INVITATION

TO

MATHEMATICAL ELASTICITY

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1 Preliminaries

1.1 Deformations

In what follows $\Omega \subset \mathbb{R}^3$ will be a bounded domain with sufficiently smooth boundary (this will be specified at particular spots).

The set $\bar{\Omega}$ represents the body before it is deformed, so we call $\bar{\Omega}$ the *reference configuration.*

A deformation of $\bar{\Omega}$ is defined through $y : \bar{\Omega} \to \mathbb{R}^3$ that is smooth enough, injective (perhaps except $\partial\Omega$) and det $\nabla y > 0$ (orientation preserving).

$y(\bar{\Omega})$ denotes the deformed configuration and we write $x^y := y(x)$.

Lemma 1.1 *(see [11, Cor. 2,p. 17]) Let $\Omega \subset \mathbb{R}^n$ be an open set and let $y \in C(\Omega; \mathbb{R}^n)$ be injective. Then $y(\Omega)$ is open.*

Proposition 1.2 *Let $\Omega \subset \mathbb{R}^n$ be an open set and let $y \in C(\bar{\Omega}; \mathbb{R}^n)$ be a mapping whose restriction on Ω is injective. Then $y(\bar{\Omega}) = \overline{y(\Omega)}$, $y(\Omega) \subset \mathrm{int} y(\bar{\Omega})$, and $y(\partial\Omega) \supset \partial y(\bar{\Omega})$.*

Proof. Take $a \in y(\bar{\Omega})$. Then there is $x \in \bar{\Omega}$ such that $y(x) = a$. Let $\lim_k x_k = x$, $\{x_k\} \subset \Omega$. Due to continuity of y, $y(x) = \lim y(x_k)$. Thus $y(\bar{\Omega}) \subset \overline{y(\Omega)}$.

Since $\bar{\Omega}$ is compact, so is $y(\bar{\Omega})$. Hence, $y(\Omega) \subset y(\bar{\Omega})$ implies

$$\overline{y(\Omega)} \subset \overline{y(\bar{\Omega})} = y(\bar{\Omega}) \ .$$

Therefore, $y(\bar{\Omega}) = \overline{y(\Omega)}$.

We see that $y(\Omega)$ is open, by the previous lemma and it is contained in $y(\bar{\Omega})$ we get $y(\Omega) \subset \mathrm{int} y(\bar{\Omega})$.

Further,

$$y(\bar{\Omega}) = \mathrm{int} y(\bar{\Omega}) \cup \partial y(\bar{\Omega})$$

and

$$\mathrm{int}\ y(\bar{\Omega}) \cap \partial y(\bar{\Omega}) = \emptyset \ .$$

On the other hand,

$$y(\bar{\Omega}) = y(\Omega \cup \partial\Omega) = y(\Omega) \cup y(\partial\Omega) \text{ and } y(\Omega) \subset \mathrm{int} y(\bar{\Omega}) \ ,$$

so that $y(\partial\Omega) \supset \partial y(\bar{\Omega})$.

□

Theorem 1.3 *Let Ω be a bounded subset in $\mathbb{R}^n$ that satisfies int $\bar{\Omega} = \Omega$ and let $y \in C(\bar{\Omega}; \mathbb{R}^n)$ be injective. Then $y(\bar{\Omega}) = \overline{y(\Omega)}$, $y(\Omega) = \mathrm{int} y(\bar{\Omega})$, $y(\partial\Omega) = \partial y(\Omega) = \partial y(\bar{\Omega})$.*

Proof. That $y(\bar{\Omega}) = \overline{y(\Omega)}$ and $y(\Omega) \subset \mathrm{int} y(\bar{\Omega})$ we already proved.

Take $a \in \mathrm{int}\ y(\bar{\Omega})$. and let $a \notin y(\Omega)$. A continuous mapping $y : \bar{\Omega} \to y(\bar{\Omega})$ is bijective and $\bar{\Omega}$ is compact, $y^{-1} : y(\bar{\Omega}) \to \bar{\Omega}$ is also continuous. By the previous lemma $y^{-1}(\mathrm{int}\ y(\bar{\Omega})$ is an open subset $\bar{\Omega}$ that contains $y^{-1}(a)$.As $y^{-1}(a) \notin \Omega$ we have the existence of an open subset of $\bar{\Omega}$ which strictly contains Ω, a contradiction.

Further, $\overline{y(\Omega)} = y(\Omega) \cup \partial y(\Omega)$. Since $y : \bar{\Omega} \to y(\bar{\Omega})$ is a bijection, we have $\partial y(\Omega) = y(\partial\Omega)$. As $y(\Omega) = \mathrm{int} y(\bar{\Omega})$ we also have

$$\overline{y(\Omega)} = y(\Omega) \cup \partial y(\bar{\Omega}) \text{ and } y(\Omega) \cap \partial y(\bar{\Omega}) = \emptyset \ ,$$

we have $\partial y(\bar{\Omega}) = \partial y(\Omega)$. □

Example 1.4 *Consider $\Omega = \{(x_1 \cos x_2, x_1 \sin x_2);\ 1 < x_1 < 2\ ,\ 0 < x_1 < 2\pi\}$. Then $\Omega \neq$ int $\bar{\Omega}$.*

1.2 Cofactor

Let $A \in \mathbb{R}^{3\times 3}$ and denote $d_{ij} = (-1)^{i+j}\det A'_{ij}$ where A'_{ij} is the submatrix of A obtained from A by removing the i-th row and the j-th column.

Then Cof $A = (d_{ij})$ is the cofactor matrix of A. Clearly, $\det A = \sum_{i=1}^{3} a_{ij}d_{ij}$ or $\det A = \sum_{j=1}^{3} a_{ij}d_{ij}$ for $j \in \{1,2,3\}$ and $i \in \{1,2,3\}$, respectively. Therefore, $I\det A = A(\mathrm{Cof}\, A)^\top = (\mathrm{Cof}\, A)^\top A$. Hence, if $A \in \mathbb{R}^{3\times 3}$ is invertible

$$\mathrm{Cof}\, A = (\det A)A^{-\top} \ .$$

Component-wise (no summation)

$$(\mathrm{Cof}\, A)_{ij} = a_{i+1,j+1}a_{i+2,j+2} - a_{i+1,j+2}a_{i+2,j+1} \ . \tag{1.1}$$

(counting the indices modulo 3)

2 Principles of elasticity

2.1 Piola transform

The *Piola transform* establishes a correspondence between tensor fields defined in the deformed and reference configurations, respectively. If $T^y(x^y)$ denotes a tensor field over $y(\bar\Omega)$ then we define $T : \bar\Omega \to \mathbb{R}^{3\times 3}$ by

$$T(x) = (\det \nabla y(x))T^y(x^y)(\nabla y(x))^{-\top} \ , \ x^y = y(x) \ . \tag{2.1}$$

Theorem 2.1 *Let $T : \bar\Omega \to \mathbb{R}^{3\times 3}$ is the Piola transform of $T^y : y(\bar\Omega) \to \mathbb{R}^{3\times 3}$. Then*

$$\mathrm{div}\, T(x) = (\det \nabla y(x)) \ \mathrm{div}\, {}^y T^y(x^y) \ \forall x^y = y(x), \ x \in \bar\Omega, \tag{2.2}$$

$$T(x)n\,dS = T^y(x^y)n^y\,dS^y \ \forall x^y = y(x), \ x \in \partial\Omega \ . \tag{2.3}$$

The area elements dS and dS^y at the points $x \in \partial\Omega$ and $x^y = y(x) \in \partial(\bar\Omega)$ with unit outer normals n and n^y, respectively are related by

$$\det \nabla y(x)|\nabla y(x)^{-\top}n|dS = |\mathrm{Cof}\, \nabla y(x)n|dS = dS^y \ . \tag{2.4}$$

Lemma 2.2 *(Piola's identity) If $y \in C^2(\Omega;\mathbb{R}^3)$ then $\mathrm{div}\ \mathrm{Cof}\, \nabla y = 0$.*

Proof. We have

$$(\mathrm{Cof}\, \nabla y)_{ij} = \frac{\partial y_{i+1}}{\partial x_{j+1}}\frac{\partial y_{i+2}}{\partial x_{j+2}} - \frac{\partial y_{i+1}}{\partial x_{j+2}}\frac{\partial y_{i+2}}{\partial x_{j+1}} \ .$$

counting the indices modulo 3. Then $\sum_j \frac{\partial}{\partial x_j}(\mathrm{Cof}\, \nabla y)_{ij} = 0$. □

Proof of Thm. 2.1 We have

$$T_{ij}(x) = (\det \nabla y(x))T^y_{ik}(y(x))(\nabla y(x)^{-\top})_{kj} \ ,$$

which implies

$$\begin{aligned}\frac{\partial}{\partial x_j}T_{ij}(x) &= (\det \nabla y(x))\frac{\partial}{\partial x_j}[T^y_{ik}(y(x))](\nabla y(x)^{-\top})_{kj} \\ &+ T^y_{ik}(y(x))\frac{\partial}{\partial x_j}[(\det \nabla y(x)(\nabla y(x)^{-\top})_{kj}] \underbrace{=}_{\text{Piola id.}} (\det \nabla y(x))\frac{\partial}{\partial x_j}[T^y_{ik}(y(x))](\nabla y(x)^{-\top})_{kj} \ .\end{aligned}$$

Using the chain rule we get

$$\frac{\partial}{\partial x_j}[T^y_{ik}(y(x))] = \frac{\partial T^y_{ik}(y(x)}{\partial y_l}\frac{\partial y_l(x)}{\partial x_j} \ .$$

Hence,

$$(\det \nabla y(x))\frac{\partial}{\partial x_j}[T^y_{ik}(y(x))](\nabla y(x)^{-\top})_{kj} = (\det \nabla y(x))\frac{\partial T^y_{ik}(y(x)}{\partial y_l}\underbrace{\frac{\partial y_l(x)}{\partial x_j}(\nabla y(x)^{-1})_{jk}}_{\delta_{kl}} \ .$$

This proves (2.2).

In order to show (2.3) we calculate for an arbitrary subdomain $A \subset \bar{\Omega}$

$$\begin{aligned}\int_{\partial A} T(x) n \,\mathrm{d}S &= \int_A \operatorname{div} T(x)\,\mathrm{d}x = \int_A (\det \nabla y(x)) \operatorname{div} {}^y T^y(y(x))\,\mathrm{d}x \\ &= \int_{y(A)} \operatorname{div} {}^y T^y(x^y)\,\mathrm{d}x^y = \int_{\partial y(A)} T^y(x^y) n^y \,\mathrm{d}S^y \ ,\end{aligned}$$

which implies (2.3) because A was arbitrary. Applying (2.3) to $T^y = I$ we get

$$\operatorname{Cof} \nabla y(x) n \mathrm{d}S = n^y \mathrm{d}S^y$$

which implies (2.4) because $|n^y| = |n| = 1$. □

Remark 2.3 *We see that if* $x^y = y(x)$

$$n^y(x^y) = \frac{\mathit{Cof}\ \nabla y(x) n}{|\mathit{Cof}\ \nabla y(x) n|} \ ,$$

i.e., this formula says how to calculate normal vectors in a deformed configuration. This might be important in many applications where one needs to apply forces in the perpendicular direction to the body surface.

2.2 Volume element in a deformed configuration

Let $\mathrm{d}x$ denotes a volume element in a point x of the reference configuration. The volume element $\mathrm{d}x^y$ in the deformed configuration is given by

$$\mathrm{d}x^y = \det \nabla y(x)\,\mathrm{d}x \ .$$

If $A \subset \bar{\Omega}$ and $A^y := y(A)$ then $|A| = \int_A \mathrm{d}x$ and $|A^y| = \int_{y(A)} \mathrm{d}x^y = \int_A \det \nabla y(x)\,\mathrm{d}x$.

2.3 Length element in a deformed configuration

If y is differentiable at $x \in \bar{\Omega}$ then we write for all points $x + \Delta x \in \bar{\Omega}$

$$y(x + \Delta x) = y(x) + \nabla y(x) \Delta x + o(|\Delta x|) \ .$$

Hence,

$$|y(x + \Delta x) - y(x)|^2 = (\Delta x)^\top (\nabla y(x))^\top \nabla y(x) \Delta x + o(|\Delta x|^2) \ .$$

The symmetric tensor

$$C = (\nabla y)^\top \nabla y \tag{2.5}$$

is called the *right Cauchy-Green strain tensor*.

Transformation of the length element

$$\mathrm{d}l = (\mathrm{d}x^\top \mathrm{d}x)^{1/2} \ , \ \mathrm{d}l^y = (\mathrm{d}x^\top C \mathrm{d}x)^{1/2} \ .$$

The *Almansi tensor* $E = (C - I)/2$ indicates how much the current deformation y differs from the rigid one[1].

2.4 Applied forces

We will consider two types applied of forces:

a/ *applied body forces* defined through the density $f^y : \Omega^y \to \mathbb{R}^3$ per unit volume in the deformed configuration,
b/ *applied surface forces* defined by $g^y : \Gamma_1^y \to \mathbb{R}^3$ on a dS^y measurable subset $\Gamma_1^y \subset \partial\Omega^y$. Then, g^y is the density per unit area in the deformed configuration.

Remark 2.4 *There are also surface forces which are only partly specified, e.g. by their normal component to the* Γ_1^y. *We will discuss them later on.*

[1] The rigid deformation means that the whole $\bar{\Omega}$ is translated and/or rotated as a rigid body, i.e. $y(x) = a + Rx$, $x \in \bar{\Omega}$, $a \in \mathbb{R}^3$, and $R \in \mathrm{SO}(3)$, and it is equivalent to $C = I$; cf. [6, Th. 1.8-1] if y is smooth.

2.5 Cauchy stress tensor

Now we are ready to postulate the existence of internal forces in the deformed specimen.

Axiom (Stress principle of Euler and Cauchy). Consider a body occupying a *fixed* deformed configuration $\bar{\Omega}^y$ and subjected to applied forces represented by densities $f^y : \bar{\Omega}^y \to \mathbb{R}^3$ and $g^y : \Gamma_1^y \to \mathbb{R}^3$. Let further $S^2 \subset \mathbb{R}^3$ denote the unit sphere centered at the origin. Then there is a vector field

$$t^y : \bar{\Omega}^y \times S^2 \to \mathbb{R}^3$$

called *Cauchy's stress vector* such that:
i/ For any subdomain $A^y \subset \bar{\Omega}^y$ and any point $x^y \in \Gamma_1^y \cap \partial A^y$ where the joint outer unit normal vector n^y exists,

$$t^y(x^y, n^y) = g^y(x^y) \ . \tag{2.6}$$

ii/ *Axiom of balance of forces.* For any subdomain $A^y \subset \bar{\Omega}^y$

$$\int_{A^y} f^y(x^y)\,\mathrm{d}x^y + \int_{\partial A^y} t^y(x^y, n^y)\,\mathrm{d}S^y = 0 \ . \tag{2.7}$$

Again, n^y denotes the outer unit normal to ∂A^y.
iii/ *Axiom of balance of monenta.* For any subdomain $A^y \subset \bar{\Omega}^y$ with the outer unit normal n^y

$$\int_{A^y} x^y \times f^y(x^y)\,\mathrm{d}x^y + \int_{\partial A^y} x^y \times t^y(x^y, n^y)\,\mathrm{d}S^y = 0 \ . \tag{2.8}$$

Remark 2.5 *The axiom asserts that there are elementary forces $t^y(x^y, n^y)\,dS^y$ along boundaries of any subdomain of $\bar{\Omega}^y$. These forces depend on A^y only through the outer unit normal to ∂A^y. Moreover, the deformed configuration $\bar{\Omega}^y$ is in the static equilibrium by ii/ and iii/.*

Theorem 2.6 *(Cauchy's theorem) Let the applied force density $f^y : \Omega^y \to \mathbb{R}^3$ be continuous and let $t^y(\cdot, n) \in C^1(\bar{\Omega}^y; \mathbb{R}^3)$ for any $n \in S^2$ and $t^y(x^y, \cdot) \in C(S^2; \mathbb{R}^3)$ for any $x^y \in \bar{\Omega}^y$. Then the axiom implies the existence of a symmetric tensor $T^y : \bar{\Omega}^y \to \mathbb{R}^{3\times 3}$ belonging to $C^1(\bar{\Omega}^y; \mathbb{R}^{3\times 3})$ such that:*

$$t^y(x^y, n) = T^y(x^y)n \ \forall x^y \in \bar{\Omega}^y \ \forall n \in S^2 \ , \tag{2.9}$$

$$-\ div\ {}^y T^y(x^y) = f^y(x^y) \ \forall x^y \in \Omega^y \ , \tag{2.10}$$

$$T^y(x^y)n^y = g^y(x^y) \ \forall x^y \in \Gamma_1^y \ , \tag{2.11}$$

where n^y is the unit outer normal vector to Γ_1^y.

Proof. Let $\{\mathbf{e}_i\}_{i=1,2,3}$ denotes a orthonormal basis. Consider a point $x^y \in \Omega^y$ and a tetrahedron V_h as in Figure 1 with three faces parallel to the coordinate planes and $h = \mathrm{dist}(v_1 v_2 v_3, x^y)$. Notice that this tetrahedron is a Lipschitz domain, so that the outer normal vector exists a.e. We suppose that the tetrahedron is contained in Ω^y, which is an open set by Lemma 1.1. Notice also that components of the normal unit vector n to the plane $v_1 v_2 v_3$ are such that $n_i > 0$, $i = 1, 2, 3$.

Let us further denote by $|v_1 v_2 v_3|$ the area of the triangle $v_1 v_2 v_3$. An analogous notation is used for other faces. The volume of the tetrahedron is proportional to $h|v_1 v_2 v_3|$ and $|v_2 x^y v_3| = n_1 |v_1 v_2 v_3|$ etc. Finally, realize that by the action-reaction principle $t^y(x^y, \nu) = -t^y(x^y, -\nu)$ for any $x^y \in \Omega^y$ and any $\nu \in S^2$.

We calculate the force balance for on the tetrahedron V_h:

$$\int_{V_h} f^y(a^y)\,\mathrm{d}a^y + \int_{\partial V_h} t^y(a^y, n^y)\,\mathrm{d}S^y = 0 \ .$$

Further, for any $i = 1, 2, 3$

$$\begin{aligned} \int_{\partial V_h} t_i^y(a^y, n^y)\,\mathrm{d}S^y &= \int_{v_1 v_2 v_3} t_i^y(a^y, n)\,\mathrm{d}S^y - \int_{v_2 x^y v_3} t_i^y(a^y, \mathbf{e}_1)\,\mathrm{d}S^y - \int_{v_1 x^y v_3} t_i^y(a^y, \mathbf{e}_2)\,\mathrm{d}S^y \\ &- \int_{v_1 x^y v_2} t_i^y(a^y, \mathbf{e}_3)\,\mathrm{d}S^y \ . \end{aligned}$$

Hence,

$$\frac{1}{|v_1v_2v_3|}\int_{v_1v_2v_3} t_i^y(a^y,n)\,\mathrm{d}S^y = \frac{n_1}{|v_2x^yv_3|}\int_{v_2x^yv_3} t_i^y(a^y,\mathbf{e}_1)\,\mathrm{d}S^y + \frac{n_2}{|v_1x^yv_3|}\int_{v_1x^yv_3} t_i^y(a^y,\mathbf{e}_2)\,\mathrm{d}S^y + \frac{n_3}{|v_1x^yv_2|}\int_{v_1x^yv_2} t_i^y(a^y,\mathbf{e}_3)\,\mathrm{d}S^y - \frac{1}{|v_1v_2v_3|}\int_{V_h} f(a^y)\,\mathrm{d}a^y\ .$$

We have by continuity

$$\lim_{h\to 0}\frac{n_1}{|v_2x^yv_3|}\int_{v_2x^yv_3} t_i^y(a^y,\mathbf{e}_1)\,\mathrm{d}S^y = t_i^y(x^y,\mathbf{e}_1)n_1$$

and similarly for other components.

Further,

$$\lim_{h\to 0}\frac{1}{|v_1v_2v_3|}|\int_{V_h} f(a^y)\,\mathrm{d}a^y| \le \lim_{h\to 0} Ch = 0\ ,$$

where $\|f^y\|_{C(V_{h_0};\mathbb{R}^3)} < C$ by our assumption.

Altogether, for $i = 1,2,3$

$$t_i^y(x^y,n) = \sum_{i=1}^{3} t_i(x^y,\mathbf{e}_j)n_j\ ,$$

or, equivalently,

$$t^y(x^y,n) = \sum_{i=1}^{3} t(x^y,\mathbf{e}_j)n_j\ . \qquad (2.12)$$

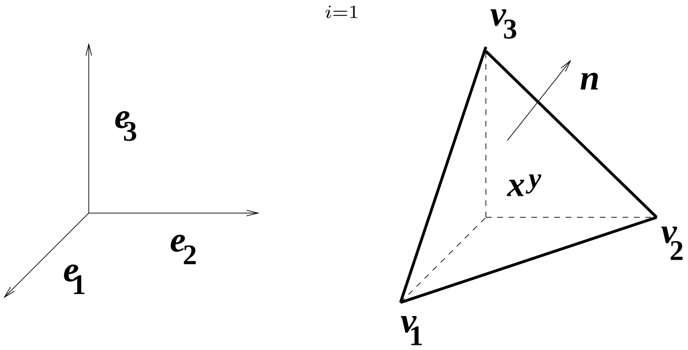

Fig. 1. Tetrahedron used in the proof of the Cauchy theorem.

As $t_i^y(x^y,\mathbf{e}_i) = -t_i^y(x^y,-\mathbf{e}_i)$ it follows that (2.12) holds even if some of $n_j \le 0$. We define $T_{ij}^y : \bar\Omega^y \to \mathbb{R}$ by $t^y(x^y,\mathbf{e}_j) = \sum_i T_{ij}^y(x^y)\mathbf{e}_i$, so that $t^y(x^y,n) = \sum_{i,j} T_{ij}^y(x^y)\mathbf{e}_i n_j$. Hence, $t_i^y(x^y,n) = \sum_j T_{ij}^y(x^y)n_j$ for all $x^y \in \bar\Omega^y$ and all $n \in S^2$, or,in other words,

$$t^y(x^y,n) = T^y(x^y)n\ .$$

The tensor $T^y = (T_{ij}^y)_{ij}$ is called the *Cauchy stress tensor.* Notice, in particular, that it shows a linear dependence of t^y on the normal n. [2]

We use the axiom of force balance and the Green theorem to infer that

$$\begin{aligned} 0 &= \int_{A^y} f^y(x^y)\,\mathrm{d}x^y + \int_{\partial A^y} t^y(x^y,n^y)\,\mathrm{d}S^y \\ &= \int_{A^y} f^y(x^y)\,\mathrm{d}x^y + \int_{\partial A^y} T^y(x^y)n^y\,\mathrm{d}S^y \\ &= \int_{A^y} f^y(x^y)\,\mathrm{d}x^y + \int_{A^y} \mathrm{div}\ {}^yT^y(x^y)\,\mathrm{d}x^y, \end{aligned}$$

which shows (2.10) because $A^y \subset \Omega^y$ was arbitrary.

Using the momentum balance we have (summation convention and the Levi-Civita symbol ε_{ijk} are used)

$$\begin{aligned} 0 &= \int_{A^y} \varepsilon_{ijk}x_j^y f_k^y(x^y)\,\mathrm{d}x^y + \int_{\partial A^y} x_j^y T_{km}^y(x^y)n_m^y\,\mathrm{d}S^y = \int_{A^y} \varepsilon_{ijk}x_j^y f_k^y(x^y)\,\mathrm{d}x^y \\ &+ \int_{A^y} \varepsilon_{ijk}\frac{\partial}{\partial x_m^y}\left(x_j^y T_{km}^y(x^y)\right)\mathrm{d}x^y = \int_{A^y} \varepsilon_{ijk}x_j^y \underbrace{\left(f_k^y(x^y) + \frac{\partial}{\partial x_m^y}T_{km}^y(x^y)\right)}_{=0 \text{ by (2.10)}}\mathrm{d}x^y \\ &+ \int_{A^y} \varepsilon_{ijk}T_{km}^y(x^y)\delta_{jm} = \int_{A^y} \varepsilon_{ijk}T_{kj}^y\,\mathrm{d}x^y\ , \end{aligned}$$

[2] See e.g. [10] that T^y is really a 2nd order tensor.

which implies symmetry of T^y. Finally, (2.11) follows from (2.6). $\square$

Let us discuss three important examples of T^y. Let us first take $\pi > 0$ and put $T^y(x^y) = -\pi I$, so that $t^y(x^y, n^y) = -\pi n^y$. This defines the pressure load on Ω^y; cf. Figure 2 a/.

Secondly,let $T^y(x^y) = \tau e \otimes e$[3] where $\tau > 0$, $|e| = 1$. Then $t^y(x^y, n^y) = T^y(x^y)n^y = \tau(e \cdot n^y)e$; cf. Figure 2 b/ and it is called pure tension.

Finally, take $\sigma > 0$ and unit mutually perpendicular vectors $e, \hat{e}$ and put $T^y(x^y) = \sigma(e \otimes \hat{e} + \hat{e} \otimes e)$. This yields $t^y(x^y, n^y) = \sigma((e \cdot n)\hat{e} + (\hat{e} \cdot n)e)$; cf. Figure 2 c/ and we call it pure shear.

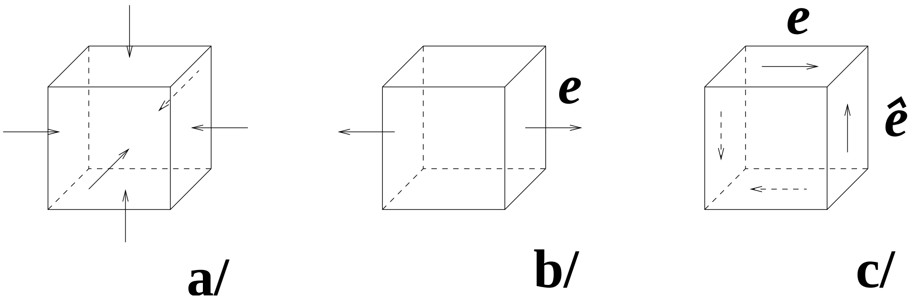

Fig. 2. a/ pressure $\pi > 0$, b/ pure tension at the direction e, c/ pure shear

The **Axiom of material frame-indifference** states that if a deformation y is composed with another deformation z of $\bar{\Omega}^y$ where $z(x) := Ry(x)$ for all $x \in \bar{\Omega}$ and some rotation $S \in \mathrm{SO}(3)$ (i.e. rotated) then for all $x \in \bar{\Omega}$ and any $n \in S^2$

$$t^z(x^z, Rn) = Rt^y(x^y, n) \ . \tag{2.13}$$

Notice that we can write

$$t^z(x^z, Rn) = T^z(x^z)Rn = Rt^y(x^y, n) = RT^y(x^y)n \ .$$

Let $m \in S^2$ be such that $Rn = m$. Then we immediately get that $T^z(x^z) = RT^y(x^y)R^\top$ for any rotation R.

2.6 Principle of virtual work, Piola-Kirchhoff and Kirchhoff stress tensors

Theorem 2.7 *(Principle of virtual work in the deformed configuration) The boundary value problem*

$$-div^y T^y = f^y \ \ in \ \Omega^y \tag{2.14a}$$
$$T^y n^y = g^y \ \ on \ \Gamma_1^y \tag{2.14b}$$

is formally equivalent to the variational equation

$$\int_{\Omega^y} T^y \cdot \nabla^y \theta^y \, dx^y = \int_{\Omega^y} f^y \cdot \theta^y \, dx^y + \int_{\Gamma_1^y} g^y \cdot \theta^y \, dS^y \tag{2.15}$$

for all smooth $\theta^y : \bar{\Omega}^y \to \mathbb{R}^3$, $\theta = 0$ on $\partial\Omega^y \setminus \Gamma_1^y$.

Proof. We formally apply the following version of Green's theorem for θ as in the theorem.

$$\int_{\Omega^y} \mathrm{div}\ ^y T^y \cdot \theta^y \, \mathrm{d}x = -\int_{\Omega^y} T^y \cdot \nabla^y \theta^y \, \mathrm{d}x^y + \int_{\Gamma_1^y} T^y n^y \cdot \theta^y \, \mathrm{d}S^y \ . \tag{2.16}$$

Thus,

$$0 = \int_{\Omega^y} (div^y T^y + f^y)\theta^y \, \mathrm{d}x^y = \int_{\Omega^y} (-T^y \cdot \nabla^y \theta^y + f^y \cdot \theta^y) \, \mathrm{d}x^y + + \int_{\Gamma_1^y} T^y n^y \cdot \theta^y \, \mathrm{d}S^y \ ,$$

which shows that (2.14) implies (2.15).

Conversely, take θ with $\theta = 0$ on $\partial\Omega$ and check that (2.15) implies (2.14a). Then (2.14b) easily follows from (2.16) and (2.14a). $\square$

[3] Recall that $(a \otimes b)_{ij} = a_i b_j$, $i, j = 1, 2, 3$.

Remark 2.8 *The formulation (2.14) equipped with the condition $T^y = T^{y\top}$ (which is automatically satisfied by Theorem 2.6 is called equilibrium equations in the deformed configuration.*

The problem is that (2.14) is formulated in the deformed configuration which is apriori unknown and is a part of a sought solution. Hence, it is desirable to transform the equilibrium equations to the reference configuration. We define the *1st Piola-Kirchhoff tensor* $T : \bar{\Omega} \to \mathbb{R}^{3\times 3}$, as the Piola transform of the Cauchy stress tensor T^y, i.e.,

$$T(x) = (\det \nabla y(x))T^y(x^y)(\nabla y(x))^{-\top} \ , \ x^y = y(x) \ , \ x \in \bar{\Omega} \ . \tag{2.17}$$

It follows from the properties of the Piola transform that

$$\text{div } T(x) = (\det \nabla y(x)) \text{ div } {}^y T^y(x^y) \ . \tag{2.18}$$

Notice that T is not symmetric in general. Instead,

$$T(x)^\top = (\nabla y(x))^{-1}(\det \nabla y(x))T^y(x^y) = (\nabla y(x))^{-1}T(x)(\nabla y(x))^\top \ . \tag{2.19}$$

The symmetric tensor $T_K : \bar{\Omega} \to \mathbb{R}^{3\times 3}$

$$T_K(x) = T(x)(\nabla y(x))^\top = (\det \nabla y(x))T^y(x^y)$$

is called the *Kirchhoff stress tensor.*

Finally, we define the *2nd Piola-Kirchhoff stress tensor* $\Sigma : \bar{\Omega} \to \mathbb{R}^{3\times 3}$ by the formula

$$\Sigma(x) = (\nabla y(x))^{-1}T(x) = (\nabla y(x))^{-1}(\det \nabla y(x))T^y(x^y)(\nabla y(x))^{-\top} \ , \tag{2.20}$$

which is clearly symmetric.

2.7 Applied forces in the reference configuration

Next we rewrite force densities from Ω^y to Ω. Having $f^y : \Omega^y \to \mathbb{R}^3$ a body force density (per volume) we look for $f : \Omega \to \mathbb{R}^3$ such that $f(x)\,\mathrm{d}x = f^y(x^y)\,\mathrm{d}x^y$. Hence,

$$f(x) = f^y(x^y) det \nabla y(x) \ , \ x^y = y(x) \ .$$

Then f is the (volume) density of body forces in the reference configuration.

If $\varrho : \Omega \to \mathbb{R}$ and $\varrho^y : \Omega^y \to \mathbb{R}$ are mass densities in the reference and deformed configurations, respectively, we have

$$\varrho(x) = \varrho(x^y)\det \nabla y(x) \ , \ x^y = y(x)$$

and $f(x) = \varrho(x)b(x)$ where $b : \Omega \to \mathbb{R}^3$ is the mass density of body forces in the reference configuration.

Similarly we proceed with surface forces. We look for $g : \Gamma_1 \to \mathbb{R}^3$, $y(\Gamma_1) = \Gamma_1^y$ such that $g(x)\,\mathrm{d}S = g^y(x^y)\,\mathrm{d}S^y$. Thus, using properties of Piola's transform

$$g(x) = g^y(x^y)|(\text{Cof } \nabla y(x))n(x)| \ , \ x^y = y(x) \ , \ x \in \Gamma_1 \ , \tag{2.21}$$

is the density of surface forces in the reference configuration.

2.7.1 Conservative forces

An applied body force is a dead load if its associated density in the reference configuration is independent of the deformation y. A simple example is a homogeneous gravity field $f(x) = (0, 0, -\text{const } \varrho(x))$, $x \in \Omega$. Likewise, an applied surface force is a dead load if its associated density in the reference configuration is independent of the deformation y.

Consider an applied surface force being a pressure load. In this situation,

$$g^y(x^y) = -\pi n^y(x^y) \ , x^y \in \Gamma_1^y \text{ and } \pi \geq 0 \ . \tag{2.22}$$

Clearly, if $\pi > 0$ then, in general, $g^{y_1}(x^{y_1}) \neq g^{y_2}(x^{y_2})$ for two different deformations $y_1, y_2 : \bar{\Omega} \to \mathbb{R}^3$. (Think e.g. about an inflated/deflated balloon.)

In order to fix ideas, we will suppose that the applied force densities are of the form

$$f(x) = \hat{f}(x, y(x)) \ , \ x \in \Omega \ , \tag{2.23}$$

and

$$g(x) = \hat{g}(x, \nabla y(x)) \ , \ x \in \Gamma_1 \ , \tag{2.24}$$

where $\hat{f} : \Omega \times \mathbb{R}^3 \to \mathbb{R}^3$ and $\hat{g} : \Gamma_1 \times \mathbb{R}^{3\times3}_+ \to \mathbb{R}^3$ are given[4].

An applied body force is *conservative* if there is $F : \{y : \bar{\Omega} \to \mathbb{R}^3\} \to \mathbb{R}$,

$$F(y) = \int_\Omega \hat{F}(x, y(x))\,\mathrm{d}x \tag{2.25}$$

with $\hat{F} : \Omega \times \mathbb{R}^3 \to \mathbb{R}$ such that

$$F'(y)\theta = \int_\Omega \hat{f}(x, y(x)) \cdot \theta(x)\,\mathrm{d}x \ .$$

Here $\hat{F}$ is called the *potential* of $\hat{f}$ and it holds that $\hat{f}(x,\xi) = \nabla_\xi \hat{F}(x,\xi)$.

In the same way we say that the applied surface force is conservative if there is $G : \{y : \bar{\Omega} \to \mathbb{R}^3\} \to \mathbb{R}$,

$$G(y) = \int_{\Gamma_1} \hat{G}(x, y(x), \nabla y(x))\,\mathrm{d}x \tag{2.26}$$

with $\hat{G} : \Gamma_1 \times \mathbb{R}^3 \times \mathbb{R}^{3\times3}_+ \to \mathbb{R}$ such that

$$G'(y)\theta = \int_{\Gamma_1} \hat{g}(x, \nabla y(x)) \cdot \theta(x)\,\mathrm{d}x \ .$$

Here $\hat{G}$ is called the potential of $\hat{g}$.

Proposition 2.9 *A pressure load is a conservative surface force.*

Proof. Combining Remark 2.3 with (2.21) we get that

$$g(x) = -\pi \mathrm{Cof}\ (\nabla y(x)) n(x) \ .$$

Therefore we look for $G : \{y : \bar{\Omega} \to \mathbb{R}^3\} \to \mathbb{R}$ such that

$$G'(y)\theta = -\pi \int_{\partial\Omega} (\mathrm{Cof}\ (\nabla y) n) \cdot \theta\,\mathrm{d}S \ . \tag{2.27}$$

Using the Green's theorem and the Piola identity we have (summation convention used)

$$\begin{aligned}\int_\Omega \det\ \nabla y\,\mathrm{d}x &= \frac{1}{3}\int_\Omega \frac{\partial y_i}{\partial x_j}(\mathrm{Cof}\ (\nabla y)_{ij}\,\mathrm{d}x = \frac{1}{3}\int_\Omega \frac{\partial}{\partial x_j}\left(y_i(\mathrm{Cof}\ (\nabla y)_{ij})\right)\mathrm{d}x \\ &= \frac{1}{3}\int_{\partial\Omega} (\mathrm{Cof}\ \nabla y) n \cdot y\,\mathrm{d}S \ .\end{aligned}$$

It is a matter of the direct calculation that $(\det\ A)'B = \frac{\mathrm{d}}{\mathrm{d}t}\det\ (A + tB)|_{t=0} = \mathrm{Cof}\ A \cdot B$.

Hence, if $\gamma(y) = \int_\Omega \det\ \nabla y\,\mathrm{d}x$ then

$$\gamma'(y)\theta = \int_\Omega (\mathrm{Cof}\ \nabla y) \cdot \nabla\theta\,\mathrm{d}x \ .$$

Applying the Green theorem analogous to (2.16) and keeping in mind the Piola identity we get

$$\int_\Omega (\mathrm{Cof}\ \nabla y) \cdot \nabla\theta\,\mathrm{d}x = \int_{\partial\Omega} (\mathrm{Cof}\ \nabla y) n \cdot \theta\,\mathrm{d}S = \gamma'(y)\theta \ . \tag{2.28}$$

Comparing (2.27) and (2.28) we see that

$$G(y) = -\pi \int_\Omega \det\ \nabla y\,\mathrm{d}x = -\frac{\pi}{3}\int_{\partial\Omega} (\mathrm{Cof}\ \nabla y) n \cdot y\,\mathrm{d}S \ .$$

□

[4] $\mathbb{R}^{3\times3}_+$ denotes 3×3 matrices with positive determinants.

2.8 Principles of virtual work in the reference configuration

We have the following boundary value problems in the reference configuration.

Theorem 2.10 *The 1st Piola-Kirchhoff tensor satisfies:*

$$-divT = f \;\; in \; \Omega \tag{2.29a}$$

$$Tn = g \;\; on \; \Gamma_1 \tag{2.29b}$$

in the reference configuration. The problem (2.29) is formally equivalent to the variational equation

$$\int_\Omega T \cdot \nabla\theta \, dx = \int_\Omega f \cdot \theta \, dx + \int_{\Gamma_1} g \cdot \theta \, dS \tag{2.30}$$

for all smooth $\theta : \bar{\Omega} \to \mathbb{R}^3$, $\theta = 0$ on $\partial\Omega \setminus \Gamma_1$.

Proof. The prof follows from (2.14) and definitions of f,g and T. □

Analogously, we have

Theorem 2.11 *The 2nd Piola-Kirchhoff tensor satisfies:*

$$-div\nabla y\Sigma = f \;\; in \; \Omega \tag{2.31a}$$

$$\nabla y\Sigma n = g \;\; on \; \Gamma_1 \tag{2.31b}$$

in the reference configuration. The problem (2.29) is formally equivalent to the variational equation

$$\int_\Omega \nabla y\Sigma \cdot \nabla\theta \, dx = \int_\Omega f \cdot \theta \, dx + \int_{\Gamma_1} g \cdot \theta \, dS \tag{2.32}$$

for all smooth $\theta : \bar{\Omega} \to \mathbb{R}^3$, $\theta = 0$ on $\partial\Omega \setminus \Gamma_1$.

The problems (2.29) and (2.31) are called equilibrium equations in the reference configurations and corresponding variational equations are referred to as *principles of virtual work in the reference configutration.*

3 Elastic materials

Looking at (2.14) we see that we have 9 unknowns (3 components of y and 6 components of T^y) but only 3 equations. Therefore, we complete (2.14) by material relations.

3.1 Response functions

We call a material *elastic* if there is a mapping

$$\hat{T}^D : \bar{\Omega} \times \mathbb{R}^{3\times 3}_+ \to \mathbb{R}^{3\times 3}_{\rm sym} \tag{3.1}$$

called a response function for the Cauchy stress such that

$$T^y(x^y) = \hat{T}^D(x, \nabla y(x)) \ , \ x^y = y(x) \ . \tag{3.2}$$

The relation (3.2) is called the constitutive equation of the material.

It can be shown that the material is *isotropic* (behaves the same way in all directions) at a point $x \in \bar{\Omega}^y$ if

$$\hat{T}^D(x, F) = \hat{T}^D(x, FR) \;\; F \in \mathbb{R}^{3\times 3}_+ \ , \ R \in \ {\rm SO}(3) \ .$$

Similarly, we can find response functions for the 1st and 2nd Piola-Kirchhoff stress tensors, respectively:

$$\hat{T}(x, F) = (\det \, F)\hat{T}^D(x, F)F^{-\top} \;\forall x \in \bar{\Omega}\, F \in \mathbb{R}^{3\times 3}_+$$

and

$$\hat{\Sigma}(x, F) = (\det \, F)F^{-1}\hat{T}^D(x, F)F^{-\top} \;\forall x \in \bar{\Omega}\, F \in \mathbb{R}^{3\times 3}_+ \ .$$

Then the P.-K. stress tensors read for all $x \in \bar{\Omega}$

$$T(x) = \hat{T}(x, \nabla y(x)) \quad , \quad \Sigma(x) = \hat{\Sigma}(x, \nabla y(x)) \ .$$

Remark 3.1 *"An elastic material" is a theoretical construction and we cannot really prove that a particular piece of matter is elastic. We can only suggest a response function and compare our predictions with experiments. A material in the reference configuration is called homogeneous if its response function does not depend on x, otherwise is called nonhomogeneous. Homogeneity is related to a particular reference configuration. T^D is related to a particular deformed configuration.*

There are theories relating $T^y(x^y)$ to the gradient ∇y in the whole Ω (nonlocal elasticity) or taking higher order gradients into considerations (nonsimple materials).

3.2 Isotropic materials

The intuitive idea of isotropy is that at a given point of our material its response is the same in all directions. As an example, we can consider polycrystalline materials or dough. On the other hand, wood is an anisotropic material because its behavior is different along and across fibers. We now give a mathematical definition of isotropy.

Consider a deformation $y : \bar{\Omega} \to y(\bar{\Omega})$. Then we have by (3.2)

$$T^y(x^y) = \hat{T}^D(x, \nabla y(x)) \ .$$

Take $x_0 \in \bar{\Omega}$ and rotate $\bar{\Omega}$ around this point by a rotation $R^\top \in \mathrm{SO}(3)$, i.e., define $\theta(z) := x_0 + R^\top(z - x_0)$ for all $z \in \bar{\Omega}$. Consider $\tilde{y} := y \circ \theta^{-1} : \theta(\bar{\Omega}) \to y(\bar{\Omega})$, so that $\tilde{y}(\tilde{x}) = y(x_0 + R(\tilde{x} - x_0))$ if $\tilde{x} \in \theta(\bar{\Omega})$. However, $x_0^y = x_0^{\tilde{y}}$ and

$$T^{\tilde{y}}(x_0^{\tilde{y}}) = \hat{T}^D(x_0, \nabla\tilde{y}(x_0)) = \hat{T}^D(x_0, \nabla y(x_0)R) \ .$$

Hence, we say that a material is isotropic at a point $x_0 \in \bar{\Omega}$ if the response function for the Cauchy stress satisfies for all $F \in \mathbb{R}^{3\times3}_+$ and all $R \in \mathrm{SO}(3)$ that

$$\hat{T}^D(x_0, F) = \hat{T}^D(x_0, FR) \ .$$

Using response functions for T and Σ we get analogously for all F, R as before that

$$\hat{T}(x_0, FR) = \hat{T}(x_0, F)R \ \text{ and } \hat{\Sigma}(x_0, FR) = R^\top\hat{\Sigma}(x_0, F)R \ .$$

3.3 Hyperelastic materials

An elastic material is *hyperelastic* if there is a *stored energy* function $\hat{W} : \bar{\Omega} \times \mathbb{R}^{3\times3}_+ \to \mathbb{R}$ such that for all $x \in \bar{\Omega}$ and all $F \in \mathbb{R}^{3\times3}_+$

$$\hat{T}(x, F) = \frac{\partial\hat{W}}{\partial F}(x, F) \ . \tag{3.3}$$

As before, a hyperelastic material is a model and its existence cannot be proven. However, it emphasizes reversibility of deformations and the idea that energy can be stored in the material and used afterwards to do work.

The Axiom of material frame-indifference asserts that for all $x \in \bar{\Omega}$, $R \in \mathrm{SO}(3)$ and any $F \in \mathbb{R}^{3\times3}_+$ we have $\hat{T}^D(x, RF) = R\hat{T}^D(x, F)R^\top$. Consequently, the response function $\hat{T}$ of the first Piola-Kirchoff stress tensor satisfies

$$R^\top\hat{T}(x, RF) = \hat{T}(x, F) \ . \tag{3.4}$$

Indeed,

$$\hat{T}(x, RF) = \det\,(RF)\hat{T}^D(x, RF)RF^{-\top} = \det\,(RF)R\hat{T}^D(x, F)R^\top RF^{-\top} = R\hat{T}(x, F) \ .$$

This means that

$$R^\top\frac{\partial\hat{W}}{\partial F}(x, RF) = \frac{\partial\hat{W}}{\partial F}(x, F) \ . \tag{3.5}$$

Fix a rotation R and denote $\hat{W}_R(x, F) := \hat{W}(x, RF)$. Then we get by the Taylor formula for $G \in \mathbb{R}^{3\times3}_+$ such that $\det\,(F + G) > 0$ that

$$\begin{aligned}
\hat{W}_R(x, F + G) &= \hat{W}(x, RF + RG) = W(x, RF) + \frac{\partial\hat{W}}{\partial F}(x, RF){:}RG + o(|G|) \\
&= \hat{W}_R(x, F) + R^\top\frac{\partial\hat{W}}{\partial F}(x, RF){:}G + o(|G|) \\
&= \hat{W}_R(x, F) + \frac{\partial\hat{W}_R}{\partial F}(x, F){:}G + o(|G|) \ .
\end{aligned} \tag{3.6}$$

Therefore, in view of (3.5) and (3.6)

$$\frac{\partial\hat{W}}{\partial F}(x, F) = \frac{\partial\hat{W}_R}{\partial F}(x, F) \ .$$

In other words, for all $F \in \mathbb{R}^{3\times3}_+$

$$\frac{\partial}{\partial F}(\hat{W}(x, F) - \hat{W}(x, RF)) = 0 \ .$$

As $\mathbb{R}_+^{3\times 3}$ is a connected set[5], we infer that there is a constant C (depending on R) such that $\hat{W}(x, RF) = \hat{W}(x, F) + C$. Testing this equality for $F := I$, $F := R$, $F := R^2$, etc. we get that $\hat{W}(x, R^n) = \hat{W}(x, I) + nC$. Thus, $\lim_{n\to\infty} |\hat{W}(x, R^n)| = +\infty$. However, the set $\{R^n\}_{n\in\mathbb{N}}$ is compact and $\hat{W}(x, \cdot)$ is differentiable and continuous. For this reason, $C = 0$. Altogether, for all rotations R and all $F \in \mathbb{R}_+^{3\times 3}$

$$\hat{W}(x, RF) = \hat{W}(x, F) \ .$$

As we can always find a decomposition $F = RU$ where $R \in \mathrm{SO}(3)$ and $U = U^\top \in \mathbb{R}_+^{3\times 3}$ with $U^2 = F^\top F$, it is clear that $\hat{W}(x, F) = \hat{w}(x, C)$ for some function $\hat{w} : \bar{\Omega} \times \{A = A^\top;\ A \in \mathbb{R}_+^{3\times 3}\} \to \mathbb{R}$ with $C = F^\top F$.

In the case of hyperelastic material and if the applied forces are conservative, a solution of elasticity equations is formally equivalent to finding a stationary point of the functional

$$I(y) = \int_\Omega \hat{W}(x, \nabla y)\,\mathrm{d}x - F(y) - G(y) \ . \tag{3.7}$$

Theorem 3.2 *Let there be given a hyperelastic material subjected to conservative applied body and surface forces. Then the equations*

$$-\ div\ \frac{\partial \hat{W}}{\partial F}(x, \nabla y(x)) = \hat{f}(x, y(x)) \ , \ x \in \Omega \tag{3.8a}$$

$$\frac{\partial \hat{W}}{\partial F}(x, \nabla y(x)) n(x) = \hat{g}(x, \nabla y(x)) \ , \ x \in \Gamma_1 \tag{3.9a}$$

are formally equivalent to the equations

$$I'(y)\theta = 0 \tag{3.10}$$

for all smooth maps $\theta : \bar{\Omega} \to \mathbb{R}^3$ vanishing on Γ_0. Here $n(x)$ is the outer unit normal to Γ_1 at x.

Proposition 3.3 *Under the assumptions of Theorem 3.2 the following holds. If $y \in \Phi := \{y : \bar{\Omega} \to \mathbb{R}^3;\ y = y_0 \text{ on } \Gamma_0\}$ is smooth enough and $I(y) = \inf_{\tilde{y}\in\Phi} I(\tilde{y})$ then y solves (3.8-3.9) and $y = y_0$ on Γ_0.*

Definition 3.4 *The functional $W(y) = \int_\Omega W(\nabla y)\, dx$ is called the strain energy, while I is called the total energy.*

Remark 3.5 *(Behavior of $\hat{W}$) For all $x \in \bar{\Omega}$, all $F \in \mathbb{R}_+^{3\times 3}$ and all $R \in$* SO(3)

$$\hat{W}(x, F) = \hat{W}(x, RF) \ .$$

This property is called principle of frame indifference.

We assume that there are positive constants α, p, q, r such that such that for each $x \in \bar{\Omega}$ and all $F \in \mathbb{R}_+^{3\times 3}$

$$\hat{W}(x, F) \geq \alpha(|F|^p + |Cof\ F|^q + (det\ F)^r) \tag{3.11}$$

and

$$W(x, F) \to \infty \ if\ det\ F \to 0_+ \ . \tag{3.12}$$

where $Cof\ A = (det\ A)A^{-\top}$.

Proposition 3.6 *There is no convex function satisfying (3.12).*

[5] Indeed, notice that if $F \in \mathbb{R}_+^{3\times 3}$ then there is an upper triangular matrix G and $R \in \mathrm{SO}(3)$ such that $F = RG$. Moreover, the diagonal components of G can be taken all positive (and then the decomposition of F is unique). Hence, $\det F = \det G = \Pi_{i=1}^3 G_{ii}$. Let for $t \in [0,1]$ $t \mapsto G_t$ be defined in such a way that G_t has the same diagonal as G but its off-diagonal elements are $(1-t)$ multiples of off-diagonal elements of G. Therefore, $G_1 = \mathrm{diag}(G_{11}, G_{22}, G_{33})$. Now we extend the mapping $t \to G_t$ to the interval $[1,2]$ in the following way: If $t \in [1,2]$ then $G_t := \mathrm{diag}((1-G_{11})t + 2G_{11} - 1, (1-G_{22})t + 2G_{22} - 1, (1-G_{33})t + 2G_{33} - 1)$. In particular, $G_2 = \mathbb{I}$ and the path $\{t \in [0,2] \mapsto G_t\} \subset \mathbb{R}_+^{3\times 3}$ and it is continuous. This means that $t \mapsto F_t := RG_t$ makes a continuous path between F and R. As R is a rotation it can be joined with the identity by a continuous path as it can be readily seen from the expression of R in terms of axial rotation angles (called Euler's decomposition). Altogether, we see that F is connected with the identity. Consequently, $\mathbb{R}_+^{3\times 3}$ is connected.

Proof. First, we notice that the convex hull of $\mathbb{R}^{3\times 3}_+$ is the whole $\mathbb{R}^{3\times 3}$. We observe that $-I$ =diag$(-1,-1,-1)$ = 0.5diag$(-3,1,-1)$ + 0.5diag$(1,-3,-1)$. Take $F \in \mathbb{R}^{3\times 3}$ and realize that $F = 0.5(\lambda I + 2F) + 0.5(-I)$ and that det $(\lambda I + 2F) > 0$ for $\lambda > 0$ large enough.

Suppose that there is $\hat{W}$ convex. We identify $\hat{W}(x;\cdot)$ with its convex extension[6] the whole $\mathbb{R}^{3\times 3}$.

There is $\mu_0 \in (0,1)$ and $F_0, G_0 \in \mathbb{R}^{3\times 3}_+$ such that $\mu_0 F_0 + (1-\mu_0)G_0 \notin \mathbb{R}^{3\times 3}$. Moreover,

$$\sup_{0\le\lambda\le 1} \hat{W}(x, \lambda F_0 + (1-\lambda)G_0) \le \max(W(x,F_0), \hat{W}(x,G_0)) \ .$$

Further, there is $\lambda_0 \in (0,\mu_0]$ such that det $(\lambda F_0+(1-\lambda)G_0) > 0$ for $\lambda \in [0,\lambda_0)$ and det $(\lambda_0 F_0+(1-\lambda_0)G_0) = 0$. But this means that $\lim_{\lambda\to(\lambda_0)_-} \hat{W}(x, \lambda F_0 + (1-\lambda)G_0) = +\infty$, a contradiction.

□

The following example shows that a minimum of an integral functional with a nonconvex term in the "gradient variable" does not necessarily exists.

Example 3.7

$$\textit{minimize } J(y) = \int_0^1 (y^2(x) + (y'^2(x)-1)^2 \, dx \ , y \in W^{1,4}(0,1) \ , \ y(0) = y(1) = 0$$

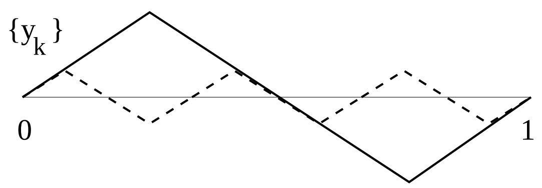

Fig. 3. Possible minimizing sequence. Lipschitz functions with derivatives ± 1 and decreasing amplitudes.

One easily sees that from the sequence depicted above that $\lim J(y_k) = \inf J = 0$. *On the other hand* $J(\text{w}-\lim y_k) = J(0) > 0$ *and no solution exists. The functional J is not sequentially weakly lower semicontinuous. A similar situation appears e.g. in models of shape memory alloys.*

Definition 3.8 *(John M. Ball's polyconvexity) Take $M \subset \mathbb{R}^{3\times 3}$. We say that $W : M \to \mathbb{R}^{3\times 3}$ is polyconvex if there exists a convex function $h : U \to \mathbb{R}$ such that*

$$W(F) = h(F, \mathit{Cof}\ F, \mathit{det}\ F) \ ,$$

where $U = \{(F, \mathit{Cof}\ F, \mathit{det}\ F); F \in M\}$.

It is clear that convex functions are polyconvex. On the other hand, $F \mapsto \det F$, $F \in \mathbb{R}^{3\times 3}$ is not convex but it is polyconvex. Hence, polyconvexity really generalizes the notion of convexity.[7].

Theorem 3.9 *(see [6]) Let $\Omega \subset \mathbb{R}^3$ be a bounded Lipschitz domain. Then for all $p \ge 2$ the mapping $y : W^{1,p}(\Omega;\mathbb{R}^3) \to \mathit{Cof}\, \nabla y \in L^{p/2}(\Omega;\mathbb{R}^{3\times 3})$ is well defined and continuous. Further, let $y^k \to y$ weakly in $W^{1,p}(\Omega;\mathbb{R}^3)$ and $\mathit{Cof}\, \nabla y^k \to H$ weakly in $L^q(\Omega;\mathbb{R}^{3\times 3})$ for some $q \ge 1$. Then $H = \mathit{Cof}\, \nabla y$.*

Proof. The good sense and continuity of the mapping in question follows by Hölder's inequality. Take $y \in C^2(\bar{\Omega};\mathbb{R}^3)$ then

$$(\text{Cof } \nabla y)_{ij} = \frac{\partial}{\partial x_{i+2}}\left(y_{j+2}\frac{\partial y_{j+1}}{\partial x_{i+1}}\right) - \frac{\partial}{\partial x_{i+1}}\left(y_{j+2}\frac{\partial y_{j+1}}{\partial x_{i+2}}\right) \ ,$$

(mod 3, no summation).

Taking, $\theta \in C_0^\infty(\Omega)$ we get (no summation)

$$\int_\Omega (\text{Cof } \nabla y)_{ij}\theta \, \mathrm{d}x = -\int_\Omega y_{j+2}\frac{\partial y_{j+1}}{\partial x_{i+1}}\frac{\partial \theta}{\partial x_{i+2}}\, \mathrm{d}x + \int_\Omega y_{j+2}\frac{\partial y_{j+1}}{\partial x_{i+2}}\frac{\partial \theta}{\partial x_{i+1}}\, \mathrm{d}x \ . \tag{3.13}$$

[6]We first extend $\hat{W}$ by $+\infty$ to $\mathbb{R}^{3\times 3}$ and then we take the pointwise supremum of all affine functions below this infinite extension.

[7]See [7] for polyconvex functions defined on $\mathbb{R}^{m\times n}$.

Both sides of the above identity are continuous in $C^2(\bar{\Omega};\mathbb{R}^3)$ equipped with the $W^{1,2}(\Omega;\mathbb{R}^3)$-norm if θ is fixed. Indeed, e.g.

$$\left|\int_\Omega (\mathrm{Cof}\ \nabla y)_{ij}\theta\,\mathrm{d}x\right| \leq \|(\mathrm{Cof}\ \nabla y)_{ij}\|_{L^1(\Omega)}\|\theta\|_{L^\infty(\Omega)} \leq c(\theta)\|y\|_{W^{1,2}(\Omega;\mathbb{R}^3)}\ .$$

We recall that $C^2(\bar{\Omega};\mathbb{R}^3)$ is dense in $W^{1,2}(\Omega;\mathbb{R}^3)$. Thus, (3.13) remains true in $W^{1,p}(\Omega;\mathbb{R}^3)$, $p\geq 2$. Due to the compact embedding of $W^{1,p}(\Omega;\mathbb{R}^3)$ to $L^r(\Omega;\mathbb{R}^3)$ if $1\leq r<p^*$[8] we can take $r<p^*$ and simultaneously $r^{-1}+p^{-1}\leq 1$. Then we have that $y^k\to y$ strongly in $L^r(\Omega;\mathbb{R}^3)$ and hence for example,

$$\int_\Omega \underbrace{y^k_{j+2}}_{\text{strongly}}\underbrace{\frac{\partial y^k_{j+1}}{\partial x_{i+1}}}_{\text{weakly}}\frac{\partial\theta}{\partial x_{i+2}}\,\mathrm{d}x \to \int_\Omega y_{j+2}\frac{\partial y_{j+1}}{\partial x_{i+1}}\frac{\partial\theta}{\partial x_{i+2}}\,\mathrm{d}x\ .$$

In other words, observing (3.13) we get

$$\lim_{k\to\infty}\int_\Omega ((\mathrm{Cof}\ \nabla y^k)_{ij}-(\mathrm{Cof}\ \nabla y)_{ij})\theta\,\mathrm{d}x = 0$$

and by our assumption $H=\mathrm{Cof}\ \nabla y$.

□

Theorem 3.10 *(see [6]) Let $\Omega\subset\mathbb{R}^3$ be a bounded Lipschitz domain. For any $p\geq 2$ and any $q\geq p/(p-1)$ the mapping $W^{1,p}(\Omega;\mathbb{R}^3)\times L^q(\Omega;\mathbb{R}^{3\times 3})\to L^s(\Omega)$, $1/s=1/p+1/q$, given by (summation over j)*

$$(y,\mathit{Cof}\ \nabla y)\mapsto \mathit{det}\ \nabla y = \frac{\partial y_1}{\partial x_j}(\mathit{Cof}\ \nabla y)_{1j}$$

is well defined and continuous. Moreover, if $y^k\to y$ weakly in $W^{1,p}(\Omega;\mathbb{R}^3)$, $\mathit{Cof}\ \nabla y^k\to H$ in $L^q(\Omega;\mathbb{R}^{3\times 3})$ and $\mathit{det}\ \nabla y^k\to\delta$ in $L^t(\Omega)$, $t\geq 1$ then $H=\mathit{Cof}\ \nabla y$ and $\delta=\mathit{det}\ \nabla y$.

Proof. The continuity of the mapping follows again by Hölder's inequality. Using the Piola identity (cf. Lemma 2.1) we have that for $y\in C^2(\bar{\Omega};\mathbb{R}^3)$

$$\frac{\partial}{\partial x_j}y_1(\mathrm{Cof}\ \nabla y)_{1j} = \det\ \nabla y\ .$$

Thus, for any $\theta\in C_0^\infty(\Omega)$

$$\int_\Omega \frac{\partial}{\partial x_j}y_1(\mathrm{Cof}\ \nabla y)_{1j}\theta\,\mathrm{d}x = -\int_\Omega y_1(\mathrm{Cof}\ \nabla y)_{1j}\frac{\partial\theta}{\partial x_j}\,\mathrm{d}x\ .$$

If $p\geq 3$ we proceed similarly as in the proof of Th. 3.9 because $y\mapsto\int_\Omega\frac{\partial}{\partial x_j}y_1(\mathrm{Cof}\ \nabla y)_{1j}\theta\,\mathrm{d}x$ is continuous with respect to the norm of $W^{1,p}(\Omega;\mathbb{R}^3)$. It remains to prove the case $p\in[2,3)$.

Notice that the bilinear form $W^{1,p}(\Omega;\mathbb{R}^3)\times L^{p'}(\Omega;\mathbb{R}^{3\times 3})\to\mathbb{R}$ defined through

$$(y,H)\mapsto\int_\Omega\frac{\partial}{\partial x_j}y_1H_{1j}\theta\,\mathrm{d}x$$

is continuous if $p'=p/(p-1)$. However,

$$\int_\Omega\frac{\partial}{\partial x_j}y_1H_{1j}\theta\,\mathrm{d}x = -\int_\Omega y_1H_{1j}\frac{\partial\theta}{\partial x_j}\,\mathrm{d}x \tag{3.14}$$

does not generally hold unless for smooth y, $H_{1j}\ \frac{\partial H_{1j}}{\partial x_j}=0$. But this is true for the cofactor as div Cof $\nabla y=0$ if $y\in C^2(\bar{\Omega};\mathbb{R}^3)$. Therefore,$\int_\Omega(\mathrm{Cof}\ \nabla y)_{1j}\frac{\partial\theta}{\partial x_j}\,\mathrm{d}x=0$ for any $\theta\in C_0^\infty(\Omega)$. Similarly as before we see that for $y\in C^2(\bar{\Omega};\mathbb{R}^3)$

$$y\mapsto\int_\Omega(\mathrm{Cof}\ \nabla y)_{1j}\frac{\partial\theta}{\partial x_j}\,\mathrm{d}x$$

is continuous with respect to the $W^{1,p}(\Omega;\mathbb{R}^3)$-norm. Subsequently, $\int_\Omega(\mathrm{Cof}\ \nabla y)_{1j}\frac{\partial\theta}{\partial x_j}\,\mathrm{d}x=0$ for any $y\in W^{1,p}(\Omega;\mathbb{R}^3)$ and any $\theta\in C_0^\infty(\Omega)$.

[8] $p^*=3p/(3-p)$ if $p<3$, or $p^*<+\infty$ if $p\geq 3$.

Having $w \in L^{p'}(\Omega;\mathbb{R}^3)$ satisfying

$$\int_\Omega w_j \frac{\partial \theta}{\partial x_j}\,\mathrm{d}x = 0 \tag{3.15}$$

for all $\theta \in C_0^\infty(\Omega)$ and any $z \in W^{1,p}(\Omega)$ we get for all $\theta \in C_0^\infty(\Omega)$

$$-\int_\Omega z w_j \frac{\partial \theta}{\partial x_j}\,\mathrm{d}x = \int_\Omega (\frac{\partial z}{\partial x_j}) w_j \theta\,\mathrm{d}x\ . \tag{3.16}$$

For fixed w, θ the above relation is linear and continuous in z, so that it is sufficient to consider $z \in C^\infty(\bar{\Omega})$ because of the density argument. Then $z\theta \in C_0^\infty$ and we see that (3.16) is implied by (3.15). Putting $z := y_1$ and $w = (\mathrm{Cof}\ \nabla y)_{1j}$ we get that (3.14) holds for $H = \mathrm{Cof}\ \nabla y$. Using the same (strong,weak) convergence argument as in the Th. 3.9 we have for $1/r + 1/p' \leq 1$

$$\int_\Omega y_1^k (\mathrm{Cof}\ \nabla y^k)_{1j} \frac{\partial \theta}{\partial x_j}\,\mathrm{d}x \to \int_\Omega y_1 (\mathrm{Cof}\ \nabla y)_{1j} \frac{\partial \theta}{\partial x_j}\,\mathrm{d}x\ .$$

This holds if $r < p^* = 3p/(3-p)$ (in our case $2 \leq p < 3$). Hence $\lim_{k\to\infty} \int_\Omega \det\ \nabla y_k \theta\,\mathrm{d}x = \int_\Omega \det\ \nabla y_k \theta\,\mathrm{d}x$ for all $\theta \in C_0^\infty$.[9] The theorem follows. □

Remark 3.11 *(i) Polyconvexity can be defined in $\mathbb{R}^{m\times n}$. If $m = n = 2$ then the convex function h representing a polyconvex function $\hat{W}$ depends on A and $\det A$, i.e. $\hat{W}(A) = h(A, \det A)$, $A \in \mathbb{R}^{2\times 2}$.*
(ii) The convex function in the definition of polyconvexity is not unique. Consider $\hat{W}(A) = |A|^2$ if $A \in \mathbb{R}^{2\times 2}$. Then $h_1(A) = |A|^2$ and $h_2(A, \det A) = (A_{11} + A_{22})^2 + (A_{12} - A_{21})^2 - 2\det A$.

3.4 Rank-one convexity of polyconvex functions

Now we derive an interesting property of polyconvex functions, namely the so-called rank-one convexity which plays a crucial role in the calculus of variations and mathematical elasticity.

Take $A \in \mathbb{R}^{3\times 3}$ and $a, b \in \mathbb{R}^3$. Consider a function $\alpha : \mathbb{R} \to \mathbb{R}$, $\alpha(t) := \det\ (A + ta \otimes b)$. We claim that for all $t \in \mathbb{R}$: $\alpha''(t) = 0$. First notice that if $r, s \in C^2(\mathbb{R})$ then $(rs)'' = r''s + 2r's' + rs''$. We can write

$$\alpha(t) = \det\ (A + ta \otimes b) = \sum_{i=1}^{3} (A + ta \otimes b)_{i1} [\mathrm{Cof}\ (A + t\alpha \otimes b)]_{i1}\ . \tag{3.17}$$

Fix $i \in \{1, 2, 3\}$ and set $r(t) := (A+ta\otimes b)_{i1}$ and $s(t) := [\mathrm{Cof}\ (A+t\alpha\otimes b)]_{i1}$. We immediately see that $r''(t) = 0$, $s''(t) = \mathrm{Cof}\ (ta\otimes b)_{i1} = 0$ because the rank of $a\otimes b$ is at most one, so that every subdeterminant of the order two must be inevitably zero. Finally, we calculate that $r'(t)s'(t) = \sum_{i=1}^3 (a \otimes b)_{i1} \frac{\mathrm{d}}{\mathrm{d}t}[\mathrm{Cof}\ (A + t\alpha \otimes b)]_{i1} = 0$. Altogether, we get that α is affine. Consequently, if $A, B \in \mathbb{R}^{3\times 3}$ such that $\mathrm{rank}(A - B) \leq 1$ (or equivalently that $\exists\, a, b \in \mathbb{R}^3$: $A - B = a \otimes b$) then for all $0 \leq \lambda \leq 1$

$$\det\ (\lambda A + (1-\lambda)B) = \lambda \det\ A + (1-\lambda)\det\ B\ . \tag{3.18}$$

An analogous result holds for "cof" because it is a matrix of 2×2 subdeterminants, i.e.,

$$\mathrm{Cof}\ (\lambda A + (1-\lambda)B) = \lambda \mathrm{Cof}\ A + (1-\lambda)\mathrm{Cof}\ B\ . \tag{3.19}$$

Assuming that $\hat{W} : \mathbb{R}^{3\times 3} \to \mathbb{R} \cup \{+\infty\}$ is polyconvex and finite on $\mathbb{R}_+^{3\times 3}$ then we get for the same λ, A, B as above that

$$\hat{W}(\lambda A + (1-\lambda)B) \leq \lambda \hat{W}(A) + (1-\lambda)\hat{W}(B)\ . \tag{3.20}$$

This property is called rank-one convexity of $\hat{W}$. We just showed that polyconvexity implies rank-one convexity of $\hat{W}$.

[9] We showed that weak convergence of $y_k \to y$ in $W^{1,p}(\Omega;\mathbb{R}^3)$, $p \geq 2$ results in the convergence of $\det \nabla y_k$ to $\det \nabla y_k$ in the sense of distributions. This is an example of *compensated compactness* studied by F. Murat and L. Tartar; cf. [13] for a survey and references therein.

3.5 Examples of hyperelastic materials

3.5.1 St Venant-Kirchhoff material

The response function of the 1st Piola-Kirchhoff stress tensor is

$$\hat{T}(F) = \lambda(\text{tr } E)I + 2\mu E \ ,$$

where $E = (F^\top F - I)/2$ and $\lambda, \mu > 0$ are *Lamé constants.* Then

$$\hat{W}(F) = \frac{\lambda}{2}(\text{tr } E)^2 + \mu|E|^2 \ . \tag{3.21}$$

Equivalently,

$$\hat{W}(F) = -\frac{3\lambda + 2\mu}{4}\text{tr } C + \frac{\lambda + 2\mu}{8}\text{tr } C^2 + \frac{\lambda}{4}\text{tr Cof } C + \text{ const.} \ , \ C = F^\top F$$

Proposition 3.12 *$\hat{W}$ given by (3.21) is not polyconvex.*

Proof. Take $\varepsilon > 0$ and two families of matrices $F_\varepsilon = \varepsilon I$ and $G_\varepsilon = \varepsilon$ diag $(1, 1, 3)$. We observe that

$$\text{Cof } \frac{1}{2}(F_\varepsilon + G_\varepsilon) = \frac{1}{2}(\text{Cof } F_\varepsilon + \text{Cof } G_\varepsilon)$$

and

$$\det \frac{1}{2}(F_\varepsilon + G_\varepsilon) = \frac{1}{2}(\det F_\varepsilon + \det G_\varepsilon) \ .$$

Suppose that $\hat{W}$ is polyconvex. It means that there is a convex function $h : \mathbb{R}^{3\times3} \times \mathbb{R}^{3\times3} \times \mathbb{R}_+ \to \mathbb{R}$ such that $\hat{W}(F) = h(F, \text{Cof } F, \det F)$. By convexity of h it means that

$$\hat{W}(\frac{1}{2}(F_\varepsilon + G_\varepsilon)) \le \frac{1}{2}\hat{W}(F_\varepsilon) + \frac{1}{2}\hat{W}(G_\varepsilon) \ . \tag{3.22}$$

The straightforward calculation shows that (3.22) does not hold if ε is small enough. $\square$

3.5.2 Compressible Mooney-Rivlin material

This material has a stored energy of the form

$$\hat{W}(F) = a|F|^2 + b|\text{Cof } F|^2 + \Gamma(\det F) \ , \tag{3.23}$$

where $a, b > 0$ and $\Gamma(\delta) = c\delta^2 - d\log\delta$, $c, d > 0$.

It can be shown [6, Th. 4.10.2] that

$$\hat{W}(F) = \frac{\lambda}{2}(\text{tr } E)^2 + \mu|E|^2 + O(|E|^3) \ , \ E = (C - I)/2 \ .$$

3.5.3 Compressible neo-Hookean material

This material has a stored energy of the form

$$\hat{W}(F) = a|F|^2 + \Gamma(\det F) \ , \tag{3.24}$$

with the constants as for compressible Mooney-Rivlin materials.

3.5.4 Ogden material

This material has a stored energy of the form

$$\sum_{i=1}^{M} a_i \text{tr } C^{\gamma_i/2} + \sum_{i=1}^{N} b_i \text{tr } (\text{Cof } C)^{\delta_i/2} + \Gamma(\det F) \ , \ F^\top F = C \tag{3.25}$$

$a_i, b_i > 0$, $\lim_{\delta\to0_+}\Gamma(\delta) = +\infty$ for $\Gamma : \mathbb{R}_+ \to \mathbb{R}$ convex growing suitably at infinity.

4 Existence results

4.1 Pure displacement and displacement-traction problem

Theorem 4.1 *(Pure displacement and displacement-traction problem) Let $\Omega \subset \mathbb{R}^3$ be a bounded Lipschitz domain and let $\hat{W}:\bar{\Omega} \times \mathbb{R}^{3\times 3}_+ \to \mathbb{R}$ be a stored energy function with the following properties:*
(a) Polyconvexity: For a.a. $x \in \Omega$ $\exists$ a convex function $h(x,\cdot) : \mathbb{R}^{3\times 3}_+ \times \mathbb{R}^{3\times 3}_+ \times \mathbb{R}_+ \to \mathbb{R}$ such that for all $F \in \mathbb{R}^{3\times 3}_+$

$$h(x, F, Cof\ F, det\ F) = \hat{W}(x, F)\ ; \tag{4.1}$$

the function $h(\cdot, F, H, \delta)$ is measurable for all $(F, H, \delta) \in \mathbb{R}^{3\times 3}_+ \times \mathbb{R}^{3\times 3}_+ \times \mathbb{R}_+$.

(b) For a.a. $x \in \Omega$ $\hat{W}(x, F) \to +\infty$ if $det\ F \to 0_+$.

(c) There are constants α, p, q, r such that $\alpha > 0$, $p \geq 2$, $q \geq \frac{p}{p-1}$, $r > 1$ such that for a.a. $x \in \Omega$ and all $F \in \mathbb{R}^{3\times 3}_+$

$$\hat{W}(x, F) \geq \alpha(|F|^p + |Cof\ F|^q + (det\ F)^r)\ . \tag{4.2}$$

Let $\Gamma = \Gamma_0 \cup \Gamma_1$ be a dA-measurable partition of $\Gamma = \partial\Omega$ with the area of $\Gamma_0 > 0$ and let $y_0 \in W^{1,p}(\Omega; \mathbb{R}^3)$ be given. Let

$$\Phi : \ = \ \{y \in W^{1,p}(\Omega; \mathbb{R}^3);\ Cof\ \nabla y \in L^q(\Omega; \mathbb{R}^{3\times 3}),\ det\ \nabla y \in L^r(\Omega),\ y = y_0 \text{ on } \Gamma_0\ det\ \nabla y > 0 \text{ a.e.}\} \tag{4.3}$$

be nonempty.
Let further $f \in L^\varrho(\Omega; \mathbb{R}^3)$ and $g \in L^\sigma(\Gamma_1; \mathbb{R}^3)$ be such that

$$W^{1,p}(\Omega; \mathbb{R}^3) \to \mathbb{R} : y \mapsto L(y) := \int_\Omega f \cdot y\, dx + \int_\Gamma g \cdot y\, dA$$

is continuous.
If there is $y \in \Phi$ such that $I(y) < +\infty$ then there exists a minimum of

$$I(y) = \int_\Omega \hat{W}(x, \nabla y)\, dx - L(y)\ . \tag{4.4}$$

on Φ.

Proof. Note that $x \mapsto \hat{W}(x, \nabla y(x), \mathrm{Cof}\ \nabla y(x), \det\ \nabla y(x))$ is measurable because $\hat{W}$ is Carathéodory function[10]
Using (c) we get

$$\begin{aligned} I(y) \ &\geq \ \alpha \int_\Omega (|\nabla y|^p + |\mathrm{Cof}\ \nabla y|^q + (\det\ \nabla y)^r)dx + \beta|\Omega| \\ &- \ \|L\|\|y\|_{W^{1,p}(\Omega;\mathbb{R}^3)} \end{aligned} \tag{4.5}$$

Applying the Poincaré inequality[11] we conclude that there are constants $c, d > 0$ and

$$I(y) \geq c(\|y\|^p_{W^{1,p}(\Omega;\mathbb{R}^3)} + \|\mathrm{Cof}\ \nabla y\|^q_{L^q(\Omega;\mathbb{R}^{3\times 3})} + \|\det\ \nabla y\|^r_{L^r(\Omega)}) + d$$

for all $y \in \Phi$ Let $\{y_k\} \subset \Phi$ be a minimizing sequence of I, i.e.,

$$\lim_{k\to\infty} I(y_k) = \inf_\Phi I < +\infty\ .$$

By (4.5) the sequence $\{(y_k, \mathrm{Cof}\ \nabla y_k, \det\ \nabla y_k)\}_{k\in\mathbb{N}}$ is bounded in the reflexive Banach space $W^{1,p}(\Omega; \mathbb{R}^3) \times L^q(\Omega; \mathbb{R}^{3\times 3}) \times L^r(\Omega)$. Hence it has a subsequence weakly converging to $(y, H, \delta) \in W^{1,p}(\Omega; \mathbb{R}^3) \times L^q(\Omega; \mathbb{R}^{3\times 3}) \times L^r(\Omega)$ and by our previous results $H = \mathrm{Cof}\ \nabla y$ and $\delta = \det\ \nabla y$. To sum up, there is a minimizing sequence $\{y_k\}$ s.t. $y_k \to y$ weakly in $W^{1,p}(\Omega; \mathbb{R}^3)$, $\mathrm{Cof}\ \nabla y_k \to \mathrm{Cof}\ \nabla y$ weakly in $L^q(\Omega; \mathbb{R}^{3\times 3})$ and $\det\ \nabla y_k \to \det\ \nabla y$ weakly in $L^r(\Omega)$.

[10] This means that $\hat{W}(x,\cdot) : \mathbb{R}^{3\times 3}_+ \to \mathbb{R}$ is continuous for almost all $x \in \Omega$ and that $\hat{W}(\cdot, F) : \bar{\Omega} \to \mathbb{R}$ is measurable for all $F \in \mathbb{R}^{3\times 3}_+$.
[11] $\int_\Omega |v|^p\, \mathrm{d}x \leq c_1 \left(\int_\Omega |\nabla v|^p\, \mathrm{d}x + \left| \int_{\Gamma_0} v\, \mathrm{d}A \right|^p \right)$

We now show that $y \in \Phi$. We must show that $\det \nabla y > 0$ a.e. in Ω and that $y = y_0$ on Γ_0. As $\det \nabla y_k \to \det \nabla y$ weakly, by Mazur's theorem there are integers $i(k) \geq k$ and numbers $\lambda_s^k \geq 0$, $\sum_{s=k}^{i(k)} \lambda_s^k = 1$, $k \leq s \leq i(k)$ such that if $k \to \infty$

$$d_k := \sum_{s=k}^{i(k)} \lambda_s^k \det \nabla y_s \to \det \nabla y$$

in $L^r(\Omega)$. Thus, a subsequence of $\{d_k\}$ converges a.e. to $\det \nabla y$. Therefore $\det \nabla y \geq 0$. Assume that $\det \nabla y = 0$ on $A \subset \Omega$, $|A| > 0$. We have

$$\int_A |\det \nabla y_k| \,\mathrm{d}x = \int_A \det \nabla y_k \,\mathrm{d}x \to \det \nabla y \,\mathrm{d}x = 0 \ ,$$

hence $\det \nabla y_k \to 0$ strongly in $L^1(A)$. Then we take a subsequence $\{\det \nabla y_m\}$ converging a.e. in A to zero. Let us define a sequence of measurable functions

$$f^m(x) = \hat{W}(x, \nabla y_m(x)) \ .$$

Note that $f^m \geq 0$ and we may apply Fatou's lemma:

$$\int_A \liminf_{m\to\infty} f^m(x) \,\mathrm{d}x \leq \liminf_{m\to\infty} \int_A f^m(x) \,\mathrm{d}x \ .$$

By our assumption, $\liminf f^m(x) = +\infty$ for a.a.$x \in \Omega$, hence $\liminf_{m\to\infty} \int_A f^m(x) \,\mathrm{d}x \to +\infty$. But this contradicts our assumption that $\lim_{m\to\infty} I(y_m) = \inf I < +\infty$. Altogether we proved that $\det \nabla y > 0$ a.e. in Ω. The fact that $y = y_0$ on Γ_0 follows from the compactness of the trace operator $W^{1,p}(\Omega;\mathbb{R}^3) \to L^p(\partial\Omega;\mathbb{R}^3)$.

We finally show that for any subsequence $\{y_m\}$ of $\{y_k\}$ it holds

$$\int_\Omega \hat{W}(x, \nabla y) \,\mathrm{d}x \leq \lim_{m\to\infty} \int_\Omega \hat{W}(x, \nabla y_m) \,\mathrm{d}x \ .$$

By Mazur's theorem we get that for any $m \in \mathbb{N}$ there is $i(m) \geq m$ such that and numbers $\lambda_s^m \geq 0$, $\sum_{s=m}^{i(m)} \lambda_s^m = 1$, $m \leq s \leq i(m)$ such that if $m \to \infty$ then

$$D^m = \sum_{s=m}^{i(m)} \lambda_s^m (\nabla y_s, \mathrm{Cof}\, \nabla y_s, \det \nabla y_s) \to (\nabla y, \mathrm{Cof}\, \nabla y, \det \nabla y)$$

in $L^p(\Omega;\mathbb{R}^{3\times 3}) \times L^q(\Omega;\mathbb{R}^{3\times 3}) \times L^r(\Omega)$.

We may assume[12] that $D^m \to (\nabla y, \mathrm{Cof}\, \nabla y, \det \nabla y)$ a.e. in Ω. By continuity of $h(x,\cdot)$ we get

$$\hat{W}(x, \nabla y(x)) = \lim_{m\to\infty} h\left(x, \sum_{s=m}^{i(m)} \lambda_s^m (\nabla y_s(x), \mathrm{Cof}\, \nabla y_s(x), \det \nabla y_s(x))\right) \ .$$

Fatou's lemma and convexity of $h(x,\cdot)$ yield

$$\begin{aligned}
\int_\Omega \hat{W}(x, \nabla y(x)) \,\mathrm{d}x &\leq \liminf_{m\to\infty} \int_\Omega h(x, \sum_{s=m}^{i(m)} \lambda_s^m (\nabla y_s(x), \mathrm{Cof}\, \nabla y_s(x), \det \nabla y_s(x))) \,\mathrm{d}x \\
&\leq \liminf_{m\to\infty} \sum_{s=m}^{i(m)} \lambda_s^m h(x, \nabla y_s(x), \mathrm{Cof}\, \nabla y_s(x), \det \nabla y_s(x)) \,\mathrm{d}x \\
&= \lim_{m\to\infty} \int_\Omega \hat{W}(x, \nabla y_m(x)) \,\mathrm{d}x \ .
\end{aligned}$$

We used a simple lemma that if $\{a_n\} \subset \mathbb{R}$ converges to $a \in \mathbb{R}$ then $b_m = \sum_{s=m}^{i(m)} \lambda_s^m a_s$ converges to a as well. Recall that $\lambda_s^m \geq 0$, $\sum_{s=m}^{i(m)} \lambda_s^m = 1$.

$\square$

[12] by extracting still further subsequence

4.2 Injectivity condition

Local invertibility of a deformation $y \in C^1(\bar{\Omega};\mathbb{R}^3)$ is ensured by the condition det $\nabla y > 0$ in $\bar{\Omega}$. On the other hand, local invertibility does not entail global one. Indeed, consider $\bar{\Omega}$ a rectangular rod of the length $2\theta l$ contained in the open half-space $x_1 > 0$, e.g. $\Omega = (1,2) \times (-\theta l, \theta l) \times (1,2)$ and the mapping $y : \bar{\Omega} \to \mathbb{R}^3$,

$$y(x_1, x_2, x_3) = (x_1 \cos(x_2/l), x_1 \sin(x_2/l), x_3) \ .$$

We see that det $\nabla y = x_1/l > 0$ but if $\theta \geq \pi$ the injectivity is lost. Indeed, we have $y(x_1, \pi l, x_3) = y(x_1, -\pi l, x_3)$ if $\theta = \pi$. If $\theta > \pi$ we even get self-penetration of the material.

In the following theorem the matrix norm is considered to be the operator norm subordinate to the Euclidean vector norm. It means that $|A| = \sup_{|x|=1} |Ax|$.

Theorem 4.2 *Let $y = \mathrm{id} + u : \Omega \subset \mathbb{R}^n \to \mathbb{R}^n$ be a mapping differentiable at a point $x \in \Omega$. Then if $|\nabla u(x)| < 1$ we have det $\nabla y(x) > 0$. Moreover, if Ω is convex then any mapping $y = \mathrm{id} + u \in C^1(\bar{\Omega};\mathbb{R}^n)$ satisfying $\sup_{x\in\bar{\Omega}} |\nabla u(x)| < 1$ is injective.*

Proof. Let $x \in \Omega$ be a point at which $|\nabla u(x)| < 1$. Then det $(I + t\nabla u(x)) \neq 0$ if $0 \leq t \leq 1$[13]. On the other hand, the function $\delta : [0,1] \to \mathbb{R}$, $\delta(t) = \det\,(I + t\nabla u(x))$ is continuous and therefore $\delta([0,1])$ is a closed interval in reals. As $\delta([0,1])$ contains $1 = \delta(0)$ but not 0 we infer that det $(I + \nabla u(x)) = \delta(1) > 0$. This proves the first statement.

As in the second assertion we suppose that Ω is convex, so is $\bar{\Omega}$. Thus, take $x_1, x_2 \in \bar{\Omega}$ and apply the mean-value theorem to y. We get

$$|y(x_1) - y(x_2) - (x_1 - x_2)| = |u(x_1) - u(x_2)| \leq \sup_{x\in]x_1,x_2[} |\nabla u(x)||x_1 - x_2| \ .$$

Hence $|y(x_1) - y(x_2) - (x_1 - x_2)| < |x_1 - x_2|$ if $x_1 \neq x_2$ and therefore $y(x_1) \neq y(x_2)$. □

In fact, we do not need the injectivity up to the boundary because we admit that the body can touch itself on the boundary. The following condition ensures the injectivity in Ω.

Theorem 4.3 *Let Ω be a bounded domain and $y \in C^1(\bar{\Omega};\mathbb{R}^3)$ be such that det $\nabla y > 0$ in Ω and*

$$\int_\Omega \det \nabla y(x)\, dx \leq |y(\Omega)| \ . \tag{4.6}$$

Then y is injective in Ω. ($|y(\Omega)|$ is the three-dimensional Lebesgue measure of $y(\Omega)$.)

Proof. Suppose that there are $x_1, x_2 \in \Omega$ such that $y(x_1) = y(x_2)$. Since $\nabla y(x_1)$ and $\nabla y(x_2)$ are invertible there are by the implicit function theorem open sets $U \subset \Omega, V \subset \Omega$, $U \cap V = \emptyset$ and $W' \subset y(\Omega)$ such that $\hat{x} \in U$, $\tilde{x} \in V$, and $y(\hat{x}) = y(\tilde{x}) \in W'$ and $y : U \to W'$ and $y \to W'$ are C^1-diffeomorphisms[14]. Hence, $\#y^{-1}(x) \geq 2$ if $x \in W'$. Since[15]

$$\int_{y(\Omega)} \#y^{-1}(x')\,\mathrm{d}x' = \int_\Omega \det \nabla y(x)\,\mathrm{d}x$$

whenever one of the integrals exists (in our case at least the right-hand side integral exists) and because $|W'| > 0$ (W' is open) it follows that

$$|y(\Omega)| = \int_{y(\Omega)} \mathrm{d}x' < \int_{y(\Omega)} \#y^{-1}(x')\,\mathrm{d}x' = \int_\Omega \det \nabla y(x)\,\mathrm{d}x \ .$$

But this contradicts (4.6). Hence, $y(x_1) \neq y(x_2)$. □

We are going to show that the injectivity condition can be imposed on any admissible deformation and an existence result similar to Theorem 4.1 still holds.

[13]You may prove this easily by contradiction.

[14]i.e. injective and $y^{-1} \in C^1(W';U)$, $C^1(W';V)$

[15]Namely, we have the general substitution formula $\int_{y(\Omega)} f(x')\#y^{-1}(x')\,\mathrm{d}x' = \int_\Omega f(y(x))\det \nabla y(x)\,\mathrm{d}x$. If y is injective then $\#y^{-1}(x') = 1$.

Theorem 4.4 *(Pure displacement and displacement-traction problem with injectivity) Let $\Omega \subset \mathbb{R}^3$ be a bounded Lipschitz domain and let $\hat{W}:\bar{\Omega}\times\mathbb{R}^{3\times3}_+ \to \mathbb{R}$ be a stored energy function with the following properties: (a) Polyconvexity: For a.a. $x \in \Omega$ $\exists$ a convex function $h(x,\cdot) : \mathbb{R}^{3\times3}_+ \times \mathbb{R}^{3\times3}_+ \times \mathbb{R}_+ \to \mathbb{R}$ such that for all $F \in \mathbb{R}^{3\times3}_+$*

$$h(x, F, Cof\, F, det\, F) = \hat{W}(x, F) \; ; \tag{4.7}$$

the function $h(\cdot, F, H, \delta)$ is measurable for all $(F, H, \delta) \in \mathbb{R}^{3\times3}_+ \times \mathbb{R}^{3\times3}_+ \times \mathbb{R}_+$.

(b) For a.a. $x \in \Omega$ $\hat{W}(x, F) \to +\infty$ if $det\, F \to 0_+$.

(c) There are constants α, p, q, r such that $\alpha > 0$, $p > 3$, $q \geq \frac{p}{p-1}$, $r > 1$ such that for a.a. $x \in \Omega$ and all $F \in \mathbb{R}^{3\times3}_+$

$$\hat{W}(x, F) \geq \alpha(|F|^p + |Cof\, F|^q + (det\, F)^r) \; . \tag{4.8}$$

Let $\Gamma = \Gamma_0 \cup \Gamma_1$ be a dA-measurable partition of $\Gamma = \partial\Omega$ with the area of $\Gamma_0 > 0$ and let y_0 be a measurable function such that

$$\begin{aligned}\tilde{\Phi} : \; = \; & \Big\{ y \in W^{1,p}(\Omega;\mathbb{R}^3);\;\; Cof\, \nabla y \in L^q(\Omega;\mathbb{R}^{3\times3}),\;\; det\, \nabla y \in L^r(\Omega),\;\; y = y_0 \;\; on\; \Gamma_0 \\ & det\, \nabla y > 0 \;\; a.e., (4.6) \;\; holds \Big\}\end{aligned} \tag{4.9}$$

is nonempty.

Let further $f \in L^\varrho(\Omega;\mathbb{R}^3)$ and $g \in L^\sigma(\Gamma_1;\mathbb{R}^3)$ be such that

$$W^{1,p}(\Omega;\mathbb{R}^3) \to \mathbb{R} : y \mapsto L(y) := \int_\Omega f \cdot y \, dx + \int_\Gamma g \cdot y \, dA$$

is continuous.

Finally, we assume that there is $y \in \Phi$ such that $I(y) < +\infty$ where I is given by (4.4). Then there exists a minimum of $I(y)$ on Φ and the minimizer is injective almost everywhere.

Proof. We only show that y obtained in the proof of Theorem 4.1 satisfies the additional condition. AS $p > 3$ we know by the embedding theorem that $y \in C(\bar{\Omega};\mathbb{R}^3)$. So a subsequence $\{y_k\}$ of the minimizing sequence converges uniformly to y. We also have that $y(\bar{\Omega})$ is compact and therefore measurable there is for any $\varepsilon > 0$ an open set O_ε such that $y(\bar{\Omega}) \subset O_\varepsilon$ and $|O_\varepsilon \setminus y(\bar{\Omega})| < \varepsilon$. We claim that there is a number $\delta(\varepsilon) > 0$ such that

$$\bigcup_{x \in y(\bar{\Omega})} B(x, \delta(\varepsilon)) \subset O_\varepsilon \; .$$

For if not, there is $\varepsilon > 0$ and sequences $\{x_k\} \in y(\bar{\Omega})$, $\{\hat{x}_k\} \notin O_\varepsilon$ and $\delta_k \to 0$ if $k \to \infty$ such that $|\hat{x}_k - x_k| < \delta_k$. By compactness we may suppose that $x_k \to x \in y(\bar{\Omega})$ and we would have also $y_k \to x$ but this means that $x \in \mathbb{R}^3 \setminus O_\varepsilon$ but it is not possible because $y(\bar{\Omega}) \subset O_\varepsilon$.

Therefore,

$$\bigcup_{x \in y(\bar{\Omega})} B(x, \delta(\varepsilon)) \subset O_\varepsilon$$

for some $\delta(\varepsilon) > 0$ and there is k_0 such that $y_k(\bar{\Omega}) \subset O_\varepsilon$ if $k \geq k_0$ because y_k converges uniformly. As $y_k \in \tilde{\Phi}$ we have for $k \geq k_0$

$$\int_\Omega \det\, \nabla y_k(x)\, \mathrm{d}x \leq |y_k(\bar{\Omega})| \leq |O_\varepsilon| \; .$$

By the weak convergence of det we also have

$$\int_\Omega \det\, \nabla y(x)\, \mathrm{d}x = \lim_{k\to\infty} \int_\Omega \det\, \nabla y_k(x)\, \mathrm{d}x \leq |O_\varepsilon| \; .$$

But $|O_\varepsilon| = |y(\bar{\Omega})| + |O_\varepsilon \setminus y(\bar{\Omega})|$ and the arbitrariness of $\varepsilon > 0$ yields

$$\int_\Omega \det\, \nabla y(x)\, \mathrm{d}x \leq |y(\bar{\Omega})| = |y(\Omega)| \; .$$

Using a generalization of (4.6) for Sobolev maps [6] we have

$$|y(\Omega)| = \int_{y(\Omega)} \mathrm{d}x^y \leq \int_{y(\Omega)} \#y^{-1}(x^y)\, \mathrm{d}x^y = \int_\Omega \det\, \nabla y(x)\, \mathrm{d}x \leq |y(\Omega)| \; .$$

Hence, $\#y^{-1}(x^y) = 1$ for almost all $x^y \in y(\Omega)$. However, as $p > n$, this implies that y is injective a.e. in Ω. $\square$

5 Linearized elasticity in brief

We know that the *Almansi tensor* E is defined as

$$E = \frac{C - \mathbb{I}}{2} \ ,$$

where $C = F^T F$ is the right Cauchy-Green tensor. If we write $F = \nabla u + \mathbb{I}$, where u is a displacement we get for $|\nabla u|$ small that

$$E = \frac{1}{2}(\nabla u + (\nabla u)^T) + o(|\nabla u|) \ .$$

Then we define the linearized strain tensor, also called *small strain tensor*,, as

$$e(u) = \frac{1}{2}(\nabla u + (\nabla u)^T) \tag{5.1}$$

If we write the boundary value problem in nonlinear elasticity in terms of the 2nd Piola-Kirchhoff stress tensor $\hat{\Sigma}$ then after employing the linearized strain tensor we get the symmetric stress tensor τ

$$\tau = \mathbb{C}e(u) \ ,$$

where $\mathbb{C}$ is the 4th-order tensor of elastic constants. In fact, it can be shown that there are only 21 independent constants in $\mathbb{C}$. If the material is homogeneous and isotropic $\mathbb{C}$ reduces to two positive quantities λ and μ called *Lamé constants* (both in [Pa]) and in this case

$$\tau = \lambda \mathrm{tr}\ e(u)\mathbb{I} + 2\mu e(u) \ . \tag{5.2}$$

The Lamé constants can be equivalently expressed in terms of the Young modulus and the Poisson ratio. We have the following assertion.

Theorem 5.1 *Let $\Gamma_0, \Gamma_1 \subset \partial\Omega$ be disjoint and of a positive Hausdorff measure Finding a solution u of the linear boundary value problem*

$$-\mathrm{div}\ \tau = f \, in\ \Omega$$

$$u = 0 \ on\ \Gamma_0$$

$$\tau\nu = g \ on\ \Gamma_1$$

if formally equivalent to finding a solution u of the equation

$$B(u,v) = L(v) \ for\ all\ v \in V$$

where

$$B(u,v) = \int_\Omega \mathbb{C}e(u) : e(v)$$

and

$$L(v) = \int_\Omega f \cdot v \, dx + \int_{\Gamma_1} g \cdot v \, dS \ .$$

V denotes the space of smooth enough vector-valued functions $\bar{\Omega} \to \mathbb{R}^3$ vanishing on Γ_0.

Proof. We use the Green formula specialized for a symmetric tensor S[16]:

$$\begin{aligned}\int_\Omega \mathrm{div}\ S \cdot v \, \mathrm{d}x &= -\int_\Omega S : \nabla v \, \mathrm{d}x + \int_{\Gamma_1} S\nu \cdot v \, \mathrm{d}S \\ &= -\int_\Omega S : e(v) \, \mathrm{d}x + \int_{\Gamma_1} S\nu \cdot v \, \mathrm{d}S \ .\end{aligned}$$

The rest of the proof is analogous to the one of Theorem 2.7. $\square$

Theorem 5.2 *Let V be a Banach space. Let $L : V \to \mathbb{R}$ be a continuous linear form and let $B : V \times V \to \mathbb{R}$ be a symmetric continuous bilinear form that is V-elliptic in the sense that there is $\beta > 0$ such that $B(u,u) \geq \beta\|u\|^2$ for all $u \in V$. Then the problem of finding $u \in V$ such that $B(u,v) = L(v)$ for all $v \in V$ has exactly one solution which is a unique minimizer of*

$$J(v) = \frac{1}{2}B(v,v) - L(v)$$

over V.

[16] Realize that if S is symmetric then $S : A = \frac{1}{2}S : (A + A^T)$

Proof. Using V-ellipticity and the continuity of B we get

$$\beta\|v\|^2 \leq B(v,v) \leq \|B\|\|v\|^2 \ .$$

Hence, B is an inner product over V making it just a Hilbert space with the norm $\|v\|_H = \sqrt{B(v,v)}$. Moreover, $\|\cdot\|_H$ and $\|\cdot\|$ are equivalent. By the Riesz theorem there is only one $u \in V$ such that $L(v) = B(u,v)$ for all $v \in V$. Thus u is the unique solution.

Notice that

$$J(u+v) - J(u) = B(u,v) - L(v) + \frac{1}{2}B(v,v) \ .$$

Therefore , if $B(u,v) = L(v)$ then $J(u+v) - J(u) \geq 0$ and u is a minimizer.

Conversely, if u is a minimizer of J and $v \in V$ is such that $B(u,v) - L(v) \neq 0$ then without loss of generality we may suppose that $B(u,v) - L(v) < 0$ (replace v by $-v$ if necessary). Then for $\theta > 0$ small enough we would have

$$0 > J(u+\theta v) - J(u) = \theta(B(u,v) - L(v)) + \frac{\theta^2}{2}B(v,v) \ ,$$

a contradiction. □

We should now decide in which spaces we will seek a solution to the problem stated in Theorem 5.1. We see that the bilinear form B is continuous with respect to the norm $\|\cdot\|_{W^{1,2}}$. Therefore, a natural candidate for V is

$$V = \{v \in W^{1,2}(\Omega;\mathbb{R}^3); v = 0 \text{ on } \Gamma_0\} \ . \tag{5.3}$$

Further we will require the following condition ensuring V-ellipticity of B:

$$\exists \alpha > 0 : \ \forall v \in V : \ B(v,v) \geq \alpha\|e(v)\|_{L^2(\Omega;\mathbb{R}^{3\times 3})} \ . \tag{5.4}$$

The V-ellipticity of B follows if we show that the seminorm $v \mapsto \|e(v)\|_{L^2}$ is a norm equivalent to $\|\cdot\|_{W^{1,2}}$ on V.

This result is a consequence of *Korn's* inequality:

Theorem 5.3 *(Korn's inequality, 1907) Let $\Omega \subset \mathbb{R}^3$ be a bounded domain with a smooth boundary. Then there is a constant $C > 0$ such that for each $v \in W^{1,2}(\Omega;\mathbb{R}^3)$*

$$\|v\|^2_{W^{1,2}(\Omega;\mathbb{R}^3)} \leq C(\|v\|^2_{L^2(\Omega;\mathbb{R}^3)} + \|e(v)\|^2_{L^2(\Omega;\mathbb{R}^{3\times 3})}) \ . \tag{5.5}$$

Hence, the norm $v \mapsto \sqrt{\|v\|^2_{L^2(\Omega;\mathbb{R}^3)} + \|e(v)\|^2_{L^2(\Omega;\mathbb{R}^{3\times 3})}}$ is equivalent to $\|\cdot\|_{W^{1,2}}$ on $W^{1,2}$.

Remark 5.4 *Assume that $u \in W^{1,2}_0(\Omega;\mathbb{R}^3)$ is smooth. Then we have*

$$\begin{aligned} \int_\Omega e(u) : e(u)\,dx &= \frac{1}{4}\int_\Omega \left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)\,dx \\ &= \frac{1}{2}\int_\Omega \sum_{i,j=1}^{3}\left(\frac{\partial u_i}{\partial x_j}\right)^2\,dx + \frac{1}{2}\int_\Omega \left(\frac{\partial u_i}{\partial x_j}\right)\left(\frac{\partial u_j}{\partial x_i}\right)\,dx \ . \end{aligned}$$

Moreover, applying twice integration by parts to the second term on the RHS we get (keep in mind that $u = 0$ on $\partial\Omega$)

$$\int_\Omega \left(\frac{\partial u_i}{\partial x_j}\right)\left(\frac{\partial u_j}{\partial x_i}\right)\,dx = -\int_\Omega u_i\left(\frac{\partial^2 u_j}{\partial x_i \partial x_j}\right)\,dx = \int_\Omega \left(\frac{\partial u_i}{\partial x_i}\right)\left(\frac{\partial u_i}{\partial x_i}\right)\,dx \geq 0 \ .$$

Hence,

$$\int_\Omega e(u) : e(u)\,dx \geq \frac{1}{2}\int_\Omega \sum_{i,j=1}^{3}\left(\frac{\partial u_i}{\partial x_j}\right)^2\,dx \ ,$$

i.e., the L^2 norm of the symmetric part of the gradient controls the L^2 norm of the whole gradient. This is surprising as the symmetric part has only 6 components whole the whole gradient has 9 components. The calculation above extends to the whole $W^{1,2}_0(\Omega;\mathbb{R}^3)$ by density of smooth maps.

To prove the general case, we will need the following lemma, cf. [8].

Lemma 5.5 *(Lion's lemma) Let $\Omega \subset \mathbb{R}^3$ be a bounded domain with a smooth boundary. Let $v \in H^{-1}(\Omega)$ and $\partial v/\partial x_i \in H^{-1}(\Omega)$ for all i. Then $v \in L^2(\Omega)$.*

Proof of Th. 5.3. We show that $W^{1,2}(\Omega;\mathbb{R}^m)$ coincides with

$$K(\Omega;\mathbb{R}^3) = \{v \in L^2(\Omega;\mathbb{R}^3); e(v) \in L^2(\Omega;\mathbb{R}^{3\times 3})\} .$$

Clearly, $K(\Omega;\mathbb{R}^3) \supset W^{1,2}(\Omega;\mathbb{R}^m)$, however, the opposite inclusion is far not obvious. The norm $v \mapsto \sqrt{\|v\|^2_{L^2(\Omega;\mathbb{R}^3)} + \|e(v)\|^2_{L^2(\Omega;\mathbb{R}^{3\times 3})}}$ makes $K(\Omega;\mathbb{R}^3)$ a Hilbert space. We have

$$\frac{\partial^2 v_i}{\partial x_j \partial x_k} = \frac{\partial}{\partial x_j} e_{ik}(v) + \frac{\partial}{\partial x_k} e_{ij}(v) - \frac{\partial}{\partial x_i} e_{jk}(v) . \tag{5.6}$$

So ,if $v \in K(\Omega;\mathbb{R}^3)$ then $e_{ij}(v) \in L^2(\Omega)$ and $\frac{\partial}{\partial x_k} e_{ij}(v) \in H^{-1}(\Omega)$. Hence $\frac{\partial^2 v_i}{\partial x_j \partial x_k} \in H^{-1}(\Omega)$ and by Lion's lemma $\nabla v \in L^2(\Omega;\mathbb{R}^{3\times 3})$. Thus, we see that $K(\Omega;\mathbb{R}^3) = W^{1,2}(\Omega;\mathbb{R}^m)$ (element-wise). Moreover, the embedding $W^{1,2}(\Omega;\mathbb{R}^m)$ into $K(\Omega;\mathbb{R}^3)$ is continuous and surjective. The proof is finished by an application of the closed-graph theorem to the identity map: $W^{1,2}(\Omega;\mathbb{R}^m) \to K(\Omega;\mathbb{R}^3)$. Notice that the identity is the bijection $W^{1,2}(\Omega;\mathbb{R}^m) \to K(\Omega;\mathbb{R}^3)$ which is continuous hence is inverse is too[17]. □

Theorem 5.6 *Let $\Omega \subset \mathbb{R}^3$ be a domain, let $\Gamma_0 \subset \partial\Omega$ be dS-measurable , meas$(\Gamma_0) > 0$. Then V is a closed subspace of $W^{1,2}(\Omega;\mathbb{R}^3)$ and $v \mapsto \|e(v)\|_{L^2}$ is a norm equivalent to the $W^{1,2}(\Omega;\mathbb{R}^3)$ norm on V.*

Proof. Closeness of V follows from the continuity of the trace operator. Let us show that $v \mapsto \|e(v)\|_{L^2}$ is a norm on V. Let $e(v) = 0$. Then (5.6) implies that v is linear in x and we get that[18]

$$v(x) = c + d \times x$$

for fixed vectors $c, d \in \mathbb{R}^3$. Now it is easy to see that if $v \in V$ and $e(v) = 0$ then $v = 0$. Namely, consider $S := \{x \in \Omega;\ v(x) = 0\}$. Then

$$S = \begin{cases} \emptyset & \text{if } d = 0 \text{ and } c \neq 0 \\ \emptyset & \text{if } d \neq 0 \text{ and } c \cdot d \neq 0 \\ x = (d \times c)/|d|^2 + dt, t \in \mathbb{R} & \text{if } c \cdot d = 0,\ d \neq 0. \end{cases}$$

It is clear that $\|e(v)\|_{L^2} \le C\|v\|_{W^{1,2}}$ for $v \in V$. Suppose now that there is $\{v_k\} \subset V$ such that $\|v_k\|_{W^{1,2}} = 1$ and $\|e(v_k)\|_{L^2} \to 0$. Thus by the compact embedding v_k converges in $L^2(\Omega;\mathbb{R}^3)$ (up to a subsequence) and because $e(v_k) \to 0$ in $L^2(\Omega;\mathbb{R}^{3\times 3})$ we get that the sequence $\{v_k\}$ is Cauchy with respect to the norm $\|v\|_{L^2(\Omega;\mathbb{R}^3)} + \|e(v)\|_{L^2(\Omega;\mathbb{R}^{3\times 3})}$. By Korn's inequality this norm is equivalent to the norm on $W^{1,2}$ and therefore it converges to $v \in V$. Then $e(v) = 0$ and by the first part $v = 0$. This in not possible because we supposed that $\|v_k\|_{W^{1,2}} = 1$. □

Theorem 5.7 *Let $\Omega \subset \mathbb{R}^3$ be as in the previous theorem. Let $f \in L^{6/5}(\Omega;\mathbb{R}^3)$, $g \in L^{4/3}(\Gamma_1)$, and (5.4) hold. Then there is one and only one solution $u \in V$ of the variational equation: for all $v \in V$ $B(u,v) = L(v)$. Moreover, u is a minimizer of J.*

Proof. It is an easy consequence of Theorems 5.1,5.2, and 5.6. □

6 Is there a linear constitutive theory in finite elasticity?

It is an interesting hypothesis whether one can derive linearized elasticity as an infinitesimal theory based on linear constitutive laws valid in finite elastisticity. A negative answer to this question was given by Fosdick and Serrin in [9]. Let $U := \{G \in \mathbb{R}^{3\times 3};\ \det\,(\mathbb{I} + G) > 0\}$. Then we have for the first Piola-Kirchoff stress tensor $T(\mathbb{I} + G) = \bar{T}(G)$ which defines $\bar{T}$ in a neighborhood of the origin. In view of (3.4) we have for any $R \in \mathrm{SO}(3)$ that

$$T(R + RG) = \bar{T}(RG + R - \mathbb{I}) = RT(\mathbb{I} + G) = R\bar{T}(G) .$$

[17] If $A : X \to Y$ is a bijective continuous linear operator between the Banach spaces X and Y, then the inverse operator $A^{-1} : Y \to X$ is continuous as well (this is sometimes called the bounded inverse theorem); cf. [12, Cor. 2.12].

[18] Here we use that for any skew symmetric tensor $T \in \mathbb{R}^{3\times 3}$ there is a vector $b \in \mathbb{R}^3$, called axial vector of T, such that $Ta = b \times a$ for any $a \in \mathbb{R}^3$.

Assume that $\bar{T}$ is linear, i.e., $\bar{T}_{ij}(G) = a_{ijkl}G_kl$ for $i, j, k, l = 1, 2, 3$. Hence, setting $G = 0$ we get for arbitrary $R \in \mathrm{SO}(3)$. $\bar{T}(R - \mathbb{I}) = 0$. We take $R_\mu := \exp(\mu A)$ for $A \in \mathbb{R}^{3\times3}$ skew symmetric and $\mu \in \mathbb{R}$. Then $\exp(\mu A) = \mathbb{I} + \sum_{i=1}^{\infty} \frac{\mu^i A^i}{i!}$. Therefore, it yields

$$\frac{\mathrm{d}^k}{\mathrm{d}\mu^k}\bar{T}(R_\mu - \mathbb{I}) = \bar{T}\big(\frac{\mathrm{d}^k}{\mathrm{d}\mu^k}R_\mu\big) = 0 \ .$$

Setting $\mu = 0$ and $k = 1, 2$ we get $\bar{T}(W) = \bar{T}(W^2) = 0$. As $W \in \mathbb{R}^{3\times3}$ is skew we have $W^2 = b \otimes b - |b|^2\mathbb{I}$ where b is the axial vector of W. Putting $b := \mathbf{e}_i$ for $i = 1, 2, 3$ and due to $\mathbf{e}_i \otimes \mathbf{e}_i = 3\mathbb{I}$ we get from the linearity of $\bar{T}$ that $\bar{T}(-2\mathbb{I}) = 0$ which implies that $\bar{T}(b \otimes b) = 0$ for any b. Consequently, as any symmetric matrix E can be ritten as $E = \lambda_i v_i \otimes v_i$ where $\lambda_i \in \mathbb{R}$ and v $v_i \in \mathbb{R}^3$ are eigenvalues and eigenvectors of v respectively. Therefore $\bar{T}(E) = 0$. This, together with $\bar{T}(W) = 0$ for any skew W yields $\bar{T} = 0$. This means that there is not a nonzero linear function assigning to a displacement gradient the first Piola-Kirchhoff stress tensor.

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