GEOMETRY OF INFINITE DIMENSIONAL CARTAN DEVELOPMENTS

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Abstract. The Cartan development takes a Lie algebra valued 1-form satisfying the Maurer-Cartan equation on a simply connected manifold M to a smooth mapping from M into the Lie group. In this paper this is generalized to infinite dimensional M for infinite dimensional regular Lie groups. The Cartan development is viewed as a generalization of the evolution map of a regular Lie group. The tangent mapping of a Cartan development is identified as another Cartan development.

1. Introduction

A regular Lie group G is one where where one can integrate smooth curves in the Lie algebra $\mathfrak{g}$ to smooth curves in the Lie group in a smooth way: The evolution mapping $\operatorname{Evol} : C^\infty(\mathbb{R}, \mathfrak{g}) \to C^\infty(\mathbb{R}, G)$ exists and is smooth; see 2.4 below. This notion is relevant for infinite dimensional Lie groups where existence results for ODEs are not available in general. A stronger version of this notion is due to Omori et al. [15, 16, 17, 18, 19, 20] and was weakened to this version for Fréchet Lie groups by Milnor [14]. It was generalized to Lie groups modeled on convenient vector spaces (i.e., locally convex vector spaces where each Mackey Cauchy sequence converges) in [12], see also [11]. Up to now, no example of a non-regular Lie group modelled on convenient vector spaces is known.

One can extend the notion of regularity to other classes of curves for certain Lie groups (e.g., modeled on Banach spaces) and ask for the existence of Evol on $C^k(\mathbb{R}, \mathfrak{g})$ for $k = 0, 1, 2, \dots$ or even $L^1(\mathbb{R}, \mathfrak{g})$. Results in these directions are due to Glöckner in [5] and Hanusch in [9, 8, 7].

In this paper we extend the domain of the evolution mapping: In order to end up in $C^\infty(M, G)$ for an even infinite dimensional manifold M we consider the spaces $\Omega^1_{\text{flat}}(M, \mathfrak{g})$ of flat $\mathfrak{g}$-valued differential forms on M in 3.1 and show that the evolution map exists and is smooth for simply connected pointed M, and we give an explicit description of its tangent mapping. This is an extension to infinite dimensions of the Cartan development.

This paper was inspired by the paper [4] on integrable surfaces which was pointed out to the authors by Oleksandr Sakhnovich. In [4] the term 'integrable' just means

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that the 1-forms considered there satisfy the Maurer-Cartan formula, and it has nothing to do with complete integrability: There is no Hamitonian aspect to this, as was pointed out to us by Boris Khesin.

2. Review of regular Lie groups

2.1. **Notation on Lie groups.** Let G be a Lie group which may be infinite dimensional, but then is supposed to be regular, with Lie algebra $\mathfrak{g}$. Let $\mu : G \times G \to G$ be the multiplication, let μ_x be left translation and μ^y be right translation, given by $\mu_x(y) = \mu^y(x) = xy = \mu(x,y)$. We denote inversion by $\nu : G \to G$, $\nu(x) = x^{-1}$. The tangent mapping $T_{(a,b)}\mu : T_aG \times T_bG \to T_{ab}G$ is given by

$$T_{(a,b)}\mu.(X_a, Y_b) = T_a(\mu^b).X_a + T_b(\mu_a).Y_b$$

and $T_a\nu : T_aG \to T_{a^{-1}}G$ is given by

$$T_a\nu = -T_e(\mu^{a^{-1}}).T_a(\mu_{a^{-1}}) = -T_e(\mu_{a^{-1}}).T_a(\mu^{a^{-1}}).$$

Let $L, R : \mathfrak{g} \to \mathfrak{X}(G)$ be the left and right invariant vector field mappings, given by $L_X(g) = T_e(\mu_g).X$ and $R_X = T_e(\mu^g).X$, respectively. They are related by $L_X(g) = R_{\mathrm{Ad}(g)X}(g)$. Their flows are given by

$$\mathrm{Fl}^{L_X}_t(g) = g.\exp(tX) = \mu^{\exp(tX)}(g), \quad \mathrm{Fl}^{R_X}_t(g) = \exp(tX).g = \mu_{\exp(tX)}(g).$$

We also need the right Maurer-Cartan form $\kappa = \kappa^r \in \Omega^1(G, \mathfrak{g})$, given by $\kappa_x(\xi) := T_x(\mu^{x^{-1}}) \cdot \xi$. It satisfies the left Maurer-Cartan equation $d\kappa - \frac{1}{2}[\kappa, \kappa]_\wedge = 0$, where $[\ ,\ ]_\wedge$ denotes the wedge product of $\mathfrak{g}$-valued forms on G induced by the Lie bracket. Note that $\frac{1}{2}[\kappa, \kappa]_\wedge(\xi, \eta) = [\kappa(\xi), \kappa(\eta)]$.

Similarly the *left Maurer-Cartan form* $\kappa^l \in \Omega^1(G, \mathfrak{g})$ is given by $\kappa^l_g = T_g(\mu_{g^{-1}}) : T_gG \to \mathfrak{g}$ and it satisfies the right Maurer-Cartan equation $d\kappa^l + \frac{1}{2}[\kappa^l, \kappa^l]^\wedge_\mathfrak{g} = 0$. We have also $(\nu^*\kappa^r)_g = -\kappa^l_g = -Ad(g^{-1})\kappa^r_g$.

The (exterior) derivative of the function $\mathrm{Ad} : G \to GL(\mathfrak{g})$ can be expressed by

$$d\,\mathrm{Ad} = \mathrm{Ad}\,.(\mathrm{ad} \circ \kappa^l) = (\mathrm{ad} \circ \kappa^r).\,\mathrm{Ad},$$

since , for $c : \mathbb{R} \to G$ smooth with $c(0) = e$ and $c'(0) = X$ like $\exp(tX)$ if exp exists,

$$d\,\mathrm{Ad}(T\mu_g.X) = \partial_t|_0\, \mathrm{Ad}(g.c(t)) = \mathrm{Ad}(g)\partial_t|_0\, \mathrm{Ad}(c(t)) = \mathrm{Ad}(g).\,\mathrm{ad}(\kappa^l(T\mu_g.X))\,.$$

Since we shall need it we also note

$$d(\mathrm{Ad} \circ \nu) = -(\mathrm{ad} \circ \kappa^l)(\mathrm{Ad} \circ \nu) = -(\mathrm{Ad} \circ \nu)(\mathrm{ad} \circ \kappa^r).$$

2.2. **The right and left logarithmic derivatives.** Let M be a manifold and let $f : M \to G$ be a smooth mapping into a Lie group G with Lie algebra $\mathfrak{g}$. We define the mapping $\delta^r f : TM \to \mathfrak{g}$ by the formula

$$\begin{aligned} \delta^r f(\xi_x) :&= T_{f(x)}(\mu^{f(x)^{-1}}).T_xf.\xi_x = \kappa^r_{f(x)}(T_xf.\xi_x) \\ &= (f^*\kappa^r)(\xi_x) \text{ for } \xi_x \in T_xM. \end{aligned}$$

Then $\delta^r f$ is a $\mathfrak{g}$-valued 1-form on M, $\delta^r f \in \Omega^1(M;\mathfrak{g})$. We call $\delta^r f$ the *right logarithmic derivative* of f, since for $f : \mathbb{R} \to (\mathbb{R}^+, \cdot)$ we have $\delta^r f(x).1 = \frac{f'(x)}{f(x)} = (\log \circ f)'(x)$.

Similarly the *left logarithmic derivative* $\delta^l f \in \Omega^1(M, \mathfrak{g})$ of a smooth mapping $f : M \to G$ is given by

$$\delta^l f.\xi_x = T_{f(x)}(\mu_{f(x)^{-1}}).T_x f.\xi_x = (f^*\kappa^l)(\xi_x)$$

Theorem. *Let $f, g : M \to G$ be smooth. Then the Leibniz rule holds:*

$$\delta^r(f.g)(x) = \delta^r f(x) + \mathrm{Ad}(f(x)).\delta^r g(x).$$

Moreover, the differential form $\delta^r f \in \Omega^1(M;\mathfrak{g})$ satifies the 'left Maurer-Cartan equation' (left because it stems from the left action of G on itself)

$$d\delta^r f(\xi,\eta) - [\delta^r f(\xi), \delta^r f(\eta)]^{\mathfrak{g}} = 0,$$

$$\textit{or} \quad d\delta^r f - \frac{1}{2}[\delta^r f, \delta^r f]^{\mathfrak{g}}_{\wedge} = 0,$$

where $\xi, \eta \in T_x M$, and where for $\varphi \in \Omega^p(M;\mathfrak{g}), \psi \in \Omega^q(M;\mathfrak{g})$ one puts

$$[\varphi, \psi]^{\mathfrak{g}}_{\wedge}(\xi_1, \dots, \xi_{p+q}) := \frac{1}{p!q!}\sum_{\sigma} \mathrm{sign}(\sigma)[\varphi(\xi_{\sigma 1}, \dots), \psi(\xi_{\sigma(p+1)}, \dots)]^{\mathfrak{g}}.$$

If $h : N \to M$ is a smooth mapping, then $\delta^r(f \circ h) = h^*(\delta^r(f))$. If $\varphi : G \to H$ is a smooth homomorphism of groups, then $\delta^r_H(\varphi \circ f) = \varphi' \circ \delta^r_G(f)$ where $\varphi' = T_e\varphi : \mathfrak{g} \to \mathfrak{h}$.

For the left logarithmic derivative the corresponding Leibniz rule is uglier, and it satisfies the 'right Maurer Cartan equation':

$$\delta^l(fg)(x) = \delta^l g(x) + Ad(g(x)^{-1})\delta^l f(x),$$

$$d\delta^l f + \frac{1}{2}[\delta^l f, \delta^l f]^{\mathfrak{g}}_{\wedge} = 0.$$

For 'regular Lie groups' a converse to this statement holds; see [11], 40.2. We shall review this in 2.3 and 2.4 below. This result has a geometric interpretation in principal bundle geometry for the trivial principal bundle $\mathrm{pr}_1 : M \times G \to M$ with right principal action. Then the submanifolds $\{(x, f(x).g) : x \in M\}$ for $g \in G$ form a foliation of $M \times G$ whose tangent distribution is complementary to the vertical bundle $M \times TG \subseteq T(M \times G)$ and is invariant under the principal right G-action. So it is the horizontal distribution of the principal connection $\omega^l \in \Omega^1(M \times G, \mathfrak{g})$ which is given by $\omega^l(\xi_x, \eta_a) := -\delta^l(f.a)(\xi_x) + \kappa^l(\eta_a)$. Thus this principal connection has vanishing curvature which translates into the result for the right logarithmic derivative.

Proof. For the Leibniz rule we compute for $\xi_x \in T_x M$, using 2.1,

$$\begin{aligned}
\delta^r(f.g)(\xi_x) &= T_{f(x).g(x)}(\mu^{(f(x).g(x))^{-1}}).T(\mu \circ (f,g)).\xi_x \\
&= T_{f(x).g(x)}(\mu^{(f(x).g(x))^{-1}}).T\mu.(T_x f.\xi_x, T_x g.\xi_x) \\
&= T(\mu^{f(x)^{-1}}).T(\mu^{g(x)^{-1}}).\big(T(\mu_{f(x)}).T_x g.\xi_x) + T(\mu^{g(x)}).T_x f.\xi_x\big) \\
&= T(\mu^{f(x)^{-1}}).T(\mu_{f(x)}).T(\mu^{g(x)^{-1}}).T_x g.\xi_x) + T(\mu^{f(x)^{-1}}).T_x f.\xi_x
\end{aligned}$$

$$= \operatorname{Ad}(f(x)).\delta^r g(\xi_x) + \delta^r f(\xi_x).$$

For the Maurer-Cartan equation we use that κ^r satisfies it:

$$\begin{aligned} d(\delta^r f) &= d(f^*\kappa^r) = f^*(d\kappa^r) = f^*(\tfrac12[\kappa^r,\kappa^r]^\wedge_{\mathfrak g}) = \tfrac12[f^*\kappa^r, f^*\kappa^r]^\wedge_{\mathfrak g} \\ &= \tfrac12[\delta^r f, \delta^r f]^\wedge_{\mathfrak g} \end{aligned}$$

Finally, $\delta^r(f\circ\varphi) = d(f\circ\varphi)^*\kappa^r = d\varphi^* f^*\kappa^r = \varphi^*\delta^r(f)$. For the left logarithmic derivative the proof is analogous. □

2.3. **The 1-dimensional evolution operator.** Let G be a possibly infinite dimensional Lie group with Lie algebra $\mathfrak g$. For a closed interval $I\subset\mathbb R$ and for $X\in C^\infty(I,\mathfrak g)$ we consider the ordinary differential equation

$$(1)\qquad \begin{cases} g(t_0) = e \\ \partial_t g(t) = T_e(\mu^{g(t)})X(t) = R_{X(t)}(g(t)) \quad \text{or } \kappa^r(\partial_t g(t)) = X(t), \end{cases}$$

for local smooth curves g in G, where $t_0\in I$. Then the following results hold; see [11], 40.2:

(2) *Local solution curves g of the differential equation* (1) *are unique.*
(3) *If for fixed X the differential equation* (1) *has a local solution near each $t_0\in I$, then it has also a global solution $g\in C^\infty(I,G)$.*
(4) *If for all $X\in C^\infty(I,\mathfrak g)$ the differential equation* (1) *has a local solution near one fixed $t_0\in I$, then it has also a global solution $g\in C^\infty(I,G)$ for each X. Moreover, if the local solutions near t_0 depend smoothly on the vector fields X then so does the global solution.*
(5) *If* (4) *holds, then the curve $t\mapsto g(t)^{-1}$ is the unique local smooth curve h in G which satifies*

$$\begin{cases} h(t_0) = e \\ \partial_t h(t) = T_e(\mu_{h(t)})(-X(t)) = L_{-X(t)}(h(t)) \\ \qquad \textit{or } \kappa^l(\partial_t h(t)) = -X(t). \end{cases}$$

2.4. **Regular Lie groups.** If for each $X\in C^\infty(\mathbb R,\mathfrak g)$ there exists $g\in C^\infty(\mathbb R,G)$ satisfying

$$(1)\qquad \begin{cases} g(0) = e, \\ \partial_t g(t) = T_e(\mu^{g(t)})X(t) = R_{X(t)}(g(t)) \\ \qquad \text{or } \kappa^r(\partial_t g(t)) = \delta^r g(\partial_t) = X(t), \end{cases}$$

then we write

$$\operatorname{evol}^r_G(X) = \operatorname{evol}_G(X) := g(1),$$
$$\operatorname{Evol}^r_G(X)(t) := \operatorname{evol}_G(s\mapsto tX(ts)) = g(t),$$

and call it the *right evolution* of the curve X in G. By 2.3 the solution of the differential equation (1) is unique, and for global existence it is sufficient that it has a local solution somewhere. Then

$$\operatorname{Evol}^r_G : C^\infty(\mathbb R,\mathfrak g)\to\{g\in C^\infty(\mathbb R,G) : g(0)=e\}$$

is bijective with inverse the right logarithmic derivative δ^r.

The Lie group G is called a *regular Lie group* if $\operatorname{evol}^r : C^\infty(\mathbb{R}, \mathfrak{g}) \to G$ exists and is smooth.

We also write

$$\begin{aligned} \operatorname{evol}^l_G(X) = \operatorname{evol}_G(X) &:= h(1), \\ \operatorname{Evol}^l_G(X)(t) := \operatorname{evol}^l_G(s \mapsto tX(ts)) &= h(t), \end{aligned}$$

if h is the (unique) solution of

$$(2) \qquad \begin{cases} h(0) = e \\ \partial_t h(t) = T_e(\mu_{h(t)})(X(t)) = L_{X(t)}(h(t)), \\ \quad \text{or } \kappa^l(\partial_t h(t)) = \delta^l h(\partial_t) = X(t). \end{cases}$$

Clearly $\operatorname{evol}^l : C^\infty(\mathbb{R}, \mathfrak{g}) \to G$ exists and is also smooth if evol^r does, since we have $\operatorname{evol}^l(X) = \operatorname{evol}^r(-X)^{-1}$ by 2.3.

Let us collect some easily seen properties of the evolution mappings. If $f \in C^\infty(\mathbb{R}, \mathbb{R})$, then we have

$$\begin{aligned} \operatorname{Evol}^r(X)(f(t)) &= \operatorname{Evol}^r(f'.(X \circ f))(t).\operatorname{Evol}^r(X)(f(0)), \\ \operatorname{Evol}^l(X)(f(t)) &= \operatorname{Evol}^l(X)(f(0)).\operatorname{Evol}^l(f'.(X \circ f))(t). \end{aligned}$$

If $\varphi : G \to H$ is a smooth homomorphism between regular Lie groups then the diagram

$$\begin{array}{ccc} C^\infty(\mathbb{R}, \mathfrak{g}) & \xrightarrow{\varphi'_*} & C^\infty(\mathbb{R}, \mathfrak{h}) \\ {\scriptstyle \operatorname{evol}_G}\downarrow & & \downarrow{\scriptstyle \operatorname{evol}_H} \\ G & \xrightarrow{\varphi} & H \end{array}$$

commutes, since $\partial_t \varphi(g(t)) = T\varphi.T(\mu^{g(t)}).X(t) = T(\mu^{\varphi(g(t))}).\varphi'.X(t)$.

Note that each regular Lie group admits an exponential mapping, namely the restriction of evol^r to the constant curves $\mathbb{R} \to \mathfrak{g}$. A Lie group is regular if and only if its universal covering group is regular.

Up to now the following statement holds:

All known Lie groups modelled on convenient vector spaces are regular.

Any Banach Lie group is regular since we may consider the time dependent right invariant vector field $R_{X(t)}$ on G and its integral curve $g(t)$ starting at e, which exists and depends smoothly on (a further parameter in) X. In particular finite dimensional Lie groups are regular.

For diffeomorphism groups the evolution operator is just integration of time dependent vector fields with compact support.

As noted in the introduction, one may ask for the existence of Evol on $C^k(\mathbb{R}, \mathfrak{g})$ for $k = 0, 1, 2, \ldots$ or even $L^1(\mathbb{R}, \mathfrak{g})$. Results in these directions are due to Glöckner in [5] and Hanusch in [9, 8, 7].

3. The general evolution operator alias the Cartan development

3.1. The space of flat differential forms. Let G be a regular Lie group with Lie algebra $\mathfrak{g}$. Let (M, x_0) be a simply connected pointed possibly infinite dimensional manifold. We consider the space

$$\Omega^1_{\text{flat}}(M, \mathfrak{g}) = \{\xi \in \Omega^1(M, \mathfrak{g}) : d\xi - \tfrac{1}{2}[\xi, \xi]^{\mathfrak{g}}_\wedge = 0\}$$

of $\mathfrak{g}$-valued 1-forms on M which are *flat* in the sense that they obey the left Maurer-Cartan equation.

The smooth structure on the space of flat differential forms. In any case, now $\Omega^1_{\text{flat}}(M, \mathfrak{g})$ inherits the smooth structure of a Frölicher space (see [11, Section 23]) from the convenient vector space $\Omega^1(M, \mathfrak{g})$, generated by all curves $c : \mathbb{R} \to \Omega^1_{\text{flat}}(M, \mathfrak{g})$ which are smooth into $\Omega^1(M, \mathfrak{g})$. This allows us to talk about smooth mappings running through it.
Moreover, it is a Frölicher space with further structure: its kinematic tangent spaces are given by $T_\xi \Omega^1(M, \mathfrak{g}) = \{\eta \in \Omega^1(M, \mathfrak{g}) : d\eta - [\xi, \eta]^{\mathfrak{g}}_\wedge = 0\}$, and the tangent bundle is again the Frölicher space

$$T\Omega^1(M, \mathfrak{g}) = \{(\xi, \eta) \in \Omega^1(M, \mathfrak{g})^2 : d\xi - \tfrac{1}{2}[\xi, \xi]^{\mathfrak{g}}_\wedge = 0 \text{ and } d\eta - [\xi, \eta]^{\mathfrak{g}}_\wedge = 0\}\,.$$

Question: If M is finite dimensional, is $\Omega^1_{\text{flat}}(M, \mathfrak{g})$ a (split) submanifold of $\Omega^1(M, \mathfrak{g})$? Can this be shown by a quasilinear version of Hodge theory?

But we shall see below that $\Omega^1_{\text{flat}}(M, \mathfrak{g})$ is diffeomorphic to $C^\infty((M, x_0), (G, e))$ and that it gets a Lie group structure via this map, at least in the case when M is compact.

3.2. The space $C^\infty(M, G)$. If M is a compact manifold, then $C^\infty(M, G)$ is a smooth manifold; see [11]. The same is true if M is locally compact for the space $C^\infty_c(M, G)$ of smooth mappings which equal the constant e off some compact subset of M; let us call these smooth mappings with compact support. But in general, there is no smooth structure admitting an atlas on $C^\infty(M, G)$, since the space $C^\infty(M, N)$ is not locally contractible in its natural topology. Thus we consider $C^\infty(M, G)$ as a Frölicher space in the general situation. It is a Frölicher space with a natural tangent bundle

$$TC^\infty(M, G) = C^\infty(M, TG) \xrightarrow{(\pi_G, \kappa^r)_*} C^\infty(M, G) \times C^\infty(M, \mathfrak{g})\,.$$

3.3. Theorem. *Let G be a regular Lie group with Lie algebra $\mathfrak{g}$. Let (M, x_0) be a simply connected pointed possibly infinite dimensional manifold. In general, there exists a unique smooth mapping, called* evolution operator*,*

$$\text{Evol} = \text{Evol}^M_G : \Omega^1_{\mathit{flat}}(M, \mathfrak{g}) \to C^\infty(M, G)$$

which satisfies

$$\delta^r \circ \text{Evol} = \text{Id} \;\; \textit{and} \;\; \text{Evol}(x_0) = e.$$

Evol *is a natural transformation with respect to M: for smooth $h : (N, y_0) \to (M, x_0)$ we have* $\text{Evol}(h^*\xi) = \text{Evol}(\xi) \circ h$, *or* $\text{Evol} \circ h^* = h^* \circ \text{Evol}$.
Evol *is also a natural transformation with respect to $(G, \mathfrak{g})$: If $\varphi : G \to H$ is a*

smooth homomorphism between regular Lie groups with $T_e\varphi = \varphi' : \mathfrak{g} \to \mathfrak{h}$ *then the following diagram commutes:*

$$\begin{array}{ccc} \Omega^1_{flat}(M,\mathfrak{g}) & \xrightarrow{\varphi'_*} & \Omega^1_{flat}(M,\mathfrak{h}) \\ \Big\downarrow{\scriptstyle \mathrm{Evol}_G} & & \Big\downarrow{\scriptstyle \mathrm{Evol}_H} \\ C^\infty(M,G) & \xrightarrow{\varphi_*} & C^\infty(M,H)\,. \end{array}$$

If one wants to avoid choosing a point x_0 in M, then Evol exists and is unique up to right translations in G. Note that a Lie group G is regular if the theorem holds for $(M, x_0) = (\mathbb{R}, 0)$. See [11, 40.2] for the main part of the proof. For finite dimensional M and Lie groups a proof can be found in [21, 22, 23], or in [2, 3] or [6] (proved with moving frames); see also [1, 5.2].

Proof. See [11, 40.2] for the first part. For completeness' sake we repeat the proof.

If we are given a 1-form $\xi \in \Omega^1(M,\mathfrak{g})$ with $d\xi - \frac{1}{2}[\xi,\xi]^{\mathfrak{g}}_\wedge = 0$ then we consider the 1-form $\omega^\xi \in \Omega^1(M \times G, \mathfrak{g})$ given by

$$\omega^\xi = \kappa^l - (\mathrm{Ad}\circ\nu).\xi\,, \quad \omega^\xi_{(x,g)}(Y_x, T\mu_g.X) = X - \mathrm{Ad}(g^{-1}).\xi_x(Y_x)$$

for $Y_x \in T_xM$ and $X \in \mathfrak{g}$. Then ω^ξ is a principal connection form on $M \times G$, since it reproduces the generators in $\mathfrak{g}$ of the fundamental vector fields for the principal right action, i.e., the left invariant vector fields $0 \times L_X$, and ω^ξ is G-equivariant:

$$\begin{aligned} ((\mu^g)^*\omega^\xi)_{(x,h)} &= \omega^\xi_{(x,hg)} \circ (\mathrm{Id}_{TM} \times T(\mu^g)) = T(\mu_{g^{-1}h^{-1}}).T(\mu^g) - \mathrm{Ad}(g^{-1}h^{-1}).\xi_x \\ &= \mathrm{Ad}(g^{-1}).\big(\kappa^l_h - \mathrm{Ad}(h^{-1}).\xi_x\big) = \mathrm{Ad}(g^{-1}).\omega^\xi_{(x,h)}. \end{aligned}$$

This connection is flat since

$$\begin{aligned} d\omega^\xi + \tfrac{1}{2}[\omega^\xi,\omega^\xi]_\wedge &= d\kappa^l + \tfrac{1}{2}[\kappa^l,\kappa^l]_\wedge - d(\mathrm{Ad}\circ\nu)\wedge\xi - (\mathrm{Ad}\circ\nu).d\xi \\ &\quad - [\kappa^l, (\mathrm{Ad}\circ\nu).\xi]_\wedge + \tfrac{1}{2}[(\mathrm{Ad}\circ\nu).\xi, (\mathrm{Ad}\circ\nu).\xi]_\wedge \\ &= -(\mathrm{Ad}\circ\nu).(d\xi - \tfrac{1}{2}[\xi,\xi]_\wedge) = 0\,. \end{aligned}$$

Since the structure group G is regular, by theorem [11, 39.2] the horizontal bundle $\ker(\omega^\xi) \subset T(M \times G)$ is integrable, and $\mathrm{pr}_1 : M \times G \to M$, restricted to each horizontal leaf, is a covering. Thus, it may be inverted over the simply connected manifold M, and the inverse $(\mathrm{Id}_M, f) : M \to M \times G$ is a horizontal section, i.e., $T(\mathrm{Id}_M, f) : TM \to \ker(\omega^\xi)$. Therefore

$$\begin{aligned} 0 = ((\mathrm{Id}, f)^*\omega^\xi)_x = (f^*\kappa^l - f^*(\mathrm{Ad}\circ\nu).\xi)_x = (\delta^l f)_x - \mathrm{Ad}(f(x)^{-1}).\xi_x \\ (\delta^r f)_x = \mathrm{Ad}(f(x))(\delta^l f)_x = \xi_x \end{aligned}$$

for $x \in U$, as required. Moreover, $(\mathrm{Id}_M \times f)$ is unique up to the choice of the branch of the covering and the choice of the leaf, i.e., f is unique up to a right translation by an element of G. We may fix $f =: \mathrm{Evol}^M_G(\xi)$ by stipulating $f(x_0) = e$.

It remain to show that Evol is smooth from the Frölicher space $\Omega^1_{\text{flat}}(M,\mathfrak{g}) \subset \Omega^1(M,\mathfrak{g})$ into $C^\infty(M, G,)$. By the principles of convenient analysis it suffices to show that $t \mapsto \mathrm{Evol}(\xi(t)) \in C^\infty(M,G)$ is smooth if $\xi : \mathbb{R} \to \Omega^1_{\text{flat}}(M,\mathfrak{g})$ is smooth into $\Omega^1(M,\mathfrak{g})$. The key result in the proof above is the use of parallel transport for a principal connection on a principal bundle with regular structure group, whose existence was proved in [11, 39.1]. There, smooth dependence of the parallel transport on the principal connection was not checked. It can easily be done and the proof will appear in the next edition of [11].

The diagram commutes: We have $\varphi^*\kappa^r_H = \varphi'.\kappa^r_G$ since

$$\begin{aligned}(\varphi^*\kappa^r_H)_g(T_e\mu^g.X) &= (\kappa^r_H)_{\varphi(g)}(T_g\varphi.T_e\mu^g.X) = T\mu^{\varphi(g)^{-1}}.T_g\varphi.T_e\mu^g.X\\ &= T_e\varphi.X = \varphi'.T_g\mu^{g^{-1}}.T_e\mu^g.X = \varphi'.(\kappa^r_G)_g(T_e\mu^g.X)\,,\end{aligned}$$

and obviously $\varphi'_* : \Omega^1_{\text{flat}}(M,\mathfrak{g}) \to \Omega^1_{\text{flat}}(M,\mathfrak{h})$, thus

$$\begin{aligned}(\varphi\circ\operatorname{Evol}_G(\xi))^*\kappa^r_H &= \operatorname{Evol}_G(\xi)^*\varphi^*\kappa^r_H = \operatorname{Evol}_G(\xi)^*(\varphi'.\kappa^r_G)\\ &= \varphi'.\operatorname{Evol}_G(\xi)^*\kappa^r_G = \varphi'.\xi = \operatorname{Evol}_H(\varphi'.\xi).\end{aligned}$$ □

3.4. **Theorem.** *For a regular Lie group G and simply connected (M,x_0) we have for $\xi,\eta\in\Omega^1_{\mathit{flat}}(M,\mathfrak{g})$:*

$$\begin{aligned}\operatorname{Evol}(\xi).\operatorname{Evol}(\eta) &= \operatorname{Evol}\Big(\xi + (\operatorname{Ad}_G\circ\operatorname{Evol}(\xi)).\eta\Big),\\ \operatorname{Evol}(\xi)^{-1} &= \operatorname{Evol}\Big((-\operatorname{Ad}_G\circ\nu\circ\operatorname{Evol}(\xi)).\xi\Big),\end{aligned}$$

so that $\operatorname{Evol} : \Omega^1_{\mathit{flat}}(M,\mathfrak{g}) \to C^\infty((M,x_0),(G,e))$ *is a bijective smooth homomorphism of Lie groups, where on $\Omega^1_{\mathit{flat}}(M,\mathfrak{g})$ the operations are given by*

$$\begin{aligned}(\xi*\eta)(x) &= \xi(x) + \operatorname{Ad}_G(\operatorname{Evol}(\xi)(x)).\eta(x),\\ \xi^{-1}(x) &= -\operatorname{Ad}_G(\operatorname{Evol}(\xi)(x)^{-1}).\xi(x).\end{aligned}$$

With this operations and with 0 as unit element $(\Omega^1_{\mathit{flat}}(M,\mathfrak{g}),*)$ becomes a regular Lie group with the manifold structure coming from $C^\infty((M,x_0),(G,e))$. Its Lie algebra is $T_0\Omega^1_{\mathit{flat}}(M,\mathfrak{g}) = \{\eta\in\Omega^1(M,\mathfrak{g})) : d\eta = 0\} =: Z(M,\mathfrak{g})$ with bracket

$$[\xi_1,\xi_2]^{Z(M,\mathfrak{g})} = [\xi_1, d^{-1}\xi_2]^{\mathfrak{g}} + [d^{-1}\xi_1,\xi_2]^{\mathfrak{g}},\quad \xi_i\in Z(M,\mathfrak{g})\,,$$

where

$$d^{-1} : \{\xi\in\Omega^1(M,\mathfrak{g}) : d\xi = 0\} =: Z(M,\mathfrak{g}) \to C^\infty((M,x_0),(\mathfrak{g},0))$$

is the bounded operator of the Poincaré lemma, the inverse of the exterior derivative.

This formula for the Lie bracket fits nicely to the 1-dimensional version derived in [11, 38.12].

Proof. We have $\mu^*\kappa^r = \operatorname{pr}_1^*\kappa^r + (\operatorname{Ad}\circ\operatorname{pr}_1)\operatorname{pr}_2^*\kappa^r$ because

$$\begin{aligned}(\mu^*\kappa^r)_{(a,b)}(\xi_a,\eta_b) &= \kappa^r_{\mu(ab)}\big(T\mu.(\xi_a,\eta_b)\big) = \kappa^r_{ab}\big(T_a\mu^b.\xi_a + T_b\mu_a.\eta_b\big)\\ &= T_{ab}\mu^{(ab)^{-1}}\big(T_a\mu^b.\xi_a + T_b\mu_a.\eta_b\big)\\ &= T\mu^{a^{-1}}.T\mu^{b^{-1}}.T\mu^b.\xi_a + T\mu^{a^{-1}}.T\mu^{b^{-1}}.T\mu_a.\eta_b\\ &= \kappa^r_a(\xi_a) + \operatorname{Ad}(a).\kappa^r_b(\eta_b).\end{aligned}$$

For $\xi,\eta\in\Omega^1_{\text{flat}}(M,\mathfrak{g})$ we have therefore

$$\delta^r\big(\operatorname{Evol}(\xi).\operatorname{Evol}(\eta)\big) = \big(\mu\circ(\operatorname{Evol}(\xi),\operatorname{Evol}(\eta))\big)^*\kappa^r$$

$$
\begin{aligned}
&= (\operatorname{Evol}(\xi), \operatorname{Evol}(\eta))^* \mu^* \kappa^r = (\operatorname{Evol}(\xi), \operatorname{Evol}(\eta))^* \big(\operatorname{pr}_1^* \kappa^r + (\operatorname{Ad} \circ \operatorname{pr}_1) \operatorname{pr}_2^* \kappa^r \big) \\
&= \operatorname{Evol}(\xi)^* \kappa^r + (\operatorname{Ad} \circ \operatorname{Evol}(\xi)). \operatorname{Evol}(\eta)^* \kappa^r = \xi + \operatorname{Ad}(\operatorname{Evol}(\xi))\eta.
\end{aligned}
$$

which implies

$$
\operatorname{Evol}(\xi). \operatorname{Evol}(\eta) = \operatorname{Evol}(\xi * \eta), \quad \operatorname{Evol}(\xi)^{-1} = \operatorname{Evol}(\xi^{-1}).
$$

Thus, $\operatorname{Evol} : \Omega^1_{\text{flat}}(M, \mathfrak{g}) \to C^\infty(M, G)$ is a group isomorphism onto the subgroup $C^\infty((M, x_0), (G, e)) := \{f \in C^\infty(M, G) : f(x_0) = e\}$ with the pointwise product, which, however, is only a Frölicher space in general, see [11, 23.1]. If M is compact then $C^\infty((M, x_0), (G, e))$ is smooth regular Lie group. If M is finite dimensional, then one has to refine the topology (control near infinity) to make it into a disjoint union of regular Lie groups, and then one has to mimick this procedure also on $\Omega^1_{\text{flat}}(M, \mathfrak{g})$.

In general, we will just take both groups as Frölicher spaces with special properties (having a tangent bundle, e.g.).

It follows that the product on $\Omega^1_{\text{flat}}(M, \mathfrak{g})$ has the properties of a group structure.

A direct proof is fun and needs naturality of Evol in an essential way. As entertainment we compute the following:

$$
\begin{aligned}
&\xi^{-1} * \xi = -\operatorname{Ad}_G(\operatorname{Evol}(\xi)^{-1}).\xi + \operatorname{Ad}_G \big(\operatorname{Evol}(-\operatorname{Ad}_G(\operatorname{Evol}(\xi))^{-1}).\xi \big) \xi = 0 \text{ since} \\
&\qquad (\operatorname{Evol}(\xi)^{-1})^* \kappa^r = (\nu \circ \operatorname{Evol}(\xi))^* \kappa^r = \operatorname{Evol}(\xi)^* \nu^* \kappa^r \\
&\qquad\quad = \operatorname{Evol}(\xi)^* (-(\operatorname{Ad} \circ \nu) \kappa^r) = -\operatorname{Ad}(\operatorname{Evol}(\xi)^{-1}) \operatorname{Evol}(\xi)^* \kappa^r \\
&\qquad\quad = -\operatorname{Ad}(\operatorname{Evol}(\xi)^{-1})\xi \text{ implies again the expression for the inverse} \\
&\qquad \operatorname{Evol}(\xi)^{-1} = \operatorname{Evol}(-\operatorname{Ad}(\operatorname{Evol}(\xi)^{-1})\xi)\,.
\end{aligned}
$$

Now we aim for the Lie bracket. Since $\delta^r : C^\infty((M, x_0), (G, e)) \to \Omega^1_{\text{flat}}(M, \mathfrak{g}) \subset \Omega^1(M, \mathfrak{g})$ is is a smooth group isomorphism, we can just take its tangent mapping at the constant e which will become a homomorphism of Lie algebras. It is a Lie derivative in the sense of [10, Chapter IX]. It has been worked out in detail in [13, 12.2–12.5] for mappings $f : M \to N$; here the only difference is that the forms are $\mathfrak{g}$-valued and that we make use the right trivilization of TG. So we choose a smooth curve $t \mapsto f(t) \in C^\infty((M, x_0), (G, e))$ with $f(0) = e$ and $\partial_t|_0 f(t) = X \in C^\infty(M, \mathfrak{g})$ with $X(x_0) = 0$; then we consider the smooth mapping $\hat{f} : \mathbb{R} \times M \to G$. By the Maurer-Cartan equation 2.2, for a vector field $Y \in \mathfrak{X}(M)$ we have

$$
\begin{aligned}
0 &= d(\delta^r \hat{f})((\partial_t, 0_M), (0_\mathbb{R}, Y)) - \big[(\delta^r \hat{f})(\partial_t, 0_M), (\delta^r \hat{f})(0_\mathbb{R}, Y)\big]^\mathfrak{g} \\
&= \partial_t \big((\delta^r \hat{f})(0_\mathbb{R}, Y)\big) - \mathcal{L}_Y \big(T\mu^{f^{-1}} \partial_t f\big) - (\delta^r \hat{f})([(\partial_t, 0_M), (0_\mathbb{R}, Y)]^{\mathfrak{X}(\mathbb{R} \times M)}) \\
&\quad - \big[(\delta^r \hat{f})(\partial_t, 0_M), (\delta^r \hat{f})(0_\mathbb{R}, Y)\big]\,.
\end{aligned}
$$

Choosing $t = 0$ and using $f(0, x) = e$ this becomes

$$
(T_e \delta^r.X)(Y) = \mathcal{L}_Y X = dX(Y).
$$

Now we can write down the Lie bracket. Since M is simply connected, its de Rham cohomology $H^1(M) = 0$, which also holds for $\mathfrak{g}$-valued 1-forms and for infinite dimensional M; see [11, Section 34]. Let

$$
d^{-1} : \{\xi \in \Omega^1(M, \mathfrak{g}) : d\xi = 0\} =: Z(M, \mathfrak{g}) \to C^\infty((M, x_0), (\mathfrak{g}, 0))
$$

be the bounded (by lemma 3.5 below) operator of the Poincaré lemma; i.e., the inverse of exterior derivative. Then for $X_i \in C^\infty((M,x_0),(\mathfrak{g},0))$ we have

$$T_e\delta^r.[X_1,X_2]^{\mathfrak{g}} = d[X_1,X_2]^{\mathfrak{g}} = [dX_1,X_2]^{\mathfrak{g}} + [X_1,dX_2]^{\mathfrak{g}}, \quad \text{thus}$$
$$[\xi_1,\xi_2]^{Z(M,\mathfrak{g})} = [\xi_1,d^{-1}\xi_2]^{\mathfrak{g}} + [d^{-1}\xi_1,\xi_2]^{\mathfrak{g}}, \text{ for } \xi_i \in Z(M,\mathfrak{g}). \qquad \square$$

3.5. **Lemma.** *For a simply connected smoothly paracompact manifold M the inverse of exterior derivative $d^{-1} : Z(M,\mathfrak{g}) \to C^\infty((M,x_0),(\mathfrak{g},0))$ is a bounded linear operator.*

Proof. For a star-shaped c^∞-open subset M in a convenient vector space E this follows from the explicit formula for the Poincaré operator $d^{-1}\xi(x) = \int_0^1 t\omega(tx)(x)dt$. In general, we cover the simply connected manifold M by open charts $u : U \to u(U) \subset E$ with $x_0 \in U$, $u(x_0) = 0$, and $u(U)$ star-shaped in E. Via the diffeomorphisms u^* the operators $d^{-1} : Z(U,\mathfrak{g}) \to C^\infty((U,x_0),(\mathfrak{g},0))$ are all bounded. Since $C^\infty((M,x_0),(\mathfrak{g},0))$ and $Z(M,\mathfrak{g})$ both carry the initial structure with respect to the restriction operators to U, the result follows. $\square$

3.6. **Theorem.** *For a regular Lie group and $\xi \in \Omega^1_{\mathit{flat}}(M,\mathfrak{g})$ we consider $f = \operatorname{Evol}(\xi) : (M,x_0) \to (G,e)$. For a mapping $h : M \to \mathfrak{g}$ we consider $\eta := \operatorname{Ad}(f)dh \in \Omega^1(M,\mathfrak{g})$. Then we have $d\eta - \frac{1}{2}[\xi,\eta]^{\mathfrak{g}}_\wedge = 0$.*

If conversely $\eta \in \Omega^1(M,\mathfrak{g})$ satisfies $d\eta - \frac{1}{2}[\xi,\eta]^{\mathfrak{g}}_\wedge = 0$ then there exists a unique smooth $h : (M,x_0) \to (\mathfrak{g},0)$ such that $dh = \operatorname{Ad}(f^{-1})\eta$.

Proof. If $h : M \to \mathfrak{g}$ exists then $d^2h = 0$ so that

$$\begin{aligned} d\eta &= d\operatorname{Ad}.Tf \wedge dh + \operatorname{Ad}(f)d^2h = (\operatorname{ad}\circ\kappa^r.Tf)\operatorname{Ad}(f)dh + 0 \\ &= \tfrac{1}{2}[f^*\kappa^r,\eta]^{\mathfrak{g}}_\wedge = \tfrac{1}{2}[\xi,\eta]^{\mathfrak{g}}_\wedge. \end{aligned}$$

Conversely, by this computation $d\eta - \frac{1}{2}[\xi,\eta]^{\mathfrak{g}}_\wedge = 0$ implies that the $\mathfrak{g}$-valued 1-form $\beta := \operatorname{Ad}(f^{-1})\eta$ is closed. Since M is simply connected, $\beta = dh$ for $h \in C^\infty(M,\mathfrak{g})$ which is unique up to addition of a constant. $\square$

3.7. **Theorem.** *Let G be a Lie group. Then via right trivialization $(\kappa^r,\pi_G) : TG \to \mathfrak{g}\times G$ the tangent group TG is isomorphic to the semidirect product $\mathfrak{g}\rtimes G$, where G acts by $\operatorname{Ad} : G \to \operatorname{Aut}(\mathfrak{g})$. So for $g,h \in G$ and $X,Y \in \mathfrak{g}$ we have:*

$$\begin{aligned} \mu_{\mathfrak{g}\rtimes G}((X,g),(Y,h)) &= (X + \operatorname{Ad}(g)Y, gh) \\ \nu_{\mathfrak{g}\rtimes G}(X,g) &= (-\operatorname{Ad}(g^{-1})X, g^{-1}), \\ [(X_1,Y_1),(X_2,Y_2)]_{\mathfrak{g}\rtimes\mathfrak{g}} &= ([Y_1,X_2]-[Y_2,X_1],[Y_1,Y_2]), \\ \operatorname{Ad}^{\mathfrak{g}\rtimes G}_{(X,g)}(Y,Z) &= (\operatorname{Ad}(g)Y - [\operatorname{Ad}(g)Z,X], \operatorname{Ad}(g)Z) \end{aligned} \tag{1}$$

If G is a regular Lie group, then so is $TG \cong \mathfrak{g}\rtimes G$ and $T\operatorname{evol}^{\mathbb{R}}_G$ corresponds to $\operatorname{evol}^{\mathbb{R}}_{TG}$, via

$$\begin{array}{ccc} TC^\infty(\mathbb{R},\mathfrak{g}) & \xrightarrow{\ \cong\ } & C^\infty(\mathbb{R},\mathfrak{g}\rtimes\mathfrak{g}) \\ {\scriptstyle T\operatorname{evol}^{\mathbb{R}}_G}\downarrow & \swarrow{\scriptstyle \operatorname{evol}^{\mathbb{R}}_{TG}} & \downarrow{\scriptstyle \operatorname{evol}^{\mathbb{R}}_{\mathfrak{g}\rtimes G}} \\ TG & \xrightarrow[\ \cong\ ]{} & \mathfrak{g}\rtimes G\,. \end{array} \tag{2}$$

In particular, for $(Y,X) \in C^\infty(\mathbb{R}, \mathfrak{g}\times\mathfrak{g}) = TC^\infty(\mathbb{R},\mathfrak{g})$*, where* X *is the footpoint, we have*

$$\operatorname{evol}^{\mathbb{R}}_{\mathfrak{g}\rtimes G}(Y,X) = \Big(\operatorname{Ad}(\operatorname{evol}^{\mathbb{R}}_G(X))\int_0^1 \operatorname{Ad}(\operatorname{Evol}^{\mathbb{R}}_G(X)(s)^{-1}).Y(s)\,ds,\ \operatorname{evol}^{\mathbb{R}}_G(X)\Big)$$

$$T_X\operatorname{evol}^{\mathbb{R}}_G.Y = T(\mu_{\operatorname{evol}^{\mathbb{R}}_G(X)}).\int_0^1 \operatorname{Ad}(\operatorname{Evol}^{\mathbb{R}}_G(X)(s)^{-1}).Y(s)\,ds, \tag{3}$$

$$T_X(\operatorname{Evol}^{\mathbb{R}}_G(\quad)(t)).Y = T(\mu_{\operatorname{Evol}^{\mathbb{R}}_G(X)(t)}).\int_0^t \operatorname{Ad}(\operatorname{Evol}^{\mathbb{R}}_G(X)(s)^{-1}).Y(s)\,ds.$$

For a pointed simply connected manifold (M,x_0) *we have*

$$\begin{aligned} T\Omega^1_{flat}(M,\mathfrak{g}) &= \{(\xi,\eta)\in\Omega^1(M,\mathfrak{g})^2 : d\xi - \tfrac12[\xi,\xi]^{\mathfrak{g}}_\wedge = 0 \text{ and } d\eta - [\xi,\eta]^{\mathfrak{g}}_\wedge = 0\}\\ &\cong \{(\eta,\xi)\in\Omega^1(M,\mathfrak{g}\rtimes\mathfrak{g}) : (d\eta - [\xi,\eta]^{\mathfrak{g}}_\wedge, d\xi - \tfrac12[\xi,\xi]^{\mathfrak{g}}_\wedge) = (0,0)\}\\ &= \Omega^1_{flat}(M,\mathfrak{g}\rtimes\mathfrak{g}) \end{aligned} \tag{4}$$

and the following diagram commutes:

$$\begin{array}{ccc} T\Omega^1_{flat}(M,\mathfrak{g}) & \xrightarrow[T\operatorname{Evol}^M_G]{} & TC^\infty((M,x_0),(G,e)) \\ \cong\downarrow & & \downarrow\cong \\ \Omega^1_{flat}(M,T\mathfrak{g}) & \xrightarrow{\operatorname{Evol}^M_{TG}} & C^\infty((M,x_0),(TG,0_e)) \\ \cong\downarrow & & \downarrow\cong \\ \Omega^1_{flat}(\mathbb{R},\mathfrak{g}\rtimes\mathfrak{g}) & \xrightarrow{\operatorname{Evol}^M_{\mathfrak{g}\rtimes G}} & C^\infty((M,x_0),(\mathfrak{g}\rtimes G,(0,e)))\,. \end{array} \tag{5}$$

Proof. The first part may be found in [11, 38.10]. We repeat the derivation of the formulas. By 2.1, for $g,h\in G$ and $X,Y\in\mathfrak{g}$, we have

$$\begin{aligned} T_{(g,h)}\mu.(R_X(g),R_Y(h)) &= T(\mu^h).R_X(g) + T(\mu_g).R_Y(h)\\ &= T(\mu^h).T(\mu^g).X + T(\mu_g).T(\mu^h).Y = R_X(gh) + R_{\operatorname{Ad}(g)Y}(gh),\\ T_g\nu.R_X(g) &= -T(\mu^{g^{-1}}).T(\mu_{g^{-1}}).T(\mu^g).X = -R_{\operatorname{Ad}(g^{-1})X}(g^{-1}), \end{aligned}$$

so that μ_{TG} and ν_{TG}, and the Lie bracket, after right trivialization, are given by

$$\begin{aligned} \mu^{\mathfrak{g}\rtimes G}((X,g),(Y,h)) &= (X+\operatorname{Ad}(g)Y, gh)\\ T\mu^{\mathfrak{g}\rtimes G}_{(X,g)}(Y',h') &= (\operatorname{Ad}(g)Y', T\mu_g h'),\quad h'\in T_hG\\ \nu^{\mathfrak{g}\rtimes G}(X,g) &= (-\operatorname{Ad}(g^{-1})X, g^{-1}),\\ (X,g).(Y,h).(X,g)^{-1} &= (X+\operatorname{Ad}(g)Y, gh).(-\operatorname{Ad}(g^{-1})X, g^{-1})\\ &= (X+\operatorname{Ad}(g)Y - \operatorname{Ad}(ghg^{-1})X, ghg^{-1})\\ \operatorname{Ad}^{\mathfrak{g}\rtimes G}(X,g)(Y',h') &= (\operatorname{Ad}(g)Y' - \operatorname{Ad}(e)[\operatorname{Ad}(g)h',X], \operatorname{Ad}(g)h'])\\ \operatorname{ad}^{\mathfrak{g}\rtimes G}(X',g')(Y',h') &= ([g',Y'] - [[g',h'],0]) - [h',X'],[g',h']\\ [(X_1,Y_1),(X_2,Y_2)]_{\mathfrak{g}\rtimes\mathfrak{g}} &= ([Y_1,X_2]-[Y_2,X_1],[Y_1,Y_2])\,. \end{aligned} \tag{6}$$

That diagram (2) commutes and equations (3) hold, has been proven in [11, 38.10]. Now we prove that diagram (5) commutes. For the bottom square this follows from theorem 3.3. We consider a curve $t\mapsto\xi(t)\in\Omega^1_{\text{flat}}(M,\mathfrak{g})$ which is smooth into

$\Omega^1(M,\mathfrak g)$. Then $\partial_t|_0\xi(t) =: \eta\in T_{\xi(0)}\Omega^1_{\text{flat}}(M,\mathfrak g)$ so that $d\eta-[\xi(0),\eta]^{\mathfrak g}_\wedge=0$. As in the beginning of the proof of theorem 3.3 we now consider the smooth curve of flat principal connections

$$t\mapsto \omega^{\xi(t)}=\kappa^l-(\operatorname{Ad}\circ\nu).\xi(t)$$

on $M\times G$, and we let $L^{\xi(t)}\subset M\times G$ be the horizontal leaf through (x_0,e) for the connection $\omega^{\xi(t)}$. Recall that

$$(\operatorname{Id}_M,\operatorname{Evol}^M_G(\xi(t)))=(\operatorname{pr}_1|L^{\xi(t)})^{-1}:M\to L^{\xi(t)}\subset M\times G$$

is the horizontal lift. Likewise, writing $\xi=\xi(0)$,

$$(\operatorname{Id}_M,\operatorname{Evol}^M_{\mathfrak g\rtimes G}(\eta,\xi)=(\operatorname{pr}_1|L^{(\eta,\xi)})^{-1}:M\to L^{(\eta,\xi)}\subset M\times(\mathfrak g\rtimes G)$$

where $L^{(\eta,\xi)}$ is the horizontal leaf through $(x_0,0,e)$ of the principal connection

$$\omega^{(\eta,\xi)}=\kappa^{l,\mathfrak g\rtimes G}-(\operatorname{Ad}^{\mathfrak g\rtimes G}\circ\nu^{\mathfrak g\rtimes G})(\eta,\xi)\,.$$

Claim. $\partial_t|_0L^{\xi(t)}$ is the horizontal leaf $L^{(\eta,\xi)}$ of the flat principal connection $\omega^{(\eta,\xi(0))}$ on the principal bundle $M\times(\mathfrak g\rtimes G)\to M$. This is sufficient to finish the proof.

$T_{(x_0,e)}L^{\xi(t)}$ consists of all $(Y,X(t))\in T_{x_0}M\times\mathfrak g$ such that $\omega^{\xi(t)}_{(x_0,e)}(Y,X(t))=X(t)-\xi(t)(Y)=0$; since $L^{\xi(t)}$ is a horizontal leaf we may fix Y. Then

$$\partial_t|_0\omega^{\xi(t)}_{(x_0,e)}(Y,X(t))=X'-\eta(Y)=0\,.$$

On the other hand $\omega^{(\eta,\xi)}_{(x_0,(0,e))}(Y,(Z,X))=(Z-\eta(Y),X-\xi(Y))$ since for $(X,g)\in\mathfrak g\rtimes G$ and $(X',g')\in T_X\mathfrak g\times T_gG$ we have

$$\begin{aligned}
&\omega^{(\eta,\xi)}_{(x,(X,g))}\big(Y,(X,X';g,g')\big)=(T\mu^{\mathfrak g\rtimes G}_{(X,g)^{-1}})(X',g')-\operatorname{Ad}^{\mathfrak g\rtimes G}((X,g)^{-1})(\eta(Y),\xi(Y))\\
&=(T\mu^{\mathfrak g\rtimes G}_{(-\operatorname{Ad}(g^{-1})X,g^{-1})})(X',g')-\operatorname{Ad}^{\mathfrak g\rtimes G}(-\operatorname{Ad}(g^{-1})X,g^{-1})(\eta(Y),\xi(Y))\\
&=(\operatorname{Ad}(g^{-1})X',T\mu_{g^{-1}}g')\\
&\qquad-(\operatorname{Ad}(g^{-1})\eta(Y)-[\operatorname{Ad}(g^{-1})\xi(Y),-\operatorname{Ad}(g^{-1})X],\operatorname{Ad}(g^{-1})\xi(Y))\\
&=\big(\operatorname{Ad}(g^{-1})X'-\operatorname{Ad}(g^{-1})\eta(Y)-\operatorname{Ad}(g^{-1})[\xi(Y),X],T\mu_{g^{-1}}g'-\operatorname{Ad}(g^{-1})\xi(Y)\big).\quad\square
\end{aligned}$$

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