BEYOND CAUCHY-KOWALEWSKY: A PICARD-LINDELÖF THEOREM FOR SMOOTH PDE

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ABSTRACT. We prove that Picard-Lindelöf iterations for an arbitrary smooth normal Cauchy problem for PDE converge if we assume a suitable Weissinger-like sufficient condition. This condition includes both a large class of non-Gevrey PDE or initial conditions, and more classical real analytic functions. The proof is based on a Banach fixed point theorem for contractions with loss of derivatives. From the latter, we also prove an inverse function theorem for locally Lipschitz maps with loss of derivatives in arbitrary graded Fréchet spaces, not necessarily of tame type or with smoothing operators.

1. INTRODUCTION

Starting from the work of H. Lewy [21], it is clear that a general Picard-Lindelöf-Cauchy-Lipschitz theorem (PLT) for Cauchy problems of the form:

$$\begin{cases} \partial_t^d y(t,x) = F\left[t, x, (\partial_x^\alpha \partial_t^\gamma y)_{\substack{|\alpha|\le L\\ \gamma\le p}}\right], \\ \partial_t^j y(t_0,x) = y_{0j}(x) \ j=0,\ldots,d-1, \end{cases} \tag{1.1}$$

is not possible (see also e.g. [5] and references therein for the more general problem of solvability of partial differential operators). In (1.1), we consider y, y_{0j}, F as arbitrary ($\mathbb{R}^m$-valued) smooth functions, $(t,x)\in T\times S\subseteq \mathbb{R}\times\mathbb{R}^s$, $\alpha\in\mathbb{N}^s$, $\gamma\in\mathbb{N}$, p, $L\in\mathbb{N}$, $d\in\mathbb{N}_{>0}$, and we assume that $p<d$. In this paper, we show the convergence of Picard-Lindelöf iterations of the general problem (1.1) under a suitable sufficient condition depending both on the initial conditions y_{0j} and the function F. We also prove that this condition includes non-trivial cases where F could be non-Gevrey, and a large class of smooth non-Gevrey initial conditions y_{0j}. These cases are not covered by the Ovsjannikov-Nirenberg-Nishida extension of the Cauchy-Kowalewsky theorem, see [28, 26, 29, 27], where only continuity in the variable t, but analyticity in x, are assumed.

According to [7, 8], one of the main problems in trying to solve (1.1) using Picard-Lindelöf iterations is that the corresponding fixed point integral operator P has $L\in\mathbb{N}$ *loss of derivatives,* i.e. satisfies $\left\|P^{n+1}(y_0)-P^n(y_0)\right\|_k \le \alpha_{kn}\left\|P(y_0)-y_0\right\|_{k+nL}$ for all k, $n\in\mathbb{N}$ (here we are using the notion of "loss of derivatives" as in [25, 7, 8], and not as e.g. in [30, 18, 19]; see Def. 4 below for a formal definition). For this reason, in Sec. 3, we first generalize the Banach fixed point theorem (BFPT)

2020 *Mathematics Subject Classification.* 35A10, 35GXX, 35G25, 47J07, 46-XX.

Key words and phrases. Cauchy-Kowalewsky theorem; Picard-Lindelöf theorem; loss of derivatives; Inverse function theorem; Nash–Moser theorem.

P. Giordano has been supported by grants P30407, P33538, P33945, P34113 and 10.55776/PAT9221023 of the Austrian Science Fund FWF.

L. Luperi Baglini has been supported by grant P30821 of the Austrian Science Fund FWF.

to contractions with loss of derivatives, and we will see that the aforementioned sufficient condition corresponds to a Weissinger-like assumption, [39]. In Sec. 4, we hence apply this BFPT to prove an inverse function theorem in arbitrary graded Fréchet spaces (not necessarily of tame type or with smoothing operators, like in Nash-Moser theorem, see [25, 15]) and for locally Lipschitz maps with loss of derivatives (non necessarily differentiable maps, like in Ekeland inverse function theorem, see [7]). In Sec. 5, this BFPT with loss of derivatives is used to prove a PLT for normal PDE. In Sec. 6, we apply this PLT to a family of PDE including both a non-Gevrey F or non-Gevrey initial conditions. Finally, in Sec. 7, we present a preliminary study of the notion of contraction with loss of derivatives.

In the following, we say that the Cauchy problem (1.1) is in *normal form* to specify that the highest derivative in t (called *normal variable*) can be isolated on the left hand side of the PDE (some authors call this problem in Kowalewskyan form).

If $y : X \longrightarrow \mathbb{R}^m$, then $y^h : X \longrightarrow \mathbb{R}$ is the $h = 1, \ldots, m$ component of y, and in $\mathbb{N} = \{0, 1, 2, \ldots\}$ we always include zero. Therefore, the notations $(\partial_x^\alpha \partial_t^\gamma y)(t, x)$ used in (1.1) include cases where some $\alpha_j = 0$, $j = 1, \ldots, s$, or $\gamma = 0$. Finally, $\mathcal{C}^k(X, \mathbb{R}^m)$ denotes the set of all the $\mathcal{C}^k$ functions $f : X \longrightarrow \mathbb{R}^m$, whereas $\mathcal{C}^k(X) := \mathcal{C}^k(X, \mathbb{R})$.

2. Main results presented in the paper

The problem (1.1) is always considered in the domain $[t_0 - a, t_0 + b] \times S =: T \times S \Subset \mathbb{R}^{1+s}$ for a, $b \in \mathbb{R}_{>0}$ and for $F \in \mathcal{C}^\infty\left(T \times S \times \mathbb{R}^{m\cdot\hat{L}}, \mathbb{R}^m\right)$, $y_{0j} \in \mathcal{C}^\infty(S, \mathbb{R}^m)$, $j = 0, \ldots, d-1$, where $\hat{L} := \mathrm{Card}\{(\alpha, \gamma) \in \mathbb{N}^s \times \mathbb{N}_{\le p} \mid |\alpha| \le L\}$ and $p \le d-1$ denotes the maximum order of derivatives $\partial_t^\gamma y(t, x) \in \mathbb{R}^m$ appearing on the right hand side of (1.1). For PDE, the starting point of the Picard-Lindelöf iterations is the function $i_0(t, x) := \sum_{j=0}^{d-1} \frac{y_{0j}(x)}{j!}(t - t_0)^j$. These iterations are defined in $\bar{B}_R(i_0) := \{u \in \mathcal{C}_t^p \mathcal{C}_x^\infty(T \times S, \mathbb{R}^m) \mid \|u - i_0\|_k \le r_k \ \forall k \in \mathbb{N}\}$, where $\mathcal{C}_t^p \mathcal{C}_x^\infty$ is the space of functions which are of class $\mathcal{C}^p$ in t and smooth in x (see Def. 9 for the precise definition) and with the related supremum norm $\|y\|_k := \max_{1\le h\le m} \max_{\substack{|\beta|\le k \\ \beta_1 \le p}} \max_{(t,x)\in T\times S} \left|\partial^\beta y^h(t, x)\right|$; the radii $r_k \in \mathbb{R}_{>0} \cup \{+\infty\}$ and $R := (r_k)_{k\in\mathbb{N}}$. We use the simplified notation $G(t, x, y) := F\left[t, x, (\partial_x^\alpha \partial_t^\gamma y)_{\substack{|\alpha|\le L \\ \gamma \le p}}(t, x)\right]$. Like in the case of ODE, we finally need a locally Lipschitz condition on F: we say that G is is Lipschitz on $\bar{B}_R(i_0)$ with loss of derivatives (LOD) L and Lipschitz factors $\Lambda_k \in \mathcal{C}^0(T \times S)$ if

$$\left|\partial_x^\nu G^h(t, x, u) - \partial_x^\nu G^h(t, x, v)\right| \le \Lambda_k(t, x) \cdot \max_{l=1,\ldots,m} \max_{\substack{|\alpha|\le k+L \\ \gamma\le p}} \left|\partial_x^\alpha \partial_t^\gamma (u^l - v^l)(t, x)\right|$$

for all $u, v \in \bar{B}_R(i_0)$, $|\nu| \le k$ and $(t, x) \in T \times S$.

The Picard-Lindelöf-Cauchy-Lipschitz theorem for PDE can be stated as follows:

Theorem 1. *In the previous notations, assume that $\mathring{S}$ is dense in S. Define $P : \bar{B}_R(i_0) \to \mathcal{C}_t^p \mathcal{C}_x^\infty(T \times S, \mathbb{R}^m)$ by*

$$P(y)(t, x) := i_0(t, x) + \int_{t_0}^{t} \mathrm{d}s_d \overset{d}{\cdots} \int_{t_0}^{s_2} G(s_1, x, y)\, \mathrm{d}s_1.$$

Assume that G is Lipschitz on $\bar{B}_R(i_0)$ with loss of derivatives L and Lipschitz factors $\Lambda_k \in \mathcal{C}^0(T \times S)$, and for all $(t,x) \in T \times S$, all $n \in \mathbb{N}$ and all $j = 1, \dots, d$, set

$$\begin{aligned} \Lambda^j_{k,0} &:= 1, \\ \Lambda^j_{k,n+1}(t,x) &:= \left| \int_{t_0}^{t} \mathrm{d}s_j \overset{j}{\dots} \int_{t_0}^{s_2} \Lambda_k(s_1,x) \cdot \max_{0<l\le d} \Lambda^l_{k+L,n}(s_1,x)\, \mathrm{d}s_1 \right|, \\ \bar{\Lambda}_{k,n} &:= \max_{\substack{x\in S \\ 0<j\le d}} \Lambda^j_{k,n}(t_0+\max(a,b),x). \end{aligned} \tag{2.1}$$

Finally, assume that the following conditions are fulfilled for all $k \in \mathbb{N}$:

(i) $P^n(i_0) \in \bar{B}_R(i_0)$ *for all* $n \in \mathbb{N}$*;*

(ii) $\sum_{n=0}^{+\infty} \bar{\Lambda}_{k,n} \cdot \|P(i_0) - i_0\|_{k+nL} < +\infty.$

Then, there exists a smooth solution $y \in \bar{B}_R(i_0) \cap \mathcal{C}^\infty(T \times S, \mathbb{R}^m)$ of the problem

$$\begin{cases} \partial_t^d y(t,x) = F\left[t, x, (\partial_x^\alpha \partial_t^\gamma y)_{\substack{|\alpha|\le L \\ \gamma \le p}}\right], \\ \partial_t^j y(t_0,x) = y_{0j}(x) \;\; j = 0, \dots, d-1, \end{cases} \tag{2.2}$$

given by $y = \lim_{n\to+\infty} P^n(i_0)$ in $\left(\mathcal{C}^p_t\mathcal{C}^\infty_x(T\times S,\mathbb{R}^m), (\|-\|_k)_{k\in\mathbb{N}}\right)$, which satisfies

$$\forall k, m \in \mathbb{N} : \;\; \|y - P^m(i_0)\|_k \le \sum_{n=m}^{+\infty} \bar{\Lambda}_{k,n} \cdot \|P(i_0) - i_0\|_{k+nL}.$$

The proof, like in the case of ODE, is based on a Banach fixed point theorem for contractions with LOD (see Def. 4 and Thm. 6).

We immediately recognize that the main assumptions of this result are (i) and (ii), e.g. because we have to avoid the possibility that $r_k \to 0$ and hence the ball $\bar{B}_R(i_0) = \{i_0\}$. We therefore show that these conditions are satisfied for the following class of examples:

$$\partial_t^d y(t,x) = p(t) \cdot \partial_x^\mu \partial_t^\gamma y(t,x) + q(t,x), \tag{2.3}$$

where $y(t,x) \in \mathbb{R}^m$, $\mu \in \mathbb{N}^s$, $|\mu| = L > 0$, $0 \le \gamma < d$, $p \in \mathcal{C}^\infty(T, \mathbb{R}^{m\times m})$ is an *arbitrary smooth function*, and q is a smooth function with uniformly bounded derivatives in x:

$$\exists Q \in \mathbb{R}_{>0}\, \forall \nu \in \mathbb{N}^s\, \forall (t,x) \in T \times S : \;\; |\partial_x^\nu q(t,x)| \le Q. \tag{2.4}$$

We have the following results:

Theorem 2. *If the initial conditions y_{0j}, $j = 0, \dots, \gamma-1$, are* arbitrary smooth *functions, whereas y_{0j} for $j = \gamma, \dots, d-1$ are analytic or they satisfy*

$$\|y_{0j}\|_{k+(n+1)L} \sim (nL)^{\sigma_j nL}, \sigma_j > 0 \quad \forall j = \gamma, \dots, d-1\, \forall k \in \mathbb{N}$$

where $d > \sigma_j L$ (therefore, y_{0j} is not an analytic function, but it is Gevrey of class $s > \sigma_j$), then there exists a smooth solution of (2.3) in $\bar{B}_R(i_0)$ for $\bar{T}$ sufficiently small and all $x \in S$. We note that, if the function p is non-Gevrey, then any *solution such that* $\partial_x^\mu \partial_t^\gamma y \neq 0$ *cannot be of Gevrey class. Moreover, if we set*

$$\begin{aligned}
I_d\left[f(t)\right] &:= \int_{t_0}^{t} \mathrm{d}s_d \overset{d}{\cdots} \int_{t_0}^{s_2} f(s_1)\,\mathrm{d}s_1 \\
\mu_{j-\gamma,0}(t) &:= p(t)\cdot(t-t_0)^{j-\gamma} \\
\mu_{j-\gamma,h+1}(t) &:= I_d\left[p(t)\cdot\partial_t^\gamma \mu_{j-\gamma,h}(t)\right] \\
\eta_0(t,x) &:= q(t,x) \\
\eta_{h+1}(t,x) &:= I_d\left[p(t)\cdot\partial_x^\mu\partial_t^\gamma \eta_h(t,x)\right],
\end{aligned}$$

the solution y *is given by the formula:*

$$y(t,x) = \sum_{j=0}^{\gamma-1} \frac{y_{0j}(x)}{j!}(t-t_0)^j + \sum_{h=0}^{+\infty}\sum_{j=\gamma}^{d-1} \frac{\partial_x^{h\mu} y_{0j}(x)}{(j-\gamma)!}\mu_{j-\gamma,h}(t) + \sum_{h=0}^{+\infty}\eta_h(t,x). \tag{2.5}$$

In particular, if the functions p *and* q *are constant, then*

$$y(t,x) = \sum_{j=0}^{\gamma-1} \frac{y_{0j}(x)}{j!}(t-t_0)^j + \sum_{h=0}^{+\infty}\sum_{j=\gamma}^{d-1} \frac{\partial_x^{h\mu} y_{0j}(x)}{(j-\gamma)!} p^{h+1} \frac{(t-t_0)^{h(d-\gamma)+j-\gamma}}{[h(d-\gamma)+j-\gamma]!}(j-\gamma)! + q.$$

Note that, even if these assumptions on the initial conditions y_{0j} are not covered by the Ovsjannikov-Nirenberg-Nishida extension of the Cauchy-Kowalewsky theorem, unfortunately in (2.3) we do not have dependence of the function p on the variable x or the unknown y.

3. A Banach fixed point theorem with loss of derivatives

The idea to extend the classical Banach fixed point theorem to sequentially complete subsets X of Hausdorff locally convex linear spaces $(E, (|-|)_{\alpha\in\Lambda})$ dates back to [3]. Here, a contraction is a map $P : X \longrightarrow X$ satisfying

$$\forall \alpha\in\Lambda\,\exists k_\alpha \in [0,1)\,\forall x,y\in X: \ |P(x)-P(y)|_\alpha \le k_\alpha |x-y|_\alpha.$$

The notion of contraction has also been extended to uniform spaces ([33, 34]) and to condensing maps on Hausdorff locally convex linear spaces via the notion of measure of non-compactness (see e.g. [2] and references therein). See also [1] for a recent survey, and [10, 37, 6, 38] for updated references framed in locally convex linear spaces.

In the present section, we want to prove a Banach fixed point theorem for contractions with loss of derivatives in graded Fréchet spaces. In this paper, by a *graded Fréchet space* $\left(\mathcal{F}, (\|-\|_k)_{k\in\mathbb{N}}\right)$ we mean a Hausdorff, complete topological vector space whose topology is defined by an increasing sequence of seminorms: $\|-\|_k \le \|-\|_{k+1}$ for all $k\in\mathbb{N}$. We denote by $B_r^k(x) := \{y\in X \mid \|x-y\|_k < r\}$ the ball of radius $r\in\mathbb{R}_{>0}$ defined by the k-norm.

A first trivial and well known result we will use is the following:

Lemma 3. *Let* $(\mathcal{F},\tau)$ *be a topological space,* $P : X \to \mathcal{F}$ *be a continuous function defined in* $X\subseteq\mathcal{F}$*, and assume that there is* $y_0\in X$ *such that* $P^n(y_0)\in X$ *for all* $n\in\mathbb{N}$ *and* $\exists \lim_{n\to+\infty} P^n(y_0)\in X$*. Then* $\lim_{n\to+\infty} P^n(y_0)$ *is a fixed point of* P*. In particular, this applies if* $\mathcal{F}$ *is a Fréchet space and* $\left(P^n(y_0)\right)_{n\in\mathbb{N}}$ *is a Cauchy sequence of points of* X*, where* $X\subseteq\mathcal{F}$ *is a Cauchy complete subspace.*

Proof. The usual proof works: $\bar{y} := \lim_{n\to+\infty} P^n(y_0) \in X$ exists by assumption, and since $P : X \longrightarrow \mathcal{F}$ is continuous we have

$$P(\bar{y}) = P\left(\lim_{n\to+\infty} P^n(y_0)\right) = \lim_{n\to+\infty} P^{n+1}(y_0) = \bar{y}. \qquad \square$$

In particular, this general Lem. 3 applies to *contractions with loss of derivatives* in Fréchet spaces: A key idea in defining this notion is that it has to depend on the starting point y_0 of the iterations:

Definition 4. Let $\left(\mathcal{F}, (\|-\|_k)_{k\in\mathbb{N}}\right)$ be a Fréchet space, X be a closed subset of $\mathcal{F}$, $y_0 \in X$ and $L \in \mathbb{N}$. We say that P *is a contraction with L loss of derivatives starting from y_0* (and we simply write $P \in \mathrm{C}(X, L, y_0)$) if the following conditions are fulfilled:

(i) $P : X \longrightarrow \mathcal{F}$ is continuous;
(ii) $P^n(y_0) \in X$ for all $n \in \mathbb{N}$;
(iii) For all k, $n \in \mathbb{N}$ there exist $\alpha_{kn} \in \mathbb{R}_{>0}$ such that

$$\left\|P^{n+1}(y_0) - P^n(y_0)\right\|_k \le \alpha_{kn} \left\|P(y_0) - y_0\right\|_{k+nL}; \tag{3.1}$$

(iv) For all $k \in \mathbb{N}$, the following *Weissinger* condition holds:

$$\sum_{n=0}^{+\infty} \alpha_{kn} \left\|P(y_0) - y_0\right\|_{k+nL} < +\infty. \tag{W}$$

Note that if we actually have only one norm $\|-\|_k = \|-\|_0$ and $L = 0$ (ODE case), then (W) reduces to the classical Weissinger condition, [39].

We first note that condition (3.1) trivially holds for $n = 0$ by taking $\alpha_{k,0} = 1$. On the other hand, thinking at (W), we are clearly interested only at the asymptotic behavior of α_{kn} as $n \to +\infty$. Secondly, Def. 4 is weaker than the usual definition of contraction because of the following first three remarks:

(i) We will see only in Sec. 6 that condition Def. 4(i) is not a simple generalization of the usual stronger $P : X \longrightarrow X$, but is essential for the choice of the radii in the PL Thm. 14.
(ii) Both conditions (3.1) and (W) depend on the initial point $y_0 \in X$. In contrast to the classical BFPT, this underscores, in an abstract setting, that for PDE the property to have a contraction with loss of derivatives depends on the initial condition $y_0 \in X$. Moreover, in this paper we are solely interested in existence results for fixed points of contractions with loss of derivatives; uniqueness results would require conditions closer to the classical BFPT (see e.g. Lemma 5).
(iii) Since we want to take $n \to +\infty$, a condition such as Def. 4(iii) intuitively implies that we have to consider all the derivatives controlled by $\|-\|_k$ for all $k \in \mathbb{N}$. It is for this reason that in the present work we deal only with smooth solutions of (1.1).
(iv) We have a loss $L \ge 0$ of derivatives. If $L = 0$, Def. 4 actually tells us that, for all $k \in \mathbb{N}$, there exists $N_k \in \mathbb{N}$ such that $\sum_{n=N_k}^{+\infty} \alpha_{kn} < 1$. If we consider only $n \ge N_k$, the following Lem. 5 states that we essentially have the classical notion of contraction.

More classically, contraction property (3.1) is implied by one of the following stronger conditions:

Lemma 5. *Let X, y_0 and L be as in Def. 4. Then, the following sufficient conditions hold:*

(i) *If $P : X \longrightarrow \mathcal{F}$ satisfies the property*

$$\forall k, n \in \mathbb{N}\, \exists \alpha_{kn} \in \mathbb{R}_{>0}\, \forall u, v \in X : \; \left\| P^n(u) - P^n(v) \right\|_k \le \alpha_{kn} \left\| u - v \right\|_{k+nL}, \tag{3.2}$$

then condition Def. 4(iii) holds for all $y_0 \in X$ with the same contraction constants α_{kn}. Moreover, $P : X \longrightarrow \mathcal{F}$ is continuous.

(ii) *If (3.2) holds only for $n = 1$ with $\alpha_k := \alpha_{k1}$, then condition Def. 4(iii) holds for all $y_0 \in X$ with contraction constants $\tilde{\alpha}_{kn} := \prod_{j=0}^{n-1} \alpha_{k+jL}$.*

Moreover, if $L = 0$ and y_1, y_2 are fixed points of P, then $\|y_1 - y_2\|_k = 0$ for all $k \in \mathbb{N}$. In particular, if at least one of $\|-\|_k$ is a norm, this entails that $y_1 = y_2$.

Proof. (i): In fact, (3.2) yields

$$\left\| P^{n+1}(y_0) - P^n(y_0) \right\|_k = \left\| P^n(P(y_0)) - P^n(y_0) \right\|_k \le \alpha_{kn} \left\| P(y_0) - y_0 \right\|_{k+nL}.$$

Taking $n = 1$ in (3.2), we have $\|P(u) - P(v)\|_k \le \alpha_{k,1}\|u - v\|_{k+L}$ and hence $P\left(B^k_{r/\alpha_{k,1}}(u) \cap X\right) \subseteq B^k_r(P(u)) \cap X$, so that P is continuous.

(ii): If (3.2) holds only for $n = 1$, then we can prove the claim by induction on n. For $n = 0$ the conclusion is trivial since $\tilde{\alpha}_{k0} = 1$. For the inductive step, we have

$$\begin{aligned}
\|P^{n+2}(y_0) - P^{n+1}(y_0)\|_k &= \|P(P^{n+1}(y_0)) - P(P^n(y_0))\|_k \\
&\le \alpha_k \|P^{n+1}(y_0) - P^n(y_0)\|_{k+L} \\
&\le \alpha_k \prod_{j=0}^{n-1} \alpha_{k+L+jL} \|P(y_0) - y_0\|_{k+L+nL} \\
&= \left(\prod_{j=0}^{n} \alpha_{k+jL} \right) \|P(y_0) - y_0\|_{k+(n+1)L}.
\end{aligned}$$

Finally, if $P(y_l) = y_l$, then $\|P^n(y_1) - P^n(y_2)\|_k = \|y_1 - y_2\|_k \le \alpha_{kn}\|y_1 - y_2\|_k \le \|y_1 - y_2\|_k$ because the convergence $\sum_{n=0}^{+\infty} \alpha_{kn} < +\infty$ implies $\alpha_{kn} \le 1$ for some $n \in \mathbb{N}$. Thereby, $\|y_1 - y_2\|_k = 0$, which entails $y_1 = y_2$ if $\|-\|_k$ is a norm. □

In Thm. 13 and in the proof of Thm. 14, we will see that for the normal smooth Cauchy problem (1.1), the corresponding fixed point integral operator P always satisfies the stronger condition (3.2). Therefore, in all these cases the real dependence on y_0 actually lies in conditions (W) and (ii).

Even in the simple case of the transport equation $\partial_t y = c \cdot \partial_x y$, where c, $y \in \mathcal{C}^\infty([0, a] \times S)$, $S \Subset \mathbb{R}$, we can recognize the appearance of a loss of derivative $L = 1$ due to the occurrence of the term $\partial_x y$ on the right hand side of the PDE. In fact, set $P(u)(t, x) := y_0(x) + \int_0^t c(s, x) \cdot \partial_x u(s, x)\, \mathrm{d}s$ for a fixed $y_0 \in \mathcal{C}^\infty(S)$ and for any $u \in \mathcal{C}^\infty([0, a] \times S)$. Considering on $\mathcal{C}^\infty([0, a] \times S)$ the family of norms

$$\|u\|_k := \max_{|\alpha|+|\beta| \le k} \max_{(t,x) \in [0,a] \times S} \left| \partial_t^\alpha \partial_x^\beta u(t, x) \right|,$$

we would like to argue in the following way (where, for simplicity, we consider only the case $n=1$ in property (3.2)):

$$\|P(u)-P(v)\|_k=\left\|\int_0^t c\cdot\partial_x(u-v)\ \mathrm{d}s\right\|_k\le$$
$$\le a\cdot\|c\|_k\cdot\|\partial_x(u-v)\|_k\le a\cdot\|c\|_k\cdot\|u-v\|_{k+1}. \tag{3.3}$$

The problem in this deduction is that the first inequality is generally *false* for these norms: if $k>0$ then any derivative ∂_t deletes the integration, so that the factor a (which is important to get a local solution) cannot appear in (3.3) (see Sec. 5.1 for more details). We will fix this problem by taking another family of norms which, anyway, respect the same basic ideas (see Def. 9), and where the estimates (3.3) hold.

Def. 4 has been tuned to allow the proof of the following result, whose proof is surprisingly simple:

Theorem 6 (BFPT with loss of derivatives)**.** *In the assumptions of Def. 4, if* $P\in\mathrm{C}(X,L,y_0)$*, then* $(P^n(y_0))_{n\in\mathbb{N}}$ *is a Cauchy sequence, and hence*

$$\overline{y}:=\lim_{n\to+\infty}P^n(y_0)\in X$$

is a fixed point of P*. Moreover, for all* k*,* $n\in\mathbb{N}$ *we have that*

$$\|\overline{y}-P^n(y_0)\|_k\le\sum_{j=n}^{+\infty}\alpha_{kj}\|P(y_0)-y_0\|_{k+jL}.$$

Proof. If we prove that $(P^n(y_0))_{n\in\mathbb{N}}$ is a Cauchy sequence, the claim follows from Lem. 3. Let m, n, $k\in\mathbb{N}$ with $m>n$. Then

$$\begin{aligned}\|P^m(y_0)-P^n(y_0)\|_k&\le\left\|P^m(y_0)-P^{m-1}(y_0)\right\|_k+\cdots+\left\|P^{n+1}(y_0)-P^n(y_0)\right\|_k\\&\le\alpha_{k,m-1}\|P(y_0)-y_0\|_{k+(m-1)L}+\cdots+\alpha_{kn}\|P(y_0)-y_0\|_{k+nL}\\&=\sum_{j=n}^{m-1}\alpha_{kj}\|P(y_0)-y_0\|_{k+jL}.\end{aligned}\tag{3.4}$$

We conclude using (W) of Def. 4. The final claim holds by taking $m\to+\infty$ in (3.4) as $\overline{y}=\lim_{m\to+\infty}P^m(y_0)$. □

Clearly, the chain of inequalities in (3.4) could be stopped in several different ways. For example, as $\|P(y_0)-y_0\|_{k+jL}\le\|P(y_0)-y_0\|_{k+(m-1)L}$, we can continue arriving at a final term of the form $\|P(y_0)-y_0\|_{k+(m-1)L}\cdot\sum_{j=n}^{+\infty}\alpha_{kj}$. Actually, this would lead us to consider a limit of the form $\lim_{\substack{n,m\to+\infty\\ n\le m}}p_m\cdot q_n$, which never exists if $p_m\to+\infty$ and $q_n=a^n$, because we can take $n\to+\infty$ depending on p_m. On the contrary, in condition (W) the summation index n links the two factors in the series; looking at Lem. 5 and next Thm. 13, we can also say that condition (W) links the right hand side F and the initial conditions y_{0j} of the Cauchy problem (1.1). This link is important because without setting growing conditions on the derivatives of y_{0j}, in general the problem (1.1) has no solution (see Sec. 6, where we prove condition (W) for a class of problems, with non-Gevrey solutions, which are not covered by the Ovsjannikov-Nirenberg-Nishida extension of the Cauchy-Kowalewsky theorem). On the other hand, it is clear that the proof of previous

Thm. 6 is quite standard, and this underscores that the key step lies in Def. 4 of contraction with loss of derivatives L starting from y_0.

4. Solutions of equations

In this Section, we want to use the Banach fixed point Thm. 6 with loss of derivatives to solve equations of the form $F(u) = v$ in arbitrary graded Fréchet spaces. We can also inscribe this problem as the proof of local surjection in inverse function theorems. In order to facilitate the comparison with other local surjection theorems, only in this section we use notations similar to those of [7, 8].

In the following, given a sequence $R = (r_k)_{k\in\mathbb{N}}$, $r_k \in \mathbb{R}_{>0} \cup \{+\infty\}$, $\ell \in \mathbb{N}$, and a point u_0 in a graded Fréchet space $\mathcal{F}$, we set

$$\bar{B}_R^{+\ell}(u_0) := \{u \in \mathcal{X} \mid \|u - u_0\|_{s+\ell} \le R_{s+\ell} \ \forall s \in \mathbb{N}\}. \tag{4.1}$$

In particular, if $\ell = 0$, we simply use the notation $\bar{B}_R(u_0) := \bar{B}_R^{+0}(u_0)$. Note that $\bar{B}_R^{+\ell}(u_0)$ is closed in $\mathcal{F}$ as it is the intersection of closed sets. Moreover, $\bar{B}_R(u_0)$ trivially generalizes the space usually used in the proof of the PLT for ODE, where we only have $r_k = r_0 < +\infty$. We also note that we can have $\bar{B}_R(u_0) = \{u_0\}$ if $r_k \to 0^+$: This underscores that a key point in these results is exactly the choice of radii r_k so that $r_k \not\to 0^+$; we will see in Thm. 21 that this is possible in a non trivial class of examples. The first trivial consequence of Thm. 6 reformulates $F(u) = v$ as a fixed point of the map $P(u) := u - F(u) + v$:

Corollary 7. *Let $\left(\mathcal{F}, (\| - \|_k)_{k\in\mathbb{N}}\right)$ be a graded Fréchet space. Let X be a closed subset of $\mathcal{F}$, $F : X \longrightarrow \mathcal{F}$ be a continuous map, $v \in \mathcal{F}$ and $L \in \mathbb{N}$. Set $P(u) := u - F(u) + v$ and assume that for all k, $n \in \mathbb{N}$, we have*

$$P^n(u_0) \in X, \tag{4.2}$$

$$\exists \alpha_{kn} \in \mathbb{R}_{>0} : \ \left\|P^{n+1}(u_0) - P^n(u_0)\right\|_k \le \alpha_{kn} \|P(u_0) - u_0\|_{k+nL}, \tag{4.3}$$

$$\sum_{n=0}^{+\infty} \alpha_{kn} \cdot \|P(u_0) - u_0\|_{k+nL} < +\infty, \tag{4.4}$$

then, there exists $u \in X$ such that $F(u) = v$.

In spite of its triviality, we will see in Sec. 6 that this result allows us to solve PDE with the same scope of the next PL Thm. 14 (which, on the other hand, already includes in its proof the verification of property (4.3)). Moreover, let us now note that in Cor. 7 we do not require differentiability of f let alone the existence of some inverse of its differential $\mathrm{d}f(u)$.

Generalizing the derivation of the inverse function theorem from the classical BFPT in Banach spaces (see e.g. [16, 4, 20]), we obtain the following theorem, where we can think $D(u_0) = DF(u_0)$ (in the case where F is differentiable) and $L(u_0)$ a right inverse of $DF(u_0)$.

Theorem 8. *Let $\left(\mathcal{X}, (\| - \|_k)_{k\in\mathbb{N}}\right)$, $\left(\mathcal{Y}, (| - |_k)_{k\in\mathbb{N}}\right)$ be graded Fréchet spaces, $u_0 \in \mathcal{X}$, and $R := (R_k)_{k\in\mathbb{N}}$, $T := (T_k)_{k\in\mathbb{N}}$ be sequences of strictly positive or infinite real numbers. Let $F : \bar{B}_R(u_0) \longrightarrow \mathcal{Y}$ be a map, $D(u_0) : \mathcal{X} \longrightarrow \mathcal{Y}$ and $L(u_0) : \mathcal{Y} \longrightarrow \mathcal{X}$ be linear maps depending only on u_0. Assume that:*

(i) $\|L(u_0)F(u+h) - L(u_0)F(u) - h\|_k \le \alpha_k \cdot \|h\|_{k+m}$ *for all $k \in \mathbb{N}$, $u \in \bar{B}_R(u_0)$, $h \in \mathcal{X}$ such that $u + h \in \bar{B}_R(u_0)$, where $\alpha_k > 0$ and $m \in \mathbb{N}$;*

(ii) $L(u_0)$ *is a right inverse of* $D(u_0)$*, i.e.* $D(u_0)L(u_0)k = k$ *for all* $k \in \mathcal{Y}$*;*
(iii) $|D(u_0)h|_k \le a_k \cdot \|h\|_{k+d}$ *for all* $k \in \mathbb{N}$ *and all* $h \in \mathcal{X}$*, where* $a_k > 0$ *and* $d \in \mathbb{N}$*;*
(iv) $\|L(u_0)k\|_k \le b_k \cdot (1 + \|u_0\|_{k+\ell})|k|_{k+\ell}$ *for all* $k \in \mathbb{N}$ *and all* $k \in \mathcal{Y}$*, where* $b_k > 0$ *and* $\ell \in \mathbb{N}$*;*
(v) $R_{k+m} \le R_k$ *and* $T_{k+\ell} \le \frac{R_{k+m}(1-\alpha_k)}{b_k \cdot (1+\|u_0\|_{k+\ell})}$ *for all* $k \in \mathbb{N}$*;*
(vi) $\sum_{n=0}^{+\infty} \prod_{j=0}^{n-1} \alpha_{k+jm} \cdot \|L(u_0)[F(u_0) - v]\|_{k+nm} < +\infty$ *for all* $k \in \mathbb{N}$ *and all* $v \in \bar{B}_T^{+\ell}(F(u_0))$*.*

Then, the following properties hold:

(a) $|F(u_0 + h) - F(u_0)|_k \le a_k (\alpha_{k+d}\|h\|_{k+d+m} + \|h\|_{k+d})$ *for all* $k \in \mathbb{N}$ *and all* $h \in \mathcal{X}$ *such that* $u_0 + h \in \bar{B}_R(u_0)$*, and therefore* F *is continuous;*
(b) $|F(u_0 + h) - F(u_0)|_{k+\ell} \ge \frac{1}{b_k \cdot (1+\|u_0\|_{k+\ell})} (\|h\|_k - \alpha_k\|h\|_{k+m})$ *for all* $k \in \mathbb{N}$ *and all* $h \in \mathcal{X}$ *such that* $u_0 + h \in \bar{B}_R(u_0)$*;*
(c) *If both* F *and* $L(u_0)$ *are continuous, then*

$$\forall v \in \bar{B}_T^{+\ell} (F (u_0)) \, \exists u \in \bar{B}_R (u_0) : \; F(u) = v.$$

(d) *For all* $u \in \bar{B}_R(u_0)$ *and all* $v \in \mathcal{Y}$*, if* $F(u) = v$*, then for all* $k \in \mathbb{N}$ *we must have*

$$\frac{\|u - u_0\|_k - \alpha_k\|u - u_0\|_{k+m}}{b_k \cdot (1 + \|u_0\|_{k+\ell})} \le |v - F(u_0)|_{k+\ell}$$
$$|v - F(u_0)|_k \le a_k (\alpha_{k+d}\|u - u_0\|_{k+d+m} + \|u - u_0\|_{k+d}).$$

In other words, if at least one of these inequalities does not hold, then the equation $F(u) = v$ *does not have a solution* u *in* $\bar{B}_R(u_0)$*.*

Note that we do not require that the spaces $\mathcal{X}$, $\mathcal{Y}$ are tame or admit smoothing operators like in the Nash-Moser theorem, see [15]; in principle, we also do not require that F is differentiable as in Ekeland inverse function theorem [7]. Moreover, its proof is a generalization of [16], so that it includes the inverse function theorem for Lipschitz maps in Banach spaces if $m = d = \ell = 0$ and $r_k = r_0$ for all $k \in \mathbb{N}$.

Proof. (a): From (ii), (iii) and linearity of $D(u_0)$, for $k \in \mathbb{N}$ we can write

$$\begin{aligned} |F(u_0 + h) - F(u_0)|_k &= |D(u_0)L(u_0)F(u_0 + h) - D(u_0)L(u_0)F(u_0)|_k \\ &\le a_k\|L(u_0)F(u_0 + h) - L(u_0)F(u_0)\|_{k+d} \\ &= a_k\|L(u_0)F(u_0 + h) - L(u_0)F(u_0) - h + h\|_{k+d} \\ &\le a_k (\alpha_{k+d}\|h\|_{k+d+m} + \|h\|_{k+d}), \end{aligned}$$

where we used (i). Considering that $\|h\|_{k+d} \le \|h\|_{k+d+m}$, from this property we have that $F : \bar{B}_R(u_0) \longrightarrow \mathcal{Y}$ is continuous.
(b): Once again from (i) and $k \in \mathbb{N}$, we get

$$\begin{aligned} \|h\|_k &\le \|L(u_0)F(u_0 + h) - L(u_0)F(u_0) - h\|_k + \|L(u_0)F(u_0 + h) - L(u_0)F(u_0)\|_k \\ &\le \alpha_k\|h\|_{k+m} + b_k \cdot (1 + \|u_0\|_{k+\ell})|F(u_0 + h) - F(u_0)|_{k+\ell} \end{aligned}$$

because of (iv) and the linearity of $L(u_0)$.

(c): Properties (iv) and (a) imply the continuity of the map $P_v(u) := u - L(u_0)[F(u) - v]$, $P_v : \bar{B}_R(u_0) \longrightarrow \mathcal{X}$, for any fixed $v \in \mathcal{Y}$. We now prove that

$P_v : \bar{B}_R(u_0) \longrightarrow \bar{B}_R(u_0)$ if $v \in \bar{B}^{+\ell}_T(F(u_0))$. Take $u \in \bar{B}_R(u_0)$ and $k \in \mathbb{N}$, then

$$\begin{aligned}
\|P_v(u) - u_0\|_k &= \|u - L(u_0)\,[F(u) - v] - u_0\|_k \\
&= \|u - u_0 - L(u_0)\,[F(u) - F(u_0) + F(u_0) - v]\,\|_k \\
&= \|u - u_0 - L(u_0)\,[F(u) - F(u_0)] - L(u_0)\,[F(u_0) - v]\,\|_k \\
&\le \|u - u_0 - L(u_0)\,[F(u) - F(u_0)]\,\|_k + \|L(u_0)F(u_0) - L(u_0)v\|_k \\
&\le \alpha_k\|u - u_0\|_{k+m} + b_k \cdot (1 + \|u_0\|_{k+\ell})|F(u_0) - v|_{k+\ell} \qquad (4.5) \\
&\le \alpha_k \cdot R_{k+m} + b_k \cdot (1 + \|u_0\|_{k+\ell}) \cdot \frac{R_{k+m}(1-\alpha_k)}{b_k \cdot (1 + \|u_0\|_{k+\ell})} = R_{k+m} \le R_k, \qquad (4.6)
\end{aligned}$$

where in (4.5) we used (i) with $h = u - u_0$, and (iv) with $k = F(u_0) - v$; in (4.6) we used $|F(u_0) - v|_{k+\ell} \le T_{k+\ell} \le \frac{R_{k+m}(1-\alpha_k)}{b_k\cdot(1+\|u_0\|_{k+\ell})}$.

For all u, $\bar{u} \in \bar{B}_R(u_0)$, property (i) yields

$$\|P_v(u) - P_v(\bar{u})\|_k = \|L(u_0)F(\bar{u}) - L(u_0)F(u) - (\bar{u} - u)\|_k \le \alpha_k \cdot \|u - \bar{u}\|_{k+m} \quad \forall k \in \mathbb{N}. \qquad (4.7)$$

Therefore, Lem. 5(ii) implies that the map P_v has contraction constants $\tilde{\alpha}_{kn} := \prod_{j=0}^{n-1} \alpha_{k+jm}$ with loss of derivatives m. Assumption (vi) is exactly Weissinger condition for this map, and hence the BFPT with loss of derivatives Thm. 6 proves claim (c).

(d): These inequalities are resp. (b) and (a) with $h := u - u_0$; □

Even if the previous statement allows us to take $r_k = +\infty$, it is now Weissinger condition (vi) that forces to take v near $F(u_0)$: the factor $\|P_v(u_0) - u_0\|_{k+nm} = \|L(u_0)\,[F(u_0) - v]\,\|_{k+nm}$ is small if v is near $F(u_0)$; note also that (vi) is implied by the stronger condition $\sum_{n=0}^{+\infty} \alpha_{kn} r_{k+nL} < +\infty$ because $P_v : \bar{B}_R(u_0) \longrightarrow \bar{B}_R(u_0)$. On the other hand, assumption (vi) is in principle compatible with growing term $\|P_v(u_0) - u_0\|_{k+nm}$ as $k + nm \to +\infty$, even if the Lipschitz factors α_{kn} must keep the series convergent. Note also that the assumption $T_{k+\ell} > 0$ and (v) imply $\alpha_k < 1$.

5. A Picard-Lindelöf theorem for PDE

In the following, considering the Cauchy problem (1.1), we always set and assume

$$\begin{aligned}
&\hat{L} := \mathrm{Card}\{(\alpha, \gamma) \in \mathbb{N}^s \times \mathbb{N}_{\le p} \mid |\alpha| \le L\} \\
&a,\ b \in \mathbb{R}_{>0}, \quad [t_0 - a, t_0 + b] \times S =: T \times S \Subset \mathbb{R}^{1+s} \qquad (5.1) \\
&F \in \mathcal{C}^\infty\left(T \times S \times \mathbb{R}^{m\cdot\hat{L}}, \mathbb{R}^m\right) \\
&y_{0j} \in \mathcal{C}^\infty(S, \mathbb{R}^m) \quad \forall j = 0, \ldots, d-1,
\end{aligned}$$

where $p \le d-1$ denotes the maximum order of derivatives $\partial_t^\gamma y(t,x) \in \mathbb{R}^m$ appearing on the right hand side of (1.1).

5.1. Supremum norms of integral functions.

We have already mentioned that the first inequality in (3.3) is generally wrong. Let us construct a counter example for the space of one variable functions $\mathcal{C}^\infty([0,a])$ with the norms $\|u\|_k :=$

$\max_{\substack{h\le k\\ t\in[0,a]}} \left|u^{(h)}(t)\right|$. Take e.g. $a=\frac{1}{2}$ and consider the straight line $y=1$. Then

$$\left\|\int_0^{(-)} y\right\|_1 = \max\left(\max_{t\in[0,\frac{1}{2}]}\left|\int_0^t 1\right|, \max_{t\in[0,\frac{1}{2}]}|1|\right) = 1$$

and

$$\|y\|_1 = \max\left(\max_{t\in[0,\frac{1}{2}]}|1|, \max_{t\in[0,\frac{1}{2}]}|0|\right) = 1,$$

therefore

$$\left\|\int_0^{(-)} 1\right\|_1 = 1 > a\cdot\|y\|_1 = \frac{1}{2}.$$

It is not hard to prove that, actually, $\|\int_0^{(-)} y\|_k > a\cdot\|y\|_k$ for all $k\ge 1$.

This remark allows us to understand, once again, why in the classical proof of the smooth PLT we consider only the space $\mathcal{C}^0([0,a])$ of continuous functions with the supremum norm $\|-\|_0$: in fact, even if we aim to get a *smooth solution* y (so that we would have to control all its derivatives), the normal form of the equation recursively yields the smoothness of y starting from a continuous solution of the corresponding integral problem.

Similarly, we can argue for normal PDE: considering the corresponding integral problem

$$y(t,x) = i_0(t,x) + \int_{t_0}^t \mathrm{d}s_d \overset{d}{\cdots} \int_{t_0}^{s_2} F\left[s_1, x, (\partial_x^\alpha\partial_t^\gamma y)_{\substack{|\alpha|\le L\\ \gamma\le p}}\right]\mathrm{d}s_1, \tag{5.2}$$

$$i_0(t,x) := \sum_{j=0}^{d-1}\frac{y_{0j}(x)}{j!}(t-t_0)^j. \tag{5.3}$$

we only need that the function y is of class $\mathcal{C}^p$ in t and smooth in x: smoothness in t recursively follows from (5.2), and we only have to control all its derivatives in x. This motivates the introduction of a space with this kind of functions.

5.2. **Spaces of separately regular functions.** As we mentioned above, instead of considering functions which are jointly regular in both variables (t,x), we need to consider separate degree of regularity in each variable.

Definition 9.

(i) If $X\subseteq\mathbb{R}^n$ is an arbitrary subset and $q\in\mathbb{N}\cup\{\infty\}$, we say that $f\in\mathcal{C}^q(X,\mathbb{R}^m)$ if for each $x\in X$ there exists an open neighborhood $x\in U\subseteq\mathbb{R}^n$ and a function $F\in\mathcal{C}^q(U,\mathbb{R}^m)$ such that $F|_{U\cap X}=f|_{U\cap X}$.

(ii) Let $T\times S\Subset\mathbb{R}^{1+s}$. Set

$$\mathbb{N}_p^{1+s} := \left\{\beta\in\mathbb{N}^{1+s} \mid \beta_1\le p\right\}, \tag{5.4}$$

and denote by $\mathcal{C}_t^p\mathcal{C}_x^\infty(T\times S,\mathbb{R}^m)$ the set of continuous functions $y\in\mathcal{C}^0(T\times S,\mathbb{R}^m)$ such that

$$\forall\beta\in\mathbb{N}_p^{1+s}:\ \exists\,\partial^\beta y\in\mathcal{C}^0(T\times S,\mathbb{R}^m).$$

The functions in $\mathcal{C}_t^p\mathcal{C}_x^\infty(T\times S,\mathbb{R}^m)$ are called *separately* $\mathcal{C}_t^p\mathcal{C}_x^\infty$ *regular.* This space is endowed with the countable family of norms $\|-\|_k$, $k\in\mathbb{N}$, defined by

$$\|y\|_k := \max_{1\le h\le m} \max_{\substack{|\beta|\le k \\ \beta\in\mathbb{N}_p^{1+s}}} \max_{(t,x)\in T\times S} \left|\partial^\beta y^h(t,x)\right|. \tag{5.5}$$

In problem (1.1), we could also consider a reduction to first order: setting $y^1 := y$, $y^{j+1} := \partial_t y^j$, $j = 1, \dots, p$, problem (1.1) is equivalent to

$$\begin{cases} \partial_t Y(t,x) = \bar{F}\left[t, x, \left(\partial_x^\alpha Y^{\gamma+1}\right)_{\substack{|\alpha|\le L \\ \gamma\le p}}\right], \\ Y(t_0,x) = Y_0(x), \end{cases} \tag{5.6}$$

where, as usual, we mean $\partial_x^\alpha Y^{\gamma+1} = \partial_x^\alpha Y^{\gamma+1}(t,x)$, and

$$\begin{aligned} Y(t,x) &:= (y^1(t,x), \dots, y^{p+1}(t,x)) \\ Y_0(x) &:= (y_0^0(x), \dots, y_0^{d-1}(x)) \\ \bar{F}^d\left[t, x, (u^{\alpha,\gamma})_{\substack{|\alpha|\le L \\ \gamma\le p}}\right] &:= F\left[t, x, (u^{\alpha,\gamma})_{\substack{|\alpha|\le L \\ \gamma\le p}}\right] \\ \bar{F}^j\left[t, x, (u^{\alpha,\gamma})_{\substack{|\alpha|\le L \\ \gamma\le p}}\right] &:= y^{j+1} \end{aligned}$$

for $j = 1, \dots, p$. In the corresponding integral problem (5.2), we could assume $d = 1$ and hence we only need that the function y is of class $\mathcal{C}_t^0\mathcal{C}_x^\infty$. On the one hand, this would simplify our next statements. However, we would obtain a PLT with assumptions that are clear only for $d = 1$, and to prove from this a corresponding result for $d > 1$ is not so easy. For this reason, we prefer to directly proceed with the generic problem (5.2) without implementing a reduction to first order.

Lemma 10. *In the notations of Def. 9, $\left(\mathcal{C}_t^p\mathcal{C}_x^\infty\left(T\times S,\mathbb{R}^m\right), \left(\|-\|_k\right)_{k\in\mathbb{N}}\right)$ is a graded Fréchet space.*

Proof. The only non trivial property to check is that the topology induced by the family of norms $\left(\|-\|_k\right)_{k\in\mathbb{N}}$ is Cauchy complete. Let $(y_n)_{n\in\mathbb{N}}$ be a Cauchy sequence in $\left(\mathcal{C}_t^p\mathcal{C}_x^\infty\left(T\times S,\mathbb{R}^m\right), \left(\|-\|_k\right)_{k\in\mathbb{N}}\right)$, so that for $k = 0$, the sequence $(y_n)_{n\in\mathbb{N}}$ converges uniformly. Let $y : T\times S \to \mathbb{R}^m$ be the continuous function defined by

$$y(t,x) := \lim_{n\to+\infty} y_n(t,x) \quad \forall (t,x)\in T\times S. \tag{5.7}$$

For all $\beta\in\mathbb{N}_p^{1+s}$, we have $\|\partial^\beta y_l - \partial^\beta y_n\|_0 \le \|y_l - y_n\|_h$ and hence $\left(\partial^\beta y_n\right)_{n\in\mathbb{N}}$ is a uniformly convergent Cauchy sequence in $\mathcal{C}^0\left(T\times S,\mathbb{R}^m\right)$ that converges to $\partial^\beta y\in\mathcal{C}^0\left(T\times S,\mathbb{R}^m\right)$. This shows that $y\in\mathcal{C}_t^p\mathcal{C}_x^\infty(T\times S,\mathbb{R}^m)$. It remains to prove that $y_n \to y$ with respect to the norms defined in (5.5). For $k = 0$, we simply recall that the limit in (5.7) is actually a uniform limit. For $k > 0$, we note that for all $\beta\in\mathbb{N}_p^{1+s}$ with $|\beta|\le k$, the sequence $\left(\partial^\beta y_n\right)_{n\in\mathbb{N}}$ converges uniformly in $T\times S$ to $\partial^\beta y$. □

Exactly because we have $\beta_1 \le p < d$ in (5.4), we can now have the desired estimate in considering the norm of an integral function:

Lemma 11. *Let $f\in\mathcal{C}_t^0\mathcal{C}_x^\infty(T\times S,\mathbb{R}^m)$ and, for every $k\in\mathbb{N}$, let $M_k\in\mathcal{C}^0(T\times S)$ be such that*

$$\left|\partial_x^\nu f^h(t,x)\right| \le M_k(t,x) \tag{5.8}$$

for all $(t,x) \in T \times S$, $h = 1, \ldots, m$, and all $\nu \in \mathbb{N}^s$ such that $|\nu| \le k$. *Set*

$$\bar{M}_{kj}(t,x) := \left| \int_{t_0}^{t} \mathrm{d}s_j \overset{j}{\cdots} \int_{t_0}^{s_2} M_k(s_1,x)\,\mathrm{d}s_1 \right| \quad \forall (t,x) \in T \times S\, \forall j = 1, \ldots, d.$$

Then, with respect to the norms in the space $\mathcal{C}^p_t\mathcal{C}^\infty_x$ *defined as in* (5.5), *we have*

(i) $$\left\| \int_{t_0}^{(-)} \mathrm{d}s_d \overset{d}{\cdots} \int_{t_0}^{s_2} f(s_1,-)\,\mathrm{d}s_1 \right\|_k \le \max_{\substack{x\in S\\ 0<j\le d}} \bar{M}_{kj}(t_0 + \max(a,b), x).$$

In particular, if $M_k = \|f\|_k$*:*

(ii) $$\left\| \int_{t_0}^{(-)} \mathrm{d}s_d \overset{d}{\cdots} \int_{t_0}^{s_2} f(s_1,-)\,\mathrm{d}s_1 \right\|_k \le \max(a,b)\cdot \|f\|_k.$$

Proof. (i): Clearly, the notation $\int_{t_0}^{(-)} \mathrm{d}s_d \overset{d}{\cdots} \int_{t_0}^{s_2} f(s_1,-)\,\mathrm{d}s_1$ denotes the function

$$\left((t,x) \in T \times S \mapsto \int_{t_0}^{t} \mathrm{d}s_d \overset{d}{\cdots} \int_{t_0}^{s_2} f(s_1,x)\,\mathrm{d}s_1 \in \mathbb{R}^m \right) \in \mathcal{C}^p_t\mathcal{C}^\infty_x(T\times S, \mathbb{R}^m)$$

(actually, this is a $\mathcal{C}^d_t\mathcal{C}^\infty_x$-function, but in the statement we are considering the norms $\|-\|_k$ of the space $\mathcal{C}^p_t\mathcal{C}^\infty_x$). For some $\beta \in \mathbb{N}^{1+s}_p$ with $|\beta| \le k$, and some $h = 1, \ldots, m$, we have

$$\left\| \int_{t_0}^{(-)} \mathrm{d}s_d \overset{d}{\cdots} \int_{t_0}^{s_2} f(s_1,-)\,\mathrm{d}s_1 \right\|_k = \\ = \max_{(t,x)\in T\times S} \left| \partial^\beta \int_{t_0}^{t} \mathrm{d}s_d \overset{d}{\cdots} \int_{t_0}^{s_2} f^h(s_1,x)\,\mathrm{d}s_1 \right|. \tag{5.9}$$

But $\beta_1 \le p < d$, and hence, setting $\nu := (\beta_2, \ldots, \beta_s)$, the operator $\partial^\beta = \partial^\nu_x \partial^{\beta_1}_t$ deletes β_1 integrals in (5.9); set $\bar{j} := d - \beta_1 > 0$. Differentiation under the integral sign yields

$$\begin{aligned}
\left\| \int_{t_0}^{(-)} \mathrm{d}s_d \overset{d}{\cdots} \int_{t_0}^{s_2} f(s_1,-)\,\mathrm{d}s_1 \right\|_k &= \\
&= \max_{(t,x)\in T\times S} \left| \int_{t_0}^{t} \mathrm{d}s_{\bar{j}} \overset{\bar{j}}{\cdots} \int_{t_0}^{s_2} \partial^\nu_x f^h(s_1,x)\,\mathrm{d}s_1 \right| \\
&\le \max_{(t,x)\in T\times S} \operatorname{sgn}(t-t_0)^{\bar{j}} \int_{t_0}^{t} \mathrm{d}s_{\bar{j}} \overset{\bar{j}}{\cdots} \int_{t_0}^{s_2} \left| \partial^\nu_x f^h(s_1,x) \right|\,\mathrm{d}s_1 \\
&\le \max_{(t,x)\in T\times S} \operatorname{sgn}(t-t_0)^{\bar{j}} \int_{t_0}^{t} \mathrm{d}s_{\bar{j}} \overset{\bar{j}}{\cdots} \int_{t_0}^{s_2} M_k(s_1,x)\,\mathrm{d}s_1 \\
&= \max_{(t,x)\in T\times S} \bar{M}_{k\bar{j}}(t,x) \\
&\le \max_{\substack{x\in S\\ 0<j\le d}} \bar{M}_{kj}(t_0 + \max(a,b), x).
\end{aligned}$$

Note that if $t > t_0$, then $t_0 < s_{\bar{j}} < t$; if $t < t_0$, then $t < s_{\bar{j}} < t_0$, and in both cases $\operatorname{sgn}(t-t_0) = \operatorname{sgn}(s_{\bar{j}} - t_0)$. Similarly, we can proceed for the other integration variables s_q.

(ii): Condition (5.8) holds if $M_k = \|f\|_k$, and we have

$$\bar{M}_{kj}(t,x) = \|f\|_k \frac{(t-t_0)^j}{j!}.$$

Thereby, $\max_{\substack{x\in S\\ j\le d}} \bar{M}_{kj}(t_0+\max(a,b),x) = \max(a,b)\|f\|_k$. □

To solve problem (1.1) or, equivalently, the integral problem (5.2), let us introduce the following simplified notation

$$G(t,x,y) := F\left[t,x,(\partial_x^\alpha \partial_t^\gamma y)_{\substack{|\alpha|\le L\\ \gamma\le p}}(t,x)\right] \in \mathbb{R}^m, \tag{5.10}$$

for all $(t,x)\in T\times S$ and all $y\in \mathcal{C}_t^p\mathcal{C}_x^\infty(T\times S,\mathbb{R}^m)$. Explicitly note that the smooth function $G(t,x,y)$ is given by composition of $F\left[t,x,(z^{\alpha,\gamma})_{\substack{|\alpha|\le L\\ \gamma\le p}}\right]$ with the derivatives $(\partial_x^\alpha\partial_t^\gamma y)(x,t) = z^{\alpha,\gamma}\in\mathbb{R}^m$ that actually appear in (1.1). On the contrary, when we use the variables $G(t,x,z)$, we mean that $z = (z^{\alpha,\gamma})_{\substack{|\alpha|\le L\\ \gamma\le p}} \in \mathbb{R}^{m\cdot\hat{L}}$.

We now introduce the following definition of Lipschitz map:

Definition 12. Let $B\subseteq \mathcal{C}_t^p\mathcal{C}_x^\infty(T\times S,\mathbb{R}^m)$. We say that a map $G: T\times S\times B\to\mathbb{R}^m$ is *Lipschitz on B with loss of derivatives* (LOD) *L and Lipschitz factors* $(\Lambda_k)_{k\in\mathbb{N}}$ if

(i) $\forall y\in B:\ G(-,-,y)\in\mathcal{C}_t^p\mathcal{C}_x^\infty(T\times S,\mathbb{R}^m)$;
(ii) $\Lambda_k\in\mathcal{C}^0(T\times S)$ for all $k\in\mathbb{N}$;
(iii) If $k\in\mathbb{N}$, $\nu\in\mathbb{N}^s$, $|\nu|\le k$, $h=1,\dots,m$, u, $v\in B$, $(t,x)\in T\times S$, then

$$\left|\partial_x^\nu G^h(t,x,u) - \partial_x^\nu G^h(t,x,v)\right| \le \Lambda_k(t,x)\cdot \max_{l=1,\dots,m}\max_{\substack{|\alpha|\le k+L\\ \gamma\le p}} \left|\partial_x^\alpha\partial_t^\gamma(u^l-v^l)(t,x)\right|. \tag{5.11}$$

We simply say that G *is Lipschitz on B with LOD L* if the previous conditions (ii) and (iii) hold for some $(\Lambda_k)_{k\in\mathbb{N}}$.

Note that if $B=\bar{B}_R(x_0)$ but $r_k\to 0^+$, then $B=\{x_0\}$ and (5.11) is trivial: Once again, this underscore that a key problem is the choice of the radii $r_k\not\to 0^+$. In the next theorem, we prove that if G is defined by (5.10) and all the radii $r_k<+\infty$, then G is always Lipschitz with respect to *constant* factors $(\Lambda_k)_{k\in\mathbb{N}}$ in the space $\bar{B}_R(i_0)\subseteq\mathcal{C}_t^p\mathcal{C}_x^\infty(T\times S,\mathbb{R}^m)$ defined in (4.1) and with loss of derivatives L given, as in (1.1), by the maximum order of derivatives in x that appears in our PDE. The space $\bar{B}_R(i_0)$ is suitable for the proof of the PLT if we are also able to prove that for these finite radii the Picard iterates $P^n(i_0)\in\bar{B}_R(i_0)$. On the other hand, in Sec. 6 we will show examples of PDE with constant Lipschitz factors Λ_k but where we are free to also take $r_k\le+\infty$. In other words, the following result is only a sufficient condition.

Theorem 13. *Let $r_k\in\mathbb{R}_{>0}$ for all $k\in\mathbb{N}$. Set $R:=(r_k)_{k\in\mathbb{N}}$, i_0 as in* (5.3) *and $\bar{B}_R(i_0)$ as in* (4.1)*, i.e:*

$$\bar{B}_R(i_0) := \{u\in\mathcal{C}_t^p\mathcal{C}_x^\infty(T\times S,\mathbb{R}^m)\mid \|u-i_0\|_k\le r_k\ \forall k\in\mathbb{N}\}. \tag{5.12}$$

Then the function G defined in (5.10) *is Lipschitz in $\bar{B}_R(i_0)$ with loss of derivatives L and constant Lipschitz factors.*

Proof. We only have to prove condition (iii) of Def. 12, so that we consider $k \in \mathbb{N}$, $\alpha \in \mathbb{N}^s$, $|\nu| \le k$, $h = 1, \dots, m$, u, $v \in \bar{B}_R(i_0)$, $(t, x) \in T \times S$. Note that

$$\partial_x^\nu G^h(t,x,u) = \partial_x^\nu \left\{ F^h \left[t, x, (\partial_x^\alpha \partial_t^\gamma u)_{\substack{|\alpha| \le L \\ \gamma \le p}} \right] \right\}. \tag{5.13}$$

We first prove the case $|\nu| = 0$. Since $u \in \bar{B}_R(i_0)$, we have $\|u - i_0\|_{L+p} \le r_{L+p}$ and hence $\partial_x^\alpha \partial_t^\gamma u(t,x) \in \overline{B_{r_{L+p}}\left(\partial_x^\alpha \partial_t^\gamma i_0(S)\right)} =: C_{0\alpha,\gamma} \subseteq \bigcup_{\substack{|\alpha| \le L \\ \gamma \le p}} C_{0\alpha,\gamma} =: C_0 \Subset \mathbb{R}^m$ for all $|\alpha| \le L$ and $\gamma \le p$ because $r_{L+p} < +\infty$. Similarly, $\partial_x^\alpha \partial_t^\gamma v(t,x) \in C_0$. Thereby, using (5.13), we have

$$\begin{aligned} \left|\partial_x^\nu G^h(t,x,u) - \partial_x^\nu G^h(t,x,v)\right| &\le \|F\|_1 \cdot \max_{l=1,\dots,m} \max_{\substack{|\alpha| \le L \\ \gamma \le p}} \left|\partial_x^\alpha \partial_t^\gamma (u^l - v^l)(t,x)\right| \le \\ &\le \|F\|_1 \cdot \max_{l=1,\dots,m} \max_{\substack{|\alpha| \le k+L \\ \gamma \le p}} \left|\partial_x^\alpha \partial_t^\gamma (u^l - v^l)(t,x)\right|, \end{aligned}$$

where the norm $\|F\|_1$ is taken on $T \times S \times C_0^{m\hat{L}} \Subset \mathbb{R}^D$, $D := \dim(\mathrm{dom}(F)) = 1 + s + m\hat{L}$. We firstly set $\tilde{\Lambda}_k(\nu) := \|F\|_1$, and now consider the case $|\nu| > 0$.

From Faà di Bruno's formula

$$\partial_x^\nu G^h(t,x,u) = \sum_{1 \le |\eta| \le |\nu|} \partial^\eta F^h \left[t, x, (\partial_x^\alpha \partial_t^\gamma u)_{\substack{|\alpha| \le L \\ \gamma \le p}} \right] \cdot B_{\eta\nu} \left[\left(\partial_x^\mu \partial_t^\gamma u(t,x)\right)_{\mu\gamma} \right], \tag{5.14}$$

where $B_{\eta\beta}((z_{\mu\gamma})_{\mu\gamma})$ are Bell's like polynomials such that $|\mu| \le |\nu| + |\alpha| \le k + L$ for all μ. For simplicity, set

$$\partial^\eta F^h\,(t,x,u) := \partial^\eta F^h \left[t, x, (\partial_x^\alpha \partial_t^\gamma u)_{\substack{|\alpha| \le L \\ \gamma \le p}} \right] \tag{5.15}$$

$$Q_{\eta\nu}(t,x,u) := B_{\eta\nu} \left[\left(\partial_x^\mu \partial_t^\gamma u(t,x)\right)_{\mu\gamma} \right], \tag{5.16}$$

so that we can estimate

$$\begin{aligned} &\left|\partial_x^\nu G^h(t,x,u) - \partial_x^\nu G^h(t,x,v)\right| = \\ &\quad = \left| \sum_\eta \partial^\eta F^h(t,x,u) \cdot Q_{\eta\nu}(t,x,u) - \sum_\eta \partial^\eta F^h(t,x,v) \cdot Q_{\eta\nu}(t,x,v) \right| \le \\ &\quad \le \left| \sum_\eta \partial^\eta F^h(t,x,u) \cdot Q_{\eta\nu}(t,x,u) - \sum_\eta \partial^\eta F^h(t,x,u) \cdot Q_{\eta\nu}(t,x,v) \right| + \\ &\quad + \left| \sum_\eta \partial^\eta F^h(t,x,u) \cdot Q_{\eta\nu}(t,x,v) - \sum_\eta \partial^\eta F^h(t,x,v) \cdot Q_{\eta\nu}(t,x,v) \right|. \end{aligned}$$

For some $L_{\eta\nu} > 0$ depending on $Q_{\eta\nu}$, the first summand yields

$$\begin{aligned} &\left|\partial^\eta F^h(t,x,u)\right| \cdot \left|Q_{\eta\nu}(t,x,u) - Q_{\eta\nu}(t,x,v)\right| \le \\ &\qquad \le \|F\|_k \cdot L_{\eta\nu} \cdot \max_{l=1,\dots,m} \max_{|\mu| \le k+L} \left|\partial_x^\mu \partial_t^\gamma (u^l - v^l)(t,x)\right|. \end{aligned}$$

The second summand gives

$$\left|\partial^\eta F^h(t,x,u)-\partial^\eta F^h(t,x,v)\right|\cdot|Q_{\eta\nu}(t,x,v)|\le$$
$$\le\|F\|_{k+1}\cdot\max_{l=1,\dots,m}\max_{|\alpha|\le k+L}\left|\partial_x^\alpha\partial_t^\gamma(u^l-v^l)(t,x)\right|\cdot N_k,$$

where $N_k:=\max_{|\eta|\le|\nu|\le k}\max_{(t,x,v)\in T\times S\times C^k_{\eta\nu}}|Q_{\eta\nu}(t,x,v)|$ and, as we did above, $v\in\bar B_R(i_0)$ yields some $C^k_{\eta\nu}\Subset\mathbb{R}^m$ such that $\partial_x^\alpha\partial_t^\gamma v(t,x)\in C^k_{\eta\nu}$ for all $\alpha\in\mathbb{N}^s$ such that $|\alpha|\le k+L$. We finally obtain

$$\left|\partial_x^\nu G^h(t,x,u)-\partial_x^\nu G^h(t,x,v)\right|\le\sum_{1\le|\eta|\le|\nu|}\left(\|F\|_k\cdot\max_{|\eta|\le|\nu|\le k}L_{\eta\nu}+\|F\|_{k+1}\cdot N_k\right)\cdot$$
$$\cdot\max_{l=1,\dots,m}\max_{|\alpha|\le k+L}\left|\partial_x^\alpha\partial_t^\gamma(u^l-v^l)(t,x)\right|.$$

Setting

$$\tilde\Lambda_k(\nu):=\sum_{1\le|\eta|\le|\nu|}\left(\|F\|_k\cdot\max_{|\eta|\le|\nu|\le k}L_{\eta\nu}+\|F\|_{k+1}\cdot N_k\right)$$
$$\Lambda_k:=\max_{|\nu|\le k}\tilde\Lambda_k(\nu),$$

we get the conclusion. □

5.3. **The Picard-Lindelöf theorem for smooth normal PDE.** A natural method to solve PDE is to transform it into an infinite-dimensional ODE and then apply a PLT, see e.g. [31]. On the other hand, our approach can be considered simpler because we do not transform partial derivatives into ordinary ones in infinite dimensional spaces.

We can now state our main local existence result for smooth normal systems of PDE (recall the general assumptions (5.1)).

Theorem 14. *Let $\Lambda_k\in\mathcal{C}^0(T\times S)$, $r_k\in\mathbb{R}_{>0}\cup\{+\infty\}$ for all $k\in\mathbb{N}$, and assume that $\mathring{S}$ is dense in S. Define $R:=(r_k)_{k\in\mathbb{N}}$, $\bar B_R(i_o)$ as in (5.12), and $P:\bar B_R(i_0)\to\mathcal{C}_t^p\mathcal{C}_x^\infty(T\times S,\mathbb{R}^m)$ by*

$$P(y)(t,x):=i_0(t,x)+\int_{t_0}^t\mathrm{d}s_d\overset{d}{\ldots}\int_{t_0}^{s_2}G(s_1,x,y)\,\mathrm{d}s_1.$$

Assume that G is Lipschitz on $\bar B_R(i_0)$ with loss of derivatives L and Lipschitz factors $(\Lambda_k)_{k\in\mathbb{N}}$, and for all $(t,x)\in T\times S$, all $n\in\mathbb{N}$ and all $j=1,\dots,d$, set

$$\begin{aligned}\Lambda^j_{k,0}&:=1,\\ \Lambda^j_{k,n+1}(t,x)&:=\left|\int_{t_0}^t\mathrm{d}s_j\overset{j}{\ldots}\int_{t_0}^{s_2}\Lambda_k(s_1,x)\cdot\max_{0<l\le d}\Lambda^l_{k+L,n}(s_1,x)\,\mathrm{d}s_1\right|,\\ \bar\Lambda_{k,n}&:=\max_{\substack{x\in S\\0<j\le d}}\Lambda^j_{k,n}(t_0+\max(a,b),x).\end{aligned}\tag{5.17}$$

Finally, assume that the following conditions are fulfilled for all $k\in\mathbb{N}$:

(i) $P^n(i_0)\in\bar B_R(i_0)$ *for all* $n\in\mathbb{N}$;

(ii) $\sum_{n=0}^{+\infty}\bar\Lambda_{k,n}\cdot\|P(i_0)-i_0\|_{k+nL}<+\infty$.

Then, there exists a smooth solution $y \in \bar{B}_R(i_0) \cap \mathcal{C}^\infty(T \times S, \mathbb{R}^m)$ of the problem

$$\begin{cases} \partial_t^d y(t,x) = F\left[t, x, (\partial_x^\alpha \partial_t^\gamma y)_{\substack{|\alpha| \le L \\ \gamma \le p}}\right], \\ \partial_t^j y(t_0, x) = y_{0j}(x) \ j = 0, \dots, d-1, \end{cases} \tag{5.18}$$

given by $y = \lim_{n \to +\infty} P^n(i_0)$ in $\left(\mathcal{C}_t^p \mathcal{C}_x^\infty (T \times S, \mathbb{R}^m), (\|-\|_k)_{k \in \mathbb{N}}\right)$, which satisfies

$$\forall k, m \in \mathbb{N}: \ \|y - P^m(i_0)\|_k \le \sum_{n=m}^{+\infty} \bar{\Lambda}_{k,n} \cdot \|P(i_0) - i_0\|_{k+nL}.$$

In particular, if $M_k \in \mathcal{C}^0(T \times S)$, we set

$$\bar{M}_{kj}(t,x) := \left| \int_{t_0}^t \mathrm{d}s_j \overset{j}{\dots} \int_{t_0}^{s_2} M_k(s_1, x) \, \mathrm{d}s_1 \right|,$$

and we also assume

(iii) $|\partial_x^\nu G(t,x,u)| \le M_k(t,x)$ *for all $u \in \bar{B}_R(i_0)$, $(t,x) \in T \times S$ and all $\nu \in \mathbb{N}^s$ such that $|\nu| \le k$;*

(iv) $\max_{\substack{x \in S \\ 0 < j \le d}} \bar{M}_{k,j}(t_0 + \max(a,b), x) \le r_k$;

Then $P : \bar{B}_R(i_0) \longrightarrow \bar{B}_R(i_0)$ and hence (i) always holds.

Proof. We prove that P actually satisfies the stronger contraction property (3.2) with contraction constants $\bar{\Lambda}_{kn}$. We firstly show, by induction on $n \in \mathbb{N}$, that for each $k \in \mathbb{N}$, u, $v \in \bar{B}_R(i_0)$, $(t,x) \in T \times S$, $h = 1, \dots, m$, and $\beta \in \mathbb{N}_p^{1+s}$ with $|\beta| \le k$, we have

$$\left|\partial^\beta \left[P^n(u)^h - P^n(v)^h\right](t,x)\right| \le \|u - v\|_{k+nL} \cdot \max_{0<j\le d} \Lambda_{kn}^j(t,x). \tag{5.19}$$

For $n = 0$, (5.19) reduces to $\left|\partial^\beta(u^h - v^h)(t,x)\right| \le \|u - v\|_k \cdot \max_{0<j\le d} \Lambda_{k,0}^j(t,x)$ which holds because $|\beta| \le k$ and $\Lambda_{k,0}^j = 1$. To prove the inductive step, we consider

$$\begin{aligned} \left|\partial^\beta \left[P^{n+1}(u)^h - P^{n+1}(v)^h\right](t,x)\right| \le \Bigg| \partial^\beta \Bigg\{ & \int_{t_0}^t \mathrm{d}s_d \overset{d}{\dots} \int_{t_0}^{s_2} G^h(s_1, x, P^n(u)) \, \mathrm{d}s_1 \\ & - \int_{t_0}^t \mathrm{d}s_d \overset{d}{\dots} \int_{t_0}^{s_2} G^h(s_1, x, P^n(v)) \, \mathrm{d}s_1 \Bigg\} \Bigg| =: (1^*) \end{aligned} \tag{5.20}$$

Since $\beta \in \mathbb{N}_p^{1+s}$, we can write $\partial^\beta = \partial_x^\nu \partial_t^{\beta_1}$, where $\nu := (\beta_2, \dots, \beta_s)$ and $\beta_1 \le p < d$. The operator $\partial_t^{\beta_1}$ deletes β_1 integrals in (5.20); set $\bar{\jmath} := d - \beta_1 > 0$, and take ∂_x^ν inside the integrals to get

$$\begin{aligned} (1^*) &\le \mathrm{sgn}(t - t_0)^{\bar{\jmath}} \int_{t_0}^t \mathrm{d}s_{\bar{\jmath}} \overset{\bar{\jmath}}{\dots} \int_{t_0}^{s_2} \left|\partial_x^\nu G^h(s_1, x, P^n(u)) - \partial_x^\nu G^h(s_1, x, P^n(v))\right| \, \mathrm{d}s_1 \\ &=: (2^*). \end{aligned}$$

Since G is Lipschitz on $\bar{B}_R(i_0)$ with factors $(\Lambda_k)_{k \in \mathbb{N}}$, we get

$$\begin{aligned} (2^*) \le \mathrm{sgn}(t - t_0)^{\bar{\jmath}} & \int_{t_0}^t \mathrm{d}s_{\bar{\jmath}} \overset{\bar{\jmath}}{\dots} \int_{t_0}^{s_2} \Lambda_k(s_1, x) \cdot \\ & \cdot \max_{l=1,\dots,m} \max_{\substack{|\alpha| \le k+L \\ \gamma \le p}} \left|\partial_x^\alpha \partial_t^\gamma \left[P^n(u)^l - P^n(v)^l\right](s_1, x)\right| \, \mathrm{d}s_1 =: (3^*). \end{aligned}$$

Using inductive hypothesis (5.19) (with $k+L$ instead of k and (γ,α) instead of β)

$$\begin{aligned}(3^*) &\le \operatorname{sgn}(t-t_0)^{\bar{j}}\int_{t_0}^{t}\mathrm{d}s_{\bar{j}}\overset{\bar{j}}{\ldots}\int_{t_0}^{s_2}\Lambda_k(s_1,x)\cdot\|u-v\|_{k+L+nL}\cdot\\ &\quad\cdot\max_{0<l\le d}\Lambda^l_{k+L,n}(s_1,x)\,\mathrm{d}s_1\\ &=\|u-v\|_{k+(n+1)L}\cdot\left|\int_{t_0}^{t}\mathrm{d}s_{\bar{j}}\overset{\bar{j}}{\ldots}\int_{t_0}^{s_2}\Lambda_k(s_1,x)\cdot\right.\\ &\quad\left.\cdot\max_{0<l\le d}\Lambda^l_{k+L,n}(s_1,x)\,\mathrm{d}s_1\right|\\ &=\|u-v\|_{k+(n+1)L}\cdot\Lambda^{\bar{j}}_{k,n+1}(t,x)\\ &\le\|u-v\|_{k+(n+1)L}\cdot\max_{0<j\le d}\Lambda^j_{k,n+1}(t,x),\end{aligned}$$

which proves our claim.

Finally, we prove (3.2): for some $\beta\in\mathbb{N}^{1+s}_p$, $|\beta|\le k$, some $h=1,\ldots,m$ and some $(t,x)\in T\times S$, from (5.19) we have

$$\begin{aligned}\|P^n(u)-P^n(v)\|_k&=\left|\partial^\beta\left[P^n(u)^h-P^n(v)^h\right](t,x)\right|\le\\ &\le\|u-v\|_{k+nL}\cdot\max_{0<j\le d}\Lambda^j_{kn}(t,x)\le\\ &\le\|u-v\|_{k+nL}\cdot\bar{\Lambda}_{kn}.\end{aligned}$$

This shows the claim on P with contraction constants $\bar{\Lambda}_{kn}$. The conclusion with $y\in\bar{B}_R(i_0)$ hence follows from Weissinger condition (ii) and Thm. 6. It only remains to prove that actually y is smooth. Since y is a fixed point of P, we have

$$\begin{aligned}y(t,x)&=i_0(t,x)+\int_{t_0}^{t}\mathrm{d}s_d\overset{d}{\ldots}\int_{t_0}^{s_2}G(s_1,x,y)\,\mathrm{d}s_1 &(5.21)\\ &=i_0(t,x)+\int_{t_0}^{t}\mathrm{d}s_d\overset{d}{\ldots}\int_{t_0}^{s_2}F\left[s_1,x,(\partial_x^\alpha\partial_t^\gamma y)_{\substack{|\alpha|\le L\\ \gamma\le p}}(s_1,x)\right]\,\mathrm{d}s_1.\end{aligned}$$

But $y\in\mathcal{C}^p_t\mathcal{C}^\infty_x(T\times S,\mathbb{R}^m)$ and hence $(\partial_x^\alpha\partial_t^\gamma y)_{\substack{|\alpha|\le L\\ \gamma\le p}}\in\mathcal{C}^0(T\times S,\mathbb{R}^m)$. By induction (5.21) proves that y is smooth at interior points of $T\times S$ and hence also at boundary points by continuity of derivatives on $\mathring{T}\times\mathring{S}$.

In particular, if we assume both (iii) and (iv), we can prove that $P:\bar{B}_R(i_0)\longrightarrow\bar{B}_R(i_0)$ using Lem. 11. In fact, for any $u\in\bar{B}_R(i_0)$ and $k\in\mathbb{N}$, from (iii) and (iv) we have

$$\begin{aligned}\|P(u)-i_0\|_k&=\left\|\int_{t_0}^{(-)}\mathrm{d}s_d\overset{d}{\ldots}\int_{t_0}^{s_2}G\left(s_1,-,u\right)\,\mathrm{d}s_1\right\|_k\\ &\le\max_{\substack{x\in S\\ 0<j\le d}}\bar{M}_k(t_0+\max(a,b),x)\le r_k.\end{aligned}$$

□

Note that, on the contrary with respect to the more classical conditions (iii) and (iv) (inherited from the classical PLT for ODE) depending both on the choice of upper bounds M_k and radii r_k, assumption (i) depends only on the radii r_k. In Sec. 6, we will see that requirement (i) leads us to the correct choice of these radii r_k (one more time, depending on the initial conditions $r_k=r_k(i_0)$).

If the radii $r_k < +\infty$, we can also consider as bounds M_k of (iii) and (iv) the minimal constant functions. This is considered in the following result, which allows us to understand that to have a local solution using the latter part of the PLT, we have to avoid that $\frac{r_k}{M_k} \to 0$:

Corollary 15. *Let $\Lambda_k \in \mathcal{C}^0(T \times S)$, $r_k \in \mathbb{R}_{>0}$ for all $k \in \mathbb{N}$. Assume that $\mathring{S}$ is dense in S and G is Lipschitz on $\bar{B}_R(i_0)$ with loss of derivatives L and Lipschitz factors $(\Lambda_k)_{k\in\mathbb{N}}$, define $\bar{\Lambda}_{kn}$ as in (5.17) and*

$$C_k := T \times S \times \bigcup_{\substack{|\nu|\le k+L\\ \gamma\le p}} \overline{B_{r_{k+L+p}}\left(\partial_x^\nu \partial_t^\gamma i_0(T \times S)\right)}$$

$$M_k := \max_{(t,x,z)\in C_k} \max_{|\nu|\le k} \left|\partial_x^\nu G(t,x,z)\right|$$

Finally, assume that the following conditions are fulfilled:

(i) $\max(a,b) \le \inf_{k\in\mathbb{N}} \frac{r_k}{M_k}$;

(ii) $\sum_{n=0}^{+\infty} \bar{\Lambda}_{kn} \cdot \|P(i_0) - i_0\|_{k+nL} < +\infty$ *for all* $k \in \mathbb{N}$.

Then, there exists a smooth solution $y \in \bar{B}_R(i_0) \cap \mathcal{C}^\infty(T \times S, \mathbb{R}^m)$ of problem (5.18).

Proof. Note explicitly that $C_k \Subset \mathbb{R}^{1+s+m}$ because $r_{k+L+p} < +\infty$. As we proved in Thm. 13, for all $u \in \bar{B}_R(i_0)$, all $|\nu| \le k+L$ and all $\gamma \le p$, we have

$$|\partial_x^\nu \partial_t^\gamma u(t,x) - \partial_x^\nu \partial_t^\gamma i_0(t,x)| \le \|u - i_0\|_{|\nu|+\gamma} \le \|u - i_0\|_{k+L+p} \le r_{k+L+p},$$

and hence $\partial_x^\nu \partial_t^\gamma u(t,x) \in C_k$. Thereby, condition Thm. 14(iii) holds for the chosen constant M_k (see also Rem. 16 just below). Therefore, $\bar{M}_{kj}(t,x) = \frac{(t-t_0)^j}{j!} M_k$ and $\max_{\substack{x\in S\\ 0<j\le d}} \bar{M}_k(t_0 + \max(a,b), x) = \max(a,b) \cdot M_k \le r_k$ for all $k \in \mathbb{N}$ by (i). We can finally apply Thm. 14. □

Remark 16. To avoid misunderstandings, we explicitly note that the simplified notation $\partial_x^\nu G(t,x,z)$ denotes the function obtained by the following process:

(i) Consider and arbitrary $u \in \bar{B}_R(i_0)$, and the derivative $\partial_x^\nu \left(G(t,x,u)\right)(t,x)$ given by (5.14);

(ii) In the formula (5.14) obtained after the computation of this derivative, substitute the variables $z_{\alpha\gamma} := (\partial_x^\alpha \partial_t^\gamma u)_{\substack{|\alpha|\le L\\ \gamma\le p}}$ and $z_{\mu\gamma} := \left(\partial_x^\mu \partial_t^\gamma u(t,x)\right)_{\mu\gamma}$ to obtain $\partial_x^\nu G(t,x,z)$, where the variable z represents all the $z_{\alpha\gamma}$ and $z_{\mu\gamma}$.

For example, for the PDE $\partial_t^d y = a(t) \cdot \partial_x^L \partial_t^\gamma y$, we calculate the derivatives as $\partial_x^\nu G(t,x,u) = a(t) \cdot \partial_x^{\nu+L} \partial_t^\gamma u(t,x)$, and here substituting z for $\partial_x^{\nu+L} \partial_t^\gamma u(t,x)$ we get $\partial_x^\nu G(t,x,z) = a(t) \cdot z$. For the PDE $\partial_t^d y = \left(\partial_x^L \partial_t^\gamma y\right)^2$, we have e.g. $\partial_x G(t,x,u) = 2\partial_x^L \partial_t^\gamma u(t,x) \partial_x^{L+1} \partial_t^\gamma u(t,x)$, and $\partial_x G(t,x,z_0,z_1) = 2z_0 z_1$.

On the contrary with respect to the Cauchy-Kowalewsky theorem, in the PL Thm. 14, it would appear that we do not need to assume $d \ge L$. However, this clearly cannot hold in general, and in Sec. 6 we show that such type of assumption is implicitly contained in the convergence request of Weissinger condition Thm. 14(ii). A first partial confirmation going in this direction, can be glimpsed by computing the iteration $P^n(i_0)(t,x)$ in case of simple linear PDE, and then taking $n \to +\infty$:

Example 17. From the Picard-Lindelöf iterations, we can also obtain the following formulas (where $a \in \mathbb{R}_{\neq 0}$; see Thm. 2 for an independent statement including all the following particular cases), we have:

(i) If $\partial_t y = a \cdot \partial_x^2 y$, then $y(t,x) = \sum_{n=0}^{+\infty} \frac{\partial_x^{2n} y_{00}(x)}{n!} a^{n+1} \cdot (t-t_0)^n$;
(ii) If $\partial_t^2 y = a \cdot \partial_x^2 y$, then $y(t,x) = \sum_{n=0}^{+\infty} \frac{\partial_x^{2n} y_{00}(x)}{(2n)!} a^{n+1}(t-t_0)^{2n} + \sum_{n=0}^{+\infty} \frac{\partial_x^{2n} y_{01}(x)}{(2n+1)!} a^{n+1}(t-t_0)^{2n+1}$;
(iii) If $\partial_t y = a \cdot \partial_x y$, then $y(t,x) = \sum_{n=0}^{+\infty} \frac{\partial_x^{n} y_{00}(x)}{n!} a^{n+1} \cdot (t-t_0)^n$;
(iv) If $\partial_t^2 y = a \cdot \partial_t \partial_x y$, then $y(t,x) = y_{00}(x) + \sum_{n=0}^{+\infty} \frac{\partial_x^{n} y_{01}(x)}{n!} a^{n+1}(t-t_0)^n$;
(v) If $\partial_t^2 y = a \cdot \partial_x y$, then $y(t,x) = \sum_{n=0}^{+\infty} \frac{\partial_x^{n} y_{00}(x)}{(2n)!} a^{n+1}(t-t_0)^{2n} + \sum_{n=0}^{+\infty} \frac{\partial_x^{n} y_{01}(x)}{(2n+1)!} a^{n+1}(t-t_0)^{2n+1}$.

The ideas of the proof of Thm. 14 are a simple generalization of the classical proof for ODE, only adapted to contractions with LOD and a countable family of norms. Indeed, for $L=0$ and $\|-\|_k = \|-\|_0$ the proof reduces to the classical proof for ODE and assumptions (iii), (iv), (ii) reduce to the usual ones for the PLT for ODE with Weissinger condition, see e.g. [35]. On the other hand, the compact set $S \Subset \mathbb{R}^s$ (with $\mathring{S}$ dense in S) is *completely arbitrary*: we can hence say that our deduction proves that, *with respect to the PLT*, PDE can be simply treated as ODE depending on a parameter $x \in S$.

If the Lipschitz factors $\Lambda_k \in \mathbb{R}$ and the upper bounds $M_k \in \mathbb{R}$ are constant (the proof of Thm. 13 and Cor. 15 show that this is not a loss of generality), then

$$\bar{M}_{kj}(t,x) = M_k \frac{|t-t_0|^j}{j!},$$
$$\Lambda_{kn}^j(t,x) = \frac{|t-t_0|^{nd+j}}{(nd+j)!} \prod_{j=0}^{n-1} \Lambda_{k+jL},$$
$$\bar{\Lambda}_{kn} = \frac{\max(a,b)^{nd}}{(nd)!} \prod_{j=0}^{n-1} \Lambda_{k+jL}.$$

Thereby, Weissinger condition Thm. 14(ii) becomes

$$\sum_{n=0}^{+\infty} \frac{\max(a,b)^{nd}}{(nd)!} \|P(i_0) - i_0\|_{k+nL} \prod_{j=0}^{n-1} \Lambda_{k+jL} < +\infty \qquad \forall k \in \mathbb{N}. \tag{5.22}$$

For ODE, we have $L=0$ and $\|-\|_k = \|-\|_0$, and (5.22) reduces to

$$\|P(i_0) - i_0\|_0 \sum_{n=0}^{+\infty} \Lambda_0^n \frac{\max(a,b)^{nd}}{(nd)!} < +\infty,$$

which always holds.

Remark 18.

(i) We believe it is worth mentioning that in a non-Archimedean setting such as that of generalized smooth functions theory and Robinson-Colombeau ring ${}^\rho\widetilde{\mathbb{R}}$, see e.g. [12, 13, 22, 14], we can repeat the proof of the PLT with exactly the same formal steps (but with the ring ${}^\rho\widetilde{\mathbb{R}}$ instead of the field $\mathbb{R}$). In addition, we can take $S = [-\iota, \iota]^s \supseteq \mathbb{R}^s$, where ι is an infinite number (this kind of sets behave as compact sets for generalized smooth functions, see [11]). In this way, we get a global solution in $x \in S$ (but clearly, we need initial conditions on S, or equivalently boundary conditions that hold for all $x \in \mathbb{R}$). Moreover, in the setting of the non-Archimedean ring ${}^{\rho}\widetilde{\mathbb{R}}$, the generalized number (equivalence class in the quotient ring ${}^{\rho}\widetilde{\mathbb{R}}$) $\mathrm{d}\rho := [\rho_\varepsilon] \in {}^{\rho}\widetilde{\mathbb{R}}$ is an infinitesimal number since $\rho_\varepsilon \to 0^+$ as $\varepsilon \to 0^+$ (and hence $\mathrm{d}\rho^{-Q} = \left[\rho_\varepsilon^{-Q}\right]$ is an infinite number for all $Q \in \mathbb{N}_{>0}$). Since in every Cauchy complete non-Archimedean ring a series converges if and only if the general term tends to zero (see e.g. [17]), condition (5.22) is equivalent to $\lim_{n\to+\infty} \frac{\max(a,b)^{nd}}{(nd)!} \|P(i_0) - i_0\|_{k+nL} \prod_{j=0}^{n-1} \Lambda_{k+jL} = 0$. Assuming that for some $Q \in \mathbb{N}_{\ge 0}$ and some $q \in \mathbb{N}_{>0}$, we have

$$\|P(i_0) - i_0\|_{k+nL} \prod_{j=0}^{n-1} \Lambda_{k+jL} \le \mathrm{d}\rho^{-Q}, \tag{5.23}$$

$$\max(a, b) \le \mathrm{d}\rho^{q},$$

then $\frac{\max(a,b)^{nd}}{(nd)!} \|P(i_0) - i_0\|_{k+nL} \prod_{j=0}^{n-1} \Lambda_{k+jL} \le \mathrm{d}\rho^{-Q} \cdot \frac{\mathrm{d}\rho^{qnd}}{(nd)!} \to 0$ as $n \to +\infty$. Thereby, since for ordinary smooth functions the left hand side of (5.23) is finite, we have that *any ordinary smooth normal Cauchy problem (even Lewy-Mizohata examples) always has a solution in an infinitesimal interval* (set $r_k = 1$ in Cor. 15 and note that we can reformulate Cor. 15(i) as $\max(a, b)M_k \le r_k = 1$ for all $k \in \mathbb{N}$, which always holds if $\max(a, b)$ is infinitesimal; see [14] for greater details).

(ii) Lewy-Mizohata examples imply that Weissinger condition in these cases does not hold. Moreover, since the non-existence of a solution does not depend on the initial condition i_0, [21, 24], and taking $a = -1$, $b = 1$, we necessarily must have

$$\exists k \in \mathbb{N} : \sum_{n=0}^{+\infty} \frac{1}{n!} \prod_{j=0}^{n-1} \Lambda_{k+j} = +\infty \tag{5.24}$$

for all Lipschitz factors $(\Lambda_j)_{j\in\mathbb{N}}$ (that always exist because of Thm. 13) (note that $d = 1 = L$, $m = 2$ for both counter-examples). Condition (5.24) strongly recall the non-analytic nature of F in these cases.

6. Examples

The main aim of this section is to show at least one example of normal PDE (1.1) where either the right hand side F or one of the initial conditions y_{0j} are not functions of Gevrey class.

The class of examples we are going to consider is

$$\partial_t^d y(t, x) = p(t) \cdot \partial_x^\mu \partial_t^\gamma y(t, x) + q(t, x), \tag{6.1}$$

where $y(t, x) \in \mathbb{R}^m$, $\mu \in \mathbb{N}^s$, $|\mu| = L > 0$, $0 \le \gamma < d$, $p \in \mathcal{C}^\infty(T, \mathbb{R}^{m\times m})$ is an *arbitrary smooth function*, and q is a smooth function with uniformly bounded derivatives in x:

$$\exists Q \in \mathbb{R}_{>0}\, \forall \nu \in \mathbb{N}^s\, \forall (t, x) \in T \times S : \ |\partial_x^\nu q(t, x)| \le Q. \tag{6.2}$$

Note that if p is not of Gevrey class, and $\partial_x^\mu \partial_t^\gamma y$ is not zero, then also y cannot be of Gevrey class. Clearly, wave, heat and Laplace equations are particular cases of (6.1). Explicitly note that also Mizohata's counterexample [24] $\partial_t y^1 = t\partial_x y^2 + q^1(t, x)$, $\partial_t y^2 = -t\partial_x y^1 + q^2(t, x)$ is exactly of the form (6.1), but $q = (q^1, q^2)$ is not analytic, so it does not satisfy condition (6.2). On the other hand, in Lewy's counterexample the coefficient $p = p(t, x_1, x_2)$ also depends on x and the term q is again not analytic.

We want to directly apply the first part of the PL Thm. 14, i.e. to check properties (i) and (ii). We start focusing on the latter, and arriving at estimates that are independent from the radii r_k.

Estimate of Lipschitz constants Λ_k. For arbitrary radii $R = (r_k)_{k\in\mathbb{N}}$ and all u, $v \in \bar{B}_R(i_0)$ and all $\nu \in \mathbb{N}^s$, with $|\nu| \le k$, we have:

$$|\partial_x^\nu G(t,x,u) - \partial_x^\nu G(t,x,v)| \le \|p\|_0 \cdot \max_{|\alpha|\le k+L} |\partial_x^\alpha u(t,x) - \partial_x^\alpha v(t,x)| .$$

We can hence consider $\Lambda_k := \|p\|_0$, so that setting $\bar{T} := \max(a, b)$, we get

$$\bar{\Lambda}_{kn} = \frac{\bar{T}^{nd}}{(nd)!} \prod_{j=0}^{n-1} \Lambda_{k+jL} = \frac{\bar{T}^{nd}}{(nd)!} \|p\|_0^n. \tag{6.3}$$

Estimate of the terms $\|P(i_0) - i_0\|_{k+nL}$. We have

$$P(i_0)(t,x) - i_0(t,x) = \int_{t_0}^{t} \mathrm{d}s_d \overset{d}{\ldots} \int_{t_0}^{s_2} \left[p(s_1)\cdot \partial_x^\mu \partial_t^\gamma i_0(s_1,x) + q(s_1,x)\right] \mathrm{d}s_1$$

$$\partial_x^\mu \partial_t^\gamma i_0(s_1,x) = \sum_{j=\gamma}^{d-1} \frac{\partial_x^\mu y_{0j}(x)}{j!} j(j-1)\cdot\ldots\cdot(j-\gamma+1)\cdot(t-t_0)^{j-\gamma}$$
$$= \sum_{j=\gamma}^{d-1} \frac{\partial_x^\mu y_{0j}(x)}{(j-\gamma)!}(t-t_0)^{j-\gamma}.$$

For $\beta \in \mathbb{N}^{1s}$, $|\beta| \le k + nL$, $\beta_x := (\beta_2, \ldots, \beta_s)$, we thus have

$$\partial^\beta \left[P(i_0)(t,x) - i_0(t,x)\right] = \sum_{j=\gamma}^{d-1} \frac{\partial_x^{\mu+\beta_x} y_{0j}(x)}{(j-\gamma)!} \int_{t_0}^{t} \mathrm{d}s_d \overset{d-\beta_1}{\ldots} \int_{t_0}^{s_2} p(s_1)\cdot(s_1-t_0)^{j-\gamma}\, \mathrm{d}s_1 +$$
$$\int_{t_0}^{t} \mathrm{d}s_d \overset{d-\beta_1}{\ldots} \int_{t_0}^{s_2} \partial_x^{\beta_x} q(s_1,x)\, \mathrm{d}s_1.$$

Using assumption (6.2):

$$\left|\partial^\beta \left[P(i_0)(t,x) - i_0(t,x)\right]\right| \le \|p\|_0 \sum_{j=\gamma}^{d-1} \frac{\left|\partial_x^{\mu+\beta_x} y_{0j}(x)\right|}{(j-\gamma)!} \frac{\bar{T}^{j-\gamma+d-\beta_1}}{(j-\gamma+d-\beta_1)!}(j-\gamma)! +$$
$$+ Q\frac{\bar{T}^{d-\beta_1}}{(d-\beta_1)!}.$$

Therefore, taking for simplicity $\bar{T} \le 1$:

$$\|P(i_0) - i_0\|_{k+nL} \le \|p\|_0 \sum_{j=\gamma}^{d-1} \frac{\|y_{0j}\|_{k+(n+1)L}}{(j-\gamma+d)!} + Q. \tag{6.4}$$

Weissinger condition. Based on (6.3) and (6.4), we can easily estimate Weissinger condition as

$$\sum_{n=0}^{+\infty} \bar{\Lambda}_{kn} \cdot \|P(i_0) - i_0\|_{k+nL} \leq$$
$$\sum_{n=0}^{+\infty} \frac{\bar{T}^{nd}}{(nd)!} \|p\|_0^{n+1} \sum_{j=\gamma}^{d-1} \frac{\|y_{0j}\|_{k+(n+1)L}}{(j-\gamma+d)!} + \sum_{n=0}^{+\infty} \frac{\bar{T}^{nd}}{(nd)!} \|p\|_0^n Q$$
$$=: S_1 + S_2.$$

Since the latter series S_2 is convergent, we focus on the first one:

$$S_1 = \|p\|_0 \sum_{n=0}^{+\infty} \left(\bar{T}^d \|p\|_0\right)^n \sum_{j=\gamma}^{d-1} \frac{\|y_{0j}\|_{k+(n+1)L}}{(nd)!} \cdot \frac{1}{(j-\gamma+d)!}. \tag{6.5}$$

In this series, the only potentially problematic terms are the fractions $\frac{\|y_{0j}\|_{k+(n+1)L}}{(nd)!}$ as $n \to +\infty$ for $j = \gamma, \ldots, d-1$, because the remaining part surely yields a convergent series for $\bar{T}$ sufficiently small. This also yields that all the possible initial conditions y_{0j} for $j = 0, \ldots, \gamma - 1$ can be freely chosen (see also Example 17(iv)).

Estimate of radii r_k. As we highlighted several times above, a key problem in using this type of results is the choice of the radii r_k. If $f = f(t)$ is a function of t, for simplicity we first set

$$I_d\left[f(t)\right] := \int_{t_0}^{t} \mathrm{d}s_d \overset{d}{\ldots} \int_{t_0}^{s_2} f(s_1)\, \mathrm{d}s_1.$$

For $j \geq \gamma$ and $h \in \mathbb{N}$, we set

$$\begin{aligned}
\mu_{j-\gamma,0}(t) &:= p(t) \cdot (t-t_0)^{j-\gamma} \\
\mu_{j-\gamma,h+1}(t) &:= I_d\left[p(t) \cdot \partial_t^\gamma \mu_{j-\gamma,h}(t)\right] \\
\eta_0(t,x) &:= q(t,x) \\
\eta_{h+1}(t,x) &:= I_d\left[p(t) \cdot \partial_x^\mu \partial_t^\gamma \eta_h(t,x)\right].
\end{aligned}$$

By induction on $n \in \mathbb{N}$, we can then prove that

$$P^n(i_0)(t,x) = \sum_{j=0}^{d-1} \frac{y_{0j}(x)}{j!}(t-t_0)^j + \sum_{h=0}^{n} \sum_{j=\gamma}^{d-1} \frac{\partial_x^{h\mu} y_{0j}(x)}{(j-\gamma)!} \mu_{j-\gamma,h}(t) + \sum_{h=0}^{n} \eta_h(t,x). \tag{6.6}$$

Thereby, we get in this way a possible definition of the radii r_k as

$$\|P^n(i_0) - i_0\|_k \leq \sum_{h=0}^{+\infty} \sum_{j=\gamma}^{d-1} \frac{\|y_{0j}\|_{k+hL}}{(j-\gamma)!} \|\mu_{j-\gamma,h}\|_k + \sum_{h=0}^{+\infty} \|\eta_h\|_k =: r_k \quad \forall k \in \mathbb{N}. \tag{6.7}$$

Using assumption (6.2), the latter series converges because

$$\|\eta_h\|_k \leq \|p\|_0^h \cdot Q \cdot \frac{\bar{T}^{(h+1)(d-\gamma)}}{[(h+1)(d-\gamma)]!}.$$

In the former series, for $h \geq 1$ we have instead

$$\|\mu_{j-\gamma,h}\|_k \leq \|p\|_0^h \cdot \frac{\bar{T}^{h(d-\gamma)+j}}{[(d-\gamma)!]^h},$$

so that

$$\sum_{h=1}^{+\infty}\sum_{j=\gamma}^{d-1}\frac{\|y_{0j}\|_{k+hL}}{(j-\gamma)!}\|\mu_{j-\gamma,h}\|_k \le \sum_{h=0}^{+\infty}\left(\frac{\|p\|_0\bar{T}^{d-\gamma}}{(d-\gamma)!}\right)^h\cdot\sum_{j=\gamma}^{d-1}\frac{\|y_{0j}\|_{k+hL}\bar{T}^j}{(j-\gamma)!}.$$

If this series converges (and this mainly depends on the growing of $\|y_{0j}\|_{k+hL}$), we can hence have $r_k < +\infty$, otherwise we simply take $r_k = +\infty$. For a non trivial term $\|y_{0j}\|_{k+hL}$, (6.7) also shows that $r_k \not\to 0^+$.

A case of exponentially growing initial conditions. If all the functions y_{0j}, $j = \gamma,\dots,d-1$, satisfy for some some $C_j \in \mathbb{R}_{>0}$

$$\|y_{0j}\|_{k+(n+1)L} \le C_j^{k+(n+1)L} \quad \forall j=\gamma,\dots,d-1\,\forall k\in\mathbb{N}, \tag{6.8}$$

then the series (6.5) converges and we have

Theorem 19. *If the initial conditions y_{0j} satisfy (6.8), whereas y_{0j} for $j = 0,\dots,\gamma-1$ are* arbitrary smooth functions, *then there exists a smooth solution of (6.1) in $\bar{B}_R(i_0)$ for $\bar{T}$ sufficiently small and all $x \in S$. In this case, we do not have constraints on d, L.*

The case of analytic initial conditions. If all the y_{0j}, $j = \gamma,\dots,d-1$, are analytic functions, then for some $C_j \in \mathbb{R}_{>0}$, we have

$$\|y_{0j}\|_{k+(n+1)L} \le C_j^{k+(n+1)L}\cdot(k+(n+1)L)! \quad \forall j=\gamma,\dots,d-1\,\forall k\in\mathbb{N}.$$

Using Stirling's approximation, we have $(k+(n+1)L)! \sim (nL)^{k+L}(nL)!$ and hence the following

Theorem 20. *If $d \ge L$, then there exists a smooth solution of (6.1) in $\bar{B}_R(i_0)$ with analytic initial conditions y_{0j} if $j=\gamma,\dots,d-1$ and* arbitrary smooth *y_{0j} if $j = 0,\dots,\gamma-1$, for $\bar{T}$ sufficiently small and all $x \in S$.*

Note explicitly that already this theorem yields more general results with respect to the classical Cauchy-Kowalewsky theorem because both the matrix coefficient $p(t)$ in (6.1) and the initial conditions y_{0j} for $j = 0,\dots,\gamma-1$ can be arbitrary smooth functions.

A case of Gevrey initial conditions and non-Gevrey solution. Now, let us assume that our initial conditions which are not analytic satisfy

$$\|y_{0j}\|_{k+(n+1)L} \sim (nL)^{\sigma_j nL}, \sigma_j > 0 \quad \forall j=\gamma,\dots,d-1\,\forall k\in\mathbb{N}. \tag{6.9}$$

Note that each function y_{0j} satisfying (6.9) cannot be an analytic function, but it is Gevrey of class $s > \sigma_j$. We have

$$\frac{\|y_{0j}\|_{k+(n+1)L}}{(nd)!} \sim \frac{e^{nd}}{\sqrt{2\pi nd}}\left(\frac{L}{d}\right)^{\sigma_j nL}\frac{1}{(nd)^{n(d-\sigma_j L)}}.$$

We therefore have the following

Theorem 21. *If the initial conditions y_{0j}, $j = 0,\dots,\gamma-1$, are* arbitrary smooth *functions, whereas y_{0j} for $j=\gamma,\dots,d-1$ are analytic or they satisfy (6.9), and if in the latter case we have $d > \sigma_j L$, then there exists a smooth solution of (6.1) in $\bar{B}_R(i_0)$ for $\bar{T}$ sufficiently small and all $x \in S$. We recall that, if the function p is non-Gevrey, then any solution such that $\partial_x^\mu \partial_t^\gamma y \ne 0$ cannot be of Gevrey class.*

Taking $n \to +\infty$ in (6.6), we also obtain the following generalization of Example 17:

Corollary 22. *In the assumptions of each one of Thm. 19, 20, 21, the solution y of Picard-Lindelöf iterations is given by the formula:*

$$y(t,x) = \sum_{j=0}^{\gamma-1} \frac{y_{0j}(x)}{j!}(t-t_0)^j + \sum_{h=0}^{+\infty}\sum_{j=\gamma}^{d-1} \frac{\partial_x^{h\mu} y_{0j}(x)}{(j-\gamma)!}\mu_{j-\gamma,h}(t) + \sum_{h=0}^{+\infty} \eta_h(t,x). \tag{6.10}$$

In particular, if the functions p and q are constant, then

$$y(t,x) = \sum_{j=0}^{\gamma-1} \frac{y_{0j}(x)}{j!}(t-t_0)^j + \sum_{h=0}^{+\infty}\sum_{j=\gamma}^{d-1} \frac{\partial_x^{h\mu} y_{0j}(x)}{(j-\gamma)!} p^{h+1} \frac{(t-t_0)^{h(d-\gamma)+j-\gamma}}{[h(d-\gamma)+j-\gamma]!}(j-\gamma)! + q.$$

Let $y(t,x;\varepsilon)$ be the solution defined by (6.10) corresponding to initial conditions $y_{0j}(x;\varepsilon)$, where $\varepsilon \in (-1,1)$. If we can exchange $\lim_{\varepsilon\to 0}$ and $\sum_{h=0}^{+\infty}$, e.g. if the sequence of derivatives $\left(\partial_x^{h\mu} y_{0j}(x;\varepsilon)\right)_{h\in\mathbb{N}}$ pointwise converges in a dominated way as $h \to +\infty$, i.e. for all $h \in \mathbb{N}$, $x \in S$, $j = 0,\dots,d-1$, and $\varepsilon \in (-1,1)$ we have

$$\begin{gathered}\exists \lim_{\varepsilon\to 0} \partial_x^{h\mu} y_{0j}(x;\varepsilon) = \partial_x^{h\mu} y_{0j}(x;0) \\ |\partial_x^{h\mu} y_{0j}(x;\varepsilon)| \le g_h(x;\varepsilon) \\ \sum_{h=0}^{+\infty} g_h(x;\varepsilon) < +\infty,\end{gathered}$$

then $\lim_{\varepsilon\to 0} y(t,x;\varepsilon) = y(t,x;0)$.

Note that our estimates above of the Weissinger condition and the radii r_k, allow us also to state that conditions (4.4) and (4.2) of Cor. 7 hold. Moreover, the proof of PL Thm. 14 shows that also (4.3) holds. Therefore, to solve (6.1) in the space $X = \bar{B}_R(i_0)$ we can also apply Cor. 7, as we stated above in Sec. 4.

We close this section by noting that for the PDE

$$\partial_t^d y(t,x) = y(t,x)\cdot \partial_x^\mu y(t,x) \tag{6.11}$$

with $|\mu| = L$, we can use ideas similar to those of Thm. 13 to show that setting

$$\begin{aligned}\bar{C}_{k+L} &:= \bigcup_{|\alpha|\le k+L} \partial_x^\alpha i_0(T\times S) \Subset \mathbb{R}^m \\ \|i_0(T\times S)\|_{k+L} &:= d(\bar{C}_{k+L},0) \\ \Lambda_k &:= 2^k\left(r_{k+L} + \|i_0(T\times S)\|_{k+L}\right),\end{aligned}$$

then (6.11) has $(\Lambda_k)_{k\in\mathbb{N}}$ as Lipschitz constants with L loss of derivatives. Thereby, $\bar{\Lambda}_{kn} = \frac{T^{nd}}{(nd)!} 2^n \prod_{j=0}^{n-1}\left(r_{k+jL} + \|i_0(T\times S)\|_{k+jL}\right)$. However, in the case $L = 1$ and y_{0j} satisfying (6.9) with $\sigma_j = 1$, we get $\bar{\Lambda}_{kn} = \frac{T^{nd}}{(nd)!}2^n H(n-1)$, where $H(n-1) = \prod_{j=0}^{n-1} j^j$ is the hyperfactorial function. Since $H(n) = O(n^{n^2/2})$, Weissinger condition never holds. This clearly left open the possibility of better estimates of different Lipschitz factors.

7. Some remarks about the loss of derivatives condition

As we discussed in the previous sections, Def. 4 of contraction with LOD is at the core of our version of the BFPT, i.e. Thm. 6. We start this section with a discussion of this notion of contraction.

Definition 23. We call *minimal loss for P from y_0* the quantity

$$L_P(y_0) := \min\{L \in \mathbb{N} \mid P \in \mathrm{C}(X, L, y_0)\}.$$

Lemma 24. *Let $(\mathcal{F}, (\|-\|_k)_{k\in\mathbb{N}})$ be a Fréchet space, X be a closed subset of $\mathcal{F}$, $y_0 \in X$ and $P : X \longrightarrow X$ be a continuous map. Assume that $\|-\|_0$ (hence, $\|-\|_k$ for every $k \in \mathbb{N}$) is a norm. If there exists $N \in \mathbb{N}_{>0}$ such that $P^N(y_0)$ is a fixed point of P, then $P \in \mathrm{C}(X, 0, y_0)$, i.e.* P is a contraction with 0 loss of derivatives starting from y_0.

Proof. Let N be the smallest number such that $P^N(y_0)$ is a fixed point of P.

If $N = 1$, $P(y_0) = y_0$ hence $\left\|P^{n+1}(y_0) - P^n(y_0)\right\|_k = 0$ for every k, $n \in \mathbb{N}$, so our claim follows just by setting each $\alpha_{kn} := 0$.

If $N > 1$, $\|P(y_0) - y_0\|_0 \neq 0$ since $\|-\|_0$ is a norm, therefore $\|P(y_0) - y_0\|_k \neq 0$ for every $k \in \mathbb{N}$, as the norms $\|-\|_k$ are increasing. For every k, $n \in \mathbb{N}$ set

$$\alpha_{kn} := \begin{cases} \frac{\left\|P^{n+1}(y_0) - P^n(y_0)\right\|_k}{\|P(y_0)-y_0\|_{k+nL}}, & \text{if } n < N; \\ \frac{1}{n^2\|P(y_0)-y_0\|_{k+nL}}, & \text{otherwise.} \end{cases} \tag{7.1}$$

With this choice, Def. 4((iii)) and Def. 4((iv)) are easily verified because $\alpha_{kn}\|P(y_0) - y_0\|_{k+nL} = \frac{1}{n^2} \geq \|P^{n+1}(y_0) - P^n(y_0)\|_k = \|y_0 - y_0\|_k$ if $n \geq N$, hence $P \in \mathrm{C}(X, 0, y_0)$. □

The following is a rather surprising fact that holds, e.g., in $\mathcal{C}^0_t\mathcal{C}^\infty_x(T \times S, \mathbb{R}^m)$.

Theorem 25. *Let $(\mathcal{F}, (\|-\|_k)_{k\in\mathbb{N}})$ be a Fréchet space, let X be a closed subset of $\mathcal{F}$, let $y_0 \in X$ and let $P : X \longrightarrow X$. Assume that $\|-\|_0$ (hence, $\|-\|_k$ for every $k \in \mathbb{N}$) is a norm. The following properties are equivalent:*

(i) *There exists $L \in \mathbb{N}$ such that $P \in \mathrm{C}(X, L, y_0)$;*

(ii) *P is continuous and for all $k \in \mathbb{N}$*

$$\sum_{n=0}^{\infty} \left\|P^{n+1}(y_0) - P^n(y_0)\right\|_k < +\infty; \tag{W'}$$

(iii) *$P \in \mathrm{C}(X, 0, y_0)$.*

Proof. (i)⇒(ii): If there exist k, $N \in \mathbb{N}$ such that $\left\|P^{N+1}(y_0) - P^N(y_0)\right\|_k = 0$, then $P^N(y_0)$ is a fixed point of P as $\|-\|_k$ is a norm. But then, for every $m \geq N$ and for every $k \in \mathbb{N}$ we have $\left\|P^{m+1}(y_0) - P^m(y_0)\right\|_k = 0$, and hence $\sum_{n=0}^{\infty} \left\|P^{n+1}(y_0) - P^n(y_0)\right\|_k = \sum_{n=0}^{N-1} \left\|P^{n+1}(y_0) - P^n(y_0)\right\|_k < +\infty$. Otherwise, $||P(y_0) - y_0||_{k+nL} \neq 0$ for every k, $N \in \mathbb{N}$, hence by Def. 4(iii) we get that $\alpha_{kn} \geq \frac{||P^{n+1}(y_0)-P^n(y_0)||_k}{||P(y_0)-y_0||_{k+nL}} \in \mathbb{R}_{>0}$ which, substituted in Def. 4(iv), gives (W').

(ii)⇒(iii): If there exist k, $N \in \mathbb{N}$ such that $\left\|P^{N+1}(y_0) - P^N(y_0)\right\|_k = 0$, then $P^N(y_0)$ is a fixed point of P as $\|-\|_k$ is a norm, and we conclude by Lemma 24. Otherwise, in particular $\|P(y_0) - y_0\|_k \neq 0$ for every $k \in \mathbb{N}$. For every k, $N \in \mathbb{N}$, we set $\alpha_{kn} := \frac{\left\Vert P^{n+1}(y_0)-P^n(y_0)\right\Vert_k}{\Vert P(y_0)-y_0\Vert_{k+nL}}$. Then Def. 4(iii) holds trivially, and Def. 4(iv) holds as, by construction

$$\sum_{n=0}^{\infty}\alpha_{kn}\left\Vert P\left(y_0\right)-y_0\right\Vert_k=\sum_{n=0}^{\infty}\left\Vert P^{n+1}\left(y_0\right)-P^n\left(y_0\right)\right\Vert_k<+\infty$$

by assumption.

(iii)⇒(i): This is trivial. □

In particular, this result shows that, if $\Vert-\Vert_k$ are norms, $L_P\left(y_0\right)=0$ whenever $P\in\mathrm{C}\left(X,L,y_0\right)$ for some L. Note that, in general, this does not entail the uniqueness of the fixed point of P, since such uniqueness would require a much stronger condition on P than Def. 4(iii) or condition (W'), see e.g. Lem. 5.

We also note that condition (W') implies that $\left(P^n\left(y_0\right)\right)_{n\in\mathbb{N}}$ is a Cauchy sequence as we did in (3.4), and this, together with the continuity of P, yields that $\overline{y}:=\lim_{n\longrightarrow+\infty}P^n\left(y_0\right)$ is a fixed point of P by Lem. 3.

On the other hand, the previous Thm. 25 *does not* imply that we can take $L=0$ in the PLT Thm. 14, because the assumption that the right hand side G of the PDE is Lipschitz on $\bar{B}_R(y_0)$ with loss of derivatives $L=0$ in general is *not* satisfied. In other words: The natural loss of derivatives $L>0$ corresponds to the maximum order of derivatives in x appearing in the PDE (1.1), and the natural Lipschitz constants α_{kn} are derived in the proof of Thm. 14, e.g. using the Lipschitz factors $\left(\Lambda_k\right)_{k\in\mathbb{N}}$ for the right hand side of the PDE derived from Thm. 13. Using these natural constants, Weissinger condition (ii) is easier to estimate than condition (W') or the use of (7.1).

8. Conclusions

Starting from the classical Kowalewsky counter-example for the heat equation or Hadamard's results on the Cauchy problem for the Laplace equation, one can think that a PDE links in a given relation $\partial_t y$ and $\partial_x y$ and hence it necessarily forces the solution, in general, in a space of functions whose derivatives growth in a restricted way, these constraints being related to the PDE itself. This implies that the initial conditions cannot be freely chosen but must be taken into another constrained space. We could say that we do not have to find a suitable space of generalized solutions for our PDE, but conditions stating when it has a solution or not; only at the philosophical level, this is similar to the point of view of nonlinear differential Galois theory, see e.g. [23], or the formal theory of differential equations, see e.g. [32].

The PLT we proved in this paper goes exactly in this direction, by showing that the existence of a solution (by Picard-Lindelöf iterations) depends on the initial conditions we start with: Def. 4 of contraction with loss of derivatives, the closure with respect to iterations (ii), Weissinger condition (W), the definition of the radii (6.6), all go in this direction. Examples considered in Sec. 6 show a first link between the syntax of the PDE (in the term $(nd)!$) and the order of growth of the derivatives of the initial conditions $\Vert y_{0j}\Vert_{k+nL}$. On the other hand, exactly as up to fourth order algebraic equations are solvable in radicals, if the order $d=1$ the method of characteristics allows one to solve a large class of PDE for *any* initial condition.

It is now natural to ask for a generalization to more singular normal (nonlinear) PDE, e.g. where the right hand side F or some of the initial conditions y_{0j} are some kind of generalized functions. In order to get this generalization by following the ideas of the present work, we would need a space of generalized functions which is closed with respect to composition and with a complete topology generated by norms; this space must clearly be non-trivial, e.g. containing all Sobolev-Schwartz distributions. In our opinion, this target can be fully accomplished in a beautiful and simple setting by considering the Grothendieck topos of non-Archimedean generalized smooth function, see e.g. [13, 22, 14]. We plan to realize this goal in future works.

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