_2, T_2)$ are orbit equivalent:

There exists an invertible, bi-measurable, measure preserving map $\phi : X_1 \to X_2$ satisfying:

$$\phi(\mathcal{O}(T(x))) = \mathcal{O}(\phi(T(x)))$$

for μ_1-a.e. $x \in X_1$.

Dye's theorem

Any two non atomic ergodic (invertible, bi-measurable) dynamical systems $(X_1, \mathcal{B}_1, \mu_1, T_1)$ and $(X_2, \mathcal{B}_2, \mu_2, T_2)$ are orbit equivalent:

There exists an invertible, bi-measurable, measure preserving map $\phi : X_1 \to X_2$ satisfying:

$$\phi(\mathcal{O}(T(x))) = \mathcal{O}(\phi(T(x)))$$

for μ_1-a.e. $x \in X_1$.

Corollary

There is an unique class of measurable orbit equivalence.

Topological framework

Framework : (X, T) minimal Cantor system.

Topological framework

Framework : (X, T) minimal Cantor system.

- $(X,\ T)$ O.E. $(Y,\ S)$:

$$\exists\ h : X \overset{homeo.}{\longrightarrow} Y; \quad h(\mathcal{O}_T(x)) = \mathcal{O}_S(h(x))$$

Topological framework

Framework : (X, T) minimal Cantor system.

- $(X,\ T)$ O.E. (Y, S):

$$\exists\ h : X \overset{homeo.}{\longrightarrow} Y; \quad h(\mathcal{O}_T(x)) = \mathcal{O}_S(h(x))$$

- unique cocycle map: $\quad n : X \to \mathbb{Z} \qquad h(Tx) = S^{n(x)}h(x).$

Topological framework

Framework : (X, T) minimal Cantor system.

- (X, T) O.E. (Y, S):

$$\exists\ h : X \stackrel{homeo.}{\longrightarrow} Y; \quad h(\mathcal{O}_T(x)) = \mathcal{O}_S(h(x))$$

- unique cocycle map: $\quad n : X \to \mathbb{Z} \quad\quad h(Tx) = S^{n(x)} h(x).$

- (X, T) S.O.E. (Y, S):
  the Cocycle map has just one point of discontinuity.

(Strong) Orbit Equivalence

[Giordano, Putnam, Skau,' 95]

- (X, T) S.O.E. (Y, S) iff

  $(K^0(X,T), K^{0^+}(X,T), [1_X]) \simeq (K^0(Y,S), K^{0^+}(Y,S), [1_Y])$.

(Strong) Orbit Equivalence

[Giordano, Putnam, Skau,' 95]

- (X, T) S.O.E. (Y, S) iff

  $(K^0(X,T), K^0{}^+(X,T), [1_X]) \simeq (K^0(Y,S), K^0{}^+(Y,S), [1_Y])$.

- (X, T) O.E. (Y, S) iff

  $(K^0_m(X,\ T), K^0_m{}^+(X,\ T), [1_X]) \simeq (K^0_m(Y,\ S), K^0_m{}^+(Y,\ S), [1_Y])$.

Topological framework

Example: For a sturmian subshift of angle α,

Topological framework

Example: For a sturmian subshift of angle α,

$K^0(X, T) = \mathbb{Z}^2$

Topological framework

Example: For a sturmian subshift of angle α,

$K^0(X, T) = \mathbb{Z}^2$

$K^{0^+}(X, T) = \{(x, y) \in K^0(X,\ T); x + \alpha y > 0\}.$

Topological framework

Example: For a sturmian subshift of angle α,

$K^0(X, T) = \mathbb{Z}^2$

$K^{0^+}(X, T) = \{(x, y) \in K^0(X, T); x + \alpha y > 0\}.$

Corollary

There is uncountably many class of (strong) orbit equivalence.

Topological framework

Example: For a sturmian subshift of angle α,

$K^0(X, T) = \mathbb{Z}^2$

$K^{0^+}(X, T) = \{(x, y) \in K^0(X,\ T); x + \alpha y > 0\}$.

Corollary

There is uncountably many class of (strong) orbit equivalence.

Question:
What are the dynamical properties perserved under OE or SOE ?

Topological framework

Theorem (Sugisaki96, Ormes97, Boyle-Handelman94)

Within a SOE class any entropy is possible.

Topological framework

Theorem (Sugisaki96, Ormes97, Boyle-Handelman94)

Within a SOE class any entropy is possible.

Theorem (Giordano-Putnam-Skau95)

Within a SOE class the set of invariant probability measures are affinely homeomorphic.

Topological framework

Theorem (Sugisaki96, Ormes97, Boyle-Handelman94)

Within a SOE class any entropy is possible.

Theorem (Giordano-Putnam-Skau95)

Within a SOE class the set of invariant probability measures are affinely homeomorphic.

Corollary

Within a SOE class, if one system is uniquely ergodic, then all systems of this class are uniquely ergodic.

Eigenvalues

$\lambda = \exp(2i\pi\alpha)$ eigenvalue of (X, T, μ):

$$f(Tx) = \lambda f(x), \mu - \text{a.e. } x \in X, \quad f \in L^2(\mu)$$

Eigenvalues

$\lambda = \exp(2i\pi\alpha)$ eigenvalue of (X, T, μ):

$$f(Tx) = \lambda f(x), \mu - \text{a.e. } x \in X, \quad f \in L^2(\mu)$$

Continuous eigenvalue if $f \in C(X)$

Group of eigenvalues

Theorem (Ormes97)

Within an OE class, any countable subgroup of the circle $\mathbb{S}^1$ *can be realized as a group of eigenvalues.*

Group of eigenvalues

Theorem (Ormes97)

Within an OE class, any countable subgroup of the circle $\mathbb{S}^1$ *can be realized as a group of eigenvalues.*

Theorem (GPS95,Ormes97)

Within an SOE class, systems share the same continuous eigenvalues that are roots of the unity.

Group of eigenvalues

Theorem (Ormes97)

Within an OE class, any countable subgroup of the circle $\mathbb{S}^1$ can be realized as a group of eigenvalues.

Theorem (GPS95,Ormes97)

Within an SOE class, systems share the same continuous eigenvalues that are roots of the unity.

Corollary

Within a SOE class, if some system has a non trivial root of unity as continuous eigenvalue, then this class has no weakly mixing systems.

Group of eigenvalues

Question: What about continuous eigenvalues that are not root of the unity?

Group of eigenvalues

Question: What about continuous eigenvalues that are not root of the unity?

$$E(X, T) = \{\alpha \in \mathbb{R} | \exp(2i\pi\alpha) \text{ is a continuous eigenvalue } \}$$

Group of eigenvalues

Question: What about continuous eigenvalues that are not root of the unity?

$$E(X,T) = \{\alpha \in \mathbb{R} | \exp(2i\pi\alpha) \text{ is a continuous eigenvalue } \}$$

$$I(X,T) = \bigcap_{\mu \in \mathcal{M}(X,T)} \left\{ \int f d\mu | f \in C(X,\mathbb{Z}) \right\}$$

Group of eigenvalues

Question: What about continuous eigenvalues that are not root of the unity?

$$E(X,T) = \{\alpha \in \mathbb{R} | \exp(2i\pi\alpha) \text{ is a continuous eigenvalue } \}$$

$$I(X,T) = \bigcap_{\mu \in \mathcal{M}(X,T)} \left\{ \int f d\mu | f \in C(X,\mathbb{Z}) \right\}$$

Theorem (Itza-Ortiz09,Bressaud-Durand-Maass10,Cortez-Durand-Petite16)

$$E(X,T) \subset I(X,T)$$

Sturmian case

Let (X, T) be a sturmian subshift of angle α.

Sturmian case

Let (X, T) be a sturmian subshift of angle α.

$$E(X,T) = \{\alpha \in \mathbb{R} | \exp(2i\pi\alpha) \text{ is a continuous eigenvalue } \} = I(X,T)$$

Sturmian case

Let (X, T) be a sturmian subshift of angle α.

$E(X, T) = \{\alpha \in \mathbb{R} | \exp(2i\pi\alpha) \text{ is a continuous eigenvalue } \} = I(X, T)$

From Ormes97, there exists (Y, S) SOE to (X, T) such that $E(X, T) = \{0\}$

Sturmian case

Let (X, T) be a sturmian subshift of angle α.

$E(X,T) = \{\alpha \in \mathbb{R} | \exp(2i\pi\alpha) \text{ is a continuous eigenvalue } \} = I(X,T)$

From Ormes97, there exists (Y, S) SOE to (X, T) such that $E(X,T) = \{0\}$

Question: What are the subgroup of $I(X, T)$ that can be realized as a $E(Y, S)$?

For the sturmian case: Can we realize $\mathbb{Z} + 2\alpha\mathbb{Z}$?

Main results

Answer: NO!

Main results

Answer: NO!

Theorem (Cortez-Durand-Petite16)

Let (X, T) be a minimal Cantor system such that infinitesimal $f \in C(X, \mathbb{Z})$ are coboundaries. Then

$$I(X, T)/E(X, T)$$

is torsion free.

Main results

Answer: NO!

Theorem (Cortez-Durand-Petite16)

Let (X, T) be a minimal Cantor system such that infinitesimal $f \in C(X, \mathbb{Z})$ are coboundaries. Then

$$I(X, T)/E(X, T)$$

is torsion free.

Theorem (Giordano-Handelman-Hosseini17)

They deleted the condition on infinitesimals.

Dimension Group: (G, G^+)

- **A Countable Partially Ordered Abelian Group with:**

  (i) $G^+ + G^+ \subset G^+$,

  (ii) $G^+ - G^+ = G$,

  (iii) $G^+ \cap -G^+ = \{0\}$,

  (iv) $a \in G$ and $na \in G^+$, $n \in \mathbb{N}$ then $a \in G^+$.

Dimension Group: $(G,\ G^+)$

- **A Countable Partially Ordered Abelian Group with:**

  (i) $G^+ + G^+ \subset G^+$,

  (ii) $G^+ - G^+ = G$,

  (iii) $G^+ \cap -G^+ = \{0\}$,

  (iv) $a \in G$ and $na \in G^+$, $n \in \mathbb{N}$ then $a \in G^+$.

- **and should satisfy the Riesz interpolation Property:**

$$\forall\ a_1, a_2, b_1, b_2,\ \ a_i \leq b_j, i, j = 1, 2 \quad \exists\ c; \quad a_i \leq c \leq b_j.$$

Dimension Group: (G, G^+)

- **A Countable Partially Ordered Abelian Group with:**

  (i) $G^+ + G^+ \subset G^+$,

  (ii) $G^+ - G^+ = G$,

  (iii) $G^+ \cap -G^+ = \{0\}$,

  (iv) $a \in G$ and $na \in G^+$, $n \in \mathbb{N}$ then $a \in G^+$.

- **and should satisfy the Riesz interpolation Property:**

$$\forall\ a_1, a_2, b_1, b_2,\ a_i \leq b_j, i, j = 1, 2 \quad \exists\ c; \quad a_i \leq c \leq b_j.$$

Example:

- any lattice ordered group, $\mathbb{Z}^r$.

- any countable dense subgroup of $\mathbb{R}^n$ with the relative ordering.

Exotic example of dimension group

G: set of real algebraic numbers

Exotic example of dimension group

G: set of real algebraic numbers

Partial ordering on K:

Exotic example of dimension group

G: set of real algebraic numbers

Partial ordering on K: for $a, b \in K$, we set $a \prec b$ if and only if $a - b$ is a root of a polynomial $p(x) \in \mathbb{R}[x]$ which is a finite sum of squares of other polynomials, $p(x) = \sum_{i=1}^{m} q_i(x)^2$

$(G, G^+, 1)$ is a dimension group

- **[G. Elliott, '76]** Any Dimension group is a direct limit of lattice ordered groups and positive homomorphisms,

$$G = \varinjlim_{n} \mathbb{Z}^{r(n)} \xrightarrow{M_n} \mathbb{Z}^{r(n+1)}.$$

$$\cdots \longrightarrow \mathbb{Z}^{r(i)} \xrightarrow{M_i} \mathbb{Z}^{r(i+1)} \xrightarrow{M_{i+1}} \cdots \longrightarrow \mathbb{G}$$

- **[G. Elliott, '76]** Any Dimension group is a direct limit of lattice ordered groups and positive homomorphisms,

$$G = \varinjlim_n \mathbb{Z}^{r(n)} \xrightarrow{M_n} \mathbb{Z}^{r(n+1)}.$$

$$\cdots \longrightarrow \mathbb{Z}^{r(i)} \xrightarrow{M_i} \mathbb{Z}^{r(i+1)} \xrightarrow{M_{i+1}} \cdots \longrightarrow \mathbb{G}$$

Direct limit:

$$G = \coprod_i (Z^{r(i)} \times \{i\}) / \sim \quad \text{and} \quad [g,\, i] \sim [h,\, j] \;\text{ iff}$$

$$\exists\ k > i, j; \quad M_i \circ M_{i+1} \circ \cdots \circ M_k(g) = M_j \circ M_{j+1} \circ \cdots \circ M_k(h).$$

- [G. Elliott, '76] Any Dimension group is a direct limit of lattice ordered groups and positive homomorphisms,

$$G = \varinjlim_n \mathbb{Z}^{r(n)} \xrightarrow{M_n} \mathbb{Z}^{r(n+1)}.$$

$$\cdots \longrightarrow \mathbb{Z}^{r(i)} \xrightarrow{M_i} \mathbb{Z}^{r(i+1)} \xrightarrow{M_{i+1}} \cdots \longrightarrow \mathbb{G}$$

Direct limit:

$$G = \prod_i (Z^{r(i)} \times \{i\}) / \sim \quad \text{and} \quad [g, i] \sim [h, j] \;\; \text{iff}$$

$$\exists\; k > i, j; \quad M_i \circ M_{i+1} \circ \cdots \circ M_k(g) = M_j \circ M_{j+1} \circ \cdots \circ M_k(h).$$

- So we have a Bratteli diagram.

Example:

- $$\mathbb{Z} \xrightarrow{\times 2} \mathbb{Z} \xrightarrow{\times 2} \cdots \longrightarrow \mathbb{G} \simeq \mathbb{Z}[\tfrac{1}{2}].$$

$$[b, m] \sim [a,\ n], m \leq n \ \Leftrightarrow\ 2^{k-m}b = 2^{k-n}a \ \Leftrightarrow\ b = \frac{a}{2^{n-m}}.$$

Example:

- $$\mathbb{Z} \xrightarrow{\times 2} \mathbb{Z} \xrightarrow{\times 2} \cdots \longrightarrow \mathbb{G} \simeq \mathbb{Z}[\tfrac{1}{2}].$$
  $$[b, m] \sim [a,\ n], m \leq n \;\Leftrightarrow\; 2^{k-m}b = 2^{k-n}a \;\Leftrightarrow\; b = \frac{a}{2^{n-m}}.$$

- $$\mathbb{Z}^2 \xrightarrow{\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}} \mathbb{Z}^2 \xrightarrow{\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}} \cdots \longrightarrow \mathbb{G} \simeq \mathbb{Z}[\tfrac{1}{2}].$$

Example:

- $$\mathbb{Z} \xrightarrow{\times 2} \mathbb{Z} \xrightarrow{\times 2} \cdots \longrightarrow \mathbb{G} \simeq \mathbb{Z}[\tfrac{1}{2}].$$

$$[b, m] \sim [a, n], m \leq n \;\Leftrightarrow\; 2^{k-m}b = 2^{k-n}a \;\Leftrightarrow\; b = \frac{a}{2^{n-m}}.$$

- $$\mathbb{Z}^2 \xrightarrow{\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}} \mathbb{Z}^2 \xrightarrow{\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}} \cdots \longrightarrow \mathbb{G} \simeq \mathbb{Z}[\tfrac{1}{2}].$$

- $$\mathbb{Z}^2 \xrightarrow{\begin{bmatrix} a_1 & 1 \\ 1 & 0 \end{bmatrix}} \mathbb{Z}^2 \xrightarrow{\begin{bmatrix} a_2 & 1 \\ 1 & 0 \end{bmatrix}} \cdots \longrightarrow \mathbb{G} \simeq \mathbb{Z} + \theta\mathbb{Z}.$$

where $\theta = [a_1, a_2, a_3, \cdots]$.

- An order unit, u, in the Dimension group G is an element in G^+ that

$$\forall\, g \in G, \quad \exists\, n \in \mathbb{N}; \quad nu \geq g.$$

- An order unit, u, in the Dimension group G is an element in G^+ that

$$\forall\ g \in G, \quad \exists\ n \in \mathbb{N}; \quad nu \geq g.$$

- A state on the dimension group $(G,\ G^+,\ u)$ is a homomorphism $p : G \to \mathbb{R}$ that

$$p(u) = 1, \quad \text{and} \quad p(G^+) \geq 0.$$

- An order unit, u, in the Dimension group G is an element in G^+ that

$$\forall\ g \in G, \quad \exists\ n \in \mathbb{N}; \quad nu \geq g.$$

- A state on the dimension group (G, G^+, u) is a homomorphism $p : G \to \mathbb{R}$ that

$$p(u) = 1, \quad \text{and} \quad p(G^+) \geq 0.$$

The set of all the states on G is denoted by $S(G, u)$.

- An order unit, u, in the Dimension group G is an element in G^+ that
$$\forall\, g \in G, \quad \exists\, n \in \mathbb{N}; \quad nu \geq g.$$
- A state on the dimension group $(G,\, G^+,\, u)$ is a homomorphism $p : G \to \mathbb{R}$ that
$$p(u) = 1, \quad \text{and} \quad p(G^+) \geq 0.$$
The set of all the states on G is denoted by $S(G,\, u)$.
- $$\mathrm{Inf}(G) = \{g \in G : \quad p(g) = 0,\ \forall p \in S(G,\, u)\}.$$

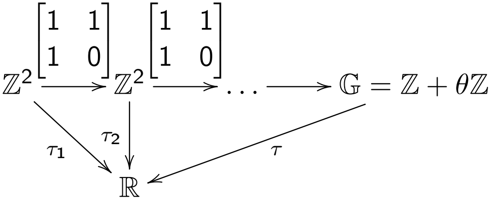
Example:

- $\theta = \frac{1+\sqrt{5}}{2} = [1, 1, 1, \cdots]$.

$$\begin{array}{ccccccc}
\mathbb{Z}^2 & \xrightarrow{\begin{bmatrix}1 & 1\\ 1 & 0\end{bmatrix}} & \mathbb{Z}^2 & \xrightarrow{\begin{bmatrix}1 & 1\\ 1 & 0\end{bmatrix}} & \dots & \longrightarrow & \mathbb{G} = \mathbb{Z} + \theta\mathbb{Z} \\
& {\scriptstyle \tau_1} \searrow & \downarrow {\scriptstyle \tau_2} & & & \swarrow {\scriptstyle \tau} & \\
& & \mathbb{R} & & & &
\end{array}$$

Example:

- $\theta = \frac{1+\sqrt{5}}{2} = [1, 1, 1, \cdots]$.

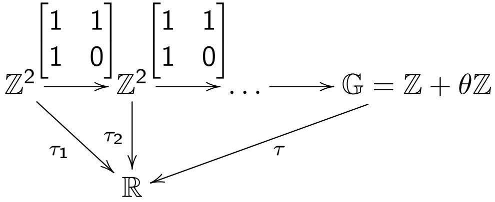


ν be the left Perron-Frobenius eigenvector.

$$\tau_k(m,\ n) = \frac{\nu \cdot (m, n)}{\theta^k},$$

Example:

- $\theta = \frac{1+\sqrt{5}}{2} = [1, 1, 1, \cdots].$



ν be the left Perron-Frobenius eigenvector.

$$\tau_k(m,\ n) = \frac{\nu \cdot (m, n)}{\theta^k}, \qquad \tau[(m,\ n),\ k] = \tau_k(m,\ n).$$

Example:

- $\theta = \frac{1+\sqrt{5}}{2} = [1, 1, 1, \cdots].$

$$\begin{array}{ccccccc} \mathbb{Z}^2 & \xrightarrow{\begin{bmatrix}1&1\\1&0\end{bmatrix}} & \mathbb{Z}^2 & \xrightarrow{\begin{bmatrix}1&1\\1&0\end{bmatrix}} & \cdots & \longrightarrow & \mathbb{G}=\mathbb{Z}+\theta\mathbb{Z} \\ & \searrow^{\tau_1} & \downarrow^{\tau_2} & & & \swarrow^{\tau} & \\ & & \mathbb{R} & & & & \end{array}$$

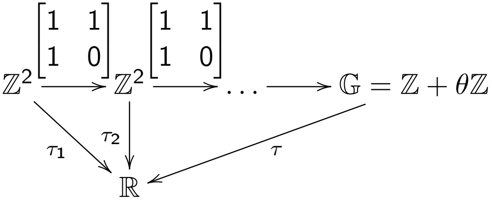
ν be the left Perron-Frobenius eigenvector.

$$\tau_k(m,\ n) = \frac{\nu \cdot (m, n)}{\theta^k}, \qquad \tau[(m,\ n),\ k] = \tau_k(m,\ n).$$

- $\ker(\tau_k) = \{(0,\ 0)\}.$

Example:

- $\theta = \frac{1+\sqrt{5}}{2} = [1, 1, 1, \cdots]$.



ν be the left Perron-Frobenius eigenvector.

$$\tau_k(m,\ n) = \frac{\nu \cdot (m, n)}{\theta^k}, \qquad \tau[(m,\ n),\ k] = \tau_k(m,\ n).$$

- $\ker(\tau_k) = \{(0,\ 0)\}$.
- G is totally ordered and so with unique state. So

$$\mathrm{Inf}(G) = \{0\}.$$

- An ordered ideal of the Dimension group G is a subgroup, J, that $J = J^+ - J^+$ and

$$\text{if } 0 \leq a \leq b \text{ and } \; b \in J \; \text{ then } \; a \in J.$$

- An ordered ideal of the Dimension group G is a subgroup, J, that $J = J^+ - J^+$ and

$$\text{if } 0 \leq a \leq b \text{ and } \; b \in J \; \text{ then } \; a \in J.$$

- A simple Dimension group is a dimension group without non-trivial ordered ideal.

- A Dimension group is simple iff it is the direct limit of a Bratteli diagram with positive incidence matrices.

Associated to any minimal Cantor system, (X, T), we have two simple dimension groups:

- (i) $K^0(X, T) := C(X, \mathbb{Z})/\{f - f \circ T : \; f \in C(X, \mathbb{Z})\}$,

Associated to any minimal Cantor system, (X, T), we have two simple dimension groups:

- (i) $K^0(X, T) := C(X, \mathbb{Z}) / \{f - f \circ T : f \in C(X, \mathbb{Z})\}$,

$$K^{0^+}(X, T) = \{[f] : \quad f \geq 0\}, \quad u = [1_X].$$

Associated to any minimal Cantor system, (X, T), we have two simple dimension groups:

- (i) $K^0(X, T) := C(X,\mathbb{Z})/\{f - f \circ T : f \in C(X,\mathbb{Z})\}$,

$$K^{0^+}(X, T) = \{[f] : \quad f \geq 0\}, \quad u = [1_X].$$

$$\mathcal{M}_T(X) = \{\mu : \ T\mu = \mu\} \sim S(K^0(X, T), u).$$

In fact, $\forall \mu \in \mathcal{M}_T(X)$, $\tau : G \to \mathbb{R}$: $[f] \mapsto \int f \, d\mu$.

Associated to any minimal Cantor system, $(X,\ T)$, we have two simple dimension groups:

- (i) $K^0(X,\ T) := C(X,\mathbb{Z})/\{f - f\circ T:\ f\in C(X,\mathbb{Z})\}$,

$$K^{0^+}(X,\ T) = \{[f]:\quad f \geq 0\},\quad u = [1_X].$$

$$\mathcal{M}_T(X) = \{\mu:\ \ T\mu = \mu\} \sim S(K^0(X,\ T),\ u).$$

In fact, $\forall \mu \in \mathcal{M}_T(X)$, $\tau : G \to \mathbb{R}$: $[f] \mapsto \int f\ d\mu$.

$$\mathrm{Inf}(K^0(X,\ T)) = \left\{[f]:\ \int f d\mu = 0,\ \forall \mu \in \mathcal{M}_T(X)\right\}$$

Associated to any minimal Cantor system, $(X,\ T)$, we have two simple dimension groups:

- (i) $K^0(X,\ T) := C(X,\mathbb{Z})/\{f - f\circ T:\ \ f\in C(X,\mathbb{Z})\}$,

$$K^{0^+}(X,\ T) = \{[f]:\ \ f\geq 0\},\quad u=[1_X].$$

$$\mathcal{M}_T(X) = \{\mu:\ \ T\mu=\mu\}\sim S(K^0(X,\ T),\ u).$$

In fact, $\forall\mu\in\mathcal{M}_T(X)$, $\tau: G\to\mathbb{R}$: $[f]\mapsto\int f\ d\mu$.

$$\mathrm{Inf}(K^0(X,\ T)) = \left\{[f]:\ \int fd\mu = 0,\ \forall\mu\in\mathcal{M}_T(X)\right\}$$

- $(ii) K^0_m(X,\ T) = K^0(X,\ T)/\mathrm{Inf}(K^0(X,\ T))$.

[Giordano, Putnam, Skau,' 95]

- (X, T) S.O.E. (Y, S) iff

  $(K^0(X, T), K^{0^+}(X, T), [1_X]) \simeq (K^0(Y, S), K^{0^+}(Y, S), [1_Y])$.

[Giordano, Putnam, Skau,' 95]

- (X, T) S.O.E. (Y, S) iff

  $(K^0(X, T), K^{0+}(X, T), [1_X]) \simeq (K^0(Y, S), K^{0+}(Y, S), [1_Y])$.

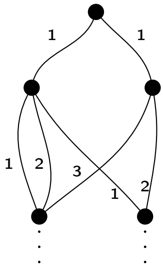
- (X, T) O.E. (Y, S) iff

  $(K^0_m(X, T), {K^0_m}^+(X, T), [1_X]) \simeq (K^0_m(Y, S), {K^0_m}^+(Y, S), [1_Y])$.

- [Herman, Putnam, Skau,'92]:
  i) $\forall$ simple dimension group $(G,\ u)$, $\exists$ CMS $(X,\ T)$;

$$K^0(X,\ T) \simeq G, \quad u = [1_X].$$

  ii) Using *Kakutani-Rokhlin partitions*, $(X,\ T)$ is conjugate to a *Vershik system* on a properly ordered *Bratteli diagram*, $(V,\ E,\ \leq)$ for G.



$$M(1) = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$M(n) = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$$

Example:

- Odometer based on $a = (a_1, a_2, \cdots)$:

$$\mathbb{Z} \xrightarrow{\times a_1} \mathbb{Z} \xrightarrow{\times a_2} \cdots \longrightarrow K^0(X, T) = \mathbb{Z}[\tfrac{1}{a}].$$

$$Z\left[\frac{1}{a}\right] = \left\{\frac{m}{a_1 a_2 \cdots a_k} : \ m \in \mathbb{Z}, k \in \mathbb{N}\right\}$$

- A Denjoy's with rotation number θ:

$$\mathbb{Z}^2 \xrightarrow{\begin{bmatrix} a_1 & 1 \\ 1 & 0 \end{bmatrix}} \mathbb{Z}^2 \xrightarrow{\begin{bmatrix} a_2 & 1 \\ 1 & 0 \end{bmatrix}} \cdots \longrightarrow K^0(X, T) = \mathbb{Z} + \theta\mathbb{Z}.$$

where $\theta = [a_1, a_2, a_3, \cdots]$.

Vershik map:

- An Odometer:

$$\{0, 1, 2\}^{\mathbb{N}} \to \{0, 1, 2\}^{\mathbb{N}}$$

$$(2,\ 2,\ 2,\ 0,\ a,\ \cdots) \mapsto (0,\ 0,\ 0,\ 0 + 1,\ a,\ \cdots).$$

Vershik map:

- An Odometer:

$$\{0,1,2\}^{\mathbb{N}} \to \{0,1,2\}^{\mathbb{N}}$$

$$(2,\ 2,\ 2,\ 0,\ a,\ \cdots) \mapsto (0,\ 0,\ 0,\ 0+1,\ a,\ \cdots).$$

- Let $(B, \leq)$ be an ordered Bratteli diagram and

$$x = (a_1,\ a_2,\ \cdots,\ a_{i_0},\ \cdots)$$

be an infinite path on it. Suppose that i_0 is the first i that a_i is not the max edge. Then

$$T(a_1,\ a_2, \cdots,\ a_{i_0},\ \cdots) = (0,\ 0,\ \cdots, 0,\ \ a_{i_0}+1,\ \cdots).$$

Recall

Dimension group

$$\cdots \longrightarrow \mathbb{Z}^{r(i)} \xrightarrow{M_i} \mathbb{Z}^{r(i+1)} \xrightarrow{M_{i+1}} \cdots \longrightarrow \mathbb{G}$$

Recall

Dimension group

$$\cdots \longrightarrow \mathbb{Z}^{r(i)} \xrightarrow{M_i} \mathbb{Z}^{r(i+1)} \xrightarrow{M_{i+1}} \cdots \longrightarrow \mathbb{G}$$

For minimal Cantor systems (X, T)

$$K^0(X,T) = C(X,\mathbb{Z})/\{f - f \circ T : \ f \in C(X,\mathbb{Z})\}$$

Kakutani-Rohlin partitions

$$\left(\mathcal{P}(n) = \{\, T^{-j} B_k(n); 1 \leq k \leq d(n),\ 0 \leq j < h_k(n)\} \ ;\ n \in \mathbb{N}\right)$$

Kakutani-Rohlin partitions

$$\left(\mathcal{P}(n) = \{T^{-j}B_k(n); 1 \leq k \leq d(n),\ 0 \leq j < h_k(n)\}\ ;\ n \in \mathbb{N}\right)$$

(KR1) $B(n+1) \subseteq B(n)$ with $B(n) = \cup_i B_i(n)$ called the base

Kakutani-Rohlin partitions

$$\left(\mathcal{P}(n) = \{T^{-j}B_k(n); 1 \leq k \leq d(n),\ 0 \leq j < h_k(n)\}\ ;\ n \in \mathbb{N}\right)$$

(KR1) $B(n+1) \subseteq B(n)$ with $B(n) = \cup_i B_i(n)$ called the base

(KR2) $\mathcal{P}(n+1) \succeq \mathcal{P}(n)$

Kakutani-Rohlin partitions

$$\left(\mathcal{P}(n) = \{T^{-j}B_k(n); 1 \leq k \leq d(n),\ 0 \leq j < h_k(n)\}\ ;\ n \in \mathbb{N}\right)$$

(KR1) $B(n+1) \subseteq B(n)$ with $B(n) = \cup_i B_i(n)$ called the base

(KR2) $\mathcal{P}(n+1) \succeq \mathcal{P}(n)$

(KR3) $\bigcap_{n\in\mathbb{N}} B(n)$ consists of a unique point

Kakutani-Rohlin partitions

$$\left(\mathcal{P}(n) = \{T^{-j}B_k(n); 1 \leq k \leq d(n),\ 0 \leq j < h_k(n)\}\ ;\ n \in \mathbb{N}\right)$$

(KR1) $B(n+1) \subseteq B(n)$ with $B(n) = \cup_i B_i(n)$ called the base

(KR2) $\mathcal{P}(n+1) \succeq \mathcal{P}(n)$

(KR3) $\bigcap_{n\in\mathbb{N}} B(n)$ consists of a unique point

(KR4) the sequence of partitions spans the topology of X

Theorem (Herman-Putnam-Skau '92)

Any minimal Cantor system has a sequence of Kakutani-Rohlin partitions.

Theorem (Herman-Putnam-Skau '92)

Any minimal Cantor system has a sequence of Kakutani-Rohlin partitions.

We define the k-th tower of $\mathcal{P}(n)$:

$$\mathcal{P}(n,k) = \bigcup_{0 \leq j < h_k(n)} T^{-j} B_k(n)$$

Theorem (Herman-Putnam-Skau '92)

Any minimal Cantor system has a sequence of Kakutani-Rohlin partitions.

We define the k-th tower of $\mathcal{P}(n)$:

$$\mathcal{P}(n,k) = \bigcup_{0 \leq j < h_k(n)} T^{-j} B_k(n)$$

Incidence matrices : $M(n) = m_{l,k}(n)$

$$m_{l,k}(n) = \#\{0 \leq j < h_l(n);\, T^{-j} B_l(n) \subseteq B_k(n-1)\}.$$

Some observations

Let $f \in C(X, \mathbb{Z})$:

Some observations

Let $f \in C(X, \mathbb{Z})$: There exists n such that f is constant on each atom of $\mathcal{P}(n)$

Some observations

Let $f \in C(X, \mathbb{Z})$: There exists n such that f is constant on each atom of $\mathcal{P}(n)$

$$f = \sum_{A \in \mathcal{P}(n)} w_A 1_A$$

Some observations

Let $f \in C(X, \mathbb{Z})$: There exists n such that f is constant on each atom of $\mathcal{P}(n)$

$$f = \sum_{A \in \mathcal{P}(n)} w_A 1_A$$

f is cohomologuous to

$$g = \sum_k \left(\sum_{A \in \mathcal{P}(n,k)} w_A \right) 1_{B_k(n)}$$

Some observations

Let $f \in C(X, \mathbb{Z})$: There exists n such that f is constant on each atom of $\mathcal{P}(n)$

$$f = \sum_{A \in \mathcal{P}(n)} w_A 1_A$$

f is cohomologuous to

$$g = \sum_k \left(\sum_{A \in \mathcal{P}(n,k)} w_A \right) 1_{B_k(n)}$$

$g - f = \beta h = h \circ T - h$, $h \in C(X, \mathbb{Z})$

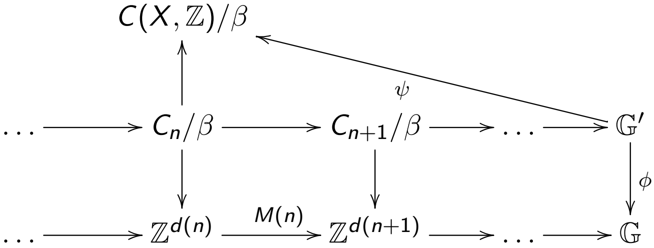
$$
\begin{array}{ccccccc}
 & & C(X,\mathbb{Z}) & & & & \\
 & & \uparrow & & & & \\
\cdots & \longrightarrow & C_n & \longrightarrow & C_{n+1} & \longrightarrow \cdots \longrightarrow \\
 & & \downarrow & & \downarrow & & \\
 & & \mathbb{Z}^{d(n)} & & \mathbb{Z}^{d(n+1)} & &
\end{array}
$$

C_n : continuous functions constant on atoms of $\mathcal{P}(n)$.

$$
\begin{array}{ccccccc}
 & & C(X,\mathbb{Z})/\beta & & & & \\
 & & \uparrow & & & & \\
\cdots & \longrightarrow & C_n/\beta & \longrightarrow & C_{n+1}/\beta & \longrightarrow & \cdots \longrightarrow \\
 & & \downarrow & & \downarrow & & \\
 & & \mathbb{Z}^{d(n)} & & \mathbb{Z}^{d(n+1)} & &
\end{array}
$$

C_n : continuous functions constant on atoms of $\mathcal{P}(n)$.

$$
\begin{array}{ccccccccc}
 & & C(X,\mathbb{Z})/\beta & & & & & & \\
 & & \uparrow & & & & & & \\
\ldots & \longrightarrow & C_n/\beta & \longrightarrow & C_{n+1}/\beta & \longrightarrow & \ldots & \longrightarrow & \\
 & & \downarrow & & \downarrow & & & & \\
\ldots & \longrightarrow & \mathbb{Z}^{d(n)} & \xrightarrow{M(n)} & \mathbb{Z}^{d(n+1)} & \longrightarrow & \ldots & \longrightarrow &
\end{array}
$$



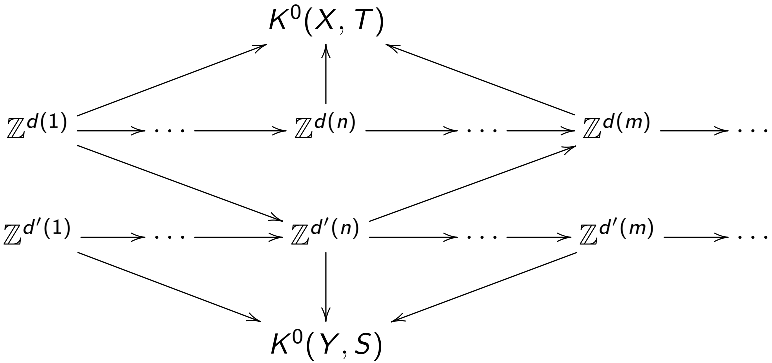
$C(X,\mathbb{Z})/\beta$
ψ
$\dots$
C_n/β
C_{n+1}/β
$\dots$
$\mathbb{G}'$
ϕ
$\dots$
$\mathbb{Z}^{d(n)}$
$M(n)$
$\mathbb{Z}^{d(n+1)}$
$\dots$
$\mathbb{G}$

Idea of the SOE proof

Suppose (X, T) and (Y, S) has the same dimension group

$$\begin{array}{ccccccccc}
 & & & & K^0(X,T) & & & & \\
 & \nearrow & & & \uparrow & & & \nwarrow & \\
\mathbb{Z}^{d(1)} & \longrightarrow & \cdots & \longrightarrow & \mathbb{Z}^{d(n)} & \longrightarrow & \cdots & \longrightarrow & \mathbb{Z}^{d(m)} & \longrightarrow & \cdots \\
 & & & & & & & & \\
\mathbb{Z}^{d'(1)} & \longrightarrow & \cdots & \longrightarrow & \mathbb{Z}^{d'(n)} & \longrightarrow & \cdots & \longrightarrow & \mathbb{Z}^{d'(m)} & \longrightarrow & \cdots \\
 & \searrow & & & \downarrow & & & \swarrow & \\
 & & & & K^0(Y,S) & & & &
\end{array}$$

Idea of the SOE proof



How to construct a (continuous) eigenfunction

$\lambda = \exp(2i\pi\alpha)$ eigenvalue of (X, T, μ).

How to construct a (continuous) eigenfunction

$\lambda = \exp(2i\pi\alpha)$ eigenvalue of (X, T, μ).

Let $r(x)$ be the first return time of x to some fixed clopen set U.

How to construct a (continuous) eigenfunction

$\lambda = \exp(2i\pi\alpha)$ eigenvalue of (X, T, μ).

Let $r(x)$ be the first return time of x to some fixed clopen set U.

We "almost" have $r(x) - r(Tx) = 1$

How to construct a (continuous) eigenfunction

$\lambda = \exp(2i\pi\alpha)$ eigenvalue of (X, T, μ).

Let $r(x)$ be the first return time of x to some fixed clopen set U.

We "almost" have $r(x) - r(Tx) = 1$

Thus $f(x) = \lambda^{r(x)}$ "almost" satisfies $f \circ T = \lambda f(x)$.

More precisely

r_n : the first return times map to $B(n)$.

More precisely

r_n : the first return times map to $B(n)$.

Proposition

Let (X, T) be a minimal Cantor systems. The following conditions are equivalent,

More precisely

r_n : the first return times map to $B(n)$.

Proposition

Let (X, T) be a minimal Cantor systems. The following conditions are equivalent,

- *λ is a continuous eigenvalue of (X, T);*

More precisely

r_n : the first return times map to $B(n)$.

Proposition

Let (X, T) be a minimal Cantor systems. The following conditions are equivalent,

- *λ is a continuous eigenvalue of (X, T);*
- *$(\lambda^{r_n(x)}; n \geq 1)$ converges uniformly in x;*

More precisely

r_n : the first return times map to $B(n)$.

Proposition

Let (X, T) be a minimal Cantor systems. The following conditions are equivalent,

- *λ is a continuous eigenvalue of (X, T);*
- *$(\lambda^{r_n(x)}; n \geq 1)$ converges uniformly in x;*
- *the sequence $(\alpha r_n(x); n \geq 1)$ converges $(\mathrm{mod}\ \mathbb{Z})$ uniformly in x.*

Heights : $H(n) = (h_l(n); 1 \leq l \leq C(n))^T$.

Heights : $H(n) = (h_l(n); 1 \leq l \leq C(n))^T$.

$$H(1) = M(1)$$

Heights : $H(n) = (h_l(n); 1 \leq l \leq C(n))^T.$

$$H(1) = M(1)$$

$$H(n) = M(n)H(n-1) = H(n) = P(n)H(1)$$

$$\text{Heights}: \; H(n) = (h_l(n); 1 \le l \le C(n))^T.$$

$$H(1) = M(1)$$

$$H(n) = M(n)H(n-1) = H(n) = P(n)H(1)$$

where

$$P(n) = M(n)M(n-1)\cdots M(2)$$

Numeration systems for minimal Cantor systems

For each $x \in X$ there exists a sequence $(s_n(x))_n$ such that

$$r_n(x) = \sum_{k=1}^{n-1} \langle s_k(x), H(k) \rangle$$

Conditions to be a continuous eigenvalue

Corollary

If λ *is a continuous eigenvalue of* (X, T) *then*

Conditions to be a continuous eigenvalue

Corollary

If λ *is a continuous eigenvalue of* (X, T) *then*

- $|||\alpha H(n)||| \to_n 0.$

Conditions to be a continuous eigenvalue

Corollary

If λ is a continuous eigenvalue of (X, T) then

- $|||\alpha H(n)||| \to_n 0$.
- $\sum_n \max_{x \in X} |||\alpha H_n||| < \infty$.

Conditions to be a continuous eigenvalue

Corollary

If λ *is a continuous eigenvalue of* (X, T) *then*

- $|||\alpha H(n)||| \to_n 0$.
- $\sum_n \max_{x \in X} |||\alpha H_n||| < \infty$.

Theorem (Durand-Frank-Maass15)

λ *is a continuous eigenvalue of* (X, T) *if and only if*

$$\sum_n \max_{x \in X} |||\langle s_n(x), \alpha H_n \rangle||| < \infty.$$

Numeration for dynamical systems

Let $\alpha \in E(X,T) \cap [0,1[$.

$$\alpha H(n) = \alpha P(n) H(1) = P(n) \alpha H(1) \to 0 \mod \mathbb{Z}$$

Numeration for dynamical systems

Let $\alpha \in E(X, T) \cap [0, 1[$.

$$\alpha H(n) = \alpha P(n) H(1) = P(n) \alpha H(1) \to 0 \mod \mathbb{Z}$$

$$\Longrightarrow \exists n_0, \; \alpha H(n_0) = v + w, \; w \in \mathbb{Z}^{C(n_0)}$$

Numeration for dynamical systems

Let $\alpha \in E(X, T) \cap [0, 1[$.

$$
\begin{aligned}
& \alpha H(n) = \alpha P(n) H(1) = P(n) \alpha H(1) \to 0 \mod \mathbb{Z} \\
\implies & \exists n_0,\ \alpha H(n_0) = v + w,\ w \in \mathbb{Z}^{C(n_0)} \\
& M(n) \ldots M(n_0 + 1) v \to 0
\end{aligned}
$$

Numeration for dynamical systems

Let $\alpha \in E(X,T) \cap [0,1[$.

$$\begin{aligned} & \alpha H(n) = \alpha P(n) H(1) = P(n) \alpha H(1) \to 0 \mod \mathbb{Z} \\ \implies & \exists n_0,\ \alpha H(n_0) = v + w,\ w \in \mathbb{Z}^{C(n_0)} \\ & M(n) \dots M(n_0+1) v \to 0 \end{aligned}$$

(We will suppose $n_0 = 1$.)

Invariant measures and eigenvalues

Let $\mu \in \mathcal{M}(X, T)$.

Invariant measures and eigenvalues

Let $\mu \in \mathcal{M}(X, T)$.

Let $\mu(n) = (\mu(B_i(n))^t$.

Invariant measures and eigenvalues

Let $\mu \in \mathcal{M}(X, T)$.

Let $\mu(n) = (\mu(B_i(n))^t$.

One has $\mu(n)^t M(n) = \mu(n-1)^t$

Invariant measures and eigenvalues

Let $\mu \in \mathcal{M}(X, T)$.

Let $\mu(n) = (\mu(B_i(n))^t$.

One has $\mu(n)^t M(n) = \mu(n-1)^t$

Thus,

$$\langle \mu(1), v \rangle = \mu(1)^t v = \mu(n)^t M(n) M(n-1) \cdots M(2) v \to_n 0$$

Invariant measures and eigenvalues

Let $\mu \in \mathcal{M}(X, T)$.

Let $\mu(n) = (\mu(B_i(n))^t$.

One has $\mu(n)^t M(n) = \mu(n-1)^t$

Thus,

$$\langle \mu(1), v \rangle = \mu(1)^t v = \mu(n)^t M(n) M(n-1) \cdots M(2) v \to_n 0$$

$$\alpha = \alpha\mu(1)^t H(1) = \mu(1)^t (v + w) = \mu(1)^t w \in I(X, T)$$

Observation

Recall : $\alpha \in I(X,T) = \bigcap_{\mu \in \mathcal{M}(X,T)} \{\mu(U) | U \subset X \text{ clopen set } \}$

Observation

Recall : $\alpha \in I(X, T) = \bigcap_{\mu \in \mathcal{M}(X,T)} \{\mu(U) | U \subset X \text{ clopen set } \}$

Proposition

For $\alpha \in E(X, T)$ there exists a clopen set U such that $\alpha = \mu(U)$ for all $\mu \in \mathcal{M}(X, T)$.

Main arguments of

Main arguments of

Theorem (Cortez-Durand-Petite16)

Let (X, T) be a minimal Cantor system such that infinitesimal $f \in C(X, \mathbb{Z})$ are coboundaries. Then

$$I(X, T)/E(X, T)$$

is torsion free.

Main arguments of

Theorem (Cortez-Durand-Petite16)

Let (X, T) be a minimal Cantor system such that infinitesimal $f \in C(X, \mathbb{Z})$ are coboundaries. Then

$$I(X, T)/E(X, T)$$

is torsion free.

- "$\alpha = \mu(U)$"

Main arguments of

Theorem (Cortez-Durand-Petite16)

Let (X,T) be a minimal Cantor system such that infinitesimal $f \in C(X,\mathbb{Z})$ are coboundaries. Then

$$I(X,T)/E(X,T)$$

is torsion free.

- "$\alpha = \mu(U)$"
- Characterization of continuous eigenvalues

Limitation of the main result of CDP

(X_α, S) : sturmian subshift with angle α, the golden mean.

Limitation of the main result of CDP

(X_α, S) : sturmian subshift with angle α, the golden mean.

$(X_{2\alpha}, S)$: ...

choose (Y, T) SOE to $(X_{2\alpha}, S)$ and weakly mixing.

Limitation of the main result of CDP

(X_α, S) : sturmian subshift with angle α, the golden mean.

$(X_{2\alpha}, S)$: ...

choose (Y, T) SOE to $(X_{2\alpha}, S)$ and weakly mixing.

$(X_\alpha \times Y, S \times T)$ is minimal and uniquely ergodic.

Limitation of the main result of CDP

(X_α, S) : sturmian subshift with angle α, the golden mean.

$(X_{2\alpha}, S)$: ...

choose (Y, T) SOE to $(X_{2\alpha}, S)$ and weakly mixing.

$(X_\alpha \times Y, S \times T)$ is minimal and uniquely ergodic.

$I(X_\alpha \times Y, S \times T) = \mathbb{Z} + 2\alpha\mathbb{Z}$

Limitation of the main result of CDP

(X_α, S) : sturmian subshift with angle α, the golden mean.

$(X_{2\alpha}, S)$: ...

choose (Y, T) SOE to $(X_{2\alpha}, S)$ and weakly mixing.

$(X_\alpha \times Y, S \times T)$ is minimal and uniquely ergodic.

$I(X_\alpha \times Y, S \times T) = \mathbb{Z} + 2\alpha\mathbb{Z}$

$E(X_\alpha \times Y, S \times T) = \mathbb{Z} + \alpha\mathbb{Z}$

Limitation of the main result of CDP

(X_α, S) : sturmian subshift with angle α, the golden mean.

$(X_{2\alpha}, S)$: ...

choose (Y, T) SOE to $(X_{2\alpha}, S)$ and weakly mixing.

$(X_\alpha \times Y, S \times T)$ is minimal and uniquely ergodic.

$I(X_\alpha \times Y, S \times T) = \mathbb{Z} + 2\alpha\mathbb{Z}$

$E(X_\alpha \times Y, S \times T) = \mathbb{Z} + \alpha\mathbb{Z}$

Thus $I/E = \mathbb{Z}/2\mathbb{Z}$ has torsion element ...

Giordano-Handelman-Hosseini theorem

Theorem (GHH17)

Let (X, T) be a minimal Cantor system. Then

$$K^0(X, T)/\Theta(E(X, T))$$

is torsion free where ...

Proposition

$\Theta : E(X, T) \to K^0(X, T)$ *defined by*

$$\Theta(\alpha) = \begin{cases} \lfloor \alpha \rfloor [1_X] + [1_{U_{\{\alpha\}}}] & \text{if } \alpha \geq 0, \\ \lceil \alpha \rceil [1_X] + [1_{U_{\{\alpha\}}}] & \text{if } \alpha < 0. \end{cases}$$

is an injective homomorphism where ...

Theorem
$\alpha \in E(X, T) \cap [0,1]$ if and only if there exists a clopen set $U = U_\alpha$ such that

$$1_{U_\alpha} - \alpha \cdot \mathbf{1}$$

is a real coboundary. Moreover, for every $\mu \in \mathcal{M}(X, T)$,

$$\mu(U_\alpha) = \alpha.$$

Realization of subgroups

(X_α, S) and (X_β, S) with α and β rationally independent with $0 < \alpha + \beta < 1$.

Realization of subgroups

(X_α, S) and (X_β, S) with α and β rationally independent with $0 < \alpha + \beta < 1$.

$(X_\alpha \times X_\beta, S \times S)$ is minimal and uniquely ergodic.

Realization of subgroups

(X_α, S) and (X_β, S) with α and β rationally independent with $0 < \alpha + \beta < 1$.

$(X_\alpha \times X_\beta, S \times S)$ is minimal and uniquely ergodic.

Realization of subgroups

(X_α, S) and (X_β, S) with α and β rationally independent with $0 < \alpha + \beta < 1$.

$(X_\alpha \times X_\beta, S \times S)$ is minimal and uniquely ergodic.

$I(X_\alpha \times Y, S \times T) = \mathbb{Z} + \alpha\mathbb{Z} + \beta\mathbb{Z} + \alpha\beta\mathbb{Z}$

Realization of subgroups

(X_α, S) and (X_β, S) with α and β rationally independent with $0 < \alpha + \beta < 1$.

$(X_\alpha \times X_\beta, S \times S)$ is minimal and uniquely ergodic.

$I(X_\alpha \times Y, S \times T) = \mathbb{Z} + \alpha\mathbb{Z} + \beta\mathbb{Z} + \alpha\beta\mathbb{Z}$

$E(X_\alpha \times Y, S \times T) = \mathbb{Z} + \alpha\mathbb{Z} + \beta\mathbb{Z}$

Realization of subgroups

(X_α, S) and (X_β, S) with α and β rationally independent with $0 < \alpha + \beta < 1$.

$(X_\alpha \times X_\beta, S \times S)$ is minimal and uniquely ergodic.

$I(X_\alpha \times Y, S \times T) = \mathbb{Z} + \alpha\mathbb{Z} + \beta\mathbb{Z} + \alpha\beta\mathbb{Z}$

$E(X_\alpha \times Y, S \times T) = \mathbb{Z} + \alpha\mathbb{Z} + \beta\mathbb{Z}$

Thus $I/E = \alpha\beta\mathbb{Z}$ ...

Recipe to realize the "maximal" group of eigenvalues

for $K^0 = \mathbb{Z} + \alpha\mathbb{Z} + \beta\mathbb{Z}$ and with 3 towers (letters)

Recipe to realize the "maximal" group of eigenvalues

for $K^0 = \mathbb{Z} + \alpha\mathbb{Z} + \beta\mathbb{Z}$ and with 3 towers (letters)

Let $v_\alpha = \alpha\mathbf{1} - e_2$, $v_\beta = \beta\mathbf{1} - e_3$

Recipe to realize the "maximal" group of eigenvalues

for $K^0 = \mathbb{Z} + \alpha\mathbb{Z} + \beta\mathbb{Z}$ and with 3 towers (letters)

Let $v_\alpha = \alpha\mathbf{1} - e_2$, $v_\beta = \beta\mathbf{1} - e_3$

We need some $(M(n))$ (3×3 integer matrices) such that

$$M(n)M(n-1)\cdots M(2)v_\alpha \to_{n\to\infty} 0$$

Recipe to realize the "maximal" group of eigenvalues

for $K^0 = \mathbb{Z} + \alpha\mathbb{Z} + \beta\mathbb{Z}$ and with 3 towers (letters)

Let $v_\alpha = \alpha\mathbf{1} - e_2$, $v_\beta = \beta\mathbf{1} - e_3$

We need some $(M(n))$ (3×3 integer matrices) such that

$$M(n)M(n-1)\cdots M(2)v_\alpha \to_{n\to\infty} 0$$

and even more

$$\sum_{2\leq n\leq N} (\max_{i,j} M(n+1)_{i,j})||M(n)M(n-1)\cdots M(2)v_\alpha|| \to_{n\to\infty} 0$$

Recipe to realize the "maximal" group of eigenvalues

for $K^0 = \mathbb{Z} + \alpha\mathbb{Z} + \beta\mathbb{Z}$ and with 3 towers (letters)

Let $v_\alpha = \alpha\mathbf{1} - e_2$, $v_\beta = \beta\mathbf{1} - e_3$

We need some $(M(n))$ (3×3 integer matrices) such that

$$M(n)M(n-1)\cdots M(2)v_\alpha \to_{n\to\infty} 0$$

and even more

$$\sum_{2\leq n\leq N} (\max_{i,j} M(n+1)_{i,j})||M(n)M(n-1)\cdots M(2)v_\alpha|| \to_{n\to\infty} 0$$

Observation: in this case

$$\cap_n \mathbb{R}^d M(n)M(n-1)\cdots M(2) = \mathbb{R}(1-(\alpha+\beta),\alpha,\beta)$$