Singular dynamical systems and Picard solutions of PDE

Applicant and PI: Paolo Giordano

National cooperation partner: Michael Kunzinger

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Abstract: The theory of continuous dynamical systems can be considered as one of the best achievements of mathematics, with applications in physics, biology, economics, social sciences, neurosciences and meteorology, to mention only a few examples. In non-smooth physics (e.g. impact mechanics or discontinuous Lagrangians in optics), neurosciences (e.g. physical modeling of firing neurons using differential equations (DE)), social sciences (e.g. opinion formation after posting of influencers on social media), the use of generalized functions (GF) to describe the dynamics represents an important and necessary generalization. In the first part of the present project (work package 1, WP 1), we propose

to start developing the theory of continuous dynamical systems using generalized smooth functions (GSF). The nonlinear theory of GSF has recently emerged as a minimal extension of Colombeau's theory that allows for more general domains for generalized functions, resulting in the closure with respect to composition, a better behavior on unbounded or purely infinitesimal sets and new general existence results for DE.

In the second part of this project, WP 2, we plan to apply this theory of singular dynamical systems to the dynamics of networks of nonlinear lumped parameter models (LPM) as used in nonlinear mechanics and in idealized models of car crash modeling, in particular with several types of singular loadings. This type of examples will also be studied as complex systems (CS) in the PI's theory of *interaction spaces* (IS), a universal mathematical theory of CS. We can e.g. represent a car as a, more or less detailed, network of LPM; these cars are subsequently represented as moving point-particles (thereby forgetting their internal structure as a network of LPM) on a simulated road where car accidents may randomly occur. We have hence a natural hierarchical description of the whole system with two CS, and the detailed one has to be considered only when an impact occurs.

In the last part of the project, WP 3, we propose to extend the recent results of the PI about the Picard-Lindelöf theorem (PLT, see [Gio-L22, Gio-LK22]) for PDE, to cases where the maximum order d of the normal derivatives in t is less than or equal to the order L of the derivatives in x (see below in Sec. 2.2). We also plan to consider not only Cauchy problems, but different type of boundary value problems (BVP). The results of this WP 3 are planned both for GSF theory and also for nonlinear ordinary smooth PDE.

1 Aims and research objectives

The main objectives of the present research project are to initiate the development of continuous dynamical systems theory using GSF, to apply this theory to examples of singular nonlinear dynamical systems in nonlinear mechanics, and finally to extend the present PLT for PDE ([Gio-L22, Gio-LK22]) to a more general class of (nonlinear, singular or smooth) BVP. In the two first parts, our main aim is to arrive at proving a Hartman-Grobman theorem for GSF ODE and to apply these results to far-reaching examples of impact mechanics and simulations of car accidents. In the last part, our main aim is to find a general definition of Picard and Newton operators with loss of derivatives, and to apply these notions to BVP where the greatest t-derivative is less or equal to the greatest x-derivative, a case that is not included in the examples of the present PLT for PDE. GSF are an extension of classical distribution theory which makes it possible to model nonlinear singular problems, while at the same time sharing a number of fundamental properties with ordinary smooth functions, such as the closure with respect to composition, a good multidimensional integration theory and several non trivial classical theorems of the calculus, see [Gio-Kun-V15, Gio-Kun-V19, Gio-Kun18a, Gio-Kun16, L-Gio-K22, LL-Gio16]. One could describe GSF as a methodological restoration of Cauchy-Dirac's original conception of generalized function (GF), see [20]. In essence, the idea of Cauchy and Dirac (as well as of Poisson, Kirchhoff, Helmholtz, Kelvin and Heaviside) was to view generalized functions as certain types of smooth set-theoretical maps obtained from ordinary smooth maps depending on suitable infinitesimal or infinite parameters. GSF are a minimal extension of Colombeau's theory of generalized functions (CGF), see [10, 31, 36]. In fact, when the domain is the set $\widetilde{\Omega}_c$ of compactly supported generalized points in the open set $\Omega \subseteq \mathbb{R}^n$, then the two spaces of GF coincide, cf. [Gio-Kun-V15]. Therefore, we expect that the directions envisaged in the present project will also exert a considerable impact on Colombeau's theory. For these reasons, the department of Mathematics of the University of Vienna, and in particular the research group of Prof. M. Kunzinger (see `http://www.mat.univie.ac.at/~mike/`), constitute the ideal place where to implement the present research project, because of the group's specific competencies on generalized functions, functional analysis and PDE.

A concise presentation of the project's main aims is as follows:

WP 1: Theory of singular dynamical systems

Problems and motivations: At present, continuous dynamical systems theory can be applied only to sufficiently regular ODE, e.g. differentiable or locally Lipschitz, and not to GF. On the other hand, several applications in different disciplines require to describe the dynamics using nonlinear ODE with GF such as different types of Dirac delta or Heaviside-like functions. See e.g. nonlinear singular mechanics [43, 42, 41, Kun-OSV, 7, 25, 26] and discontinuous Lagrangians in geometrical optics [21], to mention only a few meaningful examples. It is hence worth to start a generalization of continuous dynamical systems theory to GF. The important theoretical problem to generalize Hamiltonian mechanics to GF can be traced back to J.E. Marsden in [25, 26] (even if this attempt presented serious problems due to limitations of classical distribution theory, as proved in [Kun-OSV]).
The idea and the plan: In [L-Gio-K22], we already developed the classical theory of singular nonlinear ODE for GSF, including a PLT, Gronwall's inequality, continuous dependence on the initial conditions, etc., thereby going well beyond the classical theory of distributions. Thanks to these results and to the long list of properties shared between GSF and smooth functions, we can surely extend continuous dynamical systems theory up to its foundational results, including a generalization of the Hartman-Grobman theorem, see e.g. [50, 40].
Innovative features and deliverables: Thanks to the properties of GSF, we aim at starting a general theory including the classical one as well as singular nonlinear ODE. Besides all the potential fields of applications we mentioned above, we are in particular interested in applying our approach to the mechanics of networks of nonlinear singular LPM, see e.g. [10, 43, 42, 41, 35, 29, 19], as used for example in car crash modeling.

WP 2: Hierarchical description of car crash modeling

Problems and motivations: A new theory of generalized continuous dynamical systems will find a greater diffusion in applied sciences if we can showcase meaningful examples governed by a nonlinear ODE subject to singular initial conditions. Moreover, the dynamics of CS is frequently described as depending on a hierarchy of linked systems [15, 45, 17, 47]. It would hence be interesting to study not only coupled ODE, like in classical dynamical systems theory, but more generally systems described by cause-effect interactions between interacting entities and some hierarchical linking preserving these cause-effect relations. Some examples of hierarchical CS are: the brain, urban systems, the immune system, organisms in biology, social systems, etc. (see references above). In this way, we would have a further far-reaching generalization of dynamical systems theory outlined through this link with CS modeling and relevant examples.
The idea and the plan: First of all, CS can be studied within the applicant's mathematical theory of *interaction spaces* (IS), a universal mathematical theory of CS where we can embed cellular automata, agent based models, master equation based models, networked dynamical models, neural networks and evolutionary algorithms in a single notion, see [Gio22, V-Gio-A14, V-Gio-AA1, V-Gio-AA2]. In this

theory, a general notion of cause-effect relation between interacting entities can be defined, and hence the notion of hierarchy of CS can be introduced using category-theory functors preserving such a relation. This includes coupled ODE as a particular case, and in the first part of this WP 2 we plan to first develop this general theory because this would have applications in several interesting CS, as listed above in the statement of the problems. We then plan to apply the results of WP 1 and this theory of hierarchical CS to two classes of problems: 1) Networks of LPM to model nonlinear behavior of materials and idealized models of car crash mechanics; see e.g. [43, 42, 41] for simple LPM in the study of nonlinear material using CGF, or [35] for the use of LPM in vehicle crash studies (without the use of GF); 2) Using freely available software code (see e.g. [V-Gio-A14, 55, 8]), we plan to couple the previous description of a vehicle using LPM with a traffic flow model. The system is hence represented both at a more detailed level (first IS) with a network of LPM for each car, and also (second IS) by forgetting the detailed structure (thus using a forgetful functor) and representing cars as single point-particles. The main aim is hence to simulate car crashing in traffic flow, with clear utility in traffic planning and related economical estimates.

Innovative features and deliverables: By applying the planned general theory of WP 1, at least *for simple networks* of LPM, we aim to: 1) study (analytically or numerically) the stability and instability of this type of systems, 2) understand the properties of their stable and unstable manifolds, 3) apply the Hartman-Grobman theorem as a way to describe the linearization of the dynamics. In this WP 2, we also plan to 4) develop a general theory of hierarchy of CS, 5) show that the natural links between two hierarchically connected CS in the study of car accidents in traffic flow can be formalized using IS theory.

WP 3: Picard solutions of (singular) PDE

Problems and motivations: The PLT for PDE has been proved in [Gio-L22] for Cauchy problems in normal form $\partial_t^d y(t,x) = F\left[t, x, (\partial_x^\alpha \partial_t^\gamma y)_{\substack{|\alpha|\le L\\ \gamma\le p}}\right]$, where F is an arbitrary smooth function. Even if the proof of the PLT does not explicitly assume a condition of the form $d > L$, the investigated class of *non-analytic examples* considers arbitrary smooth initial conditions y_{0j} for $j = 0, \ldots, \gamma - 1$, but with $\|y_{0j}\|_{k+(n+1)L} \sim (nL)^{\sigma_j nL}$ as $n \to +\infty$ for $j = \gamma, \ldots, d-1$, where $d > \sigma_j L$. Similar limitations can also be considered when F and y are GSF, as in [Gio-LK22]. Can we prove an extension of the PLT to cases where $d \le L$ and to different type of BVP as well as with mixed normal derivatives $\partial_t^d \partial_x^e y(t,x)$ on the left hand side of the PDE?

The idea and the plan: The proof of the PLT is based on the Banach fixed point theorem for contractions P with loss of derivatives in a graded Fréchet space: $\left\|P^{n+1}(y_0) - P^n(y_0)\right\|_k \le \alpha_{kn} \left\|P(y_0) - y_0\right\|_{k+nL}$, see Sec. 2.2. In [13], a general definition of both Picard and Newton iterative operators is given; we first want to extend these definitions to operators with loss of derivatives. We also already numerically studied the convergence of a more general iterative process for the heat equation, noting that it is normal in both variable t and x, see Sec. 3. From the numerical perspective, this process converges to a solution satisfying both boundary conditions in x. We first want to embed this numerical example into a general definition of Picard operator with loss of derivatives, and then prove the convergence of the iterations using the Banach fixed point theorem with loss of derivatives. We also plan to proceed similarly for the wave, Laplace and Burgers equations, and then try to prove a general result including all of these as particular examples. We also plan to extend Neuberger's idea expressed in [32] to consider boundary or supplementary functional conditions that remain invariant under the iterated transformations (but using the PLT and not the Cauchy-Kowalevski theorem like in [32]). Finally, as proved in past works ([Gio-Kun-V15, Gio-Kun-V19, Gio-Kun16, Gio-Kun18a, Gio-LK22, L-Gio-K22]), the generalization to the GSF case is usually straightforward, and we only need to pay more careful attention in extending the results to hyperfinite iterations, i.e. to an infinite integer Colombeau generalized number $N = [n_\varepsilon] \in \widetilde{\mathbb{R}}$, $n_\varepsilon \in \mathbb{N}$, of iterations. In fact, hyperfinite iterations allow one to include also non-infinitesimal contraction constants α_{kn} (see Sec. 2.2).

Innovative features and deliverables: The results of this WP 3 would allow us to extend the scope of the PLT to a class of nonlinear singular PDE with $d \leq L$, which includes classical interesting cases such as the wave, heat, Laplace and Burgers PDE with the related boundary conditions, as well as many other non analytic or generalized PDE. This extension would allow one to consider, as stated in [33] for ODE, the PLT as *a position of strength from which to study a wide variety of* PDE, i.e. a new general theory for local existence both in classical smooth and in singular nonlinear PDE.

2 State of the art

2.1 State of the art in the research field

Generalized functions

J.F. Colombeau's theory of generalized functions allows one to perform nonlinear operations between embedded distributions, avoiding the difficulty of the Schwartz impossibility theorem. See e.g. [31, G-Kun-OS01, 36, 10] for an introduction with applications. This theory makes it possible to find generalized solutions of some well-known PDE which do not have solutions in the classical space of distributions, see [36], and has manifold applications, e.g. to the theory of elasticity, fluid mechanics and in the theory of shock waves (see e.g. [10, 36]), to differential geometry and relativity theory [G-Kun-OS01, Kun-OSV] and to quantum field theory. A new and fundamental step in the theory of generalized functions based on Colombeau generalized numbers, which presents several analogies with our present proposal, has first been achieved in [4, 5]. In these works, the basic idea is to generalize the derivative as a limit of an incremental ratio taken with respect to the e-norm, [4] and with increments which are asymptotic to invertible infinitesimals of the form $[\varepsilon^r] \in {}^\rho\widetilde{\mathbb{R}}$, for $r \in \mathbb{R}_{\geq 0}$. This theory extends the usual classical notion of derivative and smoothness to set-theoretical functions on Colombeau generalized numbers, e.g. of the form $f : \widetilde{\mathbb{R}}^n \longrightarrow \widetilde{\mathbb{R}}^d$, and enables one to prove that every CGF is infinitely differentiable in this new sense. Several important applications have already been achieved (see [5]) and hence the theory promises to be very relevant. As explained in greater detail in [Gio-Kun-V15, Gio-Kun-V19, Gio-Kun16], the notion of smoothness developed in [4] includes functions like $i(x) = 1$ if x is infinitesimal and $i(x) = 0$ otherwise. This makes it impossible to prove classical results like the intermediate value theorem, whereas for GSF this theorem holds. The theory of GSF, on the other hand, while fully compatible with the approach in [4], singles out a subclass of smooth functions with more favorable compatibility properties with respect to classical calculus and hence may be viewed as a refinement of that theory.

Colombeau theory can also be approached in the framework of nonstandard analysis, so as to get a field of scalars and not only a ring, and an interesting axiomatic characterization, see e.g. [51] and references therein.

Hierarchical complex systems and lumped parameter models

Most real-world dynamical systems consist of subsystems, possibly from different physical domains, modeled e.g. by PDE, ODE or algebraic equations, and combined with input and output connections. To deal with such CS, different methods can be used, from coupled continuous dynamical systems as exploited in synergetics [2, 15, 17, 18, 47, 45], agent-based models [54, 14], cellular automata models [24], up to the recent energy based modeling using the class of port-Hamiltonian systems, which are very popular in electrical, mechanical and electromechanical domains [46].

Already in the works of Colombeau [10] and Oberguggenberger [36], we can find interesting examples including suitable descriptions of elasticity and elastoplasticity in solids. The mathematical impossibility of discontinuous solutions within the classical approach and ambiguity in systems of nonconservative form have been shown. However, using CGF, [10] arrives at a detailed study of jump conditions in elastoplastic shock waves and elastic precursors. In particular, based on mass conservation, momentum conservation, energy conservation and Hooke's laws, a system of equations is obtained for describing deformation processes in solids. The elastoplatic shock in contact discontinuity has been modelled using Heaviside type functions between strain and stress, see [10].

In the works [41, 42, 43], CGF have been used for the analysis of the response of mechanical systems described by springs, dashpots and masses to a step input representing a suddenly applied force or deformation. A careful analysis leads [41, 42, 43] to several results: the full understanding of the relations between the CGF solution and classical smooth solutions far from the the singularity using the properties of Colombeau association $\approx$, a detailed limit passage in a sequence of approximating problems, and the determination of the new initial conditions after the singularity directly from the governing ODE, to name just a few. There are some analogies between the present proposal and these works. Here, we only highlight three differences: 1) In our approach we want to consider (in principle, arbitrary) networks of several types of these systems (thereby, free composition of GSF is an essential property); 2) Thanks to general existence results such as the PLT and the characterization of distributions among GSF (see [L-Gio-K22]), we plan to use directly the equality $=$ of ${}^{\rho}\widetilde{\mathbb{R}}$ rather than association $\approx$ to obtain similar results both to find out where the GSF solution is a distribution and where it is an ordinary smooth function; 3) Find the value of ε corresponding to the representative f_ε of the GSF solution $f = [f_\varepsilon(-)]$ having the best agreement with given experimental data.

Even if *nonlinear* GF do not appear to be used in classical nonlinear mechanics (outside the CGF-community), in order to understand better the present proposal, it can be useful to describe the classical approach, and to link it, where possible, to the nonlinear generalization used by [41, 42, 43]. In general, an LPM is a simplified description of a spatially distributed physical systems, such as a multibody system, into discrete entities having a well-known behavior. In classical nonlinear mechanics with LPM, the basic governing ODE relates the stress σ and the strain s, see e.g. [53]:

$$a_0\sigma + a_1\dot{\sigma} + a_2\ddot{\sigma} + ... = b_0 s + b_1\dot{s} + b_2\ddot{s} + ... \tag{2.1}$$

where the dots denote derivatives with respect to time. Practically, the form of this equation depends on the chosen model. For example, a classical lump parameter Kelvin-Voidt model could be described as a spring, a damper and a point mass attached at the end of the spring. The damper element is included to model a viscoelastic behavior. In this case, $\sigma = Es + \eta\dot{s}$ and the viscosity η is considered as a constant, which is also not true for high stress deformations. These models could be easily converted into forms suitable for description of an objects' motion (we call this *geometry structural* *formulation*) where forces replace stresses and displacement replaces strain. For example, for the Kelvin-Voidt model, we can describe the crashing test by a second order ODE $m\ddot{x} + 2\beta\dot{x} + kx = 0$, where x is the displacement and $k = \frac{EA}{L}$, $2\beta = \frac{\eta A}{L}$; here, for simplicity, we are considering a bar of length L, cross-section area A and mass m. Despite their approximate nature, this kind of models resulted in several applications, and for car crashing test modeling, we can cite: creation of an LPM for real experiments, analysis and validation, [37]; description of modeling strategies applied in replicating nonlinear vehicle crash event and occupant kinematics in an occupant protection loadcase, [35]; comparison and analysis of different LPM, [1]; a double-spring-mass-damper system, where the front mass represents the vehicle-chassis and the rear mass represents the passenger compartment has been considered in [29]; development of a three degree-of-freedom LPM for crash and support in the vehicle development process, [35].

General theories for PDE and iterative methods

In order to highlight the main differences with respect to general theories for PDE, classical iterative methods mainly used in numerical calculus and our approach based on the PLT, we can briefly cite: 1) The theory of Sobolev gradients [33], where the problem to find supplementary/boundary conditions that uniquely determine the solution of a PDE is faced in general terms. 2) The classical idea to transform a PDE into an ODE in an infinite dimensional space (typically a Hilbert space) and then to use a generalized version of the PLT *for ODE in these spaces*, see [48]; despite a similar name, it is clear that our version of the PLT is directly stated for PDE using graded Fréchet spaces and with contraction operators with loss of derivatives, without the need to transform the PDE into an equivalent infinite dimensional ODE. 3) Similar differences can also be underscored for iterative methods used to solve nonlinear equations, see e.g. [3] and references therein: usually this theory is developed in Banach spaces and not for operators with loss of derivatives in Fréchet spaces; differently with respect to the frequently considered Newton iterative method, in our Banach fixed point theorem with loss of derivatives, we do not assume that the operator is differentiable; finally, in case of iterations of Picard type, the works are only restricted to ODE, where the classical PLT can be used.

2.2 State of the art in applicant's research

In this section, we briefly introduce some of the key mathematical notions needed in the present research proposal. Some basic notations we will use in the following are: nets in the variable $\varepsilon \in I := (0, 1]$ are written as $(x_\varepsilon) \in (\mathbb{R}^n)^I$; if (x_ε) is a net of real numbers, $x = [x_\varepsilon]$ denotes the corresponding equivalence class with respect to the equivalence relation $(x_\varepsilon) \sim_\rho (y_\varepsilon)$ iff $|x_\varepsilon - y_\varepsilon| = O(\rho_\varepsilon^m)$ for every $m \in \mathbb{N} = \{0, 1, 2, \ldots\}$.

The ring of Robinson-Colombeau ${}^\rho\widetilde{\mathbb{R}}$

Given a net $\rho = (\rho_\varepsilon) : I \longrightarrow I$ such that $\rho_\varepsilon \to 0$ as $\varepsilon \to 0^+$ (which is called a *gauge*), the ring ${}^\rho\widetilde{\mathbb{R}}$ is the quotient of the ring of ρ*-moderate nets* ($\exists N \in \mathbb{N} : \ x_\varepsilon = O(\rho_\varepsilon^{-N})$) modulo ρ*-negligible nets* ($\forall n \in \mathbb{N} : \ x_\varepsilon = O(\rho_\varepsilon^n)$). The point of view of GSF is frequently that of a theory where ${}^\rho\widetilde{\mathbb{R}}$ acts as the ring of scalars for all the subsequent constructions. For example, the sharp topology is conveniently defined using the absolute value $|[x_\varepsilon]| := [|x_\varepsilon|] \in {}^\rho\widetilde{\mathbb{R}}$ and the balls $B_r(x) := \{y \in {}^\rho\widetilde{\mathbb{R}}^d \mid |y - x| < r\}$, where $r > 0$ means being a *strictly positive* generalized number, i.e. $r \in {}^\rho\widetilde{\mathbb{R}}_{\ge 0}$ and r is invertible. In this proposal, we use the notation $\mathrm{d}\rho := [\rho_\varepsilon] \in {}^\rho\widetilde{\mathbb{R}}$. We also recall that if $A_\varepsilon \subseteq \mathbb{R}^n$, then $[A_\varepsilon] :=$

$\left\{[x_\varepsilon] \in {}^\rho\widetilde{\mathbb{R}} \mid \forall^0\varepsilon:\ x_\varepsilon \in A_\varepsilon\right\}$ and $\langle A_\varepsilon\rangle := \left\{x \in {}^\rho\widetilde{\mathbb{R}} \mid \forall[x_\varepsilon] = x\,\forall^0\varepsilon:\ x_\varepsilon \in A_\varepsilon\right\}$ are resp. the internal and the strongly internal set generated by (A_ε). They are resp. closed and open in the sharp topology, [Gio-Kun-V15].

Generalized smooth functions

If $X \subseteq {}^\rho\widetilde{\mathbb{R}}^n$ and $Y \subseteq {}^\rho\widetilde{\mathbb{R}}^d$ are arbitrary subsets of generalized numbers, a GSF $f \in {}^\rho\mathcal{GC}^\infty(X,Y)$ can be simply defined as a set-theoretical map $f : X \longrightarrow Y$ such that

$$\exists(f_\varepsilon) \in \mathcal{C}^\infty(\mathbb{R}^n,\mathbb{R}^d)^I\,\forall[x_\varepsilon] \in X\,\forall\alpha \in \mathbb{N}^n:\ (\partial^\alpha f_\varepsilon(x_\varepsilon)) \text{ is } \rho-\text{moderate and } f(x) = [f_\varepsilon(x_\varepsilon)], \tag{2.2}$$

see [Gio-Kun-V15, Gio-Kun-V19]. If (2.2) holds, we say that the net (f_ε) *defines* f. If we consider a net $(f_\varepsilon) \in \mathcal{C}^p(\mathbb{R}^n,\mathbb{R}^d)^I$, we can similarly define the space of *generalized* $\mathcal{C}^p$ *functions*, ${}^\rho\mathcal{GC}^p(X,Y)$, see e.g. [G-GiorBryz-L22]. If $X = \widetilde{\Omega}_c$, the set of compactly supported points in the open set $\Omega \subseteq \mathbb{R}^n$, then ${}^\rho\mathcal{GC}^\infty(\widetilde{\Omega}_c, {}^\rho\widetilde{\mathbb{R}})$ coincides exactly with the set-theoretical maps induced by all the CGF of the algebra $\mathcal{G}^{\mathrm{s}}(\Omega)$, see [Gio-Kun-V15]. The greater flexibility in the choice of the domains X leads, e.g., to the closure of GSF with respect to composition, to the extreme value theorem on closed intervals even bounded by infinite numbers, to solutions of ODE existing only on an infinitesimal set, or also to inverses of given GSF, see [Gio-Kun-V15, Gio-Kun-V19, L-Gio-K22, Gio-Kun16]. Classical theorems like the chain rule, existence and uniqueness of primitives, integration by change of variables, the intermediate value theorem, mean value theorems, the extreme value theorem, Taylor's theorem in several forms for the remainder, suitable sheaf properties, the local inverse and implicit function theorems, some global inverse function theorems, the Banach fixed point theorem, the PLT and several results in the classical theory of the calculus of variations and optimal control, hold for these GSF, see [Gio-Kun-V15, Gio-Kun-V19, Gio-Kun16, LL-Gio16, L-Gio-K22, G-GiorBryz-L22]. One of the distinguishing properties of GSF is that these extensions of classical theorems for smooth functions have very natural statements, formally similar to the classical ones. All this underscores the different philosophical approach as compared to [4] (and to the more classical Colombeau theory), which constitutes a more general approach, but where some of these classical theorems do not hold.

We also recall that a *functionally compact set* is a sharply bounded internal set $K = [K_\varepsilon] \subseteq B_R(0)$, for some $R \in {}^\rho\widetilde{\mathbb{R}}_{>0}$, generated by a net $K_\varepsilon \Subset \mathbb{R}^n$ of compact sets. Secondly, a *solid set* is a set $S \subseteq {}^\rho\widetilde{\mathbb{R}}^n$ whose interior in the sharp topology is dense in S; the latter allows us to deal with partial derivatives at boundary points. For example, every closed interval $[a,b] \subseteq {}^\rho\widetilde{\mathbb{R}}$ is functionally compact and solid. On functionally compact sets, GSF satisfy the extreme value theorem and on measurable functionally compact sets they can be integrated: $\int_K f\,\mathrm{d}\mu = \left[\int_{K_\varepsilon} f_\varepsilon\,\mathrm{d}\mu\right] \in {}^\rho\widetilde{\mathbb{R}}$, see [Gio-Kun-V19]. The space ${}^\rho\mathcal{GC}^\infty(K, {}^\rho\widetilde{\mathbb{R}}^d)$ shares many properties with the classical Fréchet spaces of ordinary smooth functions defined on a compact set. In particular, these spaces are sharply Cauchy complete and their sharp topology can be defined using a countable family

$$\|f\|_k := \left[\max_{\substack{|\alpha|\le k\\ 1\le k\le d}} \sup_{x\in\mathbb{R}^n} |\partial^\alpha f_\varepsilon^k(x))|\right] \in {}^\rho\widetilde{\mathbb{R}} \quad \forall k \in \mathbb{N}, \tag{2.3}$$

of ${}^\rho\widetilde{\mathbb{R}}$-valued norms, see [Gio-Kun18a, Gio-LK22].

Picard-Lindelöf theorem for ODE and PDE

In [L-Gio-K22, Gio-LK22], we proved a generalization of the Banach fixed point theorem and of the corresponding PLT applicable to any normal generalized Cauchy problem for PDE, see also [Gio-L22] for the case of ordinary smooth functions. Concerning this work, we believe it is important to underscore that the version for ordinary smooth function is currently under referees' evaluation for Acta Mathematica, and Prof. I. Ekeland already reviewed this paper saying: *I was particularly interested in your paper because I have been trying for many years to do something similar myself [...] Unfortunately, my approach did not allow me to go beyond known results, whereas yours does. Congratulations!*

The basic idea is the notion of loss of derivatives: if $K \subseteq {}^\rho\widetilde{\mathbb{R}}^n$ is a solid functionally compact set, and $y_0 \in X \subseteq {}^\rho\mathcal{GC}^\infty(K, {}^\rho\widetilde{\mathbb{R}}^d)$, then we say that $P : X \longrightarrow {}^\rho\mathcal{GC}^\infty(K, {}^\rho\widetilde{\mathbb{R}}^d)$ is a *contraction on X with loss of derivatives $L \in \mathbb{N}$ starting from y_0* if: 1) P is continuous in the sharp topology defined by the norms (2.3); 2) $P^n(y_0) \in X$ for all $n \in \mathbb{N}$; 3) the following contraction inequality holds

$$\forall k, n \in \mathbb{N}\, \exists \alpha_{kn} \in {}^\rho\widetilde{\mathbb{R}}_{>0} : \ \left\|P^{n+1}(y_0) - P^n(y_0)\right\|_k \le \alpha_{kn} \left\|P(y_0) - y_0\right\|_{k+nL};$$

and finally: 4) the subsequent Weissinger condition holds $\sum_{n=0}^{+\infty} \alpha_{kn} \|P(y_0) - y_0\|_{k+nL} < +\infty$. Note that if $L = 0$, which is the case for ODE, this condition reduces to the usual Weissinger condition. Under these conditions, we proved that if $X \subseteq {}^\rho\mathcal{GC}^\infty(K, {}^\rho\widetilde{\mathbb{R}}^d)$ is a closed set, then $(P^n(y_0))_{n\in\mathbb{N}}$ is a Cauchy sequence, and hence $\bar{y} := \lim_{n\to+\infty} P^n(y_0) \in X$ is a fixed point of P. In particular, if $L = 0$, the fixed point is unique. Note explicitly that from this result we do not have the uniqueness of the fixed point $\bar{y}$ if we have a loss of $L > 0$ derivatives (which is always the case for non-trivial PDE). This Banach fixed point theorem can be used to prove both an inverse function theorem in graded Fréchet spaces more general than the classical Nash-Moser theorem, and a PLT to solve the *normal*[1] Cauchy problem:

$$\begin{cases} \partial_t^d y(t,x) = F\left[t, x, (\partial_x^\alpha \partial_t^\gamma y)_{\substack{|\alpha|\le L \\ \gamma\le p}}\right] \\ \partial_t^j y(t_0, x) = y_{0j}(x) \ j = 0, \dots, d-1. \end{cases} \tag{2.4}$$

Set $[t_0 - a, t_0 + b] =: T$[2] and assume that $S \subseteq {}^\rho\widetilde{\mathbb{R}}^m$ is a solid functionally compact set. If B is a subset of the space ${}^\rho\mathcal{GC}_t^p\mathcal{C}_x^\infty(T \times S, {}^\rho\widetilde{\mathbb{R}})$ of separately regular GSF which are of class ${}^\rho\mathcal{GC}^p$ in the variable t and of class ${}^\rho\mathcal{GC}^\infty$ in x, we first say that F is *Lipschitz on B with loss of derivatives L and Lipschitz factors* $(\Lambda_k)_{k\in\mathbb{N}}$ if the inequality

$$\left|\partial_x^\nu F^h\left[t, x, (\partial_x^\alpha\partial_t^\gamma u)_{\substack{|\alpha|\le L \\ \gamma\le p}}\right] - \partial_x^\nu F^h\left[t, x, (\partial_x^\alpha\partial_t^\gamma v)_{\substack{|\alpha|\le L \\ \gamma\le p}}\right]\right| \le \Lambda_k(t,x) \cdot \max_{l=1,\dots,m} \max_{\substack{|\alpha|\le k+L \\ \gamma\le p}} \left|\partial_x^\alpha \partial_t^\gamma (u^l - v^l)(t,x)\right|$$

holds for all $k \in \mathbb{N}$, $\nu \in \mathbb{N}^s$, $|\nu| \le k$, $h = 1, \dots, m$, u, $v \in B$. If we set $i_0(t,x) := \sum_{j=0}^{d-1} \frac{y_{0j}(x)}{j!}(t - t_0)^j$, we proved that if F is a GSF, then it is *always* Lipschitz with respect to *constant* factors $(\Lambda_k)_{k\in\mathbb{N}}$ in the space

$$B = \left\{ y \in {}^\rho\mathcal{GC}_t^p\mathcal{C}_x^\infty\left(T \times S, {}^\rho\widetilde{\mathbb{R}}^m\right) \mid \|y - i_0\|_i \le r_i \ \forall i \in \mathbb{N} \right\} \tag{2.5}$$

[1] The adjective *normal* means that we can isolate the highest order derivative $\partial_t^d y$ on the left-hand side.

[2] In general, we use the notations $[a,b] := \left\{x \in {}^\rho\widetilde{\mathbb{R}} \mid a \le x \le b\right\}$ (and similarly for (semi)-open intervals using $<$), and $[a,b]_{\mathbb{R}} := [a,b] \cap \mathbb{R}$.

which is suitable for the proof of the PLT, and with loss of derivatives L given by the maximum order of derivatives in x that appears in problem (2.4). The main assumptions of the PLT are as follows: assume that F is Lipschitz on B with loss of derivatives L and Lipschitz factors $(\Lambda_k)_{k\in\mathbb{N}}$, set $\Lambda^j_{k,0} := 1$, $\Lambda^j_{k,n+1}(t,x) := \left| \int_{t_0}^t \mathrm{d}s_j \overset{j}{\ldots} \int_{t_0}^{s_2} \Lambda_k(s_1,x) \cdot \max_{0<l\le d} \Lambda^l_{k+L,n}(s_1,x)\, \mathrm{d}s_1 \right|$, $\bar{\Lambda}_{k,n} := \max_{\substack{x\in S\\ 0<j\le d}} \Lambda^j_{k,n}(t_0+\max(a,b),x)$, and $P(y)(t,x) := i_0(t,x) + \int_{t_0}^t \mathrm{d}s_d \overset{d}{\ldots} \int_{t_0}^{s_2} F\left[s_1, x, (\partial_x^\alpha \partial_t^\gamma y)_{\substack{|\alpha|\le L\\ \gamma\le p}} \right] \mathrm{d}s_1$, then there exists a solution $y \in B \cap {}^\rho\mathcal{GC}^\infty(T\times S, {}^\rho\widetilde{\mathbb{R}}^m)$ of (2.4) if 1) $P^n(i_0) \in B$ for all $n \in \mathbb{N}$ and 2) $\sum_{n=0}^{+\infty} \bar{\Lambda}_{k,n} \cdot \| P(i_0) - i_0 \|_{k+nL} < +\infty$ (Weissinger condition). If $L = 0$, we can more simply take only any finite number of norms $(\| - \|_k)_{k=1}^p$ and radii $(r_k)_{k=1}^p$ for all fixed $p \in \mathbb{N}$, the solution is unique and condition 1) follows if we assume that $\max(a,b) \cdot \|F\|_k \le r_k$ $\forall k = 1,\ldots,p)$. It is possible to prove that the classical Lewy's counterexample satisfies these assumptions only for infinitesimal $\max(a,b) \le \mathrm{d}\rho^R$, for some $R \in \mathbb{R}_{>0}$, because $(\bar{\Lambda}_{k,n})_n \to \infty$, so that it does not have any ordinary smooth solution defined in a finite time neighborhood (see [Gio-L22] for details). Clearly, this result includes an infinite class of PDE that cannot even be formulated within the theory of Sobolev-Schwartz distributions. If the Lipschitz factors $(\Lambda_k)_k$ are all finite (this is the case if e.g. F is a polynomial or if $L = 0$, $F \in \mathcal{C}^\infty$ and $y_0 \in \mathbb{R}^m$) then $\max(a,b) \ge r \in \mathbb{R}_{>0}$ is non-infinitesimal. Note also that this kind of result is not possible for CGF either due to the absence of closure with respect to arbitrary compositions or because the domain $[t_0 - a, t_0 + b] \times S$ is not of the form $\widetilde{\Omega}_c$ (e.g. if $\max(a,b)$ is infinitesimal and the solution cannot be extended). It is also important to state that in [L-Gio-K22] the PLT has already been extended to *hyperfinite iterations* $P^n(i_0)(-) = [P_\varepsilon^{n_\varepsilon}(i_{0\varepsilon})(-)]$, where $[n_\varepsilon] \in {}^\rho\widetilde{\mathbb{R}}$ is a so-called *hypernatural number*, see [MTA-Gio], i.e. an infinite generalized number such that $n_\varepsilon \in \mathbb{N}$. This kind of numbers is also used in the theory of hyperseries [T-Gio22], hyper-power series and generalized real analytic functions [TM-Gio23]. Hyperfinite iterations allows one to include also non-infinitesimal contraction constants $(\Lambda_k)_{k\in\mathbb{N}}$ similarly to the classical case of standard contractions for ODE.

Basic theory of ODE for GSF

Using the previous existence and uniqueness result, in [L-Gio-K22] we already developed the basic theory of ODE, with continuous dependence on initial data, definition and study of the maximal set of existence and the proof that at some of its boundary points some derivatives must be sharply unbounded; we also proved Gronwall's inequalities and sufficient conditions for global solutions, and existence and uniqueness of the flow induced by an ODE. In particular, in [L-Gio-K22, Thm. 79], we proved that the space S of local solutions of the linear homogeneous ODE $\sum_{k=0}^N a_k \cdot y^{(k)} = 0$ is an ${}^\rho\widetilde{\mathbb{R}}$-module of dimension N, and all the solutions of a related non-homogeneous ODE with solution y_p can be written as a sum $y_p + S$.

Numerical calculus and visualization of GSF in Wolfram Mathematica notebooks

Since a numerical approach in the solution of ODE or PDE is sometimes either the only possible way or a path to conjecture theoretical results, we also want to briefly describe the results that have already been developed in Wolfram Mathematica during the previous FWF project *Applications of Generalized Smooth Functions*. One of the basic ideas of this project was to construct a sort of computer based laboratory with tools helping in formulating conjectures and intuition on GSF, e.g. as given by solution of differential equations. Some basic tools we developed in this toolbox are:

- We can input any Robinson-Colombeau generalized number, possibly defined by a finite number of cases in ε; for example, $x = [x_\varepsilon] \in {}^\rho\widetilde{\mathbb{R}}$, where $x_\varepsilon = f_{1\varepsilon}$ if $\varepsilon = \frac{1}{n}$ for some $n \in \mathbb{N}$ and $x_\varepsilon = f_{2\varepsilon}$ otherwise, is input as $x = \text{RC}\left[\frac{1}{n}, f_{1\varepsilon}, f_{2\varepsilon}\right]$.

- By using Wolfram Mathematica object oriented programming, we overloaded all the elementary operations, functions and relations. For example, if we input $x_1 = \text{RC}\left[\frac{1}{n}, \varepsilon^2, \frac{1}{2+\varepsilon}\right]$ and $x_2 = \text{RC}\left[\frac{1}{n^2}, \exp(\varepsilon), 1-\varepsilon\right]$, we can simply input the expression $\frac{\cos(x_1)}{x_2^3}$ obtaining by Mathematica the equivalent expression

$$\frac{\cos(x_1)}{x_2^3} = \text{RC}\left[\frac{1}{n}, \text{RC}\left[\frac{1}{n^2}, \exp(-3\varepsilon)\cos(\varepsilon^2), \frac{\cos(\varepsilon^2)}{(1-\varepsilon)^3}\right], \text{RC}\left[\frac{1}{n^2}, \exp(-3\varepsilon)\cos(\frac{1}{2+\varepsilon}), \frac{\cos(\frac{1}{2+\varepsilon})}{(1-\varepsilon)^3}\right]\right].$$

- We can visualize these numbers, and zoom as $\varepsilon \to 0^+$ using two parameters: the minimal value of ε (right boundary), or the number of subintervals we want to visualize if the number is defined by cases as above.

- We developed numerical and visual tools to conjecture when two different representatives (x_ε), (x'_ε) define the same generalized number, if the former is less than or equal than the latter, if they are invertible, etc.

- We can visualize internal sets $[A_\varepsilon] \subseteq {}^\rho\widetilde{\mathbb{R}}^2$ as $\varepsilon \to 0^+$.

- We can implement GSF of Cauchy-Dirac type, i.e. of the form $f(x) = [\phi(x_\varepsilon, p_\varepsilon)] \in {}^\rho\widetilde{\mathbb{R}}^d$, where $\phi \in \mathcal{C}^\infty(\mathbb{R}^n \times \mathbb{R}^P, \mathbb{R}^d)$ is an ordinary smooth function, and $p = [p_\varepsilon] \in {}^\rho\widetilde{\mathbb{R}}^P$ is a vector of generalized parameters. For example, $f(x) = \mathrm{d}\rho \cdot \sin\left(\frac{x}{\mathrm{d}\rho}\right)$ is an example of a Cauchy-Dirac GSF (and of the use of overloading of the product symbol $\cdot$, and of the symbol sin); we can input GSF defined by a finite number of cases in ε, and GSF computed using overloading of operations and functions.

- We can calculate (partial) derivatives, n-dimensional integrals and composition of GSF.

- We can solve ODE and Cauchy problems for PDE (with $d > L$, see (2.4)) using iterations of the PLT.

- We implemented the Dirac delta using the classical Colombeau mollifier obtained by the inverse Fourier transform of a bump function equal to 1 in a neighborhood of 0 (see e.g. [G-Kun-OS01]).

Results of GSF theory not available in a classical setting

We list here some results which are possible in GSF theory (and hence in Colombeau theory) but not in more classical approaches, like distribution theory: GF are ordinary set-theoretical maps defined on and taking values in a non-Archimedean ring extending the real field; they are closed with respect to composition so that nonlinear operations are possible; these operations coincide with the usual ones for *smooth* functions, [Gio-Kun-V15]; all classical theorems of differential and n-dimensional integral calculus hold; we have several types of sheaf properties, and GSF indeed form a Grothendieck topos, [Gio-Kun-V19]; as mentioned above, we have a full theory of nonlinear singular ODE, [LL-Gio16, L-Gio-K22], and general existence theorems for nonlinear singular PDE, e.g. the PLT; every Cauchy problem with a smooth PDE is Hadamard well-posed in the sharp topology so that, for example, the classical Lewy-Mizohata counter examples have a unique GSF solution, [Gio-LK22]; using the notion of hyperfinite Fourier transform, we generalized the classical Fourier method also to *non tempered* GF, [MTA-Gio]; we have several applications in the calculus of variations with singular Lagrangians, elastoplasticity, general relativity, quantum mechanics, singular optics, impact mechanics, etc., [10, G-Kun-OS01, G-GiorBryz-L22]. Note that, using the notion of very weak solution of PDE due to M. Ruzhansky, [16], several of these results can be reformulated in purely classical terms without the use of the language of Colombeau theory. For the aims of this proposal, it is also important to underscore that in general nonlinear ODE cannot even be formulated in distribution theory; moreover, the existence of the flow induced by a distributional ODE, as shown in [26], which is the only attempt in this setting, has been proved to be incorrect by [Kun-OSV].

Interaction spaces: a mathematical theory of complex systems

At present, different modeling methods are adopted to study CS: among the most commonly used, we may list, e.g., cellular automata, [24], agent based models, [54], master equation based models, [47], networked dynamical systems, [34], neural networks, [23], and evolutionary algorithms, [6]. However, there is no universal mathematical theory of CS, i.e. a theory sufficiently powerful to range over all of these systems, from agent based models to systems described by some type of differential equation, and, at the same time, to produce far-reaching general mathematical results applicable to large classes of systems. The problem is well-known and discussed in the literature: see e.g. [9, 11], where you can find both arguments in favor or against the possibility of such a theory. In [Gio22, V-Gio-A14, V-Gio-AA1, V-Gio-AA2], we introduced a new mathematical structure, called *interaction space* (IS), having the property to include in a single notion all the previously listed modeling frameworks. Informally:

1. An IS is made by *interacting entities* described by dynamical *state variables*. Intuitively, an interacting entity is everything able to send or receive propagator signals to interact with something else.

2. In an IS, each interaction i of type α can be described as a causally directed elementary process in which a set of *agent* entities $a_1, \ldots, a_n$, modify the state of a *patient* entity p through a *propagator* entity r:
$$a_1, \ldots, a_n \xrightarrow{r,\alpha} p \tag{2.6}$$
The propagator r can be thought of as a signal-entity activated by agents, and carrying the cause-effect relation sent by agents $a_1, \ldots, a_n$ to the patient p. We also say that a subspace of the state space of the propagator r works as a *resource space* for the changing of the state of the patient p and that the *goods* of i are (randomly) extracted from the resource space.

3. An interaction $i : a_1, \ldots, a_n \xrightarrow{r,\alpha} p$ is occurring only if *at least one* of the agents $a_1, \ldots, a_n$ *or* its propagator r *or* its patient p are *active for that interaction*. Indeed, in the state of each interacting entity e there is always a time dependent variable $\mathrm{ac}_e^i(t) \in [0, 1]$ indicating if, with respect to the given interaction i, the entity e is active or not.

4. The occurrence of an interaction and its effects depend on the history $\mathrm{n}_p x$ of the state of a set of entities called the *neighborhood of the interaction*. The neighborhood of the interaction i is intuitively defined by all the entities from which i takes the information it needs to operate. The neighborhood of an interaction always includes agent, patient and propagator entities whenever they are active for that interaction.

5. Generally speaking, interactions occur at random times, whose distribution depends on the history of the interaction's neighborhood only. The dynamics of an interaction i is specified by giving the time t_i^{s} of *starting* of i, and t_i^{a} of *arrival* of the propagator r to the patient p.

6. When some interactions i start to contemporary change the state variables of a patient p, this change is determined by a suitable *transition function* f_p (stochastically) depending on the neighborhood $\mathrm{n}_p\, x$ of p with respect to all the interactions i acting on p, and the resources ρ_p of the propagators of such interactions. The transition function satisfies general evolution equations of the form $x_p(t) = f_p(\omega, t, \mathrm{n}_p\, x, \rho_p)$ for $t \in [t_0, t_0 + \Delta_p]$ (e.g. a vector ODE described in the equivalent integral form; in this case $\Delta_p = 0$ and the equation holds for all t; in case of a cellular automata, we have $\Delta_p = 1$).

See [Gio22] for a formal mathematical definition that allows one to prove the embedding theorems for the modeling frameworks listed above.

In our works, we considered the case of Markovian IS, and we proved for them a master equation. More generally, IS theory includes also non-Markovian dynamics, which can be described using a system of mean derivative equations. We also introduced the notion of *complex adaptive systems* by condensing the original intuition of G.K. Zipf, [56] in what we call *generalized evolution principle*, consisting in the minimization of suitable cost functions and, at the same time, a maximization of the information entropy of the goods generated by the interactions. Following the classical idea of B. Mandelbrot, we also proved in a very general context, that a large class of systems that follow the generalized evolution principle, satisfy a power law. In the language of IS, the notion of interaction as described in (2.6), allows us to introduce a notion of multi-category [22] and hence the concept of a functor F preserving the cause-effect relation (2.6) between interacting entities, i.e. satisfying

$$F(a_1, \ldots, a_n) \xrightarrow{F(r,\alpha)} F(p).$$

This permits one to introduce a new, more abstract and hence more general, perspective in hierarchical modeling of CS, where different hierarchical levels are described by different IS. These levels are frequently linked by couples of adjoint functors preserving the aforementioned cause-effect relation (2.6). All of these ideas are hence related to several branches of the present proposal, e.g. when the dynamics of these hierarchical IS is described by singular nonlinear ODE, like in LPM describing impact mechanics of nonlinear materials or car crash models and their traffic flow (see below in Sec. 3 for details).

3 Work program

In this section, we describe the methods we plan to employ in carrying out the research program outlined above. For each of the two parts of the research project, we will also give a (subjective) judgment of its feasibility. Of course, this qualitative judgment of feasibility will be justified and will also be used to quantify and support the project's time planning.

3.1 WP 1: Theory of singular dynamical systems

As we already outlined above, the theory of continuous dynamical systems can be extended to GSF thanks to the large amount of properties shared between GSF and ordinary smooth functions. Among them, the most important ones for the aims of the present project, have been presented in Sec. 2.2: All the classical theorems of differential calculus, in particular Taylor's formula in different forms (Lagrange, Peano, with nilpotent infinitesimals [Gio-Kun-V19, Thm. 53-55]), all the classical results of one- and multi dimensional integral calculus [Gio-Kun-V19], inverse and implicit function theorem [Gio-Kun16], the Banach fixed point theorem for ${}^{\rho}\mathcal{GC}^{p}$ spaces, the PLT, Gronwall's inequalities, continuous dependence on initial conditions, existence and uniqueness of flow, basic results for linear ODE, [L-Gio-K22]. For these reasons, the basic method in the first part of this WP consists in carefully checking that the classical proofs can be replicated in the GSF setting by simply replacing classical theorems with the corresponding generalized ones. Specifically: 1) The ordinary Euclidean topology needs to be superseded by the sharp one; 2) Assumptions of invertibility trivially substitute the property of real numbers of being different from zero, and 3) Total order and dichotomy properties have to be replaced by working on subpoints, see [MTA-Gio, Lem. 5-7] for details.

Considering that our blueprints are [50, 40], our plan for this WP can hence be summarized as follows:

Basic theory: Regular perturbation theory; The ${}^{\rho}\widetilde{\mathbb{R}}$-matrix exponential using hyperseries $\exp(A) = {}^{\rho}\sum_{n\in{}^{\rho}\widetilde{\mathbb{N}}} \frac{A^n}{n!}$, i.e. series extended over hypernatural numbers ${}^{\rho}\widetilde{\mathbb{N}}$ (see [T-Gio22] and Sec. 2.2 above); the Jordan normal form of an ${}^{\rho}\widetilde{\mathbb{R}}$-matrix; eigenvectors, eigenvalues and Jordan form of $\exp(A)$; solution of linear first order ODE using $\exp(A)$; the case of time dependent coefficient matrix and the Wronskian; definition of one parameter Lie groups and their infinitesimal transformation and relation with hyperseries and nilpotent infinitesimals (see [Gio-Kun-V19, Thm. 55]).

Stability and orbits: Definition of orbits and their classification; definition and first study of invariant sets; the Poincaré map; Definition of (asymptotically/exponentially) stable point; theorem about exponential stability via linearization; results about stability via Liapunov's method; Poincaré–Bendixson theorem: recall that this theorem is essentially based on the Jordan curve theorem; however, in studying holomorphic GSF, we recently proved that every closed and simple generalized path $\gamma \in {}^{\rho}\mathcal{GC}^{0}([a,b], {}^{\rho}\widetilde{\mathbb{R}}^{2})$ has a representative net $\gamma = [\gamma_\varepsilon(-)]$ of closed and simple curves $\gamma_\varepsilon \in \mathcal{C}^0([a_\varepsilon, b_\varepsilon], \mathbb{R}^2)$. It is hence natural to try an interesting reformulation of the Poincaré–Bendixson theorem by considering the two strongly internal sets $\langle I_\varepsilon \rangle$ and $\langle E_\varepsilon \rangle$ (see Sec. 2.2), where $I_\varepsilon \cup E_\varepsilon = \mathbb{R}^2 \setminus \gamma_\varepsilon$ are the two connected components for each representative Jordan curve γ_ε.

The stable manifold theorem: Definition and examples of manifolds modeled in ${}^{\rho}\widetilde{\mathbb{R}}^{n}$: concerning this point, we want to extend an ordinary n-dimensional manifold M with atlas $(\phi_\alpha, U_\alpha)_{\alpha\in\mathcal{A}}$ into $\tilde{M} \supseteq M$ by using GSF of the form $\tilde{\phi}_{\tilde{\alpha}} = [\phi_{\alpha_\varepsilon}(-)] : \langle U_{\alpha_\varepsilon} \rangle \longrightarrow {}^{\rho}\widetilde{\mathbb{R}}^{n}$, where $\tilde{\alpha} := (\alpha_\varepsilon) \in \mathcal{A}^I$ and $\langle U_{\alpha_\varepsilon} \rangle \subseteq \tilde{M}$ is the strongly internal set generated by (U_{α_ε}) (see [Gio-Kun-V15] and Sec. 2.2). In this way, we should be able to extend M also with *non*-compactly supported points (see [Kun-N] to understand the importance of the open problem of extending a manifold also with non-compactly supported points) and to prove that $\tilde{\phi}_{\tilde{\alpha}} \circ \tilde{\phi}_{\tilde{\alpha}}^{-1} \in {}^{\rho}\mathcal{GC}^{\infty}({}^{\rho}\widetilde{\mathbb{R}}^{n}, {}^{\rho}\widetilde{\mathbb{R}}^{n})$; definition of (un)stable manifold of a (non)linear system and its relation with eigenvalues with negative (positive) real part; proof of the stable manifold theorem for GSF (note that the basic results used in the classical proof are a suitable use of PLT and the inverse function theorem, [50]).

Hartman-Grobman theorem: Definition of hyperbolic fixed point and the Hartman-Grobman theorem in the following form:

Conjecture: *Let $f \in {}^{\rho}\mathcal{GC}^{\infty}(M, {}^{\rho}\widetilde{\mathbb{R}}^{d})$ be a GSF vector field with $x_0 \in M$ as a hyperbolic point such that $f(x_0) = 0$. Denote by $A := df(x_0)$ the Jacobian matrix at x_0. Then, there are $\delta \in {}^{\rho}\widetilde{\mathbb{R}}_{>0}$,*

sharp neighborhoods $U \ni x_0$ and $V \ni 0$ and a generalized $\mathcal{C}^0$ homeomorphism $H \in {}^\rho\mathcal{GC}^0(U,V)$, $H^{-1} \in {}^\rho\mathcal{GC}^0(V,U)$, such that the flow $\Phi(t,x)$ of the equation $\dot{y} = f(y)$, $y(0) = x$ is equivalent to the flow of its linearization, i.e. $\forall x \in U\, \forall t \in (-\delta,\delta):\ H(\Phi(t,x)) = e^{tA}H(x)$.
For its proof, we plan to replace the use of the Banach fixed point theorem in $\mathcal{C}^0(\mathbb{R}^n,\mathbb{R}^n)$, [50, 40], with the corresponding theorem in the space ${}^\rho\mathcal{GC}^0(K,K)$ where ${}^\rho\widetilde{\mathbb{R}}^n \supseteq K \supseteq \mathbb{R}^n$ is any infinite functionally compact set. We finally want to find examples where the sharp neighborhoods U, V and $(-\delta,\delta)$ have infinitesimal measure and the ${}^\rho\mathcal{GC}^0$ homeomorphism H is not ${}^\rho\mathcal{GC}^\infty$.

Risks and solutions: Since the most problematic notions and results for the study of continuous GSF-based dynamical systems have already been proved, we do not foresee particular risks in this WP 1.
Subjective assessment of feasibility: For these reasons, in our opinion this part of the project has a *very high* assessment of feasibility.

3.2 WP 2: Hierarchical description of car crash modeling

As outlined above, we plan to start this WP 2 with a general study of cause-effect relations in arbitrary IS and hence in defining hierarchically linked arbitrary CS. This general theoretical development finds potential applications in several complex systems such as the brain, urban systems, the immune system, organisms in biology, social systems, etc., but in this project it is planned to be always exemplified only in the study of dynamics of networks of LPM and their application to (idealised) car crash modeling and car accidents in traffic flow simulations.

Study of hierarchies of CS: The cause-effect relations (2.6) representing interactions are actually primitive notions in an IS (i.e. they depend on the constructed model of the considered CS), even if our interest lies more on the possible cause-effect *graphs* that can be built up using these elementary relations. It is hence mandatory to try finding some relations between these cause-effect graphs and J. Pearl's causal calculus in Bayesian networks, see [38, 39]. For example, we want to show that Pearl's notion can be seen as a particular case of the abstract notion we plan to develop in IS theory by taking random variables as interacting entities and links in the Bayesian network as interactions. We also plan to define the multicategory generated by these causal graphs, and functors between multicategories corresponding to different IS. In particular, the case of a forgetful functor and its possible right and left adjoints can be used to define hierarchical modeling of CS. This abstract setting can find several applications in neurosciences, urban studies, artificial intelligence and biological systems, to mention just a few, see e.g. [45] and references therein. We also want to consider more general situations where we do not necessarily have a natural bijection induced by a pair of adjoint functors (which, in the sense of universal properties, represent the simplest kind of bijection between cause-effect relations (2.6) in the two multicategories corresponding to the considered IS), but more general bijections. This could represent non-optimal hierarchical link between the two considered CS.

Networks of LPM as singular dynamical systems: Already in [1, 29] different LPM are connected, but only in parallel or in series. In principle, we plan to consider an "arbitrary" number of different types of LPM and masses as links of "arbitrary" finite directed graph, even if only a few simple example can be fully understood at theoretical level and only their numerical simulation is in general possible. In the meaningful case [1, 29], the governing geometry structural equation is given by a system of coupled ODE: $m_1\ddot{x}_1+(c_1+c_2)\dot{x}_1+(k_1+k_2)x_1-c_2\dot{x}_2-k_2x_2 = 0$ and $m_2\ddot{x}_2-c_2\dot{x}_1+c_2\dot{x}_2+k_2x_2-k_2x_1 = 0$. For nonlinear material and impact mechanics, these ODE can be generalized in a system of

GSF equations $m_1\ddot{x}_1 + A\left(g_1(\frac{\dot{x}_1}{L}) + g_2(\frac{\dot{x}_1}{L})\right) + A\left(g_3(\frac{x_1}{L}) + g_4(\frac{x_1}{L})\right) - Ag_2(\frac{\dot{x}_2}{L}) - Ag_4(\frac{x_2}{L}) = 0$ and $m_2\ddot{x}_2 - Ag_2(\frac{\dot{x}_1}{L}) + Ag_2(\frac{\dot{x}_2}{L}) + Ag_4(\frac{x_2}{L}) - Ag_4(\frac{x_1}{L}) = 0$, where the functions g_1, g_2, g_3 and g_4 represent the nonlinear dependencies defined by the empirical strain-stress curves of corresponding materials. This type of generalization into a nonlinear ODE is one of the basic ideas of [41, 42, 43] using CGF. Explicitly note that experimental data are frequently available from the existing literature in works in material behavior and nonlinear mechanics, and from the public repositories about materials data base https://mechanicalc.com/calculators/materials and vehicle crash test https://www.nhtsa.gov/research-data/research-testing-databases/: we therefore do not need any further experimental data, also because of the theoretical nature of our research.

In order to link even only two LPM in series, the geometric structural formulation in the displacement x is hence very important because these displacements represent the nodes of the directed graph. A node with a single input and multiple output simply represents that the same displacement output x is input into several LPM. A more general configuration with multiple inputs and outputs is possible only if mediated by a suitable black-box pairing mechanism (e.g. the classical kinematic pairs of mechanics). In this case, we plan to consider arbitrary black-box mechanisms with a user-defined arbitrary GSF transition function. From the mathematical point of view, it is important to note that the dynamics of this type of systems requires the composition of arbitrary GF, a feature that is possible in GSF theory but not in other settings. From the perspective of numerical calculus, e.g. by discretizing the governing ODE of every LPM, it is clear that the dynamics of such networks of LPM can be simulated starting from a given GSF as initial condition, and this is clearly useful for the subsequent theoretical study. Besides their computer implementation and simulation, the general program to study this type of systems can be summarised as follows:

1. We plan to start by considering systems of LPM describing a physical problem, such as in [1, 29]. We plan to consider also different types of LPM, a different (but initially very limited) number of them, and more general (but initially very simple) graphs.

2. Find the classical system of ODE describing these systems by first using the standard relations between elements. Consider as above the generalization to nonlinear materials and singular dynamics. Here the experimental nonlinear strain-stress curves will be used.

3. Find first the numerical solution of this system of ODE and then try to use the general results of WP 1 for the theoretical study: 1) study (analytically or numerically) the stability and instability of this type of systems, 2) understand the properties of their stable and unstable manifolds, 3) apply the Hartman-Grobman theorem as a way to describe the linearization of the dynamics.

4. Start as in [41, 42, 43] by modeling the input stress $\sigma(t)$ as an H-like function with appropriate multiplication coefficient $\sigma_0 \in {}^{\rho}\widetilde{\mathbb{R}}$ representing the real-type loading $\sigma(t) = H(t) \cdot \sigma_0$. Using the governing ODE (2.1), we can then find the strain $s(t)$. It is now natural to consider other specific types of loadings such as ramp loading $\sigma(t) = \text{ramp}(t) \cdot \sigma_1$, see e.g. [44], triangular loading $\sigma(t) = \text{tri}(t) \cdot \sigma_1$ like in [49], and δ-like loading $\sigma(t) = \delta(t) \cdot \sigma_2$, see e.g. high speed tensile test in https://www.zwickroell.com/industries/materials-testing/high-speed-tensile-test/. Numerically find the values of ε corresponding to the representative x_ε of the GSF solution $x = [x_\varepsilon(-)]$ having the best agreement with given experimental data.

5. Using the experience acquired in this study of simple systems, try a generalization to a greater number of LPM or more general graphs. For arbitrary systems, at least write the numerical

code for their simulation by discretizing the dynamics of each LPM subsystem.

Simulations of car accidents in traffic flow: Since software code for traffic flow simulation is freely available (see e.g. [8] for a Python code of the classical Nagel-Schreckenberg model ([30]), or [55] for a free continuum state version of the same kind of model but based on fuzzy decision rules; we clearly also have the Matlab code of our regional traffic model [V-Gio-A14] also based on fuzzy decision rules), the following program is straightforward:

1. Represent a single car using an idealized nonlinear LPM as developed in the previous part of this WP 2 and following [35, 29].

2. Each one of these idealized cars "is represented by" a single moving vehicle in the simulation of traffic flow. This representation is exactly formalized by the hierarchical relation between two CS as developed in the initial part of this WP 2. The complete IS model (where the state of each vehicle contains all the information about the representing LPM) is linked to the simplified IS model by a forgetful functor that do not consider all this detailed information. The functor associating each point-particle vehicle to the LPM located in the same position is the functor going in the opposite direction in the adjoint pair. As it is typical in modeling of CS, this link between two hierarchical CS can also be considered only at an informal level; in our case, this represents instead the application of a general mathematical theory with potential application in several other fields.

3. When, during the stochastic traffic flow simulation, two point-particle vehicles have exactly the same position, this means that we have an accident and we can apply the dynamics of impact developed in the second part of this WP 2 using the information about the LPM in the detailed model. With respect to more classical simulation of vehicles accidents, see e.g. [28, 27] and reference therein, we can hence obtain a more detailed estimation of costs due to the use of LPM.

Risks and solutions: The only risks we can foresee are in case of very involved networks of LPM, both from the theoretical and the computational perspective. We explicitly will *not* consider these cases. Indeed, the models used in [35, 29] are one dimensional and, in case of computational problems, we can also restrict the traffic flow simulation to the simple classical one lane case. We also underscore that, mainly for the lacking of time, in the present project we do *not* aim at calibrating or validating the traffic flow simulation with real data. A possible extension in this direction can be planned in a future proposal. Moreover, we also submitted another FWF research proposal *Numerical library and nonlinear mechanics with generalized functions*, where more general LPM are considered *only* from the numerical point of view. Therefore, in case of its approval, we can also speed up the numerical analysis of the particular idealized LPM studied in the present project, and this potentially increases the feasibility of this project.

Subjective assessment of feasibility: For these reasons, and because of the detailed plan we presented above, our assessment of feasibility is *high*.

3.3 WP 3: Picard solutions of (singular) PDE

The main aims of the present WP 3 are to extend the PLT so as to also include examples with $d \leq L$ (see (2.4)), and other BVP besides the usual Cauchy one. The work plan we want to follow in this WP 3 is the following:

1. The first point is essentially a trivial extension of the PLT to mixed normal derivatives $\partial_t^d \partial_x^e y(t, x)$ on the left hand side of the PDE (2.4) with the corresponding Cauchy BVP. At a first check of the proof, this appears only a trivial generalization where, in order to transform the PDE into a fixed point problem, rather than d integrations in t, we have to also consider e integrations in x.

2. In [13] a Picard scheme for the equation $P(u) = v$ is defined as satisfying $u_{n+1} = u_n - Q(u_0)(P(u_n) - v)$, where $Q(u_0)$ is a right inverse of the differential $DP(u_0)$, so that P must be of class $\mathcal{C}^1$. A Newton scheme satisfies $u_{n+1} = u_n - Q(u_n)(P(u_n) - v)$ and need P to be of class $\mathcal{C}^2$. We first want to generalize these definitions using operators P with loss of derivatives in graded Fréchet spaces. We also want to relax the regularity condition by defining a Picard operator P as a $\mathcal{C}^0$ operator fulfilling $P^{n+1}(u_0) = u_0 + Q(P^n(u_0))$, where Q has $L \in \mathbb{N}_{\geq 0}$ loss of derivatives. Using these definitions and the Banach fixed point theorem with loss of derivatives, we want to show the convergence of these schemata under a suitable Weissinger like condition.

3. The abstract approach pursued in the previous step provides the possible foundation to study PDE which are normal with respect to different variables other than t. These include e.g. wave, heat, Laplace, viscous Burgers and, in suitable domains, also Tricomi PDE. We already considered, e.g., the unique smooth solution of the smooth heat equation $\partial_t u = \partial_x^2 u - f$ subject to $u(x, 0) = \phi(x)$, $u(0, t) = g(t)$ and $u(L, t) = h(t)$ (with $\phi(0) = g(0)$). We can hence transform the PDE into a fixed point problem in two ways: by integrating one time in t or two times in x. Since $\partial_x u(0, t) =: v(t)$ is unknown, this leads to consider the operators $P_1(u, v)(x, t) := g(t) + v(t)x + \int_0^x \mathrm{d}x_2 \int_0^{x_2} [\partial_t u(x_1, t) + f(x_1, t)]\, \mathrm{d}x_1$, $P_2(u)(t) := \frac{1}{L}[h(t) - g(t)] - \frac{1}{L} \int_0^L \mathrm{d}x_2 \int_0^{x_2} [\partial_t u(x_1, t) + f(x_1, t)]\, \mathrm{d}x_1$ and $P_3(u)(x, t) := \phi(x) + \int_0^t \left[\partial_x^2 u(x, t_1) - f(x, t_1)\right] \mathrm{d}t_1$. Note that $(P_1(u, v)(x, t), P_2(u)(t)) = (u, v)(x, t)$ is equivalent to the given PDE with the boundary conditions (but not the initial one). A preliminary numerical analysis shows that iterations with (P_1, P_2) converge to a solution satisfying only these boundary conditions. We first want to define a new iterative operator P^n by mixing the three operators P_k for different values of n. Another idea is to define $P(u, v)(x, t) := d\,(P_1(u, v)(x, t), P_2(u)(x, t), P_3(u)(x, t))$, where $d(x, y, z)$ is a distance function in $\mathbb{R}^3$. The idea to minimize a distance which considers both the solution of the PDE and the supplementary conditions has also been used in [52], and can be suitably generalized in our framework. In all of these approaches, the basic existence result will be the ones obtained in the previous step of this WP 3. Clearly, we start by considering particular PDE only as a blueprint to state and prove general results (always keeping in mind Lewy-Mizohata like counterexamples).

4. We are also in the correct setting to develop Neuberger's idea formulated in [32] (for the Cauchy-Kowalevski setting): consider boundary or supplementary functional conditions that remain invariant under the iterated transformations $P^n : B \longrightarrow B$ of the space of solutions (2.5). In B we have hence to consider possibly different supplementary conditions depending on the given BVP. This general idea can also be formulated by considering a fixed approximating schema $(f_n)_{n\in\mathbb{N}}$ for smooth functions converging to the unknown supplementary condition u_0 (e.g. the initial condition for $t = 0$ if we fix the boundary conditions for $x = 0,\, L$). In this way, the problem can be formulated recursively: given f_k, find the best f_{k+1} so that $P^{n+1}(f_{k+1}) \in B$ and $\left(P^{n+1}(f_{k+1})\right)_n$ converges to a solution y_k respecting the supplementary condition f_{k+1}. Does the corresponding diagonal process converges using the previous convergence results on Picard operators? This can be useful also to discover supplementary conditions in problems where

conditions for a unique solution seem not to be known (for example: the Tricomi equation, see [33]).

5. This WP 3 can also be linked to the previous ones by considering a simple traffic flow kinematic wave model such as the classical Lighthill and Whitham one: $\partial_t \rho + \partial_x (\rho V) = 0$, where $\rho(x,t)$ is the average spatial density, $V(x,t)$ the average velocity, and we assume a fundamental velocity-density diagram $V(x,t) = F[\rho(x,t)]$. The form of F can be taken from a statistical analysis of the simulations resulting from WP 2.

6. As we already stated above, the generalization to the GSF case is usually straightforward, and we only need a more careful attention in extending the results to hyperfinite iterations. The general approach has already been solved for ODE in [L-Gio-K22], and consists in assuming that the right hand side of the singular PDE has moderate hyperfinite iterations at a given point of the space B: the same condition for arbitrary supplementary conditions then follows for all the other points from the contraction property with loss of derivatives.

Risks and solutions: In our opinion, the work plan is formulated in a sufficiently detailed way with numerous and independent ideas. Even in case any of these ideas should present unexpected insurmountable difficulties, the remaining ones already represent a successful development of the present WP 2. Moreover, the approach has a strong and clear theoretical foundation.
Subjective assessment of feasibility: For these reasons, in our opinion this part of the project has a *very high* assessment of feasibility.

4 Scientific relevance, originality and expected benefits for potential users

The present research proposal takes place in the following international research frameworks:
• It fits well in current threads of Austrian research, in particular those of the DIANA group of Prof. M. Kunzinger at the University of Vienna, who is also one of the main developers of the theory of GSF • It also fits well into the research interests of the international community of CGF, where the interest for applications of a singular dynamical systems theory and for CS theory can greatly contribute to the dissemination of Colombeau's theory in applied sciences.

Originality, innovations and benefits of the present proposal:
• A useful and far reaching generalization of dynamical system theory with several potential applications • A general approach to nonlinear mechanics and the dynamic study of material that exploit peculiar theoretical features of GSF theory • A study of networks of LPM which is able to include a huge family of dynamical systems with connections with real-world applications • A general and innovative approach to hierarchical modeling of CS and its applications.

Potential users can hence be expected both in pure and applied mathematics, in physics and engineering applications such as the ones listed above.

Dissemination strategy: • Regular seminars of the DIANA group of the University of Vienna, with presentation of present state, open problems and new results • Contributions for two international conferences per year per person for the presentation of relevant results are planned. In particular, we are thinking of conferences such as the *ISAAC Conference* in 2025, the *International Conference on Generalized Functions* in 2024 and 2026, the *International Symposium on Plasticity and Impact Mechanics* in 2024 • We aim at journals such as: *PNAS, SIAM Journal on Applied Dynamical Systems,* *Taylor & Francis' Dynamical Systems, Springer's Journal of Dynamical and Control Systems, Nonlinearity, Advances in Nonlinear Analysis, Journal of differential equations, Int. Journal of Nonlinear Mechanics, Zeitschrift für angewandte Mathematik und Physik, Advances in Complex Systems.*

4.1 Importance for human resources

The new results achieved in the present proposal would constitute important steps to consolidate the research profiles of all the *non* permanently employed researchers involved in the present project, in particular of P. Giordano. The employment of a post-doc researcher at 70% is planned to help in developing the WP 1 and its applications to WP 2. The applicant and M. Kunzinger will work on all the WP, with a greater focus on the theoretical part of WP 2 and on WP 3. For the CV of the proposal's applicants, see below.

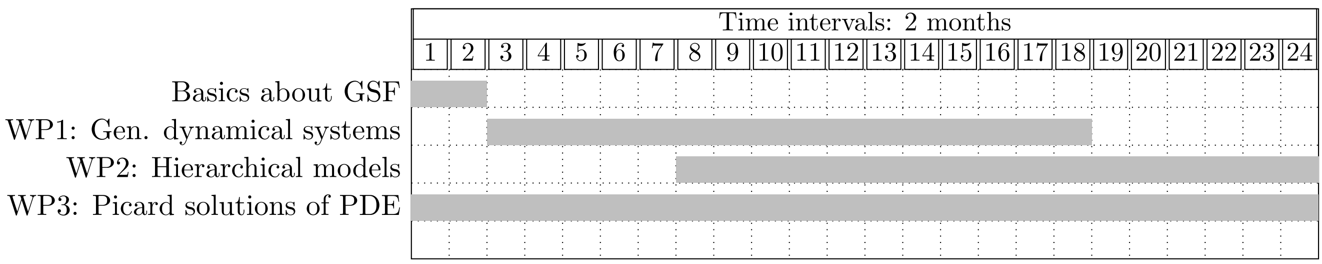
The research project has no ethical, security-related, regulatory, gender-, or sex-related implications.

5 Human resources and time planning

Work organization and human resources: Almost all the ideas of this project have been originally conceived by P. Giordano. For this reason, he will fully actively contribute to their progress during the entire project. Prof. M. Kunzinger is the national collaboration partner of this project.

P. Giordano will work on all the WP and will also supervise the post-doc researcher; M. Kunzinger will work more specifically on WP 1 and WP 3 because of his interest and expertise in dynamical systems, generalized functions and PDE.

Time planning: The research project is designed for three co-workers: P. Giordano, M. Kunzinger and one post-doc at 70%. To estimate the total amount of work to be dedicated to each of the three parts of this project, we plan 4 months to fully understand the background on GSF through weekly seminars and meetings with P. Giordano, and hence about 36 months for the development of WP 1, 34 months for WP 2 and 48 months for WP 3. The entire research project is hence planned to be concluded in 48 months and the time planning is represented as follows.



6 Nomenclature

WP work package(s)	CGF Colombeau generalized function(s)
GF generalized function(s)	DE differential equation(s)
GSF generalized smooth function(s)	PLT Picard-Lindelöf theorem
LPM lumped parameter model(s) or modeling	CS complex system(s)
IS interaction space(s)	BVP boundary value problem(s)

Annex 1: List of references

[1] Abhinav Athalye1, H. G. Phakatkar and U. S. Chavan, Mathematical modeling and Simulation of Car Crash Test, International Journal of Advanced Mechanical Engineering. ISSN 2250-3234 Volume 8, Number 1 (2018)

[2] Ajmone Marsan, G., Bellomo, N., Tosin, A., *Complex Systems and Society — Modeling and Simulation*, Springer, 2013.

[3] Amat, S., Busquier, S., (eds), *Advances in Iterative Methods for Nonlinear Equations*, SEMA SIMAI Springer Series, Vol. 10, 2016.

[4] Aragona, J., Fernandez, R., Juriaans, S.O., A Discontinuous Colombeau Differential Calculus. Monatsh. Math. 144 (2005), 13-29.

[5] Aragona, J., Fernandez, R., Juriaans, S.O., Oberguggenberger, M., Differential calculus and integration of generalized functions over membranes, Monatsh Math (2012) 166: 1. https://doi.org/10.1007/s00605-010-0275-z

[6] Bäck T., *Evolutionary algorithms in theory and practice*, Oxford University Press, 1996.

[7] Brogliato, B., *Nonsmooth Mechanics - Models, Dynamics and Control*. 2nd ed. Springer-Verlag London 1999.

[8] Butt-Shah, Y., Traffic jams (the Nagel-Schreckenberg model), Master thesis supervised by W. Just, Queen Mary University of London, 2020.

[9] Chu, D., Strand, R., Fjelland, R., Theories of complexity. Complexity, 8(3), 19-30, 2003.

[10] Colombeau, J.F., *Multiplication of Distributions; A Tool in Mathematics, Numerical Engineering and Theoretical Physics*, Lecture Notes in Mathematics, 1532, Springer-Verlag, Berlin, 1992.

[11] Crutchfield, J.T., Complex systems theory? Santa Fe Institute Retreat, 2003. See http://csc.ucdavis.edu/~chaos/chaos/talks/CSTheorySFIRetreat.pdf.

[12] Ekeland, I., An inverse function theorem in Fréchet spaces, Ann. I. H. Poincaré – AN 28 (2011) 91–105.

[13] Ekeland, I., Séré, É., A surjection theorem for maps with singular perturbation and loss of derivatives. Journal of the European Mathematical Society, Vol. 23, No. 10, 3323–3349.

[14] Epstein, J., Agent-based computational models and generative social science. Complexity 1999, 4(5), 41-60.

[15] Fuchs, A., *Nonlinear Dynamics in Complex Systems Theory and Applications for the Life-, Neuro- and Natural Sciences*, Springer-Verlag Berlin Heidelberg, 2013.

[16] Garetto, C., Ruzhansky, M., Hyperbolic Second Order Equations with Non-Regular Time Dependent Coefficients. Arch. Rational Mech. Anal. 217 (2015) 113-154.

[G-GiorBryz-L22] Gastão, S.F., Giordano, P., Bryzgalov, A., Lazo, M.J., Calculus of variations and optimal control for generalized functions. Nonlinear Analysis, Vol. 216, 2022.

[Gio-L22] Giordano, P., Luperi Baglini, L., A Picard-Lindelöf theorem for smooth PDE. Under referees evaluation for Acta Mathematica, 2023. See http://dx.doi.org/10.13140/RG.2.2.13861.99045

[Gio22] Giordano, P., Interaction spaces: towards a universal mathematical theory of complex systems. See http://www.mat.univie.ac.at/~giordap7/IS.pdf.

[Gio-Kun13] Giordano, P., Kunzinger, M., New topologies on Colombeau generalized numbers and the Fermat-Reyes theorem. Journal of Mathematical Analysis and Applications 399 (2013) 229–238. https://doi.org/10.1016/j.jmaa.2012.10.005

[Gio-Kun16] Giordano, P., Kunzinger, M., Inverse Function Theorems for Generalized Smooth Functions. Invited paper for the Special issue ISAAC - Dedicated to Prof. Stevan Pilipovic for his 65 birthday. Eds. M. Oberguggenberger, J. Toft, J. Vindas and P. Wahlberg, Springer series Operator Theory: Advances and Applications, Birkhaeuser Basel, 2016. https://doi.org/10.1007/978-3-319-51911-1_7

[Gio-Kun18a] Giordano, P., Kunzinger, M., A convenient notion of compact sets for generalized functions. Proceedings of the Edinburgh Mathematical Society, Volume 61, Issue 1, February 2018, pp. 57-92. https://doi.org/10.1017/S0013091516000559

[Gio-Kun-V19] Giordano P., Kunzinger M., Vernaeve H., The Grothendieck topos of generalized smooth functions I: basic theory. See arXiv 2101.04492 and http://www.mat.univie.ac.at/~giordap7/ToposI.pdf.

[Gio-Kun-V15] Giordano, P., Kunzinger, M., Vernaeve, H., Strongly internal sets and generalized smooth functions. Journal of Mathematical Analysis and Applications, volume 422, issue 1, 2015, pp. 56-71. https://doi.org/10.1016/j.jmaa.2014.08.036

[Gio-LK22] Giordano, P., Luperi Baglini, L., Kebiche, D.E., The Grothendieck topos of generalized smooth functions III: normal PDE. Preprint, see http://www.mat.univie.ac.at/~giordap7/ToposIII.pdf.

[G-Kun-OS01] Grosser, M., Kunzinger, M., Oberguggenberger, M., Steinbauer, R., *Geometric theory of generalized functions with applications to general relativity*, Mathematics And Its Applications, Kluwer Academic Publishers, 2001.

[17] Haag, G., *Dynamic Decision Theory: Application to Urban and Regional Topics,* Kluwer Academic, 1989.

[18] Haken, H., *Synergetics. Introduction and Advanced Topics*, Springer Berlin, 2004.

[19] Huang, M., *Vehicle Crash Mechanics,* CRC Press Boca Raton 2002.

[20] Katz, M.G., Tall, D., A Cauchy-Dirac delta function. Foundations of Science, 2012. See http://dx.doi.org/10.1007/s10699-012-9289-4 and http://arxiv.org/abs/1206.0119.

[Kun-N] Kunzinger, M., Nigsch, E., Manifold-valued generalized functions in full Colombeau spaces. Comm. Math. Univ. Carolinae 52 (4) (2011) 519-534.

[Kun-OSV] Kunzinger, M., Oberguggenberger, M., Steinbauer, R., Vickers, J.A., Generalized Flows and Singular ODEs on Differentiable Manifolds, Acta Applicandae Mathematicae, 80(2), pp 221-241, 2004.

[21] Lakshminarayanan, V. , Ghatak, A.K., Thyagarajan, K., *Lagrangian Optics*, Springer Netherlands, 2011.

[LL-Gio16] Lecke, A., Luperi Baglini, L., Giordano, P., The classical theory of calculus of variations for generalized functions. Advances in Nonlinear Analysis 2017, https://doi.org/10.1515/anona-2017-0150

[22] Leinster, T., *Higher Operads, Higher Categories*, London Mathematical Society Lecture Notes Series, Cambridge University Press, 2003.

[L-Gio-K22] Luperi Baglini, L., Giordano, P., , Kebiche, D.E., The Grothendieck topos of generalized smooth functions II: ODE, Preprint see http://www.mat.univie.ac.at/~giordap7/ToposII.pdf.

[23] Mandic, D.P., Chambers, J.A., *Recurrent Neural Networks for Prediction*, John Wiley & Sons, 2001.

[24] Margolus, N., Toffoli, T., *Cellular automata machines. A new environment for modeling.* MIT Press, MA, USA, 1987.

[25] Marsden, J.E., Generalized Hamiltonian mechanics, Arch. Rat. Mech. Anal., 28(4), pp. 323-361, 1968.

[26] Marsden, J.E., Non-smooth geodesic flows and classical mechanics, Canad. Math. Bull., 12, pp. 209-212, 1969.

[27] Marzoug, R., Lakouari, N., Ez-Zahraouy, H., Castillo Téllez, B., Castillo Téllez, M., Cisneros Villalobos, L., Modeling and simulation of car accidents at a signalized intersection using cellular automata, Physica A, 589, 2022

[28] Moussa, N., Car accidents in cellular automata models for one-lane traffic flow, Phys. Rev. E 68, 2003.

[MT-Gio22] Mukhammadiev, A., Tiwari, D., Giordano, P., A Fourier transform for all generalized functions. Under referee evaluation for Dissertationes Mathematicae, 2023. See https://arxiv.org/abs/2111.15408

[MTA-Gio] Mukhammadiev, A., Tiwari, D., Apaaboah, G., Giordano, P., Supremum, Infimum and hyperlimits of Colombeau generalized numbers. Monatshefte für Mathematik, 196, 163–190 (2021). https://doi.org/10.1007/s00605-021-01590-0

[29] Munyazikwiye, B.B., et al. Mathematical Modeling and Parameters Estimation of Car Crash Using Eigensystem Realization Algorithm and Curve-Fitting Approaches, Hindawi Publishing Corporation Mathematical Problems in Engineering Volume 2013, Article ID 262196

[30] Nagel, K., Schreckenberg, M., A cellular automaton model for freeway traffic. Journal de Physique I, EDP Sciences, 1992, 2 (12), pp.2221-2229.

[31] Nedeljkov, M., Pilipović, S., Scarpalezos, D., *Linear Theory of Colombeau's Generalized Functions*, Addison Wesley, Longman, 1998.

[32] Neuberger, J.W., An Iterative Method for Solving Nonlinear Partial Differential Equations, Advances in Mathematics 19, 245-265, 1976.

[33] Neuberger, J.W., *Sobolev Gradients and Differential Equations* 2nd ed., Springer-Verlag Berlin Heidelberg 2010.

[34] Newman, M.E.J., *Networks: An Introduction.* Oxford University Press, 2010.

[35] Noorsumar G. et al., A Novel Technique for Modeling Vehicle Crash using Lumped Parameter Models. International Conference on Simulation and Modeling Methodologies, Technologies and Applications (2021).

[36] Oberguggenberger, M., *Multiplication of distributions and applications to partial differential equations.* Pitman Research Notes in Mathematics Series, 259. Longman, Harlow, 1992.

[37] Pawlus, W., Robbersmyr, K.G., Karimi, H.R., Mathematical modeling and parameters estimation of a car crash using data-based regressive model approach/ Applied Mathematical modeling 35 (2011) 5091–5107

[38] Pearl, J., *Causality: Models, Reasoning, and Inference*, Cambridge University Press, 2000.

[39] Pearl, J., Mackenzie, D., *The Book of Why. The new science of cause and effect.* Hachette Book Group, 2018.

[40] Perko, L., *Differential Equations and Dynamical Systems.* 3rd ed. Springer-Verlag, New York, 2001.

[41] Průša, V., Rajagopal, K.R., Jump conditions in stress relaxation and creep experiments of Burgers type fluids: a study in the application of Colombeau algebra of generalized functions. Z. Angew. Math. Phys. 62, 707–740 (2011).

[42] Průša, V., Rajagopal, K.R., On the response of physical systems governed by nonlinear ordinary differential equations to step input. Int. J. nonlinearMech. 81 (2016), 207–221.

[43] Průša, V., Řehoř, M., Tůma, K., Colombeau algebra as a mathematical tool for investigating step load and step deformation of systems of nonlinear springs and dashpots, Zeitschrift für angewandte Mathematik und Physik, 2017

[44] Roman, S, Mangir, N, Hympanova, L, Chapple, CR, Deprest, J, MacNeil, S. Use of a simple in vitro fatigue test to assess materials used in the surgical treatment of stress urinary incontinence and pelvic organ prolapse. Neurourology and Urodynamics. 2019; 38: 107– 115. https://doi.org/10.1002/nau.23823

[45] Sales-Pardo, M., Guimerà, R., Moreira, A.A., Nunes Amaral, L.A., Extracting the hierarchical organization of complex systems, PNAS vol. 104, no. 39, 15224 –15229, 2007.

[46] Schaft, A.v.d., Port-Hamiltonian Systems: From Modeling to Control. In: Baillieul, J., Samad, T. (eds) Encyclopedia of Systems and Control. Springer, Cham., 2021.

[47] Schweitzer, F., *Modeling Complexity in Economic and Social Systems.* World Scientific, Singapore, 2002.

[48] Seifert, C., Trostorff, S., Waurick, M., *Evolutionary Equations - Picard's Theorem for Partial Differential Equations, and Applications*, Birkhäuser, 2022.

[49] Sjöstrand J. Some deviations from linear dynamics due to more accurate damping models, In TVSM-5000 VSM820 20171 (2017).

[50] Teschl, G., *Ordinary Differential Equations and Dynamical Systems.* American Mathematical Society, 2012.

[T-Gio22] Tiwari, D., Giordano, P., Hyperseries of Colombeau generalized numbers. Monatshefte für Mathematik, 197, 193–223 (2022). https://doi.org/10.1007/s00605-021-01647-0

[TM-Gio23] Tiwari, D., Mukhammadiev, A., Giordano, P., Hyper-power series and generalized real analytic functions, 2023. Accepted in Monatshefte für Mathematik, 2023. https://doi.org/10.1007/s00605-023-01849-8

[51] Todorov, T.D., Algebraic Approach to Colombeau Theory. San Paulo Journal of Mathematical Sciences, 7 (2013), no. 2, 127-142.

[52] Uddin, Z., Ganga, S., Asthana, R., Ibrahim, W., Wavelets based physics informed neural networks to solve non-linear differential equations, Scientific Reports (2023) 13:2882, https://doi.org/10.1038/s41598-023-29806-3

[V-Gio-A14] Vancheri, A., Giordano, P., Andrey, D., Fuzzy logic based modeling of traffic flows induced by regional shopping malls. Advances in Complex Systems Vol. 17, N. 3 & 4, 2014.https://doi.org/10.1142/S0219525914500179

[V-Gio-AA1] Vancheri A., Giordano P., Andrey D., Albeverio S., A model for urban growth processes with continuous state cellular automata, multi-agents and related differential equation. Part 1: Theory, in Environment and Planning B: Planning and Design 2008, volume 35, issue 4, pages 723-739.https://doi.org/10.1068/b31080b

[V-Gio-AA2] Vancheri A., Andrey D., Giordano P., Albeverio S., A model for urban growth processes with continuous state cellular automata, multi-agents and related differential equation. Part 2: Computer Simulations, in Environment and Planning B: Planning and Design 2008, volume 35, pages 863-880.https://doi.org/10.1068/b31080a

[53] Ward, I.M., J. Sweeney, *Mechanical properties of solid polymers*, Third edition, John Wiley & Sons, Ltd 2012

[54] Wooldridge, M.J., *An introduction to multiagent systems*, John wiley & Sons, 2002.

[55] Yeldan, Ö, A Stochastic Continuous Cellular Automata Traffic Model with Fuzzy Decision Rules. PhD thesis at Polytechnic of Milan, supervised by A. Colorni, 2012.

[56] Zipf, G.K., *Human Behavior and the Principle of Least Effort*, Addison Wesley, Cambridge, MA., 1949.

Annex 2: Financial aspects

Information on the lead research institution and the research institutions of associated research partners

Available personnel and infrastructure

The University of Vienna (AT) is the planned research institution to host the present research project. Prof. M. Kunzinger is the national collaboration partner and one of the leaders of the DIANA research group, which regularly holds weekly seminars. Further members are Prof. G. Hörmann and R. Steinbauer, who are both internationally renowned experts in generalized functions, functional analysis and PDE. All these collaborators are already employed in the respective universities.

Personnel costs

In our view, the work on the project goals can be pursued by funding one senior post-doc position for Dr. P. Giordano, and one post-doc position at 70% for 48 months. Dr. P. Giordano position will be subdivided as 50% in the present project and 50% as PI from other ongoing or future FWF projects, see the CV attached in the present proposal. The post-doc researcher is planned to help in developing the WP 1 and its applications to WP 2. The applicant and M. Kunzinger will work on all the WP, with a greater focus on the theoretical part of WP 2 and on WP 3. Therefore, on the basis of the 2023 FWF salary rates, we have the following:

- 1 senior post-doc position at 50% (20h) for 48 months = 84'430 €/y * 50% * 48 m + annual valorization = 187'016.27 €.
- 1 post-doc position at 70% (28h) for 48 months = 76'990 €/y * 70% * 48 m + annual valorization = 238'750.86 €.

This amounts to a total of 425'767.13 €.
Therefore, the total amount requested for the present proposal (considering 5% of general costs) is 447'055.49 €.

Curriculum vitae of P. Giordano

Name:	Giordano, Paolo
ORCID:	0000-0001-7653-1017
Web site with publications:	www.mat.univie.ac.at/~giordap7/
Affiliation:	University of Vienna, Oskar-Morgenstern-Platz 1 1090 Wien
Email:	paolo.giordano@univie.ac.at

Education

- *Habilitation*, University of Vienna, Austria, 2019.
- *Rheinischen Friedrich-Wilhelms-Universität Bonn* (DE), Ph.D. in Mathematics, 2009.
- *Università degli Studi di Milano* (IT), M.Sc. in Mathematics, 1997.

Academic experiences

Selected research activities

- August 2020 - present: project leader of FWF stand alone research project *Functional analysis of infinite bounded operators*. Co-investigator Prof. M. Kunzinger; 407'000 Euro.
- February 2021 - present: project leader of FWF stand alone research project *Applications of generalized smooth functions*. Co-investigator Prof. M. Kunzinger; 210'000 Euro.
- August 2021 - present: project leader of FWF stand alone research project *Fourier transforms and Cauchy-Kowalevski theorem for generalized smooth functions*. Co-investigator Prof. M. Kunzinger; 82'000 Euro.
- August 2017 - 2021: project leader of FWF stand alone research project *Hyperfinite methods for generalized smooth functions*, Wolfgang Pauli Institute, Vienna. Co-investigator Prof. M. Kunzinger; 397'000 Euro.
- December 2012 - May 2017: project leader of FWF stand alone research project *Analysis and Geometry based on generalized numbers*, Dep. of Mathematics, University of Vienna. Co-investigator Prof. M. Kunzinger; 321'000 Euro.
- June 2013 - May 2016: project leader of FWF stand alone research project *Non-Archimedean Geometry and Analysis*, Dep. of Mathematics, University of Vienna (AT). Co-investigators Prof. M. Kunzinger and Prof. V. Benci; 349'000 Euro.
- October 2010 - September 2012: project leader of the FWF Lise Meitner grant *Nilpotent Infinitesimals and Generalized Functions*, Dep. of Mathematics, University of Vienna. Co-investigator Prof. M. Kunzinger; 115'200 Euro.

- March 2002 - February 2004: Marie Curie individual fellowship of the European Commission, *A new approach to differential geometry of spaces of mappings and its applications,* Institute of Applied Mathematics, University of Bonn, 140'200 Euro.

5 selected invited lectures

1. First keynote speaker at the 9th Southeast Asian Mathematical Society UGM International Conference on Mathematics and Its Applications (Yogyakarta, Indonesia), "Artificial general intelligence based on mathematical theory of complex systems", 2023.

2. Plenary speaker at the conference "Generalized Functions 2022", "Consequences of neglecting infinitesimal (and infinite) numbers from mathematics", 2022.

3. Invited speaker at the "Seminario di Analisi Matematica", Dipartimento di Matematica *Federigo Enriques*, Università degli Studi di Milano, "The Grothendieck topos of generalized smooth functions", 2020.

4. Invited speaker at the "Seminario di Logica Matematica", Dipartimento di Matematica *Federigo Enriques*, Università degli Studi di Milano, "Interaction spaces theory: a mathematical theory of complex systems", 2020.

5. Invited speaker at the conference "Souriau 2019", May 27-31 2019, Paris-Diderot University; title: "The Grothendieck topos of generalized functions".

Supervision of Ph.D. students and postdoctoral fellows

1) 2021 - present: D.E. Kebiche, K. Islami and S. Nugraheni, Ph.D. students, University of Vienna. A. Bryzgalov, post-doc, University of Vienna. **2)** 2018 - 2022: A. Mukhammadiev and D. Tiwari, Ph.D. students, University of Vienna. **3)** 2012 - 2016: L. Luperi Baglini and E. Wu, post-docs, University of Vienna. **4)** 2006 - 2009: G.L. Ciampaglia, M. Esmaeili, Ph.D. students, University of Italian Switzerland, CH.

Reviewing activities (selection)

I am reviewer for: Acta Mathematica, Transaction of the American Mathematical Society, Proceedings of the American Mathematical Society, American Mathematical Monthly, Nonlinearity, Advances in Complex Systems, Environmental modeling and software, Physics Letters A.

International research partner (selection)

Vieri Benci, University of Pisa, Italy; Sergio Albeverio, University of Bonn, Germany; Hans Vernaeve, University of Ghent, Belgium; Alberto Vancheri, SUPSI, Switzerland; Nanang Susyanto, Universitas Gadja Mada, Indonesia.

Main areas of research and selected results

- *Theory of Generalized Smooth Functions*: a new theory of generalized functions resulting in the closure with respect to composition, a better behavior on unbounded sets and new general existence results.

- *Theory of Interaction Spaces*: a new unifying mathematical theory of complex systems which includes several types of complex systems models, with applications in urban growth, transportation and housing markets.
- *Colombeau theory*: we unified several different Colombeau-like algebras into a single general abstract notion having the same simplicity of the special algebra.
- *Theory of Fermat reals*: A non-Archimedean theory of nilpotent infinitesimals with applications in (infinite dimensional) smooth differential geometry and analysis.

Most important publications

For the links to these publications and the complete list, see www.mat.univie.ac.at/~giordap7/

1. Gastão, S.F., Giordano, P., Bryzgalov, A., Lazo, M.J., Calculus of variations and optimal control for generalized functions. Nonlinear Analysis, Vol. 216, 2022. DOI: 10.1016/j.na.2021.112718
2. Giordano P., Kunzinger M., A convenient notion of compact set for generalized functions. Proceedings of the Edinburgh Mathematical Society, Volume 61, Issue 1, February 2018, pp. 57-92. DOI: 10.1017/S0013091516000559
3. Lecke A., Luperi Baglini L., Giordano P., The classical theory of calculus of variations for generalized functions. Advances in Nonlinear Analysis, Vol. 8, issue 1, 2017. DOI: 10.1515/anona-2017-0150
4. Giordano P., Kunzinger M., Inverse Function Theorems for Generalized Smooth Functions. Chapter in "Generalized Functions and Fourier Analysis", Volume 260 of the series Operator Theory: Advances and Applications pp 95-114. DOI: 10.1007/978-3-319-51911-1_7
5. Giordano P., Wu E., Calculus in the ring of Fermat reals. Part I: Integral calculus. Advances in Mathematics 289 (2016) 888–927. DOI: 10.1016/j.aim.2015.11.021
6. Giordano P., Kunzinger M., Vernaeve H., Strongly internal sets and generalized smooth functions. Journal of Mathematical Analysis and Applications, volume 422, issue 1, 2015, pp. 56-71. DOI: 10.1016/j.jmaa.2014.08.03
7. Vancheri A., Giordano P., Andrey D., Fuzzy logic based modeling of traffic flows induced by regional shopping malls. Advances in Complex Systems Vol. 17, N. 3 & 4, 2014, (39 pages). DOI: 10.1142/S0219525914500179
8. Giordano P., Kunzinger M., Topological and algebraic structures on the ring of Fermat reals. Israel Journal of Mathematics, January 2013, Volume 193, Issue 1, pp. 459-505. DOI: 10.1007/s11856-012-0079-z
9. Giordano P., The ring of fermat reals, Advances in Mathematics 225 (2010), pp. 2050-2075. DOI: 10.1016/j.aim.2010.04.010
10. Giordano P., Fermat-Reyes method in the ring of Fermat reals. Advances in Mathematics 228, pp. 862-893, 2011. DOI: 10.1016/j.aim.2011.06.008

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Fakultät für Mathematik
Universität Wien

Prof. Dr. Michael Kunzinger
Oskar-Morgenstern-Platz 1
1090 Wien
tel. +43 1 4277 50681
michael.kunzinger@univie.ac.at
www.mat.univie.ac.at/~mike

An den
Fonds zur Förderung der
wissenschaftlichen
Haus der Forschung
Sensengasse 1
A-1090 Wien

August 2, 2023

To whom it may concern,

this letter is to confirm my interest in collaborating with Paolo Giordano on the project

Singular dynamical systems and Picard solutions of PDE,

currently submitted to FWF. The project is concerned with the development of the theory of continuous dynamical systems using generalized smooth functions (GSF), and offers a detailed plan both for showing interesting applications in nonlinear mechanics, hierarchical modeling of complex systems and innovative solution of PDE using the Picard-Lindelöf theorem, all of which will constitute major advances in mathematical analysis and will open up new avenues of research.

Due to his breakthrough results in the theory of generalized functions, Paolo Giordano is an internationally renowned leader in the field, as signified by his outstanding publication record as well as his recognition in the field (underlined by his recent plenary talk at the International Conference on Generalized Functions GF2022). It is therefore of utmost importance to the University of Vienna to retain his service.

The main focuses of my research are theories of generalized functions and mathematical general relativity and I have immensely profited from my long standing collaboration with Paolo, as underlined by a number of joint publications. I am convinced that a continued collaboration with him will be of great benefit for both sides.

The proposed project is highly ambitious and promising, and I have complete trust in his abilities as a top level mathematician to realize its full potential. I emphatically recommend the FWF project and look forward to contributing to its success.

Yours sincerely,
Prof. Dr. Michael Kunzinger

Michael Kunzinger