Discontinuous Petrov-Galerkin Methods

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Motivation

- discontinuous Petrov-Galerkin (dPG) methods invented by L. Demkowicz, J. Gopalakrishnan (2010)
- motivated by search for optimal test functions
- dPG can be viewed as mixed method with nonstandard test space (typically broken test functions) or minimal residual method
- main idea: too many test functions ensure an inf-sup condition, let computer handle redundant dofs
- applications: linear elasticity, Stokes equations, Maxwell, ...

Outline

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- $F \in Y^\star := \{G : Y \to \mathbb{R} \mid G$ linear and bounded$\}$
- for any $G \in Y^\star$, let $\|G\|_{Y^\star} := \sup_{y \in \mathcal{S}(Y)} G(y)$

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Suppose that the problem

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is well-posed with unique solution. The standard theory on mixed finite element methods [Boffi, Brezzi, Fortin] shows that this is equivalent to the inf-sup condition

$$0 < \beta := \inf_{x \in \mathcal{S}(X)} \sup_{y \in \mathcal{S}(Y)} b(x, y) \tag{H1}$$

and non-degeneracy

$$\{0\} = N := \{y \in Y \mid b(\bullet, y) = 0 \text{ in } X\}.$$

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The Discrete Problem?

Question: How to discretize this problem? For any chosen finite-dimensional subspaces $X_h \subseteq X$, $Y_h \subseteq Y$, the well-posedness of the discrete problem

$$x_h \in X_h : \quad b(x_h, \bullet) = F \quad \text{in } Y_h$$

is equivalent to the discrete inf-sup condition

$$0 < \beta_h := \inf_{x_h \in \mathcal{S}(X_h)} \sup_{y_h \in \mathcal{S}(Y_h)} b(x_h, y_h) \tag{H2}$$

and non-degeneracy

$$\{0\} = N_h := \{y_h \in Y_h \,|\, b(\bullet, y_h) = 0 \text{ in } X_h\}.$$

The Discrete Problem? (2)

$$0 < \beta_h := \inf_{x_h \in \mathcal{S}(X_h)} \sup_{y_h \in \mathcal{S}(Y_h)} b(x_h, y_h)$$

$$\{0\} = N_h := \{y_h \in Y_h \mid b(\bullet, y_h) = 0 \text{ in } X_h\}$$

Spaces X_h and Y_h need to be well-balanced to satisfy these two conditions.
For fixed X_h,

- a big Y_h makes it easier to satisfy discrete inf-sup condition
- but harder to guarantee discrete non-degeneracy.

The idea of discontinuous Petrov-Galerkin methods (dPG methods):

- choose only the discrete trial space $X_h \subseteq X$,
- compute a discrete test space $\subseteq Y$ with (H2) and non-degeneracy.

Trial-to-Test-Operator

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Define the trial-to-test-operator $T : X \to Y$ by

$$(Tx, y)_Y = b(x, y) \quad \text{for any } y \in Y.$$

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For a fixed choice of discrete trial space $X_h \subseteq X$ the *idealized dPG method* utilizes the discrete test space $T(X_h) \subseteq Y$.

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continuous problem $x \in X: \quad b(x, \bullet) = F \text{ in } Y$

dPG methods

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- Remedy: discrete trial-to-test-operator $T_h : X \rightarrow Y_h$ for some discrete space $Y_h \subseteq Y$. Define $T_h : X \rightarrow Y_h$ similar to the continuous operator T by

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- X_h and $T_h(X_h) \subseteq Y_h$ satisfy non-degeneracy but not necessarily the discrete inf-sup condition
- continuous inf-sup condition holds → Y_h big enough leads to the discrete inf-sup condition

$$0 < \inf_{x_h \in \mathcal{S}(X_h)} \sup_{y_h \in \mathcal{S}(Y_h)} b(x_h, y_h) = \inf_{x_h \in \mathcal{S}(X_h)} \sup_{y_h \in \mathcal{S}(T_h(X_h))} b(x_h, y_h).$$

dPG methods...seen as minimization

	Variational formulation		Minimal Residual Method
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A priori estimate

Theorem

Suppose (H1) and (H2) hold. Then the solution $x_h \in X_h$ to the practical dPG method satisfies

$$\|x - x_h\|_X \leq \frac{\|b\|}{\beta_h} \min_{\xi_h \in X_h} \|x - \xi_h\|_X.$$

A posteriori estimate for idealized dPG

Idealized dPG method has built-in a posteriori estimate

$$\beta \|x - x_h\|_X \leq \|b(x_h, \bullet) - F\|_{Y^\star} \leq \|b\| \|x - x_h\|_X,$$

i.e.

$$\|x - x_h\|_X \approx \|b(x_h, \bullet) - F\|_{Y^\star}.$$

Is such an estimate possible for the practical dPG method as well?

Fortin Operator

For an a posteriori estimate of the practical dPG method, we need some Fortin operator $\Pi : Y \to Y_h$.

$$\text{Existence of } \Pi : Y \to Y_h \text{ linear and bounded projection with } b(X_h, (1 - \Pi) Y) = 0 \tag{H3}$$

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(H3) $\implies$ *(H2).*

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Suppose (H1) holds. Then (H2) and (H3) are equivalent.

Proof.

(H3) $\Longrightarrow$ (H2). Continuity of Π implies $1/\|y\|_Y \leq \|\Pi\|/\|\Pi y\|_Y$ and (H1) shows

$$0 < \beta \leq \inf_{x_h \in \mathcal{S}(X_h)} \sup_{y \in Y, y \neq 0} \frac{b(x_h, y)}{\|y\|_Y} \leq \|\Pi\| \inf_{x_h \in \mathcal{S}(X_h)} \sup_{y \in Y, y \neq 0} \frac{b(x_h, \Pi y)}{\|\Pi y\|_Y}$$
$$\leq \|\Pi\| \inf_{x_h \in \mathcal{S}(X_h)} \sup_{y_h \in Y_h, y_h \neq 0} \frac{b(x_h, y_h)}{\|y_h\|_Y}.$$

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(H2) $\Longrightarrow$ *(H3).* Condition (H2) and non-degeneracy imply that $B : T_h(X_h) \to X_h^\star, BTx_h := b(\bullet, Tx_h)$ is an isomorphism. Let $G : Y \to X_h^\star, Gy := b(\bullet, y)$. Define $\Pi := B^{-1} \circ G : Y \to T_h(X_h)$ linear and bounded.

- $\Pi(Tx_h) = B^{-1} b(\bullet, Tx_h) = B^{-1} BTx_h = Tx_h$
- $y \in Y$ satisfies $b(\bullet, y) = G(y) = B(\Pi y) = b(\bullet, \Pi y)$ in X_h □

Theorem

Suppose (H1) and (H3) hold. Then any $\xi_h \in X_h$ satisfies

$$\begin{aligned}\beta\|x - x_h\|_X &\leq \|\Pi\|\|b(x_h, \bullet) - F\|_{Y_h^\star} + \|F \circ (1 - \Pi)\|_{Y^\star} \\ &\leq 2\|b\|\|\Pi\|\|x - x_h\|_X.\end{aligned}$$

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Proof.
(H1) and (H3) imply $\beta\|x - x_h\|_X \leq \|F - b(\xi_h, \bullet)\|_{Y^\star}$ and

$$\|F - b(\xi_h, \bullet)\|_{Y^\star} \leq \sup_{y \in \mathcal{S}(Y)} F((1 - \Pi)y) + \sup_{y \in \mathcal{S}(Y)} F(\Pi y) - b(\xi_h, \Pi y).$$

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Second estimate follows from continuity of b, $\|\Pi\| = \|1 - \Pi\|$ and

$$F((1 - \Pi)y) = b(x, (1 - \Pi)y) = b(x - \xi_h, (1 - \Pi)y).$$

□

Computation of dPG Solution

- $\{\Phi_1, \ldots, \Phi_J\}$ basis of X_h
- $\{\Psi_1, \ldots, \Psi_K\}$ basis of Y_h
- $A \in \mathbb{R}^{K \times J}$, $A_{kj} := b(\Phi_j, \Psi_k)$, $k = 1, \ldots, K$, $j = 1, \ldots, J$, matrix for bilinear form
- $M \in \mathbb{R}^{K \times K}$, $M_{k\ell} := (\Psi_k, \Psi_\ell)_Y$, $k, \ell = 1, \ldots, K$, matrix for scalar product on Y
- $b \in \mathbb{R}^K$, $b_k := F(\Psi_k)$, $k = 1, \ldots, K$, vector for right-hand side

For $x \in \mathbb{R}^J$ coefficient vector for solution $x_h = \sum_{j=1}^J x_j \Phi_j$, the linear system of equation reads

$$A^\top M^{-1} A x = A^\top M^{-1} b.$$

Bigger Y_h guarantees inf-sup condition, but computation will be more expensive! Concept of broken test functions leads to block-diagonal M.

Theorem (Traces of H^1-functions)

For $U \subseteq \mathbb{R}^n$ open and bounded Lipschitz domain, there exists continuous, linear $\gamma_0 : H^1(U) \to L^2(\partial U)$ with

$$\gamma_0 w = w|_{\partial U} \quad \text{for all } w \in H^1(U) \cap C^0(\overline{U}).$$

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Set

$$H^{1/2}(\partial U) := \gamma_0(H^1(U)) \text{ and } H^{-1/2}(\partial U) := (H^{1/2}(\partial U))^\star.$$

Theorem (Normal traces of $H(div)$-functions)

For $U \subseteq \mathbb{R}^n$ open and bounded Lipschitz domain, there exists continuous, linear $\gamma_\nu : H(\mathrm{div}, U) \to H^{-1/2}(\partial U)$, with

$$\gamma_\nu q = q|_{\partial U} \cdot \nu \quad \text{for all } q \in C^\infty(\overline{U}; \mathbb{R}^n).$$

Any $q \in H(\mathrm{div}, U)$ and $w \in H^1(U)$ satisfies

$$\langle \gamma_\nu q, \gamma_0 w \rangle_{\partial U} = (q, \nabla w)_U + (\mathrm{div}\, q, w)_U.$$

Traces (3)

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For $q \in H(\mathrm{div},\mathcal{T})$ define $\gamma_\nu^{\mathcal{T}} q \in \prod_{T\in\mathcal{T}} H^{-1/2}(\partial T)$ by

$$\gamma_\nu^{\mathcal{T}} q := (t_T)_{T\in\mathcal{T}}, \quad \text{with } t_T := \gamma_\nu(q|_T) \quad \text{for all } T \in \mathcal{T}.$$

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$$H^{-1/2}(\partial\mathcal{T}) := \gamma_\nu^{\mathcal{T}} H(\mathrm{div},\Omega)$$

is a Hilbert space with minimal extension norm

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For $t \in \prod_{T\in\mathcal{T}} H^{-1/2}(\partial T)$ and $v \in H^1(\mathcal{T})$ define

$$\langle t, v\rangle_{\partial\mathcal{T}} := \sum_{T\in\mathcal{T}} \langle t_T, \gamma_0 v\rangle_{\partial T}.$$

Poisson Model Problem

$\Omega \subseteq \mathbb{R}^2$ open Lipschitz domain with polygonal boundary
Seek $u : \Omega \to \mathbb{R}$ with

$$\begin{aligned} -\Delta u &= f \quad \text{in } \Omega, \\ u &= 0 \quad \text{on } \partial\Omega. \end{aligned}$$

Weak formulation

- multiply equation with test function v,
- integrate by parts on Ω
- test function $v \in H_0^1(\Omega)$

$$\int_\Omega fv \, \mathrm{d}x = \int_\Omega \nabla u \cdot \nabla v \, \mathrm{d}x - \int_{\partial\Omega} v \nabla u \cdot \nu \, \mathrm{d}s$$

Weak formulation

- multiply equation with test function v,
- integrate by parts on Ω
- test function $v \in H_0^1(\Omega)$
- boundary integral vanishes

$$\int_\Omega f v \,\mathrm{d}x = \int_\Omega \nabla u \cdot \nabla v \,\mathrm{d}x - \underbrace{\int_{\partial\Omega} v \nabla u \cdot \nu \,\mathrm{d}s}_{=0}$$

Weak formulation

- multiply equation with test function v,
- integrate by parts on $T \in \mathcal{T}$
- test function $v \in H^1(\mathcal{T})$
- boundary integral?

$$\int_T fv \, \mathrm{d}x = \int_T \nabla u \cdot \nabla v \, \mathrm{d}x - \int_{\partial T} v \nabla u \cdot \nu \, \mathrm{d}s$$

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- multiply equation with test function v,
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Weak formulation

- multiply equation with test function v,
- integrate by parts on $T \in \mathcal{T}$
- test function $v \in H^1(\mathcal{T})$
- boundary integral introduces new variable t
- sum over all elements

$$\int_T fv \, \mathrm{d}x = \int_T \nabla u \cdot \nabla v \, \mathrm{d}x - \int_{\partial T} v \underbrace{\nabla u \cdot \nu}_{t_T} \mathrm{d}s$$

Primal dPG formulation seeks $u \in H_0^1(\Omega), t \in H^{-1/2}(\partial\mathcal{T})$ with

$$(f, v)_\Omega = (\nabla u, \nabla_{NC} v)_\Omega - \langle t, v \rangle_{\partial\mathcal{T}} \text{ for all } v \in H^1(\mathcal{T}).$$

Duality lemma

Theorem

Any $t \in H^{-1/2}(\partial\mathcal{T})$ *satisfies*

$$\|t\|_{H^{-1/2}(\partial\mathcal{T})} \leq \sup_{v \in H^1(\mathcal{T}), v \neq 0} \frac{\langle t, v \rangle_{\partial\mathcal{T}}}{\|v\|_{H^1(\mathcal{T})}}.$$

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Proof.

Let $v \in H^1(\mathcal{T})$ on each $T \in \mathcal{T}$ weak solution to

$$-\Delta v + v = 0 \text{ in } T \text{ and } \nabla v \cdot \nu = t_T \text{ on } \partial T.$$

With $q := \nabla_{NC} v \in H(\operatorname{div}, \Omega)$, it holds $\|t\|_{H^{-1/2}(\partial\mathcal{T})} \leq \|q\|_{H(\operatorname{div})}$, $\operatorname{div} q = v$, and $\|q\|_{H(\operatorname{div})} = \|v\|_{H^1(\mathcal{T})}$. Integration by parts shows

$$\langle t, v\rangle_{\partial\mathcal{T}} = (q, \nabla_{NC} v)_\Omega + (\operatorname{div} q, v)_\Omega = \|q\|^2_{H(\operatorname{div})}. \qquad \square$$

Theorem

The spaces $X := H^1_0(\Omega) \times H^{-1/2}(\partial\mathcal{T})$, $Y := H^1(\mathcal{T})$ and the bilinear form $b : X \times Y$, $b(u,t;v) := (\nabla u, \nabla_{NC} v)_\Omega - \langle t, v\rangle_{\partial\mathcal{T}}$ satisfy (H1).

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Proof.

The Friedrichs inequality implies

$$\|\nabla u\|_{L^2(\Omega)} \lesssim \sup_{v \in H^1_0(\Omega), v \neq 0} \frac{(\nabla u, \nabla v)_\Omega}{\|v\|_{H^1(\mathcal{T})}} = \sup_{v \in H^1_0(\Omega), v \neq 0} \frac{b(u, t; v)}{\|v\|_{H^1(\mathcal{T})}}.$$

The duality lemma and the triangle inequality show

$$\|t\|_{H^{-1/2}(\partial\mathcal{T})} \leq \sup_{v \in Y, v \neq 0} \frac{b(u, t; v)}{\|v\|_{H^1(\mathcal{T})}} + \sup_{v \in Y, v \neq 0} \frac{(\nabla u, \nabla_{NC} v)_\Omega}{\|v\|_{H^1(\mathcal{T})}}. \qquad \square$$

Recall

$$X = H^1_0(\Omega) \times H^{-1/2}(\partial\mathcal{T}),$$
$$Y = H^1(\mathcal{T}).$$

The discrete spaces read

$$X_h := S^1_0(\mathcal{T}) \times P_0(\mathcal{E}) \subseteq X,$$
$$Y_h := P_1(\mathcal{T}) \subseteq Y.$$

Discrete Inf-Sup Condition

Theorem

The discrete spaces X_h and Y_h satisfy (H2).

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Proof.

- given $x_h = (u_C, t_0) \in X_h$, let $q_{RT} \in RT_0(\mathcal{T})$, $\gamma_\nu^{\mathcal{T}}(q_{RT}) = t_0$
- choose $v_1 = -\operatorname{div} q_{RT} + (\nabla u_C - \Pi_0 q_{RT}) \cdot (\bullet - \operatorname{mid}(\mathcal{T})) \in Y_h$
- integration by parts shows

$$\begin{aligned} b(u_C, t_0; v_1) &= (\nabla u_C - q_{RT}, \nabla_{NC} v_1)_\Omega - (\operatorname{div} q_{RT}, v_1)_\Omega \\ &= \|\nabla u_C - \Pi_0 q_{RT}\|_{L^2(\Omega)}^2 + \|\operatorname{div} q_{RT}\|_{L^2(\Omega)}^2 \end{aligned}$$

- Since $P_0(\mathcal{T})$ is orthogonal to $(\bullet - \operatorname{mid}(\mathcal{T}))$ in $L^2(\Omega)$, $\|v_1\|_{H^1(\mathcal{T})}^2 \leq (1 + h_{\max}^2) b(u_C, t_0; v_1)$.

Proof.

- recall $b(u_C, t_0; v_1) = \|\nabla u_C - \Pi_0 q_{RT}\|^2_{L^2(\Omega)} + \|\operatorname{div} q_{RT}\|^2_{L^2(\Omega)}$
- Helmholtz decomposition leads to $\alpha_C \in S^1_0(\mathcal{T})$, $\beta_{CR} \in CR^1(\mathcal{T})$ with $\nabla u_C - \Pi_0 q_{RT} = \nabla \alpha_C + \operatorname{Curl}_{NC} \beta_{CR}$. Orthogonality in Helmholtz and integration by parts shows

$$\begin{aligned}\|\nabla(u_C - \alpha_C)\|^2_{L^2(\Omega)} &= (\nabla(u_C - \alpha_C), q_{RT})_\Omega = -(u_C - \alpha_C, \operatorname{div} q_{RT})_\Omega \\ &\lesssim \|\nabla(u_C - \alpha_C)\|_{L^2(\Omega)} \|\operatorname{div} q_{RT}\|_{L^2(\Omega)}.\end{aligned}$$

- triangle inequality implies $\|\nabla u_C\|_{L^2(\Omega)} \le \|\nabla(u_C - \alpha_C)\|_{L^2(\Omega)} + \|\nabla \alpha_C\|_{L^2(\Omega)} \lesssim \|\operatorname{div} q_{RT}\|_{L^2(\Omega)} + \|\nabla u_C - \Pi_0 q_{RT}\|_{L^2(\Omega)} \lesssim b(u_C, t_0; v_1)^{1/2}$ and $\|t_0\|_{H^{-1/2}(\partial\mathcal{T})} \le \|q_{RT}\|_{H(\operatorname{div})} \lesssim \|\Pi_0 q_{RT}\|_{L^2(\Omega)} + \|\operatorname{div} q_{RT}\|_{L^2(\Omega)} \lesssim b(u_C, t_0; v_1)^{1/2}$. □

Summary

- idea of dPG: choose discrete trial space, compute discrete test space
- idealized dPG: inf-sup stable, but not practical
- practical dPG inf-sup stable for Y_h big enough
- practical dPG has built-in a priori and a posteriori error control
- application to Poisson as primal dPG with broken test functions
- continuous inf-sup follows from stability of non-broken functions
- discrete inf-sup utilizes discrete Helmholtz decomposition