Workshop on CENTRAL Trends in PDEs

Vienna, November 12-13, 2015

Venue: Sky Lounge, 12th floor, Faculty of Mathematics, Oskar-Morgenstern-Platz 1, Vienna

Thursday, November 12, 2015

14:00-14:30	Registration
14:30-15:15	Carsten Carstensen (HU Berlin) *Axioms of Adaptivity: Rate optimality of adaptive algorithms with separate marking*
15:15-15:45	Manuel Friedrich (Univ. Vienna) *A quantitative geometric rigidity result in SBD and the derivation of linearized models from nonlinear Griffith energies in fracture mechanics*
15:45-16:15	Jens Markus Melenk (TU Vienna) *hp-FEM for singular perturbations: balanced norms and multiple scales*
16:15-16:45	break
16:45-17:30	Petr Knobloch (Charles Univ. Prague) *Analytical and Numerical Results for Algebraic Flux Correction Schemes*
17:30-18:00	Anton Arnold (TU Vienna) *Entropy method for hypocoercive & non-symmetric Fokker-Planck equations with linear drift*
19:30	social dinner

Friday, November 13, 2015

9:00-9:30	Francesca Bonizzoni (Univ. Vienna) *Padé approximation for the parametric Helmholtz equation*
9:30-10:00	Edoardo Mainini (Univ. Vienna) *Carbon nanostructures as optimal configurations for classical interacting particles*
10:00-10:30	Scott Congreve (Univ. Vienna) *hp-Version Trefftz Discontinuous Galerkin Method for the Homogeneous Helmholtz Equation*
10:30-11:00	break
11:00-11:45	Marita Thomas (WIAS Berlin) *From adhesive contact to brittle delamination in visco-elastodynamics*
11:45-12:30	Martin Kružík (Academy of Sciences, Prague) *Boundary effects and weak* lower semicontinuity for signed integral functionals on BV*
12:30-14:00	lunch

Abstracts

Anton Arnold (joint work with Jan Erb)

Entropy method for hypocoercive & non-symmetric Fokker-Planck equations with linear drift

In the last 15 years the entropy method has become a powerful tool for analyzing the large-time behavior of the Cauchy problem for linear and non-linear Fokker-Planck type equations (advection-diffusion equations, kinetic Fokker-Planck equation of plasma physics, e.g.). In particular, this entropy method can be used to analyze the rate of convergence to the equilibrium (in relative entropy and hence in L1). The essence of the method is to first derive a differential inequality between the first and second time derivative of the relative entropy, and then between the entropy dissipation and the entropy.

For degenerate parabolic equations, the entropy dissipation may vanish for states other than the equilibrium. Hence, the standard entropy method does not carry over. For hypocoercive Fokker-Planck equations (with drift terms that are linear in the spatial variable) we introduce an auxiliary functional (of entropy dissipation type) to prove exponential decay of the solution towards the steady state in relative entropy. We show that the obtained rate is indeed sharp (both for the logarithmic and quadratic entropy). Finally, we extend the method to the kinetic Fokker-Planck equation (with non-quadratic potential) and non-degenerate, non-symmetric Fokker-Planck equations. For the latter examples the "hypocoercive entropy method" yields the sharp global decay rate (as an envelope for the relative entropy function), while the standard entropy method only yields the sharp local decay rate.

References:

1) A. Arnold, J. Erb. Sharp entropy decay for hypocoercive and non-symmetric Fokker-Planck equations with linear drift, arXiv 2014.

2) A. Arnold, P. Markowich, G. Toscani, A. Unterreiter. On logarithmic Sobolev inequalities and the rate of convergence to equilibrium for Fokker-Planck type equations. Comm. PDE 26/1-2 (2001) 43-100.

Francesca Bonizzoni (with Fabio Nobile and Ilaria Perugia)

Padé approximation for the parametric Helmholtz equation

Let D be an open bounded domain in $\mathbb{R}^d$ ($d = 1, 2, 3$) and consider the parametric Helmholtz problem

$$-\Delta u - k^2 u = f \quad \text{in } D \qquad k^2 \in [k^2_{min}, k^2_{max}] \subset \mathbb{R}^+, \tag{1}$$

with either Dirichlet or Neumann homogeneous boundary conditions on ∂D. Problem (1) is well-posed provided that $k^2 \notin \Lambda$, Λ being the set of eigenvalues of the Laplacian with the considered boundary conditions.

Since the solution u is a function of the space variable $\mathbf{x} \in D$, as well as of the wavenumber k^2, we can introduce the following solution map

$$\begin{array}{rcl} \mathcal{S} : [k^2_{min}, k^2_{max}] & \to & V \\ k^2 & \mapsto & u(k^2, \cdot), \end{array} \tag{2}$$

where V denotes either $H^1(D)$ or $H^1_0(D)$.

We extend the solution map (2) to the complex half-plane $\mathbb{C}^+ := \mathbb{R}_{>0} + i\mathbb{R}$, and prove that $\mathcal{S} : \mathbb{C}^+ \to V$ is a meromorphic map with a simple pole in each $\lambda \in \Lambda$. The rational Padé approximant $\mathcal{S}_P$ of the solution map $\mathcal{S}$ is constructed, and an upper bound on the approximation error $\|\mathcal{S}(z) - \mathcal{S}_P(z)\|_V$ is derived.

Carsten Carstensen

Axioms of Adaptivity: Rate optimality of adaptive algorithms with separate marking

Mixed finite element methods with flux errors in $H(\mathrm{div})$-norms and div-least-squares finite element methods require the separate marking strategy in obligatory adaptive mesh-refining. The refinement indicator $\sigma_\ell^2(K) = \eta_\ell^2(K) + \mu^2(K)$ of a finite element domain K in a triangulation $\mathcal{T}_\ell$ on the level ℓ consists of some residual-based error estimator η_ℓ with some reduction property under local mesh-refining and some data approximation error μ_ℓ. Separate marking (SAFEM) means either Dörfler marking if $\mu_\ell^2 \le \kappa\eta_\ell^2$ or otherwise an optimal data approximation algorithm run with controlled accuracy as established in [CR11, Rab15] and reads as follows

for $\ell = 0, 1, \ldots$ **do**

  COMPUTE $\eta_\ell(K)$, $\mu(K)$ for all $K \in \mathcal{T}_\ell$

  if $\mu_\ell^2 := \mu^2(\mathcal{T}_\ell) \le \kappa\eta_\ell^2 \equiv \kappa\eta_\ell^2(\mathcal{T}_\ell)$ **then**

    $\mathcal{T}_{\ell+1} := \texttt{Dörfler\_marking}(\theta_A, \mathcal{T}_\ell, \eta_\ell^2)$

  else

    $\mathcal{T}_{\ell+1} := \mathcal{T}_\ell \oplus \texttt{approx}(\rho_B\mu_\ell^2, \mathcal{T}_0, \mu_\ell^2)$.

The enfolded set of axioms (A1)–(A4) and (B1)-(B2) plus (QM) simplifies and generalizes [CFPP14] for collective marking, treats separate marking in an axiomatic framework for the first timex, generalizes [CP15] for least-squares schemes, and extends [CR11] to the mixed FEM with flux error control in $H(\mathrm{div})$.

The presented set of axioms guarantees rate optimality for AFEMs based on collective and separate marking and covers existing literature of rate optimality of adaptive FEM. Separate marking is necessary for least-squares FEM and mixed FEM with convergence rates in $H(\mathrm{div}, \Omega) \times L^2(\Omega)$.

This is ongoing joint work with Hella Rabus.

References

[CFPP14] C. Carstensen, M. Feischl, M. Page, and D. Praetorius. Axioms of adaptivity. *Comput. Methods Appl. Math.*, 67(6):1195–1253, 2014.

[CR11] C. Carstensen and H. Rabus. An optimal adaptive mixed finite element method. *Math. Comp.*, 80(274):649–667, 2011.

[CR15] C. Carstensen and H. Rabus. Axioms of adaptivity for separate marking. *in preparation.*

[CP15] C. Carstensen and E.-J. Park. Convergence and optimality of adaptive least squares finite element methods. *SIAM J. Numer. Anal.*, 53:43–62, 2015.

[Rab15] H. Rabus. Quasi-optimal convergence of AFEM based on separate marking – Part I and II. *Journal of Numerical Analysis*, 23(2):137–156, 57–174, 2015.

[Ste08] R. Stevenson. The completion of locally refined simplicial partitions created by bisection. *Math. Comp.*, 77:227–241, 2008.

Scott Congreve

hp-Version Trefftz Discontinuous Galerkin Method for the Homogeneous Helmholtz Equation

We consider a Trefftz discontinuous Galerkin finite element (TDG) approximation of the solution to the homogeneous Helmholtz equation $-\nabla u - k^2 u = 0$. The TDG method uses (local) solutions to the Helmholtz equation as basis functions (such as the plane waves $\mathrm{e}^{ikd_l\cdot(x-x_K)}$, where d_l, $l = 1, \ldots, p_K$, are distinct propagation directions), rather than polynomial basis functions.

The general TDG formulation includes three different flux parameters, $\alpha > 0$, $\beta > 0$ and $0 < \delta < \frac{1}{2}$. Different functions for these flux parameters (constant and mesh-dependent functions) have been proposed in the literature. We study the effects of various selections of these parameters on the error in both mesh-dependent and L^2-norms, for both uniform and non-uniform meshes, with varying h (mesh size) and p (effective polynomial degree). We also compare the numerical results to a (polynomial) discontinuous Galerkin approximation, where the polynomial degree is the same as the effective polynomial order of the TDG approximation. Furthermore, we consider adaptive mesh refinement for the TDG method based on *a posteriori* error analysis.

Manuel Friedrich

A quantitative geometric rigidity result in SBD and the derivation of linearized models from nonlinear Griffith energies in fracture mechanics

We derive Griffith functionals in the framework of linearized elasticity from nonlinear and frame indifferent energies via Gamma-convergence. The convergence is given in terms of rescaled configurations measuring the displacement of the deformations from piecewise rigid motions which are constant on each connected component of the cracked body. The key ingredient to establish a compactness result is a quantitative geometric rigidity result for special functions of bounded deformation (SBD). This estimate generalizes the result of Friesecke, James, Mller in nonlinear elasticity theory and the piecewise rigidiy result of Chambolle, Giacomini, Ponsiglione for SBV functions which do not store elastic energy.

Petr Knobloch

Analytical and Numerical Results for Algebraic Flux Correction Schemes

We introduce a nonlinear algebraic flux correction scheme for a general linear boundary value problem in several space dimensions. Then we prove the existence of a solution under reasonable assumptions on the limiters and, for a particular choice of such limiters, we present results on the discrete maximum principle. In addition, we formulate the algebraic flux correction scheme in a variational form and derive an abstract error estimate. After that we apply the abstract theory to steady-state linear convection-diffusion-reaction equations and derive a particular error estimate. Finally, we present results of our numerical studies which show that the derived error estimate is sharp if only minimal assumptions on the limiters are considered. We also demonstrate how convergence rates are influenced by the used mesh and compare these numerical results with our theory.

Martin Kružík

Boundary effects and weak lower semicontinuity for signed integral functionals on BV*

We characterize lower semicontinuity of integral functionals with respect to weak* convergence in BV, including integrands whose negative part has linear growth. In addition, we allow for sequences without a fixed trace at the boundary. In this case, both the integrand and the shape of the boundary play a key role. This is made precise in our newly found condition – quasi-sublinear growth from below at points of the boundary – which compensates for possible concentration effects generated by the sequence. Our work extends some recent results by J. Kristensen and F. Rindler (Arch. Rat. Mech. Anal. 197 (2010), 539–598 and Calc. Var. 37 (2010), 29–62). It is a joint work with B. Benešová (Würzburg) and S. Krömer (Cologne).

Edoardo Mainini

Carbon nanostructures as optimal configurations for classical interacting particles

We investigate local and global minimality for classical particle interaction potentials including two- and three-body nearest-neighbor terms. We give conditions for the arising of the typical hexagonal periodicity of carbon covalent systems. We focus on the characterization of global minimizers in dimension two, together with a quantification of the lower-order surface energy contribution. Then we turn to the local minimality issue for carbon rolled up structures, and we give some remarks on their mechanical response under uniaxial load within the same variational framework. This is joint work with Hideki Murakawa, Paolo Piovano and Ulisse Stefanelli.

Jens Markus Melenk (with B. Pichler and C. Xenophontos)

hp-FEM for singular perturbations: balanced norms and multiple scales

We consider the approximation of solutions of singularly perturbed elliptic equations. Two classes are studied. The first class consists of elliptic-elliptic problems. There, the classical Galerkin approximation is optimal in the so-called energy norm, which is, however, so weak that features within the layer are not well resolved. We show that robust exponential convergence of a high order FEM on suitable meshes is achieved also in a stronger, "balanced" norm. This latter norm is balanced in the sense that layer contributions are of size $O(1)$ uniformly in the perturbation parameter. Thus, the numerical approximation is also reliable in the layer. The second class consists of systems of singularly perturbed ODEs of elliptic-hyperbolic type with multiple scales. We discuss well-posedness of the variational formulations and robust convergence of an h-version Discontinuous Galerkin method on Shishkin meshes.

Reference:

[1] J.M. Melenk and C. Xenophontos. Robust exponential convergence of *hp*-fem in balanced norms for singularly perturbed reaction-diffusion equations. Technical report, `arXiv:1408.3328`, 2014. in press in *Calcolo*.

Marita Thomas

From adhesive contact to brittle delamination in visco-elastodynamics

This contribution addresses two models describing the rate-independent fracture of a material compound along a prescribed interface in a visco-elastic material. This unidirectional process is modeled in the framework of Generalized Standard Materials with the aid of an internal delamination parameter. In the context of (fully) rate-independent systems within the energetic formulation it has become a well-established procedure to obtain solutions of the brittle model via an adhesive-contact approximation based on tools of evolutionary Gamma-convergence. This means that the non-smooth, local brittle constraint, confining displacement jumps to the null set of the delamination parameter, is approximated by a smooth, non-local surface energy term. Here, we discuss the extention of this approach for systems that couple the rate-independent evolution of the delamination parameter with a viscous and dynamic evolution of the displacements in the bulk. This is joint work with Riccarda Rossi (Brescia).