Extremal Properties of Central Half-Spaces
for Product Measures

F. Barthe

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Abstract

We deal with the isoperimetric and the shift problem for subsets of measure one half in product probability spaces. We prove that the canonical central half-spaces are extremal in particular cases: products of log-concave measures on the real line satisfying precise conditions and products of uniform measures on spheres, or balls. As a corollary, we improve the known log-Sobolev constants for Euclidean balls. We also give some new results about the related question of estimating the volume of sections of unit balls of ℓ_p-sums of Minkowski spaces.

1 Introduction

Among subsets of measure $1/2$ in the unit cube $[0,1]^n$, the half cube $[0,1/2] \times [0,1]^{n-1}$ has minimal boundary measure [18]. A new proof of this fact appears in [7]. It is based on the comparison of the isoperimetric function of the set $[0,1]$ with the one of the Gaussian space. Our aim here is to extend this method to other settings: products of uniform measures on spheres, on balls, products of log-concave measures on the real line. We will also develop a similar approach to get sharp solutions to shift problems; we will put emphasis on the formal similarities between the two questions.

Our results give a new look to the following result of Meyer and Pajor [25] about the volume of hyperplane sections of the unit balls of ℓ_p^n. For $p \in [1, \infty)$, and $x = (x_i)_{i=1}^n \in \mathbb{R}^n$, let $\|x\|_p = (\sum_{i=1}^n |x_i|^p)^{1/p}$ and $\|x\|_\infty = \sup\{|x_i|;\ i = 1, \ldots, n\}$. Let $B_p^n = \{x \in \mathbb{R}^n;\ \|x\|_p \leq 1\}$. If $h \in \mathbb{R}^n$ is a unit vector, and $e_1 = (1, 0, \ldots, 0)$, then

$$|h^- \cap B_p^n|_{n-1} \geq |e_1^- \cap B_p^n|_{n-1}, \text{ for } 2 \leq p \leq \infty,$$

$$|h^- \cap B_p^n|_{n-1} \leq |e_1^- \cap B_p^n|_{n-1}, \text{ for } 1 \leq p \leq 2.$$

In these formulas $|\,.\,|_{n-1}$ is the Lebesgue measure in the corresponding hyperplane. Their proof uses the probability measures on $\mathbb{R}^n$:

$$d\mu_p^n(x) = \exp(-\|\alpha_p x\|_p^p)\,dx.$$

They prove that

$$\int_{h^\perp} e^{-\|\alpha_p x\|_p^p}\,d^{n-1}x \geq 1 = \int_{e_1^\perp} e^{-\|\alpha_p x\|_p^p}\,d^{n-1}x,$$

for $p \geq 2$ and the reverse inequality when $p \in [1,2]$. This means that among the sets $(h^-)_+ = \{x;\ \langle x,h\rangle \geq 0\}$ (which have measure $1/2$), the set $(e_1^-)_+$ has minimal μ_p^n-boundary measure. Our results will imply that $(e_1^-)_+$ has minimal boundary among all Borel subsets A such that $\mu_p^n(A) = 1/2$.

We will generalize the reverse inequality for $p \in [1,2]$ in the following way: if $A \subset \mathbb{R}^n$ is a smooth domain with finite boundary measure and such that $\mu_p^n(A) = 1/2$, then denoting by $n_A(x)$ the outer normal of A at x, the Euclidean norm

$$\left| \int_{\partial A} n_A(x) e^{-\|\alpha_p x\|_p^p} \right|$$

is always less than for $A = (e_1^-)_+$. Notice that when $A = (h^-)_+$, the normal vector is constant: $n_A(x) = -h$ for all x in the boundary. Thus the quantity $|\int_{\partial A} n_A(x) \exp(-\alpha_p \|x\|_p^p)|$ is equal to the μ_p^n-measure of the boundary of A.

This work is divided into two technically independent parts. However, both of them contain statements of extremality of canonical half-spaces for product measures, proved by comparison with the Gaussian case. In the first part, we compare isoperimetric and shift functionals; the tensorization devices, which allow to go to product measures, are Bobkov-type functional forms of the geometric inequalities. In the second part, we get more from a method of Vaaler [29]. This time, one compares the values of measures on symmetric convex sets and the tensorizing device is a result of Kanter [19] about the peaked order on unimodal measures. These tools were also the basis in [25]. We will complete this second part by an extension of their theorem to ℓ_p-sums of arbitrary finite dimensional normed spaces.

As the reader will see, the two methods give quite similar results. Nevertheless, they are efficient in very different settings. The first one is convenient for the general isoperimetric problem on manifolds. For example we solve it for sets of measure $1/2$ in a product of k-dimensional spheres. The second method requires a linear setting but it can be applied to non log-concave measures, where the first method would fail.

2 Comparing isoperimetric and shift functions

Let us introduce some notation. We start with the isoperimetric problem. It consists in finding subsets of prescribed measure, whose measure increases

the less under enlargement. Let (M, ρ) be a Riemannian manifold, let d be the geodesic distance and μ be a probability measure on M. For a Borel set $A \subset M$ and for $\varepsilon > 0$, the ε-enlargement of A is $A_\varepsilon = \{x \in M; \ d(x, A) \leq \varepsilon\}$. The boundary measure of A is

$$\mu^+(A) = \liminf_{\varepsilon \to 0^+} \frac{\mu(A_\varepsilon) - \mu(A)}{\varepsilon}.$$

The isoperimetric function of (M, μ) is defined for $a \in [0, 1]$ by

$$I_\mu(a) = \inf\{\mu^+(A); \ \mu(A) = a\}.$$

It vanishes at 0 and 1.

For convenience, we will use some rescallings to ensure that $I_\mu(1/2) = 1$. When μ is a measure on $\mathbb{R}^n$, one defines μ_λ for $\lambda > 0$, by $\mu_\lambda(A) = \mu(\lambda A)$. One easily checks that $I_{\mu_\lambda} = \lambda I_\mu$. In the case of the Euclidean n-dimensional sphere of radius r, $rS^n \subset \mathbb{R}^{n+1}$, we consider the Riemannian struture induced by $\mathbb{R}^{n+1}$. Then, if σ_{rS^n} is the uniform probability on rS^n, one has $r I_{\sigma_{rS^n}} = I_{\sigma_{S^n}}$.

We turn now to the shift problem. Its aim is to find the sets of given measure whose measure varies the most under translations. For references, one can see [10]. The natural setting will be the Euclidean space $(\mathbb{R}^n, \langle \cdot, \cdot \rangle, |\cdot|)$, with a probability measure μ, with density ρ_μ with respect to the Lebesgue measure. The shift function of μ can be defined for $a \in [0, 1]$ as

$$S(a) = \sup\left\{\sup_{|h|=1} \limsup_{\varepsilon \to 0^+} \frac{|\mu(A + \varepsilon h) - \mu(A)|}{\varepsilon}; \ \mu(A) = a\right\}.$$

When μ has a smooth density ρ_μ and $A \subset \mathbb{R}^n$,

$$\frac{1}{\varepsilon}(\mu(A + \varepsilon h) - \mu(A)) = \int_A \frac{\rho_\mu(y + \varepsilon h) - \rho_\mu(y)}{\varepsilon}\, dy$$

tends to $\int_A \langle \nabla \rho_\mu(y), h \rangle \, dy$. Thus the shift function is

$$S_\mu(a) = \sup\left\{\left|\int_A \nabla \rho_\mu(x)\, dx\right|; \ \mu(A) = a\right\}.$$

This makes sense in the more general case when the distributional gradient of ρ_ν is a signed measure with density with respect to μ ([10]). Notice that when ρ_μ is smooth, Green's formula yields $\int_A \nabla \rho_\mu(y)\, dy = \int_{\partial A} \rho_\nu n_A$, where n_A is the outer normal af A and the integral is with respect to the Lebesgue measure on the boundary of A. The latter quantity exists in the more general setting of embedded manifolds. So when μ is a probability with smooth density ρ_μ on an embedded manifold $M \subset \mathbb{R}^k$, we can define the shift function $S_\mu(a)$, for $a \in [0, 1]$, by

$$S_\mu(a) = \sup\left\{\left|\int_{\partial A} \rho_\mu n_A\right|; \ A \subset M, \mu(A) = a\right\},$$

where the supremum is over the open sets with smooth boundary for which the integral is absolutely convergent.

The Gaussian measure will be of particular importance in the following. Notice that we do not choose the usual convention. Let γ be the probability measure on $\mathbb{R}$ with density $\rho_\gamma(t) = \exp(-\pi t^2)\,dt$. For a measure ν on $\mathbb{R}$ we denote by R_ν the distribution function $R_\nu(t) = \nu(]-\infty, t])$. The isoperimetric problem for the measures $\gamma^{\otimes n}$ was solved in [14] and in [28]. The solution to the shift problem for these measures is in [20]. Half-spaces are always extremal. This remarkable property of the Gaussian measure can be stated as: $I_{\gamma^{\otimes n}} = I_\gamma = S_{\gamma^{\otimes n}} = S_\gamma = \rho_\gamma \circ R_\gamma^{-1}$, where R_γ^{-1} is the reciprocal of the distribution function of γ.

When studying the isoperimetric or shift function of a product measure $\mu^{\otimes n}$, it will be useful to compare I_μ or S_μ with $I_\gamma = S_\gamma$. It turns out that such comparisons are equivalent to Bobkov or reverse Bobkov-type inequalities (see [9], [6]):

Theorem 1 ([7]) *Let M be a Riemannian manifold and μ a density probability on M. Let $c > 0$. Then the following assertions are equivalent:*

i) $\quad I_\mu \geq c\, I_\gamma$

ii) $\quad$ *For all locally Lipschitz functions $f : M \to [0, 1]$,*

$$I_\gamma\left(\int f\,d\mu\right) \leq \int \sqrt{I_\gamma^2(f) + \frac{1}{c^2}|\nabla f|^2}\,d\mu$$

Now, we extend to manifolds a result of [6]:

Theorem 2 *Let μ a density probability measure on an embedded manifold $M \subset \mathbb{R}^k$. Let $c > 0$. Then the following assertions are equivalent:*

i) $\quad S_\mu \leq c\, I_\gamma$

ii) $\quad$ *For all smooth and compactly supported functions $f : M \to [0, 1]$,*

$$I_\gamma\left(\int f\,d\mu\right) \geq \sqrt{\left(\int I_\gamma(f)\,d\mu\right)^2 + \frac{1}{c^2}\left|\int \nabla f\,d\mu\right|^2}.$$

Proof: We show first that *ii)* implies *i)*. Notice that *ii)* can be extended to continuous piecewise C^1 functions with compact support. Let A be a smooth compact domain in M. For $\varepsilon > 0$, let $f_\varepsilon : M \to [0, 1]$ be defined by

$$f_\varepsilon(x) = \left(1 - \frac{1}{\varepsilon}d(x, A)\right)_+$$

where d is the geodesic distance. Applying *ii)* to f_ε, one gets

$$\left|\frac{1}{\varepsilon}\int_{A_\varepsilon - A} \nabla d(\cdot, A)\,d\mu\right| \leq c\, I_\gamma\left(\int f_\varepsilon\right).$$

4

Notice that ∇d has norm one. Close to the boundary of A, it becomes orthogonal to it. Thus, letting ε to zero, we get $|\int_{\partial A} n_A \rho_\mu| \leq c\, I_\gamma(\mu(A))$.

Next, we assume i) and show ii). Let f be smooth and compactly supported. Let ν be the distribution of f under μ. We may assume that ν is absolutely continuous with respect to the Lebesgue measure on $[0,1]$ and has positive density on its support. By the co-area formula

$$\int \nabla f\, d\mu = \int_0^1 \int_{\partial A_t} \frac{\nabla f}{|\nabla f|}\, d\sigma_t\, dt,$$

where $A_t = \{x;\ f(x) \geq t\}$ and σ_t is the measure on ∂A_t of density ρ_μ. Since $\nabla f / |\nabla f|$ is a unit inner normal of A_t, we get by i)

$$\left| \int \nabla f\, d\mu \right| \leq \int_0^1 \left| \int_{\partial A_t} n_{A_t} \rho_\mu \right|\, dt \leq c \int_0^1 I_\gamma(\mu(A_t))\, dt.$$

Let $N(t) = \mu(\{f \leq t\}) = \nu([0,t])$. Define $k = N^{-1} \circ R_\gamma$, and apply the reverse Bobkov inequality of [1] to k and the measure γ_1:

$$I_\gamma^2 \left(\int_{\mathbb{R}} k\, d\gamma_1 \right) \geq \left| \int_{\mathbb{R}} k'\, d\gamma_1 \right|^2 + \left(\int_{\mathbb{R}} I_\gamma(k)\, d\gamma_1 \right)^2.$$

By the change of variable $t = k(x)$,

$$\int_{\mathbb{R}} k'(x)\, d\gamma_1(x) = \int_0^1 \rho_\gamma(k^{-1}(t))\, dt = \int_0^1 I_\gamma(N(t))\, dt.$$

Since the law of k under γ_1 is equal to the one of f with respect to μ, we get

$$\left(\int I_\gamma(\mu(A_t))\, dt \right)^2 = \left(\int I(\mu(\{f \leq t\})\, dt \right)^2 \leq I_\gamma^2 \left(\int f\, d\mu \right) - \left(\int I(f)\, d\mu \right)^2$$

where we have used $I(p) = I(1-p)$. Thus we get ii) and the proof is complete. $\qquad\square$

As stated in [12], [7] and [6] the functional inequalities in the latter two theorems have the tensorisation property: if they are true for μ, then they hold for $\mu^{\otimes n}$ for all $n \geq 1$. This remark yields the following result, which is the basis of our comparison method.

Corollary 3 *Let μ be a probability measure on M and let $\mu^{\otimes n}$ be the product measure on M^n.*

i) If $I_\mu \geq c\, I_\gamma$, then for $n \geq 1$ one has $I_{\mu^{\otimes n}} \geq c\, I_\gamma$.

Assume M embedded.

ii) If $S_\mu \leq d\, I_\gamma$, then for $n \geq 1$, $S_{\mu^{\otimes n}} \leq d\, I_\gamma$.

Let us emphasize that we consider on M^n the canonical Riemannian product structure. For the shift problem, if M is embedded in $\mathbb{R}^k$, we consider the canonical product embedding of M^n in $\mathbb{R}^{nk}$. Let us give a few comments:

1) It is clear that $I_\mu \geq I_{\mu \otimes 2} \geq I_{\mu \otimes 3} \ldots$ and $S_\mu \leq S_{\mu \otimes 2} \leq S_{\mu \otimes 3} \ldots$. Moreover, if μ is on $\mathbb{R}$ and has finite variance, classical cantral limit arguments about the sets $\{x \in \mathbb{R}^n; \ \sum_{i=1}^{n} \leq t\sqrt{n}\}$ show that $\inf_n I_{\mu \otimes n} \leq C I_\gamma$ and $\sup_n S_{\mu \otimes n} \leq C I_\gamma$ for some constant C depending on the variance. Thus, in terms of behaviour close to zero, I_γ is maximal for $i)$ and minimal for $ii)$.

2) Similar results were established earlier. Let $J(t) = \min(t, 1-t)$, Bobkov and Houdré [12] showed that $I_\mu \geq cJ$ implies $I_{\mu \otimes n} \geq c/(2\sqrt{6}) J$ for all $n \geq 1$. In [10], Bobkov shows, for $H(t) = t \log(1/t) + (1-t) \log(1/(1-t))$, that $S_\mu \leq cH$ implies $S_{\mu \otimes n} \leq 24c H$ for $n \geq 1$.

3) The remarkable fact about I_γ is that there is no loss on the constant c when going to product measures. We shall show that I_γ is almost the only one with this property. Let $K : (0,1) \to (0, \infty)$ be a positive, concave function such that for all t, $K(1-t) = K(t)$. Assume that for every probability measure μ on $\mathbb{R}$, $I_\mu \geq K$ implies $I_{\mu \otimes 2} \geq K$. By [8], there exists an even log-concave probability ν on $\mathbb{R}$ such that $I_\nu = \rho_\nu \circ R_\nu^{-1} = K$. But $K \leq I_\nu$ implies $K \leq I_{\nu \otimes 2}$ which is clearly less than I_ν. So $I_\nu = I_{\nu \otimes 2} = \rho_\nu \circ R_\nu^{-1}$, which means that half-spaces of the form $\{x_1 \leq \alpha\}$ are solution to the isoperimetric problem. By [11], this implies that ν is either a Gaussian (and $I_\nu = \lambda I_\gamma$) or a Dirac mass at a point, which is excluded.

The situation is the same for the shift problem. Let K is concave positive and symmetric as before and such that $S_\mu \leq K$ implies $S_{\mu \otimes 2} \leq K$. Consider again the log-concave probability ν on $\mathbb{R}$ such that $K = \rho_\nu \circ R_\nu^{-1}$. We show in the next section $S_\nu = K$. Again we can deduce from this that $S_{\nu \otimes 2} = \rho_\nu \circ R_\nu^{-1}$, which means that half-spaces $\{x_1 \leq \alpha\}$ are solution to the shift problem. One can check that all the steps of the proof in [11] can be carried out in this new situation. This is due to the fact that their argument only uses half-spaces for which boundary measure and norm of the integral of the outer normal coincide. The result is again that K is a multiple of I_γ.

Next, we take advantage of the previous property of I_γ to get exact solutions of isoperimetric problems. The shift case is similar. Assume that μ is such that $I_\mu \geq cI_\gamma$ and that there exists $a \in (0,1)$ such that $I_\mu(a) = cI_\gamma(a)$ (i.e. c is maximal so that $I_\mu \geq cI_\gamma$). By the previous results and remarks, we have

$$I_\mu(a) = cI_\gamma(a) \leq I_{\mu \otimes n}(a) \leq I_\mu(a).$$

Thus $I_{\mu \otimes n}(a) = I_\mu(a)$. Let A be a solution of measure a of the isoperimetric problem for μ, $\mu(A) = a$, $\mu^+(A) = I_\mu(a)$. Then $A \times M^{n-1} \subset M^n$ satisfies $\mu^{\otimes n}(A \times M^{n-1}) = \mu(A) = a$ and

$$(\mu^{\otimes n})^+(A \times M^{n-1}) = \mu^+(A) = I_\mu(a) = I_{\mu \otimes n}(a).$$

So, for all n, $A \times M^{n-1}$ is a solution of the isoperimetric problem of measure a.

In the next sections we give applications of this methods in concrete cases. Each time, we try to have exact comparisons with the Gaussian case.

2.1 Products of log-concave measures on the real line

The isoperimetric problem for log-concave measures on the real line was solved by Bobkov [8]. In particular, he proves:

Proposition 4 *Let μ be a log-concave measure on the real line. Then, μ is symmetric around its median if and only if for all $0 < p < 1$ and all $h > 0$, the infimum of $\mu(A + [-h, h])$ over the sets A such that $\mu(A) = p$ is achieved for an interval of the form $(-\infty, a]$.*

For convenience, we will always assume that 0 is a median of our measures. The previous result has the following infinitesimal corollary. Recall that ρ_μ is the density of μ and R_μ is its distribution function.

Proposition 5 *Let μ be a log-concave even probability measure on $\mathbb{R}$, then its isoperimetric function is given by $I_\mu(0) = I_\mu(1) = 0$ and for $t \in]0, 1[$,*

$$I_\mu(t) = \rho_\mu(R_\mu^{-1}(t)).$$

We will need a similar statement for the shift problem. The results have the same form.

Lemma 6 *Let ν be a log-concave measure on the real line, with positive density $\rho_\mu = e^{-N}$. Let $0 < p < 1$ and $h > 0$, then*

$$\sup\{\nu(A + h); \ \nu(A) = p\}$$

is achieved for intervals of the form $(-\infty, a]$.

Proof: One can see from the formula

$$\nu(A + h) = \int_A e^{-(N(x+h) - N(x))} \, d\nu(x)$$

that, given p and h, the supremum is achieved for $A_0 = \{x; \ N(x + h) - N(x) \leq \alpha\}$ where α is chosen so that $\nu(A_0) = p$. Since N is convex, the function $x \to N(x + h) - N(x)$ is non-decreasing. Thus one can take $A_0 = (-\infty, R_\mu^{-1}(p)]$. $\quad\square$

Notice that $\inf\{\nu(A + h); \ \nu(A) = p\}$ is achieved on sets of the form $[b, +\infty)$. And one has reversed results when h is negative. Letting h to zero, one easily gets

Proposition 7 *Let ν be a log-concave even probability measure on $\mathbb{R}$ with positive density, then its shift function is given by $S_\nu(0) = S_\nu(1) = 0$ and for $0 < t < 1$,*

$$S_\nu(t) = \rho_\nu\big(R_\nu^{-1}(t)\big).$$

We have computed isoperimetric and shift functions. The next statement is usefull in comparing them.

Lemma 8 *Let μ and ν be even log-concave probability measures on $\mathbb{R}$, with densities ρ_μ and ρ_ν. Let $m, n \in [0, \infty]$ be the supremums of the supports of μ and ν. Assume that ρ_μ is decreasing on $\mathbb{R}^+$, that $\rho_\mu(0) = \rho_\nu(0) = 0$ and $\lim_{m\perp} \rho_\mu \geq \lim_{n\perp} \rho_\nu$.*
If $\rho_\nu^{-1} \circ \rho_\mu$ is convex, then for $0 < t < 1$, $\rho_\mu \circ R_\mu^{-1}(t) \geq \rho_\nu \circ R_\nu^{-1}(t)$.

Proof: Notice that $\rho_\nu^{-1} \circ \rho_\mu$ is well defined. By symmetry of the measures, we can restrict to $t \in [1/2, 1)$. The announced inequality is equivalent to $t \geq R_\nu \circ \rho_\nu^{-1} \circ \rho_\mu \circ R_\mu^{-1}(t)$, for $t \in [0, m)$. Setting $t := R_\mu(y)$, we have to show that for $y \in [0, m)$,

$$f(y) := R_\mu(y) - R_\nu\big((\rho_\nu^{-1} \circ \rho_\mu)(y)\big)$$

is non-negative. Obviously,

$$f'(y) = \rho_\mu(y)\Big(1 - (\rho_\nu^{-1} \circ \rho_\mu)'(y)\Big),$$

where $'$ stands for right-derivative. By hypothesis $(\rho_\nu^{-1} \circ \rho_\mu)'$ is non-decreasing. Thus, f' can either be of constant sign on $[0, m)$ or be non-negative on some $[0, a)$ and then non-positive on (a, m). Since f is continuous and $f(0) = \lim_{m\perp} f = 0$, we are in the second case and f is non-negative. $\quad\square$

Combining this lemma with Theorem 3 and the preceding computations of isoperimetric and shift functions, we get

Theorem 9 *Let μ be an even absolutely continuous log-concave probability measure on $\mathbb{R}$. We write $d\mu = e^{-M}$, where $M : \mathbb{R} \to [0, \infty]$ is convex. Assume that $M(0) = 0$.*
  i) If $\sqrt{M}$ is convex, then for every integer n, one has $I_{\mu^{\otimes n}} \geq I_\gamma$. In particular, among sets of measure $1/2$ for $\mu^{\otimes n}$, the half-space $\mathbb{R}_+ \times \mathbb{R}^{n-1}$ is solution to the isoperimetric problem.
  ii) If $\sqrt{M}$ is concave, then for n integer, $S_{\mu^{\otimes n}} \leq S_\gamma$. In particular, among sets of measure $1/2$ for $\mu^{\otimes n}$, the half-space $\mathbb{R}_+ \times \mathbb{R}^{n-1}$ is solution to the shift problem.

Notice that the hypothesis "M convex and $\sqrt{M}$ concave" implies that the density $\rho_\mu = e^{-M}$ is positive and decreasing on $\mathbb{R}^+$. This theorem can be applied to the probability measures $d\mu_p = e^{-|\alpha_p t|^p}$. They are in the case $i)$ when $p \geq 2$ and in the case $ii)$ when $1 \leq p \leq 2$.

Remark: If ν is the push forward of a measure μ by a Lipschitz map f, it is well-known that $\|f\|_{\text{Lip}} I_\nu \geq I_\mu$. In particular, if μ is a probability on $\mathbb{R}^n$,

$$\inf_{(0,1)} \frac{I_\mu}{I_\gamma} \geq \sup \left\{ 1/\|f\|_{\text{Lip}}; \; f(\gamma^{\otimes n}) = \mu \right\}.$$

When $n = 1$, the latter is a equality [22]. The optimal map is then given by the canonical monotone transportation defined by $R_\gamma = R_\mu \circ f$. Notice that f is a contraction if and only if $1 \geq |f'| = \rho_\gamma / (\rho_\mu \circ f) = \rho_\gamma / (\rho_\mu \circ R_\mu^{-1} \circ R_\gamma)$, and we recover the condition $\rho_\gamma \circ R_\gamma^{-1} \leq \rho_\mu \circ R_\mu^{-1}$. When this holds, the map $f_n(x_1, \ldots, x_n) = (f(x_1), \ldots, f(x_n))$ is a contraction of $\gamma^{\otimes n}$ onto $\mu^{\otimes n}$, thus $I_{\mu^{\otimes n}} \geq I_{\gamma^{\otimes n}}$. These classical arguments provide a slightly simpler proof of the statement $i)$ in the previous theorem. However, they do not work for the shift problem. In larger dimensions, building transportations is more difficult and, a priori, does not give the optimal constants in comparisons of isoperimetric functions.

2.2 Products of spherical measures

For $n \in \mathbb{N}$, let $S^n \subset \mathbb{R}^{n+1}$ be the Euclidean unit sphere and let s_n denote its n-dimensional Lebesgue measure, by convention $s_0 = 2$. Let $r_n = s_{n-1}/s_n$. We consider $r_n S^n \subset \mathbb{R}^{n+1}$ with the Riemannian structure induced by $\mathbb{R}^{n+1}$. Let σ_n be the uniform probability on this special sphere.

The measure of a spherical cap $C_t = \{ x \in r_n S^n; \; \langle x, e_1 \rangle \leq t \}$ is, for $|t| \leq r_n$

$$\Phi_n(t) = \int_{-r_n}^t \left(1 - \left(\frac{u}{r_n} \right)^2 \right)^{\frac{n-2}{2}} du,$$

whereas the boundary measure of C_t is

$$\varphi_n(t) = \left(1 - \left(\frac{t}{r_n} \right)^2 \right)^{\frac{n-1}{2}}.$$

Since spherical caps are solution to the isoperimetric problem [24], [27], the isoperimetric function of σ_n is

$$I_{r_n S^n}(t) = \varphi_n(\Phi_n^{-1}(t)), \quad t \in [0, 1].$$

It is obviouly symmetric with respect to $1/2$ and decreasing on $[1/2; 1]$.

Next, we compute the total normal $\int_{\partial C_t} n_{C_t}$ for these caps. By rotationnal invariance, it is parallel to e_1. At any boundary point, one has $\langle n_{C_t}, e_1 \rangle = -\sqrt{1 - (t/r_n)^2}$. Thus

$$\psi_n(t) := \left| \int_{\partial C_t} n_{C_t} \right| = \sqrt{1 - \left(\frac{t}{r_n} \right)^2} \; \varphi_n(t) = \left(1 - \left(\frac{t}{r_n} \right)^2 \right)^{\frac{n}{2}}.$$

Our next result asserts that caps are also solution to the shift problem.

9

Theorem 10 *The shift function of the sphere is*

$$S_{r_n S^n} = \psi_n \circ \Phi_n^{-1}.$$

For $n \geq 2$, $S_{r_n S^n} = (I_{r_n S^n})^{\frac{n}{n+1}}$.

Proof: Let $a \in [0,1]$. We consider only smooth functions $f : r_n S^n \to [0,1]$. By rotationnal invariance (of the norm and of the sphere),

$$\sup_{\int f = a} \left| \int \nabla f \, d\sigma_n \right| = \sup_{\int f = a} \int \langle \nabla f, -e_1 \rangle \, d\sigma_n.$$

By Green's formula

$$\int \langle \nabla f, -e_1 \rangle \, d\sigma_n = n \int f(x) \langle -e_1, \frac{x}{r_n} \rangle \, d\sigma_n(x).$$

Under the condition $\int f = a$, the latter integral is maximal when f is the characteristic function of the cap $C_{\Phi_n^{-1}(a)}$. This implies that for smooth f,

$$\left| \int \nabla f \, d\sigma_n \right| \leq S_n \left(\int f \, d\sigma_n \right).$$

Applying this to approximations of characteristic functions of sets, as in section 2, we get the result for sets: when $\sigma_n(A) = \sigma_n(C_t)$, one has $| \int_{\partial A} n_A | \leq | \int_{\partial C_t} n_{C_t} |$. $\qquad\square$

Proposition 11 *Let $n \geq 1$, then for $t \in [0,1]$*

$$S_{r_n S^n}(t) \leq S_{r_{n+1} S^{n+1}}(t) \leq I_\gamma(t) \leq I_{r_{n+1} S^{n+1}}(t) \leq I_{r_n S^n}(t),$$

with equality only at $t = 0, 1/2$ and 1.

When $n = 2$, we recover the inequalities $2\sqrt{t(1-t)} \geq I_\gamma \geq 4t(1-t)$ which were noticed respectively in [21] and [6].

Proof: We show first the right hand side inequality. By symmetry, it is enough to prove it on $[1/2, 1]$. Notice that, by construction, there is equality at the end points of this interval. We want to show that for $x \in [1/2, 1]$,

$$\left(1 - \left(\frac{\Phi_{n+1}^{-1}(x)}{r_{n+1}} \right)^2 \right)^{\frac{n}{2}} \leq \left(1 - \left(\frac{\Phi_n^{-1}(x)}{r_n} \right)^2 \right)^{\frac{n+1}{2}}.$$

Since Φ_{n+1} is increasing, this is equivalent to

$$\Phi_{n+1} \left(r_{n+1} \left[1 - \left(1 - \left(\frac{\Phi_n^{-1}(x)}{r_n} \right)^2 \right)^{\frac{n+1}{n}} \right]^{\frac{1}{2}} \right) \leq x,$$

for $x \in [1/2, 1]$. Setting $x = \Phi_n(r_n y), y \in [0, 1]$, we have to check that for $y \in [0, 1]$, the following function is non-negative:

$$f(y) = \Phi_n(r_n y) - \Phi_{n+1}\left(r_{n+1}\left[1 - (1 - y^2)^{\frac{n+1}{n}}\right]^{\frac{1}{2}}\right)$$

For $y \in (0, 1)$, its derivative is

$$
\begin{aligned}
f'(y) &= r_n(1 - y^2)^{\frac{n+2}{2}} - r_{n+1}(1 - y^2)^{\frac{(n+1)^2}{2n}} \frac{\partial}{\partial y}\left(1 - (1 - y^2)^{\frac{n+1}{n}}\right)^{\frac{1}{2}} \\
&= (1 - y^2)^{\frac{n+2}{2}}\left[r_n - \frac{n-1}{n}r_{n+1}y(1 - y^2)^{-\frac{1}{2n}}\left(1 - (1 - y^2)^{\frac{n+1}{n}}\right)^{-\frac{1}{2}}\right].
\end{aligned}
$$

So $f'(y) \geq 0$ is equivalent to:

$$\frac{(1 - y^2)^{\frac{1}{n}} - 1}{y^2} \geq \left(\frac{(n-1)r_{n+1}}{nr_n}\right)^2 - 1$$

Since $t \to t^{1/n}$ is concave, the left quantity is decreasing on $(0, 1)$. Thus, there exits a such that f increases on $(0, a)$ and decreases on $(a, 1)$. Since $f(0) = f(1) = 0$, f is non-negative.

The inequality $I_\gamma \leq I_{r_n S^n}$ can be proved with the same method. It can be understood by the Poincaré limit argument: the sequence $(I_{r_n S^n})_{n \geq 1}$ is non-increasing, and I_γ is its limit. Indeed, for a fixed $x \in \mathbb{R}$, when n tends to infinity,

$$\psi_n(x) = \exp\left(\frac{n-2}{2}\ln\left(1 - \frac{x^2}{r_n^2}\right)\right) \sim e^{-\frac{n+2}{2r_n^2}x^2} \sim e^{-\pi x^2},$$

and in the same way $\Phi_n(x) \to R_\gamma(x)$. The inequalities involving the shift functions have a similar proof. $\qquad\square$

By the previous comparisons and by the results of Section 2, we have

Theorem 12 *Let $f : (r_n S^n)^k \to [0, 1]$ be smooth, then*

$$I_\gamma\left(\int f \, d\sigma_n^{\otimes k}\right) \leq \int \sqrt{I_\gamma^2(f) + |\nabla f|^2} \, d\sigma_n^{\otimes k},$$

and

$$I_\gamma\left(\int f \, d\sigma_n^{\otimes k}\right) \geq \sqrt{\left(\int I_\gamma(f) \, d\sigma_n^{\otimes k}\right)^2 + \left|\int \nabla f \, d\sigma_n^{\otimes k}\right|^2}.$$

In particular $I_{(r_n S^n)^k} \geq I_\gamma \geq S_{(r_n S^n)^k}$.

The latter inequality appeared for $k = 1$, in a slightly different form, in [6].

Corollary 13 *Let $S_+^n = \{(x_i)_{i=1}^{n+1} \in S^n; \, x_1 \geq 0\}$. Among subsets of measure $1/2$ in a product of k spheres of dimension n, the set $S_+^n \times (S^n)^{k-1}$ is solution to the isoperimetric and to the shift problem.*

2.3 Products of uniform measures on Euclidean balls

The isoperimetric problem for the uniform distribution on the Euclidean ball
was solved by Burago and Zalgaller [16]. The case of dimension 1 is simple.
From now on we work in dimension $n \geq 2$. Solution sets are intersections
with orthogonal balls or their complements. Let v_n be the volume of the
Euclidean unit ball B_2^n. Set $R_n = v_{n-1}/v_n$. We will consider the uniform
probability λ_n on $R_n B_2^n$. Now we give a description for the solutions of
measure larger than $1/2$. Let $m \geq R_n$ and $\rho \in [m - R_n, m]$. The ball
$m e_1 + \rho B_2^n$ crosses B_2^n. The intersection lies in the hyperplane $\{x_1 = a\}$
where $a \geq 0$ satisfies $R_n^2 - a^2 = \rho^2 - (m - a)^2$. The boundaries of the two
balls intersect orthogonaly if $m^2 = R_n^2 + \rho^2$. In that case, $B_2^n \setminus (m e_1 + \rho B_2^n)$
is a solution to the isoperimetric problem, with measure larger than $1/2$,
and all solutions for measure $\geq 1/2$ are isometric to such a set. The solution
for volume $1/2$ is the half-ball; this corresponds to m and ρ infinite, when
the other ball becomes a half-space.

These sets can be viewed as a one-parameter family indexed by $\alpha :=
a/R_n \in [0,1]$. One easily checks that it is an increasing function of α, in
the sense of the inclusion order. For a given α, we express λ_n-measure and
boundary measure. First the volume

$$
\begin{aligned}
V(\alpha) &= \int_{-R_n}^{a} v_{n-1}(R_n^2 - t^2)^{\frac{n-1}{2}} \frac{dt}{v_n R_n^n} - \int_{m-a}^{\rho} v_{n-1}(\rho^2 - s^2)^{\frac{n-1}{2}} \frac{ds}{v_n R_n^n} \\
&= R_n \left[\int_{-1}^{\alpha} (1 - \tau^2)^{\frac{n-1}{2}} \, d\tau - \left(\frac{\sqrt{1 - \alpha^2}}{\alpha} \right)^n \int_{\sqrt{1-\alpha^2}}^{1} (1 - \sigma^2)^{\frac{n-1}{2}} \, d\sigma \right],
\end{aligned}
$$

where we have used the relations $m = R_n^2/a$, $\rho = R_n \sqrt{R_n^2 - a^2}/a$, the
definition of R_n and the change of variables $t = R_n \tau$ and $s = \rho \sigma$. In the
same way the boundary measure is

$$
\begin{aligned}
S(\alpha) &= \int_{m-a}^{\rho} s_{n-2}(\rho^2 - s^2)^{\frac{n-3}{2}} \rho \frac{ds}{v_n R_n^n} \\
&= (n-1) \left(\frac{\sqrt{1 - \alpha^2}}{\alpha} \right)^{n-1} \int_{\sqrt{1-\alpha^2}}^{1} (1 - \sigma^2)^{\frac{n-3}{2}} \, d\sigma.
\end{aligned}
$$

Clearly, the right parameter is $\theta \in (0, \pi/2)$ such that $\alpha = \cos(\theta)$. Then

$$
v(\theta) := V(\cos \theta) = R_n \left[\int_{\theta}^{\pi} \sin^n u \, du - \tan^n \theta \int_{\theta}^{\pi/2} \cos^n u \, du \right]
$$

$$
s(\theta) := S(\cos \theta) = (n-1) \tan^{n-1} \theta \int_{\theta}^{\pi/2} \cos^{n-2} u \, du.
$$

These functions can be extended by continuity: $s(0) = 0, s(\pi/2) = 1$, and
$v(0) = 1, v(\pi/2) = 1/2$. The isoperimetric function of λ_n is $I_{R_n B_2^n} = S \circ
V^{-1} = s \circ v^{-1}$. We have the following comparison with the Gaussian case

Theorem 14 *Let $n \geq 1$, then for $t \in [0,1]$*

$$I_{R_n B_2^n}(t) \geq I_\gamma(t),$$

with equality only at $t = 0, 1/2$ and 1.

Proof: We start with some preliminary calculations. Notice that

$$v'(\theta) = -nR_n \frac{\sin^{n-1}\theta}{\cos^{n+1}\theta} \int_\theta^{\pi/2} \cos^n u \, du$$

$$s'(\theta) = (n-1)\frac{\sin^{n-2}\theta}{\cos^n\theta} \left[(n-1)\int_\theta^{\pi/2} \cos^{n-2} u \, du - \cos^{n-1}\theta \sin\theta \right].$$

Writing $\int \cos^n u \, du = \int \cos^{n-2} u \, du + \int -\sin u \cos^{n-2} u \times \sin u \, du$ and integrating by parts in the latter integral shows that

$$n \int_\theta^{\pi/2} \cos^n u \, du = (n-1) \int_\theta^{\pi/2} \cos^{n-2} u \, du - \cos^{n-1}\theta \sin\theta,$$

thus $\dfrac{s'(\theta)}{v'(\theta)} = -\dfrac{n-1}{R_n} \cdot \dfrac{\cos\theta}{\sin\theta}$. In particular s is increasing in θ, and thus remains less than 1. By symmetry, it is enough to show that $\rho_\gamma \circ R_\gamma^{-1} \leq s \circ v^{-1}$ on $(1/2, 1)$. This is equivalent to the non-negativity on $(0, \pi/2)$ of the function

$$f(\theta) = v(\theta) - R_\gamma \left(\sqrt{-\frac{1}{\pi} \ln s(\theta)} \right).$$

Here we have used $s \in [0,1]$. Since $f = 0$ at the end points of this interval, we are done if we can prove that f' is first positive and then negative. After simplification one gets

$$f'(\theta) = v'(\theta) + \frac{s'(\theta)}{2\sqrt{-\pi \ln s(\theta)}}.$$

Thus $f'(\theta) \geq 0$ is equivalent to

$$g(\theta) := 4\pi \ln s(\theta) + \left(\frac{s'(\theta)}{v'(\theta)} \right)^2 \geq 0.$$

Here one had to be careful about signs, which depend on the choice of parameters. When θ tends to $\pi/2$, $s(\theta)$ goes to 1 and $\cos\theta$ to 0, thus $\lim_{(\pi/2)^\perp} g = 0$. When y goes to zero,

$$\frac{\cos\theta}{\sin\theta} \sim \frac{1}{\theta} \qquad \text{and} \qquad s(\theta) \sim (n-1)\theta^{n-1} \int_0^{\pi/2} \cos^{n-2} u \, du.$$

13

So $\lim_{0+} g = +\infty$. It would be enough to show that g' is first negative and then positive. Clearly

$$g'(\theta) = 4\pi \frac{s'(\theta)}{s(\theta)} - 2\left(\frac{n-1}{R_n}\right)^2 \frac{\cos\theta}{\sin^3\theta}.$$

This quantity has the same sign as

$$
\begin{aligned}
&2\pi \sin^3\theta \, s'(\theta) - \left(\frac{n-1}{R_n}\right)^2 \cos\theta \, s(\theta) \\
={}& 2\pi(n-1)\frac{\sin^{n+1}\theta}{\cos^n\theta}\left[(n-1)\int_\theta^{\pi/2} \cos^{n-2} u \, du - \cos^{n-1}\theta \, \sin\theta\right] \\
&- \left(\frac{n-1}{R_n}\right)^2 (\cos\theta)(n-1)\tan^{n-1}\theta \int_\theta^{\pi/2} \cos^{n-2} u \, du \\
={}& \left[\left(2\pi(n-1)\frac{\sin^{n+1}\theta}{\cos^n\theta} - \left(\frac{n-1}{R_n}\right)^2 \frac{\sin^{n-1}\theta}{\cos^{n-2}\theta}\right)\left(\int_\theta^{\pi/2}\cos^{n-2} u \, du\right)\right. \\
&\left. - 2\pi\frac{\sin^{n+2}\theta}{\cos\theta}\right] \times (n-1).
\end{aligned}
$$

Multiplying by $\cos^n\theta / ((n-1)\sin^{n-1}\theta)$, we get that g' has the same sign as

$$
\begin{aligned}
h_n(\theta) \;:=\; &\left(\int_\theta^{\pi/2}\cos^{n-2} u \, du\right)\left(2\pi(n-1)\sin^2\theta - \left(\frac{n-1}{R_n}\right)^2 \cos^2\theta\right) \\
&- 2\pi \sin^3\theta \, \cos^{n-1}\theta.
\end{aligned}
$$

Let $\alpha_n = 2\pi(n-1)$, $\beta_n = ((n-1)/R_n)^2$ and $\theta_n = \arctan(\sqrt{\beta_n/\alpha_n})$. If $\theta \in [0, \theta_n]$ then $h_n(\theta) < 0$. On $(\theta_n, \pi/2]$, $h_n(\theta)$ has the sign of

$$j_n(\theta) = \int_\theta^{\pi/2}\cos^{n-2} u \, du - \frac{2\pi\sin^3\theta\,\cos^{n-1}\theta}{\alpha_n\sin^2\theta - \beta_n\cos^2\theta}.$$

Notice that $\lim_{\theta_n^+} j_n = -\infty$ and $j_n(\pi/2) = 0$. We are done if we can prove that on $(\theta_n, \pi/2)$, j_n is first negative and then positive. A straightforward computation yields

$$j_n'(\theta) = \frac{\cos^{n-2}\theta}{(\alpha_n\sin^2\theta - \beta_n\cos^2\theta)^2}P_n(\sin^2\theta),$$

where P_n is a polynomial of degree 3, with leading term $(\alpha_n+\beta_n).(\alpha_n+2\pi) > 0$. Moreover, P_n satisfies

$$P_n\left(\frac{\beta_n}{\alpha_n+\beta_n}\right) > 0 \qquad \text{and} \qquad P_n(1) = 0.$$

14

To study the variations of j_n, we just need to study P_n on $[x_n, 1]$, where we have set $x_n = \beta_n/(\alpha_n + \beta_n)$. Because of its degree, P_n can decrease only on a bounded interval. Since $P_n(x_n) > P_n(1)$, this interval has to intersect the interval we are working on. If we can prove that $P_n'(1)$ is positive, then clearly P_n is positive on (x_n, η_n), and negative on $(\eta_n, 1)$ for some η_n between x_n and 1. In this case, h_n is first negative and then positive and the theorem is proved. One easily checks that

$$
\begin{aligned}
P_n'(1) &= \alpha_n^2 + 2\pi \alpha_n - 4\pi \beta_n - \alpha_n \beta_n \\
&= \frac{2\pi(n-1)}{R_n^2}(2\pi n R_n^2 - n^2 + 1).
\end{aligned}
$$

Recall thet $R_n = v_{n-1}/v_n$ with $v_n = \pi^{\frac{n}{2}}/\Gamma(1+\frac{n}{2})$. So $P_n'(1) > 0$ is equivalent to

$$
1 + 2n\left(\frac{\Gamma\left(1+\frac{n}{2}\right)}{\Gamma\left(1+\frac{n-1}{2}\right)}\right)^2 > n^2.
$$

But this follows from the next lemma. $\qquad\square$

Lemma 15 *For all $x > 1/2$, one has*

$$
\frac{\Gamma\left(x+\frac{1}{2}\right)}{\Gamma(x)} > \sqrt{x - \frac{1}{2}}.
$$

Proof: It is classical that when x tends to infinity,

$$
\frac{\Gamma\left(x+\frac{1}{2}\right)}{\Gamma(x)} \sim \sqrt{x} \sim \sqrt{x - \frac{1}{2}}.
$$

This implies that

$$
f(x) = \log\Gamma\left(x+\frac{1}{2}\right) - \log\Gamma(x) - \frac{1}{2}\log\left(x - \frac{1}{2}\right)
$$

tends to zero when $x \to \infty$. The lemma will be proved if we show that f is decreasing. Using the formula

$$
(\log\Gamma)'(x) = -\frac{1}{x} - C - \sum_{k=1}^{\infty}\left(\frac{1}{x+k} - \frac{1}{k}\right).
$$

we get

$$
\begin{aligned}
f'(x) &= \sum_{k=0}^{\infty}\left(\frac{1}{x+k} - \frac{1}{x+\frac{1}{2}+k}\right) - \frac{1}{2}\cdot\frac{1}{x-\frac{1}{2}} \\
&= \frac{1}{2}\left(\sum_{k=0}^{\infty}\left(\frac{1}{(x+k)(x+\frac{1}{2}+k)}\right) - \frac{1}{x-\frac{1}{2}}\right)
\end{aligned}
$$

15

Next, by convexity of $y \mapsto y^{-2}$,

$$\frac{1}{(x+k)(x+\frac{1}{2}+k)} < \frac{1}{(x+k)^2} \leq \int_{x+k-\frac{1}{2}}^{x+k+\frac{1}{2}} \frac{dy}{y^2}$$

Eventually we get that f is decreasing

$$f'(x) < \frac{1}{2} \left(\int_{x-\frac{1}{2}}^{\infty} \frac{dy}{y^2} - \frac{1}{x-\frac{1}{2}} \right) = 0.$$

$\square$

¿From Theorems 1 and 14, we derive Bobkov's inequality with optimal constant on $R_n B_2^n$. Let $f : R_n B_2^n \to [0,1]$ be smooth, then

$$I_\gamma \left(\int_{R_n B_2^n} f \, d\lambda_n \right) \leq \int_{R_n B_2^n} \sqrt{I_\gamma^2(f) + |\nabla f|^2} \, d\lambda_n. \tag{1}$$

As explained before this yields an exact solution to the isoperimetric problem in $(B_2^n)^k$ for sets containing half of the whole volume. Let $B_{2,+}^n = \{(x_1, \ldots, x_n) \in B_2^n \, ; \, x_1 \geq 0\}$, and μ_n be the uniform probability on B_2^n. Among sets of probability $1/2$ in $(B_2^n)^k$, the set $B_{2,+}^n \times (B_2^n)^{k-1}$ has minimal boundary measure.

For a probability μ on $\mathbb{R}^n$ and $f : \mathbb{R}^n \to [0, +\infty)$, denote

$$\mathrm{Ent}_\mu(f) = \int f \log f \, d\mu - \left(\int f \, d\mu \right) \log \left(\int f \, d\mu \right).$$

By [1] or by Beckner's limit argument (see [23]), inequality (1) implies a log-Sobolev inequality for $R_n B_2^n$. Let us state it for the unit ball B_2^n. Easy scaling arguments give that for every smooth $f : B_2^n \to [0, +\infty)$,

$$\mathrm{Ent}_{\mu_n}(f^2) \leq \left(\frac{\Gamma\left(\frac{n+1}{2}\right)}{\Gamma\left(\frac{n+2}{2}\right)} \right)^2 \int_{B_2^n} |\nabla f|^2 d\mu_n \leq \frac{2}{n} \int_{B_2^n} |\nabla f|^2 d\mu_n.$$

Here, we improve a result of Bobkov and Ledoux [13]: using a rotation-symmetric transportation of the Gaussian measure onto μ_n, they got the constant $\Gamma(1 + n/2)^{-n/2} \sim_{n \to \infty} 2e/n$. Our constant is asymptotically sharp when n goes to infinity (notice that the previous log-Sobolev inequality implies the Gaussian sharp log-Sobolev inequality).

3 Unimodality and sections of product measures

In [29], Vaaler proved that the volume of the sections of the cube $[-1/2, 1/2]^n$ by k-dimensional subspaces through the origin is always bigger than 1.

Peaked order and unimodal measures [19] were the main ingredients of his proof. His method was pushed forward by several authors: Meyer and Pajor [25] proved that for any k-dimensional subspace $K \subset \mathbb{R}^n$, the function

$$s_K(p) := \frac{|K \cap B_p^n|}{|B_p^k|}$$

is non-decreasing for $p \geq 1$. They actually derived a more general statement for ℓ_p sums of Euclidean spaces. Next Caetano [17] established $s_K(p) \leq s_K(1)$ for $p \in (0,1)$. In [3], we showed that s_K is non-decreasing on $(0, +\infty]$. Our aim here is to extend these results to ℓ_p-sums of arbitrary finite dimensional spaces and to apply the peaked order method to the study of the isoperimetric and the shift problem in the case of half-spaces. This partial approach nevertheless enables to deal with non log-concave product measures.

3.1 Some preliminaries

Our definitions slightly differ from [19]. They lead to less technical proofs; for details we refer to [3].

Let $\mathcal{C}_n$ be the set of all bounded origin-symmetric convex Borell subsets of $\mathbb{R}^n$. A function f on $\mathbb{R}^n$ is said to be unimodal if it is the increasing limit of a sequence of functions of the form:

$$\sum_{j=1}^{J} a_j \mathbf{1}_{C_j}$$

where $J \in \mathbb{N}$, $a_j \geq 0$ and $C_j \in \mathcal{C}_n$. One easily checks that even non-negative quasi-concave functions, and a fortiori even log-concave functions are unimodal. On the real line, a function is unimodal if and only if it is even and non-increasing on $\mathbb{R}_+$.

One says that a Radon measure on $\mathbb{R}^n$ is unimodal if it is absolutely continuous with respect to Lebesgue's measure and admits some unimodal density. When μ and ν are unimodal measures, so is the product measure $\mu \otimes \nu$; this is due to the fact that when $C \in \mathcal{C}_n$ and $D \in \mathcal{C}_m$, one has $C \times D \in \mathcal{C}_{n+m}$ and $\mathbf{1}_C(x)\mathbf{1}_D(y) = \mathbf{1}_{C \times D}(x,y)$.

Let μ, ν be Radon measures on $\mathbb{R}^n$. One says that μ is more peaked than ν and writes $\mu \succ \nu$ when $\mu(C) \geq \nu(C)$ holds for every $C \in \mathcal{C}_n$. It is remarkable that the inequalities for $\succ$ can be tensorised as soon as they involve unimodal measures:

Theorem 16 (Kanter) *For $1 \leq i \leq k$, let μ_i and ν_i be unimodal measures on $\mathbb{R}^{n_i}$ such that $\mu_i \succ \nu_i$. Then, the following inequality between measures on $R^{n_1 + \cdots + n_k}$ holds:* $\mu_1 \otimes \cdots \otimes \mu_k \succ \nu_1 \otimes \cdots \otimes \nu_k$.

3.2 Sections of product measures and of unit balls

Lemma 17 *Let ϕ_1, ϕ_2, f be continuous functions from $\mathbb{R}^+$ to $\mathbb{R}^+$. Assume that f vanishes at most at zero and that ϕ_1/ϕ_2 is non-decreasing. If*

$$\int_0^\infty f\, e^{-\phi_1} \geq \int_0^\infty f\, e^{-\phi_2}\,,$$

then for all $a \geq 0$

$$\int_0^a f\, e^{-\phi_1} \geq \int_0^a f\, e^{-\phi_2}\,.$$

This is obvious by differentiation. Notice that the statement can be extended to the case when ϕ_1 and ϕ_2 have values in $[0, \infty]$.

Proposition 18 *Let $\varphi : \mathbb{R}_+ \rightarrow [0, \infty]$, be non-decreasing. Assume that $\varphi(0) = 0$ and $\int \exp[-\varphi(|t|)]dt = 1$. Let $E \subset \mathbb{R}^n$ be a k-dimensional subspace.*
  i) If $\varphi(t)/t^2$ is non-increasing, then

$$\int_E \prod_{i=1}^n e^{-\varphi(|x_i|)} d^k(x) \leq \int_{R^k} \prod_{i=1}^n e^{-\varphi(|x_i|)} d^k(x) = 1.$$

ii) If $\varphi(t)/t^2$ is non-decreasing, then

$$\int_E \prod_{i=1}^n e^{-\varphi(|x_i|)} d^k(x) \geq \int_{R^k} \prod_{i=1}^n e^{-\varphi(|x_i|)} d^k(x) = 1.$$

Proof: Assume the hypothesis of i). Lemma 17 implies that

$$d\nu(t) := \exp[-\varphi(|t|)]dt \prec \exp[-\pi t^2]dt.$$

By Theorem 16, the inequality holds for the nth powers of these unimodal measures. Let $(u_{k+1}, \ldots, u_n)$ be an orthonormal basis of E^-. For $\varepsilon > 0$, let

$$E(\varepsilon) = \{x \in \mathbb{R}^n;\ |\langle x, u_i \rangle| \leq \varepsilon/2,\ i = k+1, \ldots, n\}.$$

Then $\nu^{\otimes n}(E(\varepsilon)) \leq \gamma^{\otimes n}(E(\varepsilon)) = \gamma^{\otimes n}(R^k \times [-\varepsilon/2, \varepsilon/2]^{n-k})$, where we have used the definition of the peaked order for the sets $E(\varepsilon) \cap r B_2^n, r \rightarrow \infty$ and the rotationnal invariance of Gaussian measures. The conclusion follows from a standard limit argument. The proof of ii) is similar. $\qquad\square$

Let $C \subset \mathbb{R}^n$ be a symmetric convex body and let $\|\cdot\|_C$ be the corresponding norm on $\mathbb{R}^n$. For $p > 0$, we set

$$\alpha_{p,C} = \left[|C| \cdot \Gamma\left(1 + \frac{n}{p}\right)\right]^{\frac{1}{n}}.$$

Notice that n only depends on C. When $C = [-1, 1] \subset \mathbb{R}$, we simply write α_p. We are ready to state our extension of the results of Meyer-Pajor and Caetano.

18

Theorem 19 *Let $N, m, (n_i)_{i=1}^{m}$ be positive integers such that $\sum_{i=1}^{m} n_i = N$. For $i \leq m$, let C_i be a symmetric convex body in $\mathbb{R}^{n_i}$. Identifying $\mathbb{R}^N$ with $\mathbb{R}^{n_1} \times \cdots \times \mathbb{R}^{n_m}$, we write every $x \in \mathbb{R}^N$ as $x = (x_1, \ldots, x_m)$.*

For $0 < p \leq \infty$, let us consider the sets

$$\mathcal{B}_p = \left\{ x \in \mathbb{R}^N ; \sum_{i=1}^{m} \|\alpha_{i,p} x_i\|_{C_i}^{p} \leq 1 \right\} \text{ and } \mathcal{D}_p = \left\{ x \in \mathbb{R}^N ; \sum_{i=1}^{m} \|x_i\|_{C_i}^{p} \leq 1 \right\}.$$

Let E be a k-dimensional subspace of $\mathbb{R}^N$. Then the quantity

$$\Gamma(1 + k/p) |E \cap \mathcal{B}_p|$$

is a non-decreasing function of $p \in (0, +\infty]$. Under the additionnal condition $n_1 = \cdots = n_m = n$,

$$\left(\frac{|E \cap \mathbf{D}_p|}{|B_p^k|} \right)^{\frac{1}{k}} |B_p^n|^{\frac{1}{n}}$$

is a non-decreasing function of $p \in (0, +\infty]$.

An application of this result appears in [26]. See also [25] for applications to Siegel-type lemmas. We start with some preliminary statements. Following Meyer and Pajor, we define the measure $\mu_{p,C}$ on $\mathbb{R}^n$ by

$$d\mu_{p,C}(x) = \exp\left(-\|\alpha_{p,C}\, x\|_C^p \right) d^n x.$$

It is a probability measure. Since the level sets of its density are convex and symmetric, it is unimodal.

Proposition 20 *Let C be a symmetric convex body in $\mathbb{R}^n$. If $p > q > 0$ then $\mu_{p,C} \succ \mu_{q,C}$.*

Proof: If $n = 1$, the statement follows from Lemma 17 applied to $f = 1, \psi_1(t) = t^p, \psi_2(t) = t^q$. Assume now $n \geq 2$. It is enough to consider sets C with C^∞ norm on $\mathbb{R}^n \setminus \{0\}$. In this case the boundary ∂C of C is a submanifold. For $\omega \in \partial C$, let $n(\omega)$ be the outer normal of C at ω and let $d\sigma$ be the surface measure on ∂C. We will use the diffeomorphism Θ from $\mathbb{R}_+^* \times \partial C$ onto $\mathbb{R}^n \setminus \{0\}$ which maps (r, ω) to the vector $r\omega$.

Since we work with absolutely continuous measures, it is enough to compare their values on symmetric convex bodies. Let $K \subset \mathbb{R}^n$ be such a set. One has

$$
\begin{aligned}
\mu_p(K) &= \int_{\mathbb{R}^n} \mathbf{1}_{\{\|x\|_K \leq 1\}} e^{-\|\alpha_{p,C}\, x\|_C^p} \, dx \\
&= \iint_{\mathbb{R}_+^n \times \partial C} \mathbf{1}_{\{\|r\omega\|_K \leq 1\}}\, e^{-\|\alpha_{p,C}\, r\omega\|_C^p} \langle \omega, n(\omega) \rangle\, r^{n-1} \, dr\, d\sigma(\omega) \\
&= \int_{\partial C} \langle \omega, n(\omega) \rangle \left(\int_{r=0}^{1/\|\omega\|_K} e^{-\alpha_{p,C}^p r^p} r^{n-1} \, dr \right) d\sigma(\omega).
\end{aligned}
$$

Taking $K = \mathbb{R}^n$ in this formula shows that

$$\int_0^\infty e^{-\alpha_p^p r^p} r^{n-1}\, dr$$

does not depend on p. For each ω, we apply Lemma 17 with $f(r) = r^{n-1}$, $\phi_1(r) = r^p$ and $\phi_2(r) = r^q$; the hypothesis $p > q$ ensures that ϕ_1/ϕ_2 is non-decreasing. Since $\langle \omega, n(\omega) \rangle$ is always non-negative, one gets $\mu_p(K) \geq \mu_q(K)$.
$\square$

Lemma 21 *Given E a subspace of $\mathbb{R}^n$ of dimension k and $(u_{k+1}, \ldots u_n)$ an orthonormal basis of E^-, we consider*

$$E(\varepsilon) = \{x \in \mathbb{R}^n;\ |\langle x, u_i \rangle| \leq \varepsilon/2,\ i = k+1, \ldots, n\}.$$

Let $N : \mathbb{R}^n \to \mathbb{R}_+$ be a continuous homogeneous fonction, vanishing only at the origin. Then the set $B = \{x;\ N(x) \leq 1\}$ is a symmetric star-shaped body and for $p > 0$, one has

$$\Gamma\left(1 + \frac{k}{p}\right)|E \cap B|_k = \int_K e^{-N(x)^p} d^k x = \lim_{\varepsilon \to 0^+} \varepsilon^{k-n} \int_{E(\varepsilon)} e^{-N(y)^p} d^n y.$$

The first equality is obvious by level-sets integration. The second one follows from dominated convergence (notice that there exists $d > 0$ such that $d|x| \leq N(x) \leq |x|/d$ for all $x \in \mathbb{R}^n$.)

Proof of Theorem 19: Let $p > q > 0$. By Lemma 21 and with the same notation, the following relation holds for $r > 0$:

$$\Gamma(1 + k/r) \cdot |E \cap \mathcal{B}_r| = \lim_{\eta \to 0} \eta^{k-N} \mu_{r,C_1} \otimes \cdots \otimes \mu_{r,C_m}\left(E(\eta)\right).$$

The previous proposition and Theorem 16 yield

$$\mu_{p,C_1} \otimes \cdots \otimes \mu_{p,C_m} \succ \mu_{q,C_1} \otimes \cdots \otimes \mu_{q,C_m}.$$

Since $E(\eta)$ is convex and symmetric, the latter relation implies that

$$\Gamma(1 + k/p) \cdot |E \cap \mathcal{B}_p| \geq \Gamma(1 + k/q) \cdot |E \cap \mathcal{B}_q|.$$

When $n_1 = \cdots = n_m = n$,

$$
\begin{aligned}
\mathcal{B}_p &= \left\{ x \in \mathbb{R}^N;\ \sum_{i=1}^m \left\| \Gamma\left(1 + \frac{n}{p}\right)^{\frac{1}{n}} |C_i|^{\frac{1}{n}} x_i \right\|_{C_i}^p \leq 1 \right\} \\
&= \Gamma\left(1 + \frac{n}{p}\right)^{-\frac{1}{n}} \left\{ x \in \mathbb{R}^N;\ \sum_{i=1}^m \left\| |C_i|^{\frac{1}{n}} x_i \right\|_{C_i}^p \leq 1 \right\}.
\end{aligned}
$$

The linear mapping T defined on $\mathbb{R}^N$ by $T(x) = \left(|C_1|^{\frac{1}{n}} x_1, \ldots, |C_m|^{\frac{1}{n}} x_m \right)$ is bijective. Thus

$$\mathcal{B}_p \cap E = \Gamma \left(1 + \frac{n}{p} \right)^{-\frac{1}{n}} T^{-1}(\mathbf{D}_p \cap TE)$$

Notice that T multiplies volumes by $|\det(T)|$ and that TE can be an arbitrary k-subspace. Hence for any F of dimension k

$$p \to \frac{\Gamma(1 + k/p)}{\Gamma(1 + n/p)^{\frac{k}{n}}} |F \cap \mathbf{D}_p|$$

is non-decreasing on $(0, +\infty]$. $\qquad\square$

3.3 Remarks on the Brascamp-Lieb inequalities

We are going to expose an alternative proof of the first statement in Proposition 18. It uses an inequality due to Brascamp and Lieb [15] (see also [2], [5]). Let E be a k-dimensional subspaces of $\mathbb{R}^n$ and let P be the orthogonal projection onto E. Then $\sum_{i=1}^n Pe_i \otimes Pe_i = P$ is the identity when restricted to E. Set $c_i = |Pe_i|^2$ and $u_i = Pe_i/|Pe_i|$. As linear mappings of E, $\sum_{i=1}^n c_i u_i \otimes u_i = Id_E$. The Brascamp-Lieb inequality yields

$$\int_E \prod_{i=1}^n e^{-\psi(\langle x, e_i \rangle)} dx = \int_E \prod_{i=1}^n \left(e^{-\psi(\sqrt{c_i} \langle x, u_i \rangle)/c_i} \right)^{c_i} dx \le \prod_{i=1}^n \left(\int_{\mathbb{R}} e^{-\psi(\sqrt{c_i} t)/c_i} dt \right)^{c_i}.$$

Assume that ψ is even and $\psi(t)/t^2$ is non-increasing on R_+. Since $c_i \in (0, 1]$, one has $\psi(\sqrt{c_i} t)/c_i \ge \psi(t)$ for all t. Hence the previous integral is smaller than

$$\left(\int_{\mathbb{R}} e^{-\psi(t)} dt \right)^{\sum_{i=1}^n c_i} = \int_{\mathbb{R}^k} \prod_{i=1}^k e^{-\psi(x_i)} dx,$$

where we have used $\sum c_i = k$. Thus among k-dimensional subspaces, the canonical subspaces are extremal.

If one takes $\psi(t) = |\alpha_p t|^p$, one can use homogeneity to improve on the latter argument and extend one of Ball's volume estimates on sections of the unit cube [2]:

$$\int_E e^{-\alpha_p^p \|x\|_p^p} dx \le \prod_{i=1}^n \left(\int_{\mathbb{R}} e^{-|\alpha_p c_i^{1/2 \perp 1/p} t|^p} dt \right)^{c_i} = \left(\prod_{i=1}^n c_i^{c_i} \right)^{1/p - 1/2}.$$

Since $c_i \in (0, 1]$ and $\sum_{i=1}^n c_i = k$, one has $(k/n)^k \le \prod c_i^{c_i} \le 1$. If $p \le 2$, the integral is bounded by one as before. If $p \ge 2$, Lemma 21 and the previous estimate give

$$\frac{|B_p^n \cap E|}{|B_p^k|} \le \left(\frac{n}{k} \right)^{k(1/2 - 1/p)}.$$

21

One can check that this is optimal when k divides n. In this case, let $d = n/k$ and for $j = 1, \ldots, k$, let $v_j = e_{1+j(d-1)} + \cdots + e_{jd}$. Then $\text{span}\{v_1, \ldots, v_k\} \cap B_p^n$ is isometric to $(n/k)^{1/2-1/p} B_p^k$.

The second statement in Proposition 18 is a reverse form of the first statement. One can wonder whether it is provable via the reverse form of the Brascamp-Lieb inequality ([4], [5]). The answer seems to be negative: the duality between the Brascamp-Lieb inequality and its converse corresponds to duality of convex sets. It turns sections into projections. Since projections are larger than sections, this provides weaker results. Let us give an example with $\psi(t) = \exp(-|\alpha_p t|^p)$: By Lemma 21, and the reverse Brascamp-Lieb inequality, one can estimate from below the volume of the orthogonal projection of B_p^n onto a k-dimensional subspace E. With the previous notation

$$
\begin{aligned}
\frac{|P_E(B_p^n)|}{|B_p^k|} &= \int_E e^{-\alpha_p^p \inf\{\sum_{i=1}^n |\lambda_i|^p \, ; \, x = P(\sum_{i=1}^n \lambda_i e_i)\}} \, dx \\
&= \int_E \sup_{\sum_{i=1}^n c_i \theta_i u_i = x} \prod_{i=1}^n \left(e^{-|\alpha_p \theta_i c_i^{1/2 \perp 1/p}|^p} \right)^{c_i} dx \\
&\geq \prod_{i=1}^n \left(\int_{\mathbb{R}} e^{-|c_i^{1/2 \perp 1/p} \alpha_p t|^p} dt \right)^{c_i} = \left(\prod_{i=1}^n c_i^{c_i} \right)^{1/p - 1/2}.
\end{aligned}
$$

If $p \geq 2$, this is bigger than 1. This result was implied by the one on sections, because $E \cap B_p^n \subset P_E(B_p^n)$. If $0 < p \leq 2$, we get

$$
\frac{|P_E(B_p^n)|}{|B_p^k|} \geq \left(\frac{k}{n} \right)^{k(1/p - 1/2)}.
$$

By duality, this is optimal when k divides n and $p \geq 1$. The equality is achieved for the same subspace as for sections.

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