Tilings, Smale Spaces, C^*-algebras

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Smale spaces Def. (Ruelle) X is compact metric, $\phi : X \to X$ homeo. Then (X, ϕ) is a Smale space if $\exists\ \varepsilon_X > 0$ and $\lambda \in (0,1)$ and a bracket map on $\{(x,y) \in X^2 : d(x,y) < \varepsilon_X\} \longmapsto [x,y] \in X$ satisfying

B1) $[x,x] = x$

B2) $\cancel{[x,y]}\ [x,[y,z]] = [x,z]$

B3) $[[x,y],z] = [x,z]$

B4) $\phi([x,y]) = [\phi(x), \phi(y)]$

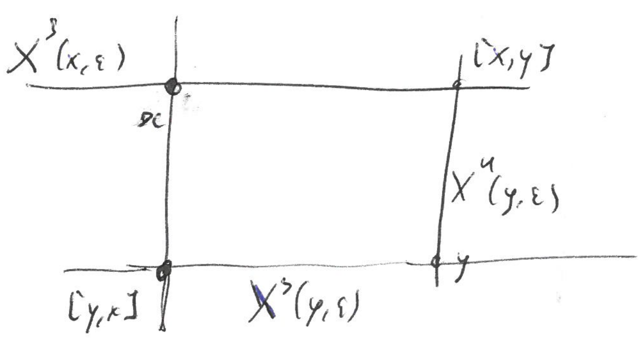
$\Big\}$ wherever defined

and contraction axioms

C1) For $(x,y) \in X^2$ s.t. $[x,y] = y$ we have
$$d(\phi(x), \phi(y)) < \lambda\, d(x,y)$$
contracting direction

C2) For $(x,y) \in X^2$ s.t. $[x,y] = x$ we have
$$d(\phi^{-1}(x), \phi^{-1}(y)) < \lambda\, d(x,y)$$

Let us define for $0 < \varepsilon < \varepsilon_X$:
$$X^s(x,\varepsilon) = \{ y \in X : [x,y] = y,\ d(x,y) < \varepsilon\}$$
$$X^u(x,\varepsilon) = \{ y \in X : [x,y] = x,\ d(x,y) < \varepsilon\}$$



There are global equivalence relations

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For $x \in X$ define $X^s(x) = \{ y \in X : d(\phi^n x, \phi^n y) \to 0$ as $n \to \infty \}$

$X^u(x) = \{ y \in X : d(\phi^{-n} x, \phi^{-n} y) \to 0$ as $n \to \infty \}$

Examples ① SFT $\Sigma_G = \{$ bi-infinite seq. of adjacent vertices of a graph $\}$

with left shift σ acting on it.

Here $\varepsilon_{\Sigma_G} = \frac{1}{2}$. if $d(x,y) < 1/2$ then $[x,y] = \cdots y_{-2} y_{-1} x_0 x_1 x_2 \cdots$

② Hyperbolic toral automorphisms.

③ William solenoid, Basic sets of Axiom A

Wieler solenoid (generalization of William solenoid)

Tiling spaces (substitution tilings)

Prototiles $\mathcal{P} = \{p_1, \ldots, p_n\}$ (e.g. polytopes in $\mathbb{R}^n$)

Tiling T = countable collection of translated prototiles $t_i = p + x$, $p \in \mathcal{P}$, $x \in \mathbb{R}^n$

$\bigcup t_i = \mathbb{R}^n$ $t_i^\circ \cap t_j^\circ = \emptyset$.

Properties: (i) Finite local complexity: ~~local~~ finitely many local patches up to translation

(ii) Aperiodic: $T + x = T \Rightarrow x = 0$.

(iii) Repetitivity: $\forall$ patch A $\exists R$ s.t. $\forall x \in \mathbb{R}^n$ $B_R(x)$ contains A.

Tiling metric: $d(T, T') = \inf \{ \varepsilon > 0 : T + u = T + v'$ on a ball of radius $R = 1/\varepsilon$ around the origin and $\|u\|, \|v\| < \varepsilon \} \vee 1$.

The hull of a tiling T is the completion of $T + \mathbb{R}^d$ in tiling metric.

A substitution ω on $\mathcal{P}$ consists of a constant $r > 1$ s.t. $\omega(p)$ = collection of translated prototiles and $\mathrm{supp}(\omega(p)) = \mathrm{supp}(r \cdot p)$

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(Solomyak)
Fact$^{\vee}$ $\omega : \Omega \to \Omega$ is homeo and topological mixing if ω is primitive.

The bracket map for $d(T,T') < \varepsilon < \varepsilon_\Omega$:

$[T, T'] = T' + v - u$ where $T+u$ & $T+v$ agree on a ball $B_{1/\varepsilon}(0)$ and $\|u\|, \|v\| < \varepsilon$

In this case: $X^s(T, \varepsilon) = \{ T' \in \Omega \mid T'$ and T agree on $B_{1/\varepsilon}(0) \}$

$X^u(T, \varepsilon) = \{ T' \in \Omega \mid T' = T + w$ for $\|w\| < 2\varepsilon \}$

Thm (Anderson & Putnam '98)

(Ω, d, ω) is a Smale space.

$X^s(T) = \{ T' : \exists n \in \mathbb{N}$ s.t. $\omega^n T$ & $\omega^n T'$ agree on a ball of radius 1/2 (or so) $\}$

$X^u(T) = \{ T' : T' = T + w$ for $w \in \mathbb{R}^d \}$

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C^*-algebras of dyn. systems

Def A C^*-algebra A is a Banach $*$-algebra such that

$$\|a^*a\| = \|a\|^2 \quad \forall a \in A$$

Here $a \mapsto a^*$ has the properties
$(a^*)^* = a$
$(a+b)^* = a^* + b^*$
$\lambda a^* = \bar{\lambda} a^* \quad \forall \lambda \in \mathbb{C}$
$(ab)^* = b^* a^*$

Thm (Gelfand-Naimark) Every C^* algebra is $*$-isomorphic to a closed $*$-subalgebra of bounded operators on a Hilbert space.

Thm (Gelfand-Naimark) If A is a commutative C^*-algebra then A is $*$-isomorphic to $C_0(X)$, for X a locally compact Hausdorff space.

compact support

Examples 1) $C(X)$ for X compact Hausdorff.

$$\|f\|_{C^*} = \sup_{x \in X} |f(x)| = \|f\|_\infty \qquad f(x)^* = \overline{f(x)} \text{ for } f: X \to \mathbb{C}$$

2) $\mathcal{B}(H)$ = bounded operators on a Hilbert space.
and $*$ is adjoint and operator norm.

3) $M_n(\mathbb{C})$ = $n \times n$ complex matrices, $A^* = \overline{A^T}$.

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Connes dictionary

Commutative	non-commutative
top. space	C^*-algebra
measure space	von Neumann algebras
range of function	spectrum of an operator
complex variables	operator
real variable	self-adjoint operator
infinitesimal	compact operator
integral	trace

Non-commutative quotient constructions

Suppose X is a top. space and $\sim$ an equivalence relation on X.
Typically we view $X/\sim$ as "forgetting" part of X.
The idea of non-communicative quotients is to "remember" the data from equivalent classes. To do this we use groupoids à la Renault (1980 PhD thesis).

Def A groupoid G is a set with a (distinguished) subset, ~~called~~ $X \subset G$, called the unit space, such that

- ~~r, s~~ $r, s : G \longrightarrow X$ are maps (range and source map)
- there is a set $G^{(2)} := G_r \times_s G := \{ (g,h) \in G \times G : r(g) = s(g) \}$ and a partially defined product
$$(g,h) \in G^{(2)} \longmapsto gh \in G$$
- there is an inverse map $g \mapsto g^{-1}$

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satisfying some relations

NB In addition we need a topology on G so that r, s, and $g \mapsto g^{-1}$ are continuous.

A groupoid is <u>étale</u> if the range and source maps are local homeomorphisms.

<u>Remark</u>
Done by Kellendonk for tiling spaces.

<u>Remark</u>
If $\sim$ is an equivalence relation, define G by $(x,y) \in G \iff x \sim y$.
Let the range map $r(x,y) = x$ (think of the diagonal of $G \times G$ as unit space X.)

and the source map $s(x,y) = y$

- $(x,y)(y,z) := (x,z)$ <u>Rem</u> $(x,y)(u,z)$ undefined if $y \neq u$.
- $(x,y)^{-1} = (y,x)$

Given a groupoid G, we take $C_{compact}(G)$ $= C_c(G)$ and this defines a $*$-algebra structure, with the <u>product</u>

$$f \cdot g(x,y) = \sum_{z \sim x} f(x,z) \cdot g(z,y)$$

and $*$-operation $f^*(x,y) = \overline{f(y,x)}$

To make this a C^*-algebra, we represent $C_r(G)$ as bounded operators on a suitable Hilbert space.

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Example let $A = \{a, b, c\}$

Define $a \sim b$. Consider $A/\sim$ and C^*-algebra on $C(A/\sim)$ that is isomorphic to $\mathbb{C}^2$.

Alternatively (instead) we define a C^*-algebra as above: The groupoid

$$G = \{(a,a), (a,b), (b,a), (b,b), (c,c)\}$$

Consider $e_{(i,j)} = \mathbb{1}_{(i,j)}$

Then $e_{(a,b)} \cdot e_{(b,a)}(a,a) = \sum_{x \sim a} e_{(a,b)}(a,x) \cdot e_{(b,a)}(x,a)$

$$= e_{(a,b)}(a,b)\, e_{(b,a)}(b,a) = 1$$

this gives $e_{(i,j)} e_{(j,k)} = e_{(i,k)}$

The C^*-algebra emerging is

$$C^*(G) = \left\{ \begin{pmatrix} \lambda_1 e_{(a,a)} & \lambda_2 e_{(a,b)} & 0 \\ \lambda_3 e_{(b,a)} & \lambda_4 e_{(b,b)} & 0 \\ 0 & 0 & \lambda_5 e_{(c,c)} \end{pmatrix}, \ \lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5 \in \mathbb{C} \right\}$$

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C^*-algebras of tilings and applications

Kellendonk

Suppose Ω is the hull of a tiling and that every $T \in \Omega$ is aperiodic.

There is a basis of the metric topology of Ω_{punc}

Let P be a patch in $T \in \Omega_{punc}$ and t is a tile in P.

Let $\mathcal{U}(P,t) = \{ T \in \Omega : P - x(t) \subset T \}$

where $x(t)$ is a distinguished point in t

Kellendonk's idea was to consider a subset of Ω. To do this, we want to puncture the prototiles: To each prototile $p \in \mathcal{P}$ define a point $x(p) \in p^\circ$. Extend this ~~to every~~ to every tile $t = p + y$ as $x(t) = x(p) + y$. The punctured hull

$$\Omega_{punc} := \{ T \in \Omega : \text{the tile at the origin has its puncture } x(t) = 0 \}$$

This excludes tiles with $0 \in \partial t$ for some $t \in T$

Lemma (Kellendonk) Ω_{punc} is a Cantor set.

We want to define an equivalence relation groupoid. Let

$$R_{punc} := \{ (T_1, T_2) \in \Omega_{punc}^2 : T_2 = T_1 - x(t) \text{ for some } t \in T \}$$

Define a metric on R_{punc} by

$$d\big((T, T - x(t)), (T', T' - x(t')) \big) = d(T, T') + \| x(t) - x(t') \|$$

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A basis for the topology of this metric is given by: suppose P is a patch and t, t' tiles in P and define

$$V(P,t,t') = \{ (T,T') : P - x(t) \subset T,\ P - x(t) \subset T' \text{ and } T' = T - (x(t') - x(t)) \}$$

We have a range & source map

$$r(V(P,t,t')) = \mathcal{U}(P,t)$$
$$s(V(P,t,t')) = \mathcal{U}(P,t')$$

these are local homeomorphisms.

Consider $C_c(R_{punc})$. For $f, g \in C_c(R_{punc})$ we have

$$f \cdot g\,(T,T') = \sum_{T'' \sim T} f(T,T'') \cdot g(T'',T')$$

$$g^*(T,T') = \overline{g(T',T)}$$

and we want to represent $C_r(R_{punc})$ on a Hilbert space $\ell^2(Q)$ for some tiling $Q \in \Omega_{punc}$

Define $\pi_Q : C_c(R_{punc}) \to \mathcal{B}(\ell^2(Q))$ by

$$(\pi_Q(f)\,\xi)(t) = \sum_{t'' \in Q} f(Q - x(t), Q - x(t'))\,\xi(t')$$

where $\xi \in \ell^2(Q)$, $t \in Q$, $f \in C_c(R_{punc})$

To define a C^*-algebra, we let

$$\|f\| := \sup_{Q \in \Omega_{punc}} \|\pi_Q(f)\|_{\text{operator-norm}}$$

We complete $C_c(R_{punc})$ in this norm to get Kellendonk's algebra.

Consider the continuous functions

$$e(P, t, t') := \mathbb{1}_{V(P,t,t')}$$

these elements multiply as follows:

$$e(P_1, t_1, t_1') \cdot e(P_2, t_2, t_2') = \begin{cases} e(P_1 \cup P_2, t_1, t_2') & \text{if } P_1 \text{ and } P_2 \text{ agree on } P_1 \cap P_2 \text{ and } t_2 = t_1' \\ 0 & \text{otherwise} \end{cases}$$

$$e(P, t, t')^* = e(P, t', t)$$

Let $\mathcal{E} = \{ e(P, t, t') \}$, the span$(\mathcal{E})$ is dense in Kellendonk's algebra.