# A $q$-PRODUCT TUTORIAL FOR A $q$-SERIES MAPLE PACKAGE

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ABSTRACT. This is a tutorial for using a new $q$-series Maple package. The package includes facilities for conversion between $q$-series and $q$-products and finding algebraic relations between $q$-series. Andrews found an algorithm for converting a $q$-series into a product. We provide an implementation. As an application we are able to effectively find finite q-product factorisations when they exist thus answering a question of Andrews. We provide other applications involving factorisations into theta-functions and eta-products.

Dedicated to George E. Andrews on the occasion of his 60th Birthday

CONTENTS



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## 1. Introduction

In the study of $q$-series one is quite often interested in identifying generating functions as infinite products. The classic example is the Rogers-Ramanujan identity

$$\sum_{n=0}^{\infty} \frac{q^{n^2}}{(q;q)_n} = \prod_{n=1}^{\infty} \frac{1}{(1-q^{5n-1})(1-q^{5n-4})}.$$

Here we have used the notation in (2.2). It can be shown that the left-side of this identity is the generating function for partitions whose parts differ by at least two. The identity is equivalent to saying the number of such partitions of $n$ is equinumerous with partitions of $n$ into parts congruent to $\pm 1 \pmod 5$.

The main goals of the package are to provide facility for handling the following problems.

1. Conversion of a given $q$-series into an "infinite" product.
2. Factorization of a given rational function into a finite $q$-product if one exists.
3. Find algebraic relations (if they exist) among the $q$-series in a given list.

A $q$-product has the form

$$\prod_{j=1}^{N} (1-q^j)^{b_j}, \tag{1.1}$$

where the $b_j$ are integers.

In [4, §10.7], George Andrews also considered Problems 1 and 2, and asked for an easily accessible implementation. We provide implementations as well as considering factorisations into theta-products and eta-products. The package provides some basic functions for computing $q$-series expansions of eta functions, theta functions, Gaussian polynomials and $q$-products. It also has a function for sifting out coefficients of a $q$-series. It also has the basic infinite product identities: the triple product identity, the quintuple product identity and Winquist's identity.

### 1.1. Installation instructions.

The `qseries` package can be downloaded via the WWW. First use your favorite browser to access the URL:

`http://www.math.ufl.edu/~frank/qmaple.html` then follow the directions on that page. There are two versions: one for UNIX and one for WINDOWS.

## 2. Basic functions

We describe the basic functions in the package which are used to build $q$-series.

### 2.1. **Finite $q$-products.**

2.1.1. *Rising $q$-factorial.* `aqprod(a,q,n)` returns the product

$$(a; q)_n = (1 - a)(1 - aq) \cdots (1 - aq^{n-1}). \tag{2.2}$$

We also use the notation

$$(a; q)_\infty = \prod_{n=1}^{\infty} (1 - aq^{n-1}).$$

2.1.2. *Gaussian polynomials.* When $0 \le m \le n$, `qbin(q,m,n)` returns the Gaussian polynomial (or $q$-binomial coefficient)

$$\begin{bmatrix} n \\ m \end{bmatrix}_q = \frac{(q)_n}{(q)_m (q)_{n-m}},$$

otherwise it returns 0.

### 2.2. **Infinite products.**

2.2.1. *Dedekind eta products.* Suppose $\Re\tau > 0$, and $q = \exp(2\pi i\tau)$. The Dedekind eta function [27, p. 121] is defined by

$$\begin{aligned}\eta(\tau) &= \exp(\pi i\tau/12) \prod_{n=1}^{\infty} (1 - \exp(2\pi i n\tau)) \\ &= q^{1/24} \prod_{n=1}^{\infty} (1 - q^n).\end{aligned}$$

`etaq(q,k,T)` returns the $q$-series expansion (up to $q^T$) of the eta product

$$\prod_{n=1}^{\infty} (1 - q^{kn}).$$

This corresponds to the eta function $\eta(k\tau)$ except for a power of $q$. Eta products occur frequently in the study of $q$-series. For example, the generating function for $p(n)$, the number of partitions of $n$, can be written as

$$\sum_{n=0}^{\infty} p(n) q^n = \frac{1}{\prod_{n=1}^{\infty} (1 - q^n)}.$$

See [1, pp. 3–4]. The generating function for the number of partitions of $n$ that are $p$-cores [19], $a_p(n)$, can be written

$$\sum_{n=0}^{\infty} a_p(n) q^n = \prod_{n=1}^{\infty} \frac{(1-q^{pn})^p}{(1-q^n)}.$$

Recently, Granville and Ono [21] were able to prove a long-standing conjecture in group representation theory using elementary and function-theoretic properties of the eta product above.

2.2.2. *Theta functions.* Jacobi [24, Vol I, pp. 497–538] defined four theta functions $\theta_i(z,q)$, $i = 1,2,3,4$. See also [42, Ch. XXI]. Each theta function can be written in terms of the others using a simple change of variables. For this reason, it is common to define

$$\theta(z,q) := \sum_{n=-\infty}^{\infty} z^n q^{n^2}.$$

`theta(z,q,T)` returns the truncated theta-series

$$\sum_{i=-T}^{T} z^i q^{i^2}.$$

The case $z = 1$ of Jacobi's theta functions occurs quite frequently. We define

$$\theta_2(q) := \sum_{n=-\infty}^{\infty} q^{(n+1)^2/2}$$
$$\theta_3(q) := \sum_{n=-\infty}^{\infty} q^{n^2}$$
$$\theta_4(q) := \sum_{n=-\infty}^{\infty} (-1)^n q^{n^2}$$

`theta2(q,t)`, `theta3(q,t)`, `theta4(q,t)` (resp.) returns the $q$-series expansion to order $O\left(q^T\right)$ of $\theta_2(q)$, $\theta_3(q)$, $\theta_4(q)$ respectively. Let $a$, and $b$ be positive integers and suppose $|q| < 1$. Infinite products of the form

$$(q^a; q^b)_\infty (q^{b-a}; q^b)_\infty$$

occur quite frequently in the theory of partitions and $q$-series. For example the right side of the Rogers-Ramanujan identity is the reciprocal of the product with $(a,b) = (1,5)$. In (3.4) we will see how the function `jacprodmake` can be used to identify such products.

## 3. Product Conversion

In [1, p. 233], [4, §10.7] there is a very nice and useful algorithm for converting a $q$-series into an infinite product. Any given $q$-series may be written formally as an infinite product

$$1+\sum_{n=1}^{\infty} b_n q^n = \prod_{n=1}^{\infty} (1-q^n)^{-a_n}.$$

Here we assume the $b_n$ are integers. By taking the logarithmic derivative of both sides we can obtain the recurrence

$$nb_n = \sum_{j=1}^{n} b_{n-j} \sum_{d|j} d a_d.$$

Letting $a_n = 1$ we obtain the well-known special case

$$np(n) = \sum_{j=1}^{n} p(n-j)\sigma(j).$$

We can also easily construct a recurrence for the $a_n$ from the recurrence above.

The function `prodmake` is an implementation of Andrews' algorithm. Other related functions are `etamake` and `jacprodmake`.

### 3.1. `prodmake`.

`prodmake(f,q,T)` converts the $q$-series $f$ into an infinite product that agrees with $f$ to $O(q^T)$. Let's take a look at the left side of the Rogers-Ramanujan identity.

```
>  with(qseries):
>  x:=1:
>  for n from 1 to 8 do
        x := x + q^(n*n)/aqprod(q,q,n):
   od:
>  x := series(x,q,50);
```

$$x := 1+q+q^2+q^3+2q^4+2q^5+3q^6+3q^7+4q^8+5q^9+6q^{10} +7q^{11}+9q^{12}+10q^{13}+12q^{14}+14q^{15}+17q^{16}+19q^{17}+23q^{18} +26q^{19}+31q^{20}+35q^{21}+41q^{22}+46q^{23}+54q^{24}+61q^{25}+70q^{26}+79q^{27}+91q^{28}+102q^{29}+117q^{30}+131q^{31}+149q^{32}+167q^{33}+189q^{34}+211q^{35}+239q^{36}+266q^{37}+299q^{38}+333q^{39}+374q^{40}+415q^{41}+465q^{42}+515q^{43}+575q^{44}+637q^{45}+709q^{46} +783q^{47}+871q^{48}+961q^{49}+O(q^{50})$$

```
>  prodmake(x,q,40);
```

$$1/\big((1-q)(1-q^4)(1-q^6)(1-q^9)(1-q^{11})(1-q^{14})(1-q^{16})(1-q^{19}) (1-q^{21})(1-q^{24})(1-q^{26})(1-q^{29})(1-q^{31})(1-q^{34})(1-q^{36})(1-q^{39})\big)$$

We have rediscovered the right side of the Rogers-Ramanujan identity!

**Exercise 1**. Find (and prove) a product form for the $q$-series

$$\sum_{n=0}^{\infty} \frac{q^{n^2}}{(q;q)_{2n}}.$$

The identity you find is originally due to Rogers [34](p.330). See also Andrews [2](pp.38–39) for a list of some related papers.

3.2. `qfactor`. The function `qfactor` is a version of `prodmake`. `qfactor(f,T)` attempts to write a rational function $f$ in $q$ as a $q$-product, i.e., as a product of terms of the form $(1-q^i)$. The second argument $T$ is optional. It specifies an an upper bound for the exponents of $q$ that occur in the product. If $T$ is not specified it is given a default value of $4d+3$ where $d$ is the maximum of the degree in $q$ of the numerator and denominator. The algorithm is quite simple. First the function is factored as usual, and then it uses `prodmake` to do further factorisation into $q$-products. Thus even if only part of the function can be written as a $q$-product `q`factor is able to find it.

As an example we consider some rational functions $T(r,h)$ introduced by Andrews [4](p.14) to explain Rogers's [34] first proof of the Rogers-Ramanujan identities. The $T(r,n)$ are defined recursively as follows:

$$T(r,0) = 1, \tag{3.3}$$

$$T(r,1) = 0, \tag{3.4}$$

$$T(r,N) = -\sum_{1\le 2j\le N} \begin{bmatrix} r+2j \\ j \end{bmatrix} T(r+2j, N-2j). \tag{3.5}$$

```
> T:=proc(r,j)
>    option remember;
>    local x,k;
>    x:=0;
>    if j=0 or j=1 then
>      RETURN((j-1)^2):
>    else
>       for k from 1 to floor(j/2) do
>           x:=x-qbin(q,k,r+2*k)*T(r+2*k,j-2*k);
>        od:
>        RETURN(expand(x));
>    fi:
```

```
> end:
>  t8:=T(8,8);
```

$$t8 := 3\,q^9 + 21\,q^{16} + 36\,q^{25} + 9\,q^{36} + q^6 + q^7 + 2\,q^8 + 5\,q^{10} + 6\,q^{11} + 9\,q^{12} + 11\,q^{13} + 15\,q^{14} + 17\,q^{15} + 33\,q^{28} + 34\,q^{27} + 37\,q^{26} + 38\,q^{24} + 36\,q^{23} + 37\,q^{22} + 34\,q^{21} + 33\,q^{20} + 29\,q^{19} + 28\,q^{18} + 23\,q^{17} + 5\,q^{38} + 6\,q^{37} + 11\,q^{35} + 15\,q^{34} + 17\,q^{33} + 21\,q^{32} + 23\,q^{31} + 28\,q^{30} + 29\,q^{29} + 3\,q^{39} + q^{42} + q^{41} + 2\,q^{40}$$

```
>  factor(t8);
```

$$q^6\left(q^4+q^3+q^2+q+1\right)\left(q^4-q^3+q^2-q+1\right)\left(q^{10}+q^9+q^8+q^7+q^6+q^5+q^4+q^3+q^2+q+1\right)\left(q^4+1\right)\left(q^6+q^3+1\right)\left(q^8+1\right)$$

```
>  qfactor(t8,20);
```

$$\frac{(1-q^9)\,(1-q^{10})\,(1-q^{11})\,(1-q^{16})\,q^6}{(1-q)\,(1-q^2)\,(1-q^3)\,(1-q^4)}$$

Observe how we used **factor** to factor **t8** into cyclotomic polynomials. However, **qfactor** was able to factor **t8** as a $q$-product. We see that

$$T(8,8) = \frac{(q^9;q)_3(1-q^{16})q^6}{(q;q)_4}.$$

**Exercise 2**. Use **qfactor** to factorize $T(r,n)$ for different values of $r$ and $n$. Then write $T(r,n)$ (defined above) as a $q$-product for general $r$ and $n$.

For our next example we examine the sum

$$\sum_{k=-\infty}^{\infty} (-1)^k q^{k(3k+1)/2} \begin{bmatrix} b+c \\ c+k \end{bmatrix} \begin{bmatrix} c+a \\ a+k \end{bmatrix} \begin{bmatrix} a+b \\ b+k \end{bmatrix}.$$

```
> dixson:=proc(a,b,c,q)
>    local x,k,y;
>    x:=0:  y:=min(a,b,c):
>    for k from -y to y do
>      x:=x+(-1)^k*q^(k*(3*k+1)/2)*
>      qbin(q,c+k,b+c)*qbin(q,a+k,c+a)*qbin(q,b+k,a+b);
>    od:
>    RETURN(x):
> end:
```

```
> dx := expand(dixson(5,5,5,q)):
> qfactor(dx,20);
```

$$(1-q^6)(1-q^7)(1-q^8)(1-q^9)(1-q^{10})(1-q^{11})(1-q^{12})(1-q^{13})(1-q^{14})(1-q^{15})/((1-q)^2(1-q^2)^2(1-q^3)^2(1-q^4)^2(1-q^5)^2)$$

We find that

$$\sum_{k=-\infty}^{\infty}(-1)^k q^{k(3k+1)/2}\left[\begin{array}{c}10\\5+k\end{array}\right]^3=\frac{(q^6;q)_{10}}{(q;q)_5^2}. \tag{3.6}$$

**Exercise 3.** Write the sum

$$\sum_{k=-\infty}^{\infty}(-1)^k q^{k(3k+1)/2}\left[\begin{array}{c}2a\\a+k\end{array}\right]^3$$

as a $q$-product for general integral $a$. The identity you obtain is a special case of [4](Eq.(4.24), p.38).

3.3. `etamake`. Recall from (2.2.1) that `etaq` is the function to use for computing $q$-expansions of eta products. If one wants to apply the theory of modular forms to $q$-series it is quite useful to determine whether a given $q$-series is a product of eta functions. The function in the package for doing this conversion is `etamake`. `etamake(f,q,T)` will write the given $q$-series $f$ as a product of eta functions which agrees with $f$ up to $q^T$. As an example, let's see how we can write the theta functions as eta products.

```
> theta2(q,100)/q^(1/4);
```

$$2q^{132}+2q^{110}+2q^{90}+2q^{72}+2q^{56}+2q^{42}+2q^{30}+2q^{20}+2q^{12}+2q^6+2q^2+2+q^{156}$$

```
> etamake(",q,100);
```

$$2\,\frac{\eta(4\tau)^2}{q^{1/4}\,\eta(2\tau)}$$

```
> theta3(q,100);
```

$$2q^{121}+2q^{100}+2q^{81}+2q^{64}+2q^{49}+2q^{36}+2q^{25}+2q^{16}+2q^9+2q^4+2q+1$$

```
> etamake(",q,100);
```

$$\frac{\eta(2\tau)^5}{\eta(4\tau)^2\,\eta(\tau)^2}$$

```
> theta4(q,100);
```

$$-2\,q^{121}+2\,q^{100}-2\,q^{81}+2\,q^{64}-2\,q^{49}+2\,q^{36}-2\,q^{25}+2\,q^{16}-2\,q^{9}+2\,q^{4}-2\,q+1$$

```
> etamake(",q,100);
```

$$\frac{\eta(\,\tau\,)^2}{\eta(\,2\,\tau\,)}$$

We are led to the well-known identities:

$$\theta_2(q) = 2\frac{\eta(4\tau)^2}{\eta(2\tau)},$$
$$\theta_3(q) = \frac{\eta(2\tau)^5}{\eta(4\tau)^2\eta(\tau)^2},$$
$$\theta_4(q) = \frac{\eta(\tau)^2}{\eta(2\tau)}.$$

The idea of the algorithm is quite simple. Given a $q$-series $f$ (say with leading coefficient 1) one just keeps recursively multiplying by powers of the right eta function until the desired terms agree. For example, suppose we are given a $q$-series

$$f = 1 + cq^k + \cdots .$$

Then the next step is to multiply by `etaq(q,k,T)^(-c)`.

**Exercise 4**. Define the $q$-series

$$a(q) := \sum_{n=-\infty}^{\infty}\sum_{m=-\infty}^{\infty} q^{n^2+nm+m^2}$$
$$b(q) := \sum_{n=-\infty}^{\infty}\sum_{m=-\infty}^{\infty} \omega^{n-m} q^{n^2+nm+m^2}$$
$$c(q) := \sum_{n=-\infty}^{\infty}\sum_{m=-\infty}^{\infty} q^{(n+1/3)^2+(n+1/3)(m+1/3)+(m+1/3)^2}$$

where $\omega = \exp(2\pi i/3)$. Two of the three functions above can be written as eta products. Can you find them?
*Hint*: It would be wise to define

```
> omega := RootOf(z^2 + z + 1 = 0);
```

See [12] for the answer and much more.

3.4. `jacprodmake`. In (2.2.2) we observed that the right side of the Rogers-Ramanujan identity could be written in terms of a Jacobi product. The function `jacprodmake` converts a $q$-series into a Jacobi-type product if one exists. Given a $q$-series $f$, `jacprodmake(f,q,T)` attempts to convert $f$ into a product of theta functions that agrees with $f$ to order $O(q^T)$. Each theta-function has the form $\mathrm{JAC}(a, b, \infty)$, where $a$, $b$ are integers and $0 \le a < b$. If $0 < a$, then $\mathrm{JAC}(a, b, \infty)$ corresponds to the theta-product

$$(q^a; q^b)_\infty (q^{b-a}; q^b)_\infty (q^b; q^b)_\infty.$$

We call this a theta product because it is $\theta(-q^{(b-2a)/2}, q^{b/2})$. The `jacprodmake` function is really a variant of `prodmake`. It involves using `prodmake` to compute the sequence of exponents and then searching for periodicity.

If $a = 0$, then $\mathrm{JAC}(0, b, \infty)$ corresponds to the eta-product

$$(q^b; q^b)_\infty.$$

We note that this product can also be thought of as a theta-product since it can be written

$$(q^b; q^b)_\infty = (q^b; q^{3b})_\infty (q^{2b}; q^{3b})_\infty (q^{3b}; q^{3b})_\infty.$$

Let's re-examine the Rogers-Ramanujan identity.

```
>  with(qseries):
>  x:=1:
>  for n from 1 to 8 do
>     x:=x+q^(n*n)/aqprod(q,q,n):
>  od:
>  x:=series(x,q,50):
>  y:=jacprodmake(x,q,40);
```

$$y := \frac{JAC(0, 5, \infty)}{JAC(1, 5, \infty)}$$

```
>  z:=jac2prod(y);
```

$$z := \frac{1}{(q, q^5)_\infty (q^4, q^5)_\infty}$$

Note that we were able to observe that the left side of the Rogers-Ramanujan identity (at least up through $q^{40}$) can be written as a quotient of theta functions. We used the function `jac2prod`, to simplify the result and get it into a more recognizable form. The function `jac2prod(jacexpr)` converts a product of theta functions into $q$-product form; i.e., as a product of functions of the form $(q^a; q^b)_\infty$. Here `jacexpr` is a product (or quotient) of terms $\mathrm{JAC}(i, j, \infty)$, where $i$, $j$ are integers and $0 \le i < j$.

A related function is `jac2series`. This converts a Jacobi-type product into a form better for computing its $q$-series. It simply replaces each Jacobi-type product with its corresponding theta-series.

```
> with(qseries):
> x:=0:
> for n from 0 to 10 do
  x := x + q^(n*(n+1)/2)*aqprod(-q,q,n)/aqprod(q,q,2*n+1):
  od:
> x:=series(x,q,50):
> jacprodmake(x,q,50);
```

$$\mathrm{JAC}(0,\,14,\,\infty)^6 \Big/ (\mathrm{JAC}(1,\,14,\,\infty)^2\,\mathrm{JAC}(3,\,14,\,\infty)\,\mathrm{JAC}(4,\,14,\,\infty)\,\mathrm{JAC}(5,\,14,\,\infty)\,\mathrm{JAC}(6,\,14,\,\infty)\sqrt{\frac{\mathrm{JAC}(7,\,14,\,\infty)}{\mathrm{JAC}(0,\,14,\,\infty)}})$$

```
> jac2series(",500);
```

$$(q^{364} - q^{210} + q^{98} - q^{28} + 1 - q^{14} + q^{70} - q^{168} + q^{308} - q^{490})^6 \Big/ (( -q^{621} + q^{496} - q^{385} + q^{288} - q^{205} + q^{136} - q^{81} + q^{40} - q^{13} + 1 - q + q^{16} - q^{45} + q^{88} - q^{145} + q^{216} - q^{301} + q^{400} - q^{513})^2 (-q^{603} + q^{480} - q^{371} + q^{276} - q^{195} + q^{128} - q^{75} + q^{36} - q^{11} + 1 - q^3 + q^{20} - q^{51} + q^{96} - q^{155} + q^{228} - q^{315} + q^{416} - q^{531})(-q^{594} + q^{472} - q^{364} + q^{270} - q^{190} + q^{124} - q^{72} + q^{34} - q^{10} + 1 - q^4 + q^{22} - q^{54} + q^{100} - q^{160} + q^{234} - q^{322} + q^{424} - q^{540})(-q^{585} + q^{464} - q^{357} + q^{264} - q^{185} + q^{120} - q^{69} + q^{32} - q^9 + 1 - q^5 + q^{24} - q^{57} + q^{104} - q^{165} + q^{240} - q^{329} + q^{432} - q^{549})(-q^{576} + q^{456} - q^{350} + q^{258} - q^{180} + q^{116} - q^{66} + q^{30} - q^8 + 1 - q^6 + q^{26} - q^{60} + q^{108} - q^{170} + q^{246} - q^{336} + q^{440} - q^{558}) ((-2\,q^{567} + 2\,q^{448} - 2\,q^{343} + 2\,q^{252} - 2\,q^{175} + 2\,q^{112} - 2\,q^{63} + 2\,q^{28} - 2\,q^7 + 1)/( q^{364} - q^{210} + q^{98} - q^{28} + 1 - q^{14} + q^{70} - q^{168} + q^{308} - q^{490}))^{1/2})$$

It seems that the $q$-series

$$\sum_{n\geq 0} \frac{(-q;q)_n q^{n(n+1)/2}}{(q;q)_{2n+1}}$$

can be written as Jacobi-type product. Assuming that this is the case we used `jac2series` to write this $q$-series in terms of theta-series at least up to $q^{500}$. This should provide an efficient method for computing the $q$-series expansion and also for computing the function at particular values of $q$.

**Exercise 5**. Use `jacprodmake` and `jac2series` to compute the $q$-series expansion of

$$\sum_{n\geq 0} \frac{(-q;q)_n q^{n(3n+1)/2}}{(q;q)_{2n+1}}$$

up to $q^{1000}$, assuming it is Jacobi-type product. Can you identify the infinite product? This function occurs in Slater's list [36](Eq.(46), p.156).

## 4. The Search for Relations

The functions for finding relations between $q$-series are `findhom`, `findhomcombo`, `findnonhom`, `findnonhomcombo`, and `findpoly`.

### 4.1. `findhom`.

If the $q$-series one is concerned with are modular forms of a particular weight, then theoretically these functions will satisfy homogeneous polynomial relations. See [18, p. 263], [9] for more details and examples. The function `findhom(L,q,n,topshift)` returns a set of potential homogeneous relations of order $n$ among the $q$-series in the list $L$. The value of `topshift` is usually taken to be zero. However if it appears that spurious relations are being generated then a higher value of `topshift` should be taken.

The idea is to convert this into a linear algebra problem. This program generates a list of monomials of degree $n$ of the functions in the given list of $q$-series $L$. The $q$-expansion (up to a certain point) of each monomial is found and converted into a row vector of a matrix. The set of relations is then found by computing the kernel of the transpose of this matrix. As an example, we now consider relations between the theta functions $\theta_3(q)$, $\theta_4(q)$, $\theta_3(q^2)$, and $\theta_4(q^2)$.

```
> with(qseries):
> findhom([theta3(q,100),theta4(q,100),theta3(q^2,100),
theta4(q^2,100)],q,1,0);
```

```
# of terms , 25
-----RELATIONS-----of order---, 1
```

$$\{\{\}\}$$

```
> findhom([theta3(q,100),theta4(q,100),theta3(q^2,100),
theta4(q^2,100)],q,2,0);
```

```
# of terms , 31
-----RELATIONS-----of order---, 2
```

$$\left\{{X_1}^2+{X_2}^2-2\,{X_3}^2, -X_1X_2+{X_4}^2\right\}$$

From the session above we see that there is no linear relation between the functions $\theta_3(q)$, $\theta_4(q)$, $\theta_3(q^2)$ and $\theta_4(q^2)$. However, it appears that there are two quadratic relations:

$$\theta_3(q^2) = \left(\frac{\theta_3^2(q) + \theta_4^2(q)}{2}\right)^{1/2}$$

and

$$\theta_4(q^2) = \left(\theta_3^2(q)\theta_4^2(q)\right)^{1/2}.$$

This is Gauss' parametrization of the arithmetic-geometric mean iteration. See [13, Ch. 2] for details.

> **Exercise 6**. Define $a(q)$, $b(q)$ and $c(q)$ as in **Exercise 2**. Find homogeneous relations between the functions $a(q)$, $b(q)$, $c(q)$, $a(q^3)$, $b(q^3)$, $c(q^3)$. In particular, try to get $a(q^3)$ and $b(q^3)$ in terms of $a(q)$, $b(q)$. See [12] for more details. These results lead to a cubic analog of the AGM due to Jon and Peter Borwein [10], [11].

4.2. `findhomcombo`. The function `findhomcombo` is a variant of `findhom`. Suppose $f$ is a $q$-series and $L$ is a list of $q$-series. `findhomcombo(f,L,q, n,topshift,etaoption)` tries to express $f$ as a homogeneous polynomial in the members of $L$. If `etaoption=yes` then each monomial in the combination is "converted" into an eta-product using `etamake`.

We illustrate this function with certain Eisenstein series. For $p$ an odd prime define

$$\chi(m) = \left(\frac{m}{p}\right) \qquad \text{(the Legendre symbol).}$$

Suppose $k$ is an integer, $k \geq 2$, and $(p-1)/2 \equiv k \pmod 2$. Define the Eisenstein series

$$U_{p,k}(q) := \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} \chi(m) n^{k-1} q^{mn}.$$

Then $U_{p,k}$ is a modular form of weight $k$ and character $\chi$ for the congruence subgroup $\Gamma_0(p)$. See [28], [20] for more details. The classical result is the following identity found by Ramanujan [32, Eq. (1.52), p. 354]:

$$U_{5,2} = \frac{\eta(5\tau)^5}{\eta(\tau)}.$$

Kolberg [28] has found many relations between such Eisenstein series and certain eta products. The eta function $\eta(\tau)$ is a modular form of weight $\frac{1}{2}$ [27, p. 121]. Hence the modular forms

$$B_1 := \frac{\eta(5\tau)^5}{\eta(\tau)}, \qquad B_2 := \frac{\eta(\tau)^5}{\eta(5\tau)}$$

are modular forms of weight $\frac{(5-1)}{2} = 2$. In fact, it can be shown that they are modular forms on $\Gamma_0(5)$ with character $\left(\frac{\cdot}{5}\right)$. We might therefore expect that $U_{5,6}$ can be written as a homogeneous cubic polynomial in $B_1$, $B_2$. We write a short MAPLE program to compute the Eisenstein series $U_{p,k}$.

```
> with(numtheory):
> UE:=proc(q,k,p,trunk)
>  local x,m,n:
>  x:=0:
>  for m from 1 to trunk do
>   for n from 1 to trunk/m do
>    x:=x + L(m,p)*n^(k-1)*q^(m*n):
>   od:
>  od:
> end:
```

The function `UE(q,k,p,trunk)` returns the $q$-expansion of $U_{p,k}$ up through $q^{\text{trunk}}$. We note that `L(m,p)` returns the Legendre symbol $\left(\frac{m}{p}\right)$. We are now ready to study $U_{5,6}$.

```
> with(qseries):
> f := UE(q,6,5,50):
> B1 := etaq(q,1,50)^5/etaq(q,5,50):
> B2 := q*etaq(q,5,50)^5/etaq(q,1,50):
> findhomcombo(f,[B1,B2],q,3,0,yes);
                    # of terms , 25
  -----possible linear combinations of degree------, 3
```

$$\left\{\eta(5\,\tau)^3\eta(\tau)^9 + 40\,\eta(5\,\tau)^9\eta(\tau)^3 + 335\,\frac{\eta(5\,\tau)^{15}}{\eta(\tau)^3}\right\}$$

$$\left\{{X_1}^2X_2 + 40\,X_1{X_2}^2 + 335\,{X_2}^3\right\}$$

It would appear that

$$U_{5,6} = \eta(5\,\tau)^3\eta(\tau)^9 + 40\,\eta(5\,\tau)^9\eta(\tau)^3 + 335\,\frac{\eta(5\,\tau)^{15}}{\eta(\tau)^3}.$$

The proof is a straightforward exercise using the theory of modular forms.

**Exercise 7**. Define the following eta products:

$$C_1 := \frac{\eta(7\tau)^7}{\eta(\tau)}, \qquad C_2 := \eta(\tau)^3\,\eta(7\tau)^3, \qquad C_3 := \frac{\eta(\tau)^7}{\eta(7\tau)}.$$

What is the weight of these modular forms?

Write $U_{7,3}$ in terms of $C_1$, $C_2$, $C_3$.

The identity that you should find was originally due to Ramanujan. Also see Fine [15, p. 159] and [19, Eq. (5.4)].

If you are ambitious find $U_{7,9}$ in terms of $C_1$, $C_2$, $C_3$.

4.3. **`findnonhom`.** In section 4.1 we introduced the function `findhom` to find homogeneous relations between $q$-series. The nonhomogeneous analog is `findnonhom`.

The syntax of `findnonhom` is the same as `findhom`. Typically (but not necessarily) `findhom` is used to find relations between modular forms of a certain weight. To find relations between modular functions we would use `findnonhom`. We consider an example involving theta functions.

```
> with(qseries):
> F := q -> theta3(q,500)/theta3(q^5,100):
> U := 2*q*theta(q^10,q^25,5)/theta3(q^25,20);
```

$$U := \frac{2\,q\,(q^{575}+q^{360}+q^{195}+q^{80}+q^{15}+1+q^{35}+q^{120}+q^{255}+q^{440}+q^{675})}{2\,q^{625}+2\,q^{400}+2\,q^{225}+2\,q^{100}+2\,q^{25}+1}$$

```
> EQNS := findnonhom([F(q),F(q^5),U],q,3,20);
                    # of terms , 61
           -----RELATIONS-----of order---, 3
```

$$\mathtt{EQNS} := \left\{-1 - X_1X_2X_3 + {X_2}^2 + {X_3}^2 + X_3\right\}$$

```
> ANS:=EQNS[1];
```

$$\mathtt{ANS} := -1 - X_1X_2X_3 + {X_2}^2 + {X_3}^2 + X_3$$

```
> CHECK := subs(X[1]=F(q),X[2]=F(q^5),X[3]=U,ANS):
> series(CHECK,q,500);
```

$$O\left(q^{500}\right)$$

We define

$$F(q) := \frac{\theta_3(q)}{\theta_3(q^5)},$$

and

$$U(q) := 2\frac{\sum_{n=-\infty}^{\infty} q^{25n^2+10n+1}}{\theta_3(q^{25})}.$$

We note that $U(q)$ and $F(q)$ are modular functions since they are ratios of theta series. From the session above we see that it appears that

$$1 + F(q)F(q^5)U(q) = F(q^5)^2 + U(q)^2 + U(q).$$

Observe how we were able to verify this equation to high order. When `findnonhom` returns a set of relations the variable $X$ has been declared *global*. This is so we can manipulate the relations. It this way we were able to assign `ANS` to the relation found and then use `subs` and `series` to check it to order $O(q^{500})$.

4.4. `findnonhomcombo`. The syntax of `findnonhomcombo` is the same as `findhomcombo`. We consider an example involving eta functions. First we define

$$\xi := \frac{\eta(49\,\tau)}{\eta(\tau)},$$

and

$$T := \left(\frac{\eta(7\,\tau)}{\eta(\tau)}\right)^4,$$

Using the theory of modular functions it can be shown that one must be able to write $T^2$ in terms of $T$ and powers of $\xi$. We now use `findnonhomcombo` to get $T^2$ in terms of $\xi$ and $T$.

```
> xi:=q^2*etaq(q,49,100)/etaq(q,1,100):
> T:=q*(etaq(q,7,100)/etaq(q,1,100))^4:
> findnonhomcombo(T^2,[T,xi],q,7,-15);
                    # of terms , 42
                 matrix is , 37, x, 42
  -----possible linear combinations of degree------, 7
```

$$\{147\,{X_2}^5 + 343\,{X_2}^7 + 343\,{X_2}^6 + X_2 + 49\,{X_2}^4 + 49\,X_1{X_2}^3 + 7\,X_1X_2 + 21\,{X_2}^3 + 7\,{X_2}^2 + 35\,X_1{X_2}^2\}$$

Then it seems that

$$T^2 = \left(35\,\xi^2 + 49\,\xi^3 + 7\,\xi\right)T + 343\,\xi^7 + 343\,\xi^6 + 147\,\xi^5 + 49\,\xi^4 + 21\,\xi^3 + 7\,\xi^2 + \xi.$$

This is the modular equation used by Watson[41] to prove Ramanujan's partition congruences for powers of 7. Also see [5] and [26], and see [16] for an elementary treatment.

**Exercise 8**. Define

$$\xi := \frac{\eta(25\,\tau)}{\eta(\tau)},$$

and

$$T := \left(\frac{\eta(5\,\tau)}{\eta(\tau)}\right)^{6}.$$

Use `findnonhomcombo` to express $T$ as a polynomial in $\xi$ of degree 5. The modular equation you find was used by Watson to prove Ramanujan's partition congruences for powers of 5. See [23] for an elementary treatment.

**Exercise 9**. Define $a(q)$ and $c(q)$ as in **Exercise 2**. Define

$$x(q) := \frac{c(q)^3}{a(q)^3},$$

and the classical Eisenstein series (usually called $E_3$; see [35, p. 93])

$$N(q) := 1 - 504\sum_{n=1}^{\infty} \frac{n^5 q^n}{1-q^n}.$$

Use `findnonhomcombo` to express $N(q)$ in terms of $a(q)$ and $x(q)$.

*HINT*: $N(q)$ is a modular form of weight 6 and $a(q)$ and $c(q)$ are modular forms of weight 1. See [8] for this result and many others.

4.5. `findpoly`. The function `findpoly` is used to find a polynomial relation between two given $q$-series with degrees specified.

`findpoly(x,y,q,deg1,deg2,check)` returns a possible polynomial in $X$, $Y$ (with corresponding degrees deg1, deg2) which is satisfied by the $q$-series $x$ and $y$.

If check is assigned then the relation is checked to $O\left(q^{\text{check}}\right)$.

We illustrate this function with an example involving theta functions and the function $a(q)$ and $c(q)$ encountered in **Exercises 2** and **7**. It can be shown that

$$a(q) = \theta_3(q)\theta_3(q^3) + \theta_2(q)\theta_2(q^3).$$

See [12] for details. This equation provides a better way of computing the $q$-series expansion of $a(q)$ than the definition. In **Exercise 2** you would have found that

$$c(q) = 3\ \frac{\eta^3(3\,\tau)}{\eta(\tau)}.$$

See [12] for a proof. Define

$$y := \frac{c^3}{a^3},$$

and

$$x := \left(\frac{\theta_2(q)}{\theta_2(q^3)}\right)^2 + \left(\frac{\theta_3(q)}{\theta_3(q^3)}\right)^2.$$

We use `findpoly` to find a polynomial relation between $x$ and $y$.

```
> with(qseries):
> x1 := radsimp(theta2(q,100)^2/theta2(q^3,40)^2):
> x2 := theta3(q,100)^2/theta3(q^3,40)^2:
> x := x1+x2:
> c := q*etaq(q,3,100)^9/etaq(q,1,100)^3:
> a := radsimp(theta3(q,100)*theta3(q^3,40)+theta2(q,100)
  *theta2(q^3,40)):
> c := 3*q^(1/3)*etaq(q,3,100)^3/etaq(q,1,100):
> y := radsimp(c^3/a^3):
> P1:=findpoly(x,y,q,3,1,60);
WARNING: X,Y are global.
```

dims , 8, 18

The polynomial is

$$(X+6)^3\,Y - 27\,(X+2)^2$$

Checking to order, 60

$$O\left(q^{59}\right)$$

$$\mathtt{P1} := (X+6)^3\,Y - 27\,(X+2)^2$$

It seems that $x$ and $y$ satisfy the equation

$$p(x,y) = (x+6)^3\,y - 27(x+2)^2 = 0.$$

Therefore it would seem that

$$\frac{c^3}{a^3} = 27\,\frac{(x+2)^2}{(x+6)^3}.$$

See [8, pp. 4237–4240] for more details.

**Exercise 10.** Define

$$m := \left(\frac{\theta_3(q)}{\theta_3(q^3)}\right)^2.$$

Use `polyfind` to find $y = \frac{c^3}{a^3}$ as a rational function in $m$. The answer is Eq.(12.8) in [8].

## 5. Sifting coefficients

Suppose we are given a $q$-series

$$A(q) = \sum_{n=0}^{\infty} a_n q^n.$$

Occasionally it will turn out the generating function

$$\sum_{n=0}^{\infty} a_{mn+r} q^n$$

will have a very nice form. A famous example for $p(n)$ is due to Ramanujan:

$$\sum_{n=0}^{\infty} p(5n+4)q^n = 5\prod_{n=1}^{\infty} \frac{(1-q^{5n})^6}{(1-q^n)^5}.$$

See [1, Cor. 10.6]. In fact, G.H. Hardy and Major MacMahon [31, p. xxxv] both agreed that this is Ramanujan's most beautiful identity.

Suppose $s$ is the $q$-series

$$\sum a_i q^i + \mathrm{O}(q^T)$$

then `sift(s,q,n,k,T)` returns the $q$-series

$$\sum a_{ni+k} q^i + \mathrm{O}(q^{T/n}).$$

We illustrate this function with another example from the theory of partitions. Let $pd(n)$ denote the number of partitions of $n$ into distinct parts. Then it is well known that

$$\sum_{n=0}^{\infty} pd(n)q^n = \prod_{n=1}^{\infty}(1+q^n) = \prod_{n=1}^{\infty} \frac{(1-q^{2n})}{(1-q^n)}.$$

We now examine the generating function of $pd(5n+1)$ in MAPLE.

```
> PD:=series(etaq(q,2,200)/etaq(q,1,200),q,200):
```

```
> sift(PD,q,5,1,199);
```

$$1 + 4\,q + 5010688\,q^{26} + 53250\,q^{15} + 668\,q^{7} + 12\,q^{2} + 165\,q^{5} + 12076\,q^{12} + 1087744\,q^{22} + 109420549\,q^{35} + 76\,q^{4} + 32\,q^{3} + 340\,q^{6} + 1260\,q^{8} + 2304\,q^{9} + 4097\,q^{10} + 7108\,q^{11} + 20132\,q^{13} + 32992\,q^{14} + 84756\,q^{16} + 133184\,q^{17} + 206848\,q^{18} + 317788\,q^{19} + 728260\,q^{21} + 20792120\,q^{30} + 2368800\,q^{24} + 483330\,q^{20} + 1611388\,q^{23} + 3457027\,q^{25} + 7215644\,q^{27} + 10327156\,q^{28} + 14694244\,q^{29} + 29264960\,q^{31} + 40982540\,q^{32} + 57114844\,q^{33} + 79229676\,q^{34} + 150473568\,q^{36} + 206084096\,q^{37} + 281138048\,q^{38} + 382075868\,q^{39}$$

```
> PD1:=":
> etamake(PD1,q,38);
```

$$\frac{\eta(\,5\,\tau\,)^3\,\eta(\,2\,\tau\,)^2}{q^{5/24}\,\eta(\,10\,\tau\,)\,\eta(\,\tau\,)^4}$$

So it would seem that

$$\sum_{n=0}^{\infty} pd(5n+1)q^n = \prod_{n=1}^{\infty} \frac{(1-q^{5n})^3(1-q^{2n})^2}{(1-q^{10n})(1-q^n)^4}.$$

This result was found originally by Rødseth [33].

**Exercise 11**. Rødseth also found the generating functions for $pd(5n+r)$ for $r = 0$, 1, 2, 3 and 4. For each $r$ use `sift` and `jacprodmake` to identify these generating functions as infinite products.

## 6. Product Identities

At present, the package contains the Triple Product identity, the Quintuple Product identity and Winquist's identity. These are the most commonly used of the Macdonald identities [30], [37], [38]. The Macdonald identities are the analogs of the Weyl denominator for affine roots systems. Hopefully, a later version of this package will include these more general identities.

6.1. **The Triple Product Identity.** The triple product identity is

$$\sum_{n=-\infty}^{\infty} (-1)^n z^n q^{n(n-1)/2} = \prod_{n=1}^{\infty} (1-zq^{n-1})(1-q^n/z)(1-q^n), \tag{6.7}$$

where $z \neq 0$ and $|q| < 1$. The Triple Product Identity is originally due to Jacobi [24, Vol I]. The first combinatorial proof of the triple product identity is due to Sylvester [39]. Recently, Andrews [3] and Lewis [29] have found nice combinatorial proofs. The triple product occurs frequently in the theory of partitions. For instance, most proofs of the Rogers-Ramanujan identity crucially depend on the triple product identity.

`tripleprod(z,q,T)` returns the $q$-series expansion to order $O(q^T)$ of Jacobi's triple product (6.7). This expansion is found by simply truncating the right side of (6.7).

```
>    tripleprod(z,q,10);
```

$$\frac{q^{21}}{z^6}-\frac{q^{15}}{z^5}+\frac{q^{10}}{z^4}-\frac{q^6}{z^3}+\frac{q^3}{z^2}-\frac{q}{z}+1-z+z^2q-z^3q^3+z^4q^6-z^5q^{10}+z^6q^{15}$$

```
>    tripleprod(q,q^3,10);
```

$$q^{57}-q^{40}+q^{26}-q^{15}+q^7-q^2+1-q+q^5-q^{12}+q^{22}-q^{35}+q^{51}$$

The last calculation is an illustration of Euler's Pentagonal Number Theorem [1, Cor. 1.7 p.11]:

(6.8)
$$\prod_{n=1}^{\infty}(1-q^n)=\prod_{n=1}^{\infty}(1-q^{3n-1})(1-q^{3n-2})(1-q^{3n})=\sum_{n=-\infty}^{\infty}(-1)^n q^{n(3n-1)/2}.$$

6.2. **The Quintuple Product Identity.** The following identity is the Quintuple Product Identity:

(6.9)
$$(-z,q)_\infty(-\frac{q}{z},q)_\infty(z^2q,q^2)_\infty(\frac{q}{z^2},q^2)_\infty(q,q)_\infty = \sum_{m=-\infty}^{\infty}\left((-z)^{-3\,m}-(-z)^{3\,m-1}\right)q^{\frac{m(3\,m+1)}{2}}.$$

Here $|q|<1$ and $z\neq 0$. This identity is the $BC_1$ case of the Macdonald identities [30]. The quintuple product identity is usually attributed to Watson [40]. However it can be found in Ramanujan's lost notebook [32, p. 207]. Also see [7] for more history and some proofs.

The function `quinprod(z,q,T)` returns the quintuple product identity in different forms:

(i) If $T$ is a positive integer it returns the $q$-expansion of the right side of (6.9) to order $O(q^T)$.
(ii) If $T=$ *prodid* then `quinprod(z,q,prodid)` returns the quintuple product identity in product form.
(iii) If $T=$ *seriesid* then `quinprod(z,q,seriesid)` returns the quintuple product identity in series form.

```
>    quinprod(z,q,prodid);
```

$$(-z,\, q)_\infty\, (-\frac{q}{z},\, q)_\infty\, (z^2\, q,\, q^2)_\infty\, (\frac{q}{z^2},\, q^2)_\infty\, (q,\, q)_\infty = (\frac{q^2}{z^3},\, q^3)_\infty\, (q\, z^3,\, q^3)_\infty\, (q^3,\, q^3)_\infty + z\, (\frac{q}{z^3},\, q^3)_\infty\, (q^2\, z^3,\, q^3)_\infty\, (q^3,\, q^3)_\infty$$

```
>    quinprod(z,q,seriesid);
```

$$(-z,\, q)_\infty \quad (-\frac{q}{z},\, q)_\infty \quad (z^2\, q,\, q^2)_\infty \quad (\frac{q}{z^2},\, q^2)_\infty \quad (q,\, q)_\infty = \sum_{m=-\infty}^{\infty} ((-z)^{(-3\,m)} - (-z)^{(3\,m-1)})\, q^{(1/2\,m\,(3\,m+1))}$$

```
>    quinprod(z,q,3);
```

$$(z^{12} + \frac{1}{z^{11}})\, q^{22} + (-z^9 - \frac{1}{z^8})\, q^{12} + (z^6 + \frac{1}{z^5})\, q^5 + (-z^3 - \frac{1}{z^2})\, q + 1 + z + (-\frac{1}{z^3} - z^4)\, q^2 + (\frac{1}{z^6} + z^7)\, q^7 + (-\frac{1}{z^9} - z^{10})\, q^{15} + (\frac{1}{z^{12}} + z^{13})\, q^{26}$$

Let's examine a more interesting application. Euler's infinite product may be dissected according to the residue of the exponent of $q$ mod 5:

$$\prod_{n=1}^{\infty}(1-q^n) = E_0(q) + qE_1(q^5) + q^2E_2(q^5) + q^3E_3(q^5) + q^4E_2(q^5).$$

By (6.8) we see that $E_3 = E_4 = 0$ since $n(3n-1)/2 \equiv 0$, 1 or 2 mod 5. Let's see if we can identify $E_0$.

```
>   with(qseries):
>   EULER:=etaq(q,1,500):
>   E0:=sift(EULER,q,5,0,499);
```

$$E0 := 1 + q - q^3 - q^7 - q^8 - q^{14} + q^{20} + q^{29} + q^{31} + q^{42} - q^{52} - q^{66} - q^{69} - q^{85} + q^{99}$$

```
>   jacprodmake(E0,q,50);
```

$$\frac{\mathrm{JAC}(\,2, 5, \infty\,)\, \mathrm{JAC}(\,0, 5, \infty\,)}{\mathrm{JAC}(\,1, 5, \infty\,)}$$

```
>   jac2prod(");
```

$$\frac{(\,q^5, q^5\,)_\infty \;\; (\,q^2, q^5\,)_\infty \;\; (\,q^3, q^5\,)_\infty}{(\,q, q^5\,)_\infty \;\; (\,q^4, q^5\,)_\infty}$$

```
>   quinprod(q,q^5,20):
>   series(",q,100);
```

$$1+q-q^3-q^7-q^8-q^{14}+q^{20}+q^{29}+q^{31}+q^{42}-q^{52}-q^{66}-q^{69}-q^{85}+O\left(q^{99}\right)$$

From our MAPLE session it appears that

$$E_0 = \frac{(q^5;q^5)_\infty (q^2;q^5)_\infty (q^3;q^5)_\infty}{(q;q^5)_\infty (q^4;q^5)_\infty}, \tag{6.10}$$

and that this product can be gotten by replacing $q$ by $q^5$ and $z$ by $q$ in the product side of the quintuple product identity (6.9).

**Exercise 12**. (i) Use the quintuple product identity (6.9) and Euler's pentagonal number theorem to prove (6.10) above.
(ii) Use MAPLE to identify and prove product expressions for $E_1$ and $E_2$.
(iii) This time see if you can repeat (i), (ii) but split the exponent mod 7.
(iv) Can you generalize these results to arbitrary modulus?
Atkin and Swinnerton-Dyer found a generalization. See Lemma 6 in [6].

6.3. **Winquist's Identity.** Back in 1969, Lasse Winquist [43] discovered a remarkable identity

(6.11)

$$\begin{aligned}&(a;q)_\infty (q/a;q)_\infty (b;q)_\infty (q/b;q)_\infty (ab;q)_\infty (q/(ab);q)_\infty (a/b;q)_\infty \\ &\qquad (b/(aq);q)_\infty (q;q)_\infty^2 \\ &\quad = \sum_{n=0}^{\infty} \sum_{m=-\infty}^{\infty} (-1)^{n+j}((a^{-3n} - a^{3n+3})(b^{-3m} - b^{3m+1}) \\ &\qquad\qquad + (a^{-3m+1} - a^{3m+2})(b^{3n+2} - b^{-3n-1}))q^{3n(n+1)/2+m(3m+1)/2}.\end{aligned}$$

By dividing both sides by $(1-a)(1-b)$ and letting $a$, $b \to 1$ he was able to express the product $\prod_{n=1}^{\infty}(1-q^n)^{10}$ as a double series and prove Ramanujan's partition congruence

$$p(11n+6) \equiv 0 \pmod{11}.$$

This was probably the first truly elementary proof of Ramanujan's congruence modulo 11. The interested reader should see Dyson's article [14] for some fascinating history on identities for powers of the Dedekind eta function and how they led to the Macdonald identities. A new proof of Winquist's identity has been found recently by S.-Y. Kang [25]. Mike Hirschhorn [22] has found a four-parameter generalization of Winquist's identity.

The function `winquist(a,b,q,T)` returns the $q$-expansion of the right side of (6.11) to order $O(q^T)$.

We close with an example. For $1 < k < 33$ define

$$Q(k) = \prod_{n=1}^{\infty}(1-q^k)(1-q^{33-k})(1-q^{33}).$$

Now define the following functions:

$$A_0 = Q(15), \quad A_3 = Q(12), \quad A_7 = Q(6), \quad A_8 = Q(3), \quad A_9 = Q(9);$$

$$\begin{aligned} B_0 &= Q(16) - q^2 Q(5),\\ B_1 &= Q(14) - qQ(8),\\ B_2 &= Q(13) - q^3 Q(2),\\ B_4 &= Q(7) + qQ(4),\\ B_7 &= Q(10) + q^3 Q(1). \end{aligned}$$

These functions occur in Theorem 6.7 of [17] as well as the function $A_0B_2 - q^2A_9B_4$.

```
> with(qseries):
> Q:=n->tripleprod(q^n,q^33,10):
> A0:=Q(15):   A3:=Q(12):   A7:=Q(6):
> A8:=Q(3):   A9:=Q(9):
> B2:=Q(13)-q^3*Q(2):   B4:=Q(7)+q*Q(4):
> IDG:=series( ( A0*B2-q^2*A9*B4),q,200):
> series(IDG,q,10);
```

$$1 - q^2 - 2\,q^3 + q^5 + q^7 + q^9 + O\left(q^{11}\right)$$

```
> jacprodmake(IDG,q,50);
```

$$\frac{\mathrm{JAC}(2,11,\infty)\,\mathrm{JAC}(3,11,\infty)^2\,\mathrm{JAC}(5,11,\infty)}{\mathrm{JAC}(0,11,\infty)^3}$$

```
> jac2prod(");
```

$$(q^2,q^{11})_\infty\,(q^9,q^{11})_\infty\,(q^{11},q^{11})_\infty\,{(q^3,q^{11})_\infty}^2\,{(q^8,q^{11})_\infty}^2\,(q^5,q^{11})_\infty\,(q^6,q^{11})_\infty$$

```
> series(winquist(q^5,q^3,q^11,10),q,20);
```

$$1 - q^2 - 2\,q^3 + q^5 + q^7 + \mathrm{O}(q^9)$$

```
> series(IDG-winquist(q^5,q^3,q^11,10),q,60);
```

$$\mathrm{O}(q^{49})$$

From our MAPLE session it seems that

$$A_0B_2 - q^2A_9B_4 = (q^2;q^{11})_\infty(q^9;q^{11})_\infty(q^{11};q^{11})_\infty(q^3;q^{11})_\infty^2(q^8;q^{11})_\infty^2 (q^5;q^{11})_\infty(q^6;q^{11})_\infty, \tag{6.12}$$

and that this product appears in Winquist's identity on replacing $q$ by $q^{11}$ and letting $a = q^5$ and $b = q^3$.

**Exercise 13**. (i) Prove (6.12) by using the triple product identity (6.7) to write the right side of Winquist's identity (6.11) as a sum of two products.

(ii) In a similar manner find and prove a product form for

$$A_0B_0 - q^3A_7B_4.$$

## References


[1] George E. Andrews. *The theory of partitions*. Addison-Wesley Publishing Co., Reading, Mass.-London-Amsterdam, 1976. Encyclopedia of Mathematics and its Applications, Vol. 2.

[2] George E. Andrews. *Partitions: yesterday and today*, With a foreword by J. C. Turner. New Zealand Math. Soc., Wellington, 1979.

[3] George E. Andrews. Generalized Frobenius partitions. *Mem. Amer. Math. Soc.*, 49(301):iv+44, 1984.

[4] George E. Andrews. *q-series: their development and application in analysis, number theory, combinatorics, physics, and computer algebra*. Published for the Conference Board of the Mathematical Sciences, Washington, D.C., 1986.

[5] A. O. L. Atkin. Ramanujan congruences for $p_{-k}(n)$. *Canad. J. Math. 20 (1968), 67-78; corrigendum, ibid.*, 21:256, 1968.

[6] A. O. L. Atkin and H. P. F. Swinnerton-Dyer. Some properties of partitions. *Proc. London Math. Soc.*, 4:84–106, 1954.

[7] Bruce C. Berndt. Ramanujan's theory of theta-functions. In *Theta functions: from the classical to the modern*, volume 1 of *CRM Proc. Lecture Notes*, pages 1–63. Amer. Math. Soc., Providence, RI, 1993.

[8] Bruce C. Berndt, S. Bhargava, and Frank G. Garvan. Ramanujan's theories of elliptic functions to alternative bases. *Trans. Amer. Math. Soc.*, 347(11):4163–4244, 1995.

[9] J. Borwein, P. Borwein, and F. Garvan. Hypergeometric analogues of the arithmetic-geometric mean iteration. *Constr. Approx.*, 9(4):509–523, 1993.

[10] J. M. Borwein and P. B. Borwein. A remarkable cubic mean iteration. In *Computational methods and function theory (Valparaíso, 1989)*, volume 1435 of *Lecture Notes in Math.*, pages 27–31. Springer, Berlin, 1990.

[11] J. M. Borwein and P. B. Borwein. A cubic counterpart of Jacobi's identity and the AGM. *Trans. Amer. Math. Soc.*, 323(2):691–701, 1991.

[12] J. M. Borwein, P. B. Borwein, and F. G. Garvan. Some cubic modular identities of Ramanujan. *Trans. Amer. Math. Soc.*, 343(1):35–47, 1994.

[13] Jonathan M. Borwein and Peter B. Borwein. *Pi and the AGM.* Canadian Mathematical Society Series of Monographs and Advanced Texts. John Wiley & Sons Inc., New York, 1987. A study in analytic number theory and computational complexity, A Wiley-Interscience Publication.

[14] Freeman J. Dyson. Missed opportunities. *Bull. Amer. Math. Soc.*, 78:635–652, 1972.

[15] N. J. Fine. On a system of modular functions connected with the Ramanujan identities. *Tohoku Math. J. (2)*, 8:149–164, 1956.

[16] F. G. Garvan. A simple proof of Watson's partition congruences for powers of 7. *J. Austral. Math. Soc. Ser. A*, 36(3):316–334, 1984.

[17] F. G. Garvan. New combinatorial interpretations of Ramanujan's partition congruences mod $5, 7$ and 11. *Trans. Amer. Math. Soc.*, 305(1):47–77, 1988.

[18] Frank Garvan. Cubic modular identities of Ramanujan, hypergeometric functions and analogues of the arithmetic-geometric mean iteration. In *The Rademacher legacy to mathematics (University Park, PA, 1992)*, volume 166 of *Contemp. Math.*, pages 245–264. Amer. Math. Soc., Providence, RI, 1994.

[19] Frank Garvan, Dongsu Kim, and Dennis Stanton. Cranks and $t$-cores. *Invent. Math.*, 101(1):1–17, 1990.

[20] Frank G. Garvan. Some congruences for partitions that are $p$-cores. *Proc. London Math. Soc. (3)*, 66(3):449–478, 1993.

[21] Andrew Granville and Ken Ono. Defect zero $p$-blocks for finite simple groups. *Trans. Amer. Math. Soc.*, 348(1):331–347, 1996.

[22] Michael D. Hirschhorn. A generalisation of Winquist's identity and a conjecture of Ramanujan. *J. Indian Math. Soc. (N.S.)*, 51:49–55 (1988), 1987.

[23] Michael D. Hirschhorn and David C. Hunt. A simple proof of the Ramanujan conjecture for powers of 5. *J. Reine Angew. Math.*, 326:1–17, 1981.

[24] C. G. J. Jacobi. *Gesammelte Werke. Bände I–VIII.* Chelsea Publishing Co., New York, 1969. Herausgegeben auf Veranlassung der Königlich Preussischen Akademie der Wissenschaften. Zweite Ausgabe.

[25] Soon-Yi Kang. A new proof of Winquist's identity. *J. Combin. Theory Ser. A*, 78(2):313–318, 1997.

[26] Marvin I. Knopp. *Modular functions in analytic number theory.* Markham Publishing Co., Chicago, Ill., 1970.

[27] Neal Koblitz. *Introduction to elliptic curves and modular forms*, volume 97 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, second edition, 1993.

[28] O. Kolberg. Note on the Eisenstein series of $\gamma_0\,(p)$. *Arbok Univ. Bergen Mat.-Natur. Ser.*, 1968(6):20 pp. (1969), 1968.

[29] R. P. Lewis. A combinatorial proof of the triple product identity. *Amer. Math. Monthly*, 91(7):420–423, 1984.

[30] I. G. Macdonald. Affine root systems and Dedekind's $\eta$-function. *Invent. Math.*, 15:91–143, 1972.

[31] S. Ramanujan. *Collected papers of Srinivasa Ramanujan.* Chelsea Publishing Co., New York, 1962. Edited with notes by G. H. Hardy, P. V. Sesu Aiyar and B. M. Wilson.

[32] Srinivasa Ramanujan. *The lost notebook and other unpublished papers.* Springer-Verlag, Berlin, 1988. With an introduction by George E. Andrews.

[33] Øystein Rødseth. Dissections of the generating functions of $q(n)$ and $q_0(n)$. *Årbok Univ. Bergen Mat.-Natur. Ser.*, (12):12 pp. (1970), 1969.

[34] L. J. Rogers. Second memoir on the expansion of certain infinite products. *Proc. London Math. Soc.*, 25:318–343, 1894.

[35] J.-P. Serre. *A course in arithmetic*. Springer-Verlag, New York, 1973. Translated from the French, Graduate Texts in Mathematics, No. 7.

[36] L. J. Slater. Further identities of the Rogers-Ramanujan type. *Proc. London Math. Soc.*, 54:147–167, 1952.

[37] Dennis Stanton. Sign variations of the Macdonald identities. *SIAM J. Math. Anal.*, 17(6):1454–1460, 1986.

[38] Dennis Stanton. An elementary approach to the Macdonald identities. In *q-series and partitions (Minneapolis, MN, 1988)*, volume 18 of *IMA Vol. Math. Appl.*, pages 139–149. Springer, New York, 1989.

[39] J. J. Sylvester. A constructive theory of partitions, arranged in three acts, an interact and an exodion. *Amer. J. Math.*, 5:251–331, 1882.

[40] G. N. Watson. Theorems stated by Ramanujan (vii): Theorems on continued fractions. *J. London Math. Soc.*, 4:39–48, 1929.

[41] G. N. Watson. Ramanujans Vermutung über Zerfällungsanzahlen. *J. Reine Angew. Math.*, 179:97–128, 1938.

[42] E. T. Whittaker and G. N. Watson. *A course of modern analysis*. Cambridge Mathematical Library. Cambridge University Press, Cambridge, 1996. An introduction to the general theory of infinite processes and of analytic functions; with an account of the principal transcendental functions, Reprint of the fourth (1927) edition.

[43] Lasse Winquist. An elementary proof of $p(11m+6) \equiv 0 \pmod{11}$. *J. Combinatorial Theory*, 6:56–59, 1969.

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