Math Finance (cont. time), SS25 , Sheet 1

1. Let $\mathcal{G}$ be a finite σ-algebra and $A_1, \ldots, A_n$ be the corresponding partition. Show that a random variable X is $\mathcal{G}$ measurable if and only if X is constant on each cell of the partition.
2. Recall the concept of conditional expectation and prove two of the properties discussed in the lecture.
3. Show that if H is a bounded trading strategy and X is a martingale, then the process $((H \cdot X)_t)_{t=0}^T$ is a martingale as well.
4. Show that X is a martingale iff $\mathbb{E}(H \cdot X)_T = 0$ for all bounded trading strategies H.
5. Show that if X is a martingale, then X satisfies NA.
6. Construct a model which admits an arbitrage opportunity and a model which satisfies NA. (Ideally specify $\Omega, \mathcal{F}, \mathbb{P}, (\mathcal{F}_t)_{t=0}^T, (X_t)_{t=0}^T$.)

Math Finance (cont. time), SS25 , Sheet 2

In the problems below, assume that we are working on a finite space Ω, etc.

1. A sub-martingale is an adapted integrable process X such that $\mathbb{E}[X_{t+1}|\mathcal{F}_t] \geq X_t$, $t < T$. Show the Doob-martingal decomposition: There exist unique processes M, A such that
   - $A_0 = 0$, $t \mapsto A_t$ is increasing, A is predictable.
   - M is a martingale.
   - $X = M + A$.

2. Construct a model which admits an arbitrage opportunity and a model which satisfies NA. (Specify $\Omega, \mathcal{F}, \mathbb{P}, (\mathcal{F}_t)_{t=0}^T, (X_t)_{t=0}^T$.)

3. Geometric interpretation of the NA-condition: Show that NA is equivalent to $K = \{(H \cdot X)_T : H \in \mathcal{H}\} \cap L^+ = \{0\}$, where L^+ denotes the positive orthand, i.e. the set of all non negative RV.

   Geometric interpretation of the martingale condition: Show that $\mathbb{Q}$ is an equivalent martingale measure if and only if $\mathbb{Q} \in P_+ \cap \{\sigma \in L(\Omega, \mathbb{P})^d : \langle Z, \sigma \rangle = 0 \ \forall Z \in K\}$, where P_+ denotes the set of probability measures with support Ω.

   Draw pictures!

4. Show that the set of all absolutely continuous martingale measures is compact and convex in $L(\Omega, \mathbb{P})^d$. ($L(\Omega, \mathbb{P})^d$ is equipped with the topology stemming from the identification with $\mathbb{R}^N$.)

5. Give examples showing that $M^e(X)$ may be compact but is not necessarily compact.

Math Finance (cont. time), SS25, Sheet 3

1. Construct a martingale model in 2 dimensions explicitly.

2. Construct a model in 2 dimensions which does not have an equivalent martingale measure.

3. Assume that the underlying model X satisfies NA. Show that if a derivative Z is attainable through a strategy (a, H) at a price p then p is an arbitrage free price.

4. Assume that the underlying model X satisfies NA. Show that if a derivative Z is attainable through a strategy (a, H) at a price p then any $q \neq p$ is not an arbitrage free price.

5. Assume that the underlying model X satisfies NA and let $\mathbb{Q}$ be an equivalent martingale measure. Show that $\mathbb{E}_{\mathbb{Q}}[Z]$ is an arbitrage free price for the derivative Z.

6. Assume that the underlying model X satisfies NA and that every derivative Z is attainable. Show that there exists only one equivalent martingale measure.

Math Finance (cont. time), SS25, Sheet 4

1. If $\mathcal{M}^e = \{\mathbb{Q}\}$ then every $\mathbb{Q}$-martingale M has the representation

$$M_t = M_0 + \sum_{k=1}^{t} H_k \cdot (X_k - X_{k-1})$$

for some predictable process H.

2. Conversely, if every $\mathbb{Q}$-martingale M has the representation

$$M_t = M_0 + \sum_{k=1}^{t} H_k \cdot (X_k - X_{k-1})$$

for some predictable process H, then $\mathcal{M}^e = \{\mathbb{Q}\}$.

3. Let $\Omega = \{-1, 1\}^N$, $Y_n(y_1, \ldots, y_N) = y_n$, and $\mathcal{F}_n = \sigma(Y_1, \ldots, Y_n)$. Furthermore, let $X_n = c + Y_1 + \ldots + Y_n$, $c \in \mathbb{R}$, be the stock price process in our model.

   Show that the model is complete and that the unique martingale measure $\mathbb{Q}$ satisfies the following conditions:

   - $\mathbb{Q}(Y_n = 1) = 1/2, \quad n \leq N$
   - $\mathcal{F}_n$ and Y_{n+1} are independent for $n < N$.

   (Hint: An event A is independent of the σ-algebra $\mathcal{G}$ if and only if $\mathbb{P}[A|\mathcal{G}] = \mathbb{P}[A]$.)

4. Consider again the model from the previous example. Let a derivative $f(X_N)$ be given. Find a recursion for the value process $v_t(X_t) := V_t := \mathbb{E}[f(X_N)|\mathcal{F}_t]$ and the strategy $H_t = h_t(X_{t-1})$ which satisfies

$$\mathbb{E}[f(X_N)] + (H \cdot X)_N = f(X_N).$$