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REDUCTION OF SASAKIAN MANIFOLDS

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Abstract. We show that the contact reduction can be specialized to Sasakian manifolds. We link this Sasakian reduction to Kähler reduction by considering the Kähler cone over a Sasakian manifold. We present examples of Sasakian manifolds obtained by S^1 reduction of standard Sasakian spheres.

1. Introduction

Reduction technique was naturally extended from symplectic to contact structures by H. Geiges in [7]. Even earlier, Ch. Boyer, K. Galicki and B. Mann defined in [3] a moment map for 3-Sasakian manifolds, thus extending the reduction procedure for nested metric contact structures. Quite surprisingly, a reduction scheme for Sasakian manifolds (contact manifolds endowed with a compatible Riemannian metric satisfying a curvature condition), was still missing.

In this note we fill the gap by defining a Sasakian moment map and constructing the associated reduced space. We then relate Sasakian reduction to Kähler reduction *via* the Kähler cone over a Sasakian manifold.

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2. Sasakian manifolds

Let us briefly recall the notion of a Sasakian manifold. The definition we give is not the standard one but is suited for our purpose. For more details, we refer to [2] and [4] .

Definition 2.1. A *Sasakian* manifold is a $(2n+1)$-dimensional Riemannian manifold (N, g) endowed with a unitary Killing vector field ξ such that the curvature tensor of g satisfies the equation:

$$R(X,\xi)Y = \eta(Y)X - g(X,Y)\xi \tag{2.1}$$

where η is the metric dual 1-form of ξ: $\eta(X) = g(\xi, X)$.

Let $\varphi = \nabla\xi$, whwre ∇ is the Levi-Civita connection of g. The following formulae are then easily deduced:

$$\varphi\xi = 0, \quad g(\varphi Y, \varphi Z) = g(Y,Z) - \eta(Y)\eta(Z). \tag{2.2}$$

It can be seen that η is a contact form on N, whose Reeb field is ξ (it is also called the characteristic vector field). Moreover, the restriction of φ to the contact distribution $\eta = 0$ is a complex structure.

The simplest example is the standard sphere $S^{2n+1} \subset \mathbb{C}^{n+1}$, with the metric induced by the flat one of $\mathbb{C}^{n+1}$. The characteristic Killing vector field is $\xi_p = -i\overrightarrow{p}$, i being the imaginary unit. Other Sasakian structures on the sphere can be obtained by D-homothetic transformations (cf. [9]). Also, the unit sphere bundle of any space form is Sasakian.

More generally, the quantization bundle of a compact Kähler manifold naturally carries a Sasakian structure. The converse construction, possible when the characteristic field is regular, is known as the Boothby-Wang fibration. Precisely, the following result (the metric part is due to Morimoto and Hatakeyama) is available (cf. [11] or [4]):

Theorem 2.1. *Let (P, h) be a Hodge manifold. There exists a principal circle bundle $\pi : N \to P$ and a connection form η in it, with curvature form the pull-back of the Kähler form of P, which is a contact form on S. Let ξ be the vector field dual to η with respect to the metric $g = \pi^*h + \eta \otimes \eta$. Then (N, g, ξ) is Sasakian.*

The following equivalent definition puts Sasakian geometry in the framework of holonomy groups. Let $C(N) = N \times \mathbb{R}_+$ be the cone over (N, g). Endow it with the warped-product cone metric $C(g) = r^2g + dr^2$. Let $R_0 = r\partial r$ and define on $C(N)$ the complex structure J acting like this (with obvious identifications): $JY = \varphi Y - \eta(Y)R_0$, $JR_0 = \xi$. We have:

Theorem 2.2. [4] *(N, g, ξ) is Sasakian if and only if the cone over N $(C(N), C(g), J)$ is Kählerian.*

3. Main results

Theorem 3.1. *Let (N, g, ξ) be a compact $2n+1$ dimensional Sasakian manifold and G a compact d-dimensional Lie group acting on N by contact isometries. Suppose $0 \in \mathfrak{g}^*$ is a regular value of the associated moment map μ. Then the reduced space $M = N//G := \mu^{-1}(0)/G$ is a Sasakian manifold of dimension $2(n-d)+1$.*

Proof. By [7], the contact moment map $\mu : N \to \mathfrak{g}^*$ is defined by

$$< \mu(x), \underline{X} >= \eta(X)$$

for any $\underline{X} \in \mathfrak{g}$ and X the corresponding field on N. We know that the reduced space is a contact manifold, *loc. cit.* Hence we only need to check that (1) the Riemannian metric is projected on M and (2) the field ξ projects to a unitary Killing field on M such that the curvature tensor of the projected metric satisfies formula (2.1).

To this end, we first describe the metric geometry of the Riemannian submanifold $\mu^{-1}(0)$.

Let $\{\underline{X}_1, ..., \underline{X}_d\}$ be a basis of $\mathfrak{g}$ and let $\{X_1, ..., X_d\}$ be the corresponding vector fields on N. Since 0 is a regular value of μ, $\{X_{i_x}\}$ is a linearly independent system in each $T_x\mu^{-1}(0)$. From the very definition of the moment map we have $\eta_p(X_i) = \mu(p)(X_i) = 0$ hence $X_i - \xi$. As G acts by contact isometries, we have

$$\mathcal{L}_{X_i} g = 0, \quad \mathcal{L}_{X_i}\eta = 0 \quad i = 1, ..., d. \tag{3.1}$$

Note that these also imply $[X_i, \xi] = \mathcal{L}_{X_i}\xi = 0$.

Observe that $\mu^{-1}(0)$ is an isometrically immersed submanifold of N (we denote the induced metric also with g) whose tangent space in each point is described by: $Y \in T_x\mu^{-1}(0)$ if and only if $d\mu_x(Y) = 0$. Hence, by the definition of the moment map, the vector fields ξ and X_i are tangent to $\mu^{-1}(0)$. Moreover, for any Y tangent to $\mu^{-1}(0)$, one has $g(\varphi X_i, Y) = d\eta(Y, X_i) = d\mu(Y) = 0$, hence the vector fields $\{X_i\}$ produce a local basis (not necessarily orthogonal) of the normal bundle of $\mu^{-1}(0)$. The shape operators $A_i := A_{\varphi X_i}$ of this submanifold in N are computed as follows (we let ∇, ∇^N be the Levi Civita covariant derivatives of $\mu^{-1}(0)$, resp. N):

(3.2)
$$\begin{aligned} g(A_iY,Z) &= -g(\nabla^N_Y(\|X_i\|^{-1}\varphi X_i),Z) = \\ &= -g(Y(\|X_i\|^{-1})\varphi X_i,Z) - g(\|X_i\|^{-1}\nabla^N_Y(\varphi X_i),Z) = \\ &= -\|X_i\|^{-1}g(\nabla^N_Y(\varphi X_i),Z) = \\ &= -\|X_i\|^{-1}g(\nabla^N_Y(\varphi)X_i + \varphi\nabla^N_Y X_i,Z) = \\ &= -\|X_i\|^{-1}g(\eta(X_i)Y - g(X_i,Y)\xi + \varphi\nabla^N_Y X_i,Z) = \\ &= \|X_i\|^{-1}\{g(X_i,Y)\eta(Z) - g(\varphi\nabla^N_Y X_i,Z\}. \end{aligned}$$

In particular, for the corresponding quadratic second funadamental forms we get:

$$h_i(Y,\xi) = \|X_i\|^{-1}g(X_i,Y), \quad h_i(\xi,\xi) = 0. \tag{3.3}$$

Consequently, one easily obtains: *the restriction of the vector field ξ is Killing on $\mu^{-1}(0)$ too.*

Using the Gauss equation of a submanifold

$$\begin{aligned} R^N(X,Y,Z,W) = R^{\mu^{-1}(0)}(X,Y,Z,W) \\ + g(h(X,Z),h(Y,W)) - g(h(X,W),h(Y,Z)) \end{aligned}$$

and the formula (3.2) we now compute the needed part of the curvature tensor of $\mu^{-1}(0)$ at a fixed point $p \in \mu^{-1}(0)$. We take X, Y, Z orthogonal to ξ_p and obtain:

(3.4)
$$\begin{aligned} &g(R^{\mu^{-1}(0)}(X,\xi)Y,Z) - g(R^N(X,\xi)Y,Z) = \\ &\quad = -\sum_{i=1}^d \|X_i\|^{-2}\{h_i(X,Y)h_i(\xi,Z) - h_i(X,Z)h_i(\xi,Y)\} \\ &\quad = -\sum_{i=1}^d \|X_i\|^{-2}\left\{g(X_i,Z)g(\nabla^N_X X_i,\varphi Y) - g(X_i,Y)g(\nabla^N_X X_i,\varphi Z)\right\} \end{aligned}$$

(Note that $\nu_i = \|X_i\|^{-1}\varphi X_{ip}$ are chosen to be orthonormal in p; this is always possible pointwise by appropriate choice of the initial $\underline{X}_i$).

Let now $\pi : \mu^{-1}(0) \to M$ and endow M with the projection g^M of the metric g such that π becomes a Riemannian submersion. This is possible because G acts by isometries. In this setting, the vector fields X_i span the vertical distribution of the submersion, whilst ξ is horizontal and projectable (because $\mathcal{L}_{X_i}\xi = 0$). Denote with ζ its projection on M. ζ is obviously unitary. To prove that ζ is Killing on M, we just observe that $\mathcal{L}_\zeta g(Y,Z) = \mathcal{L}_\xi g(Y^h,Z^h)$, where Y^h denotes the horizontal lift of of Y. Finally, to compute the values $R^M(X,\zeta)Y$ of the curvature tensor of g^M, we use O'Neill formula (cf. [1], (9.28f))

$$g^M(R^M(X,\zeta)Y,Z) = g(R^{\mu^{-1}(0)}(X^h,\xi)Y^h,Z^h) + g(A(X^h,\xi),A(Y^h,Z^h)) - g(A(\xi,Y^h),A(X^h,Z^h)) + g(A(X^h,Z^h),A(\xi,Z^h))$$

where X,Y,Z are unitary, normal to ζ and the O'Neill $(1,2)$ tensor A is defined as: $A(Z^h,X^h) =$ *vertical part of* $\nabla_{Z^h}X^h$. Using Gauss formula and (3.3), we obtain

$$g(\nabla_{Z^h}\xi,X_i) = g(\varphi Z^h,X_i) = -g(Z^h,\varphi X_i) = 0$$

hence $\nabla_{Z^h}\xi$ has no vertical part and $A(Z^h,\xi)=0$. Thus

$$R^M(X,\zeta)Y = R^{\mu^{-1}(0)}(X^h,\xi)Y^h = R^N(X^h,\xi)Y^h$$

because of (3.4) and the fact that X^h,Y^h are normal to all X_i. Hence

$$R^M(X,\zeta)Y = g(\xi,Y^h)X^h - g(X^h,y^h)\xi = g^M(\zeta,Y)X - g^M(X,Y)\zeta$$

which proves that (M,g^M,ζ) is a Sasakian manifold. □

In the following we relate Sasakian reduction to Kähler reduction by using the cone construction. Roughly speaking, we prove that reduction and taking the cone are commuting operations.

Let $\omega = dr^2\wedge\eta + r^2 d\eta$ be the Kähler form of the cone $C(N)$ over a Sasakian manifold (N,g,ξ). If ρ_t are the translations acting on $C(N)$ by $(x,r)\mapsto(x,tr)$, then the vector field $R_0 = r\partial r$ is the one generated by $\{\rho_t\}$. Moreover, the following two relations are useful:

$$\mathcal{L}_{R_0}\omega = \omega, \quad \rho_t^*\omega = t\omega. \tag{3.5}$$

Suppose a compact Lie group G acts on $C(N)$ by holomorphic isometries, commuting with ρ_t. This ensures a corresponding action of G on N. In fact, we can consider $G\cong G\times\{Id\}$ acting as $(g,(x,r))\times(gx,r)$.

Suppose that a moment map $\Phi : C(N)\to\mathfrak{g}$ exists.

As above, let $\{\underline{X}_1,...,\underline{X}_d\}$ be a basis of $\mathfrak{g}$ and let $\{X_1,...,X_d\}$ be the corresponding vector fields on $C(N)$. We see that X_i are independent on r, hence can be considered as vector fields on N. Furthermore, the commutation of G with ρ_t implies

$$\Phi(\rho_t(p)) = t\Phi(p). \tag{3.6}$$

Now imbed N in the cone as $N\times\{1\}$ and let $\mu := \Phi|_{N\times\{1\}}$. This is the moment map of the action of G on N. To see this, recall the definition of the symplectic moment map $\Phi = (\Phi_1,...,\Phi_d)$: Φ_i is given up to constant by $d\Phi_i(Y) = \omega(X_i,Y)$. Here we uniquely determine Φ_i by imposing the condition $\eta(X_i) = \Phi|_{N\times\{1\}}$. This immediately implies that the Reeb field of N is orthogonal to the vector fields X_i since $g(\xi, X_i) = \eta(X_i) = 0$. As G acts by isometries on $C(N)$, we may project the cone metric to a metric on $N'//G \times \mathbb{R}_+$ which we denote by g_0. Then $g_0(Y, Z) = C(g)(Y^h, Z^h)$, where Y^h, Z^h are the unique vector fields on $\Phi^{-1}(0)$ orthogonal to all of X_i which project on Y, Z (we call them horizontal).

Let $P = \Phi^{-1}(0)/G$ be the reduced Kähler manifold. The key remark is that because of (3.6), $\Phi^{-1}(0)$ is the cone $N' \times \mathbb{R}_+$ over $N' = \{x \in N \; ; \; (x, 1) \in \Phi^{-1}(0)\}$. Moreover, since the actions of G and ρ_t commute, one has an induced action of G on N'. Then

$$\Phi^{-1}(0)/G \cong (N' \times \mathbb{R}_+)/G \cong N'/G \times \mathbb{R}_+$$

The manifold $N'//G \times \mathbb{R}_+$ is Kähler, as reduction of a Kähler manifold, but we still have to check that this Kähler structure is a cone one. For the more general, symplectic case, this was done in [5]. Let g_0 be the reduced Kähler metric and g' be the Sasakian reduced metric on $N'//G$. It is easily seen that the lift of g_0 to $\Phi^{-1}(0)$ coincides with the lift of the cone metric $r^2 g' + dr^2$ on horizontal fields. This implies that the cone metric coincides with g_0.

Summing up we have proved:

Theorem 3.2. *Let (N, g, ξ) be a Sasakian manifold and let $(C(N), C(g), J)$ be the Kähler cone over it. Let a compact Lie group G act by holomorphic isometries on $C(N)$ and commuting with the action of the 1-parameter group generated by the field R_0. If a moment map with regular value 0 exists for this action, then a moment map with regular value 0 exists also for the induced action of G on N. Moreover, the reduced space $C(N)//G$ is the Kähler cone over the reduced Sasakian manifold $N//G$.*

The advantage of defining the Sasakian reduction *via* Kähler reduction, as done in [3] for 3-Sasakian manifolds, is the avoiding of curvature computations.

4. Examples: S^1 actions on Sasakian spheres

Example 4.1. Start with $S^7 \subset \mathbb{C}^4$ with its standard Sasakian structure. Let the complex coordinates of $\mathbb{C}^4$ be $(z_0, ..., z_3)$, with $z_j = x_j + iy_j$. The contact form on S^7 can then be written

$$\eta = \sum_{j=0}^{3} (x_j dy_j - y_j dx_j)$$

and its Reeb field is

$$\xi = \sum_{j=0}^{3}(x_j\partial y_j - y_j\partial x_j).$$

Let S^1 act on S^7 by $e^{it} \mapsto (e^{-it}z_0, e^{-it}z_1, e^{it}z_2, e^{it}z_3)$. The associated field of this action is (in real coordinates)

$$X_0 = -(x_0\partial y_0 - y_0\partial x_0) - (x_1\partial y_1 - y_1\partial x_1) + \\ + (x_2\partial y_2 - y_2\partial x_2) + (x_3\partial y_3 - y_3\partial x_3).$$

The moment map $\mu : S^7 \to \mathbb{R}$ reads:

$$\mu(z) = \eta_z(X_0) = -|z_0|^2 - |z_1|^2 + |z_2|^2 + |z_3|^2$$

with zero level set

$$\{z \in S^7\ ;\ |z_0|^2 + |z_1|^2 = |z_2|^2 + |z_3|^2\} = S^3(\frac{1}{\sqrt{2}}) \times S^3(\frac{1}{\sqrt{2}}).$$

Clearly μ is nondegenerate on $\mu^{-1}(0)$.

The reduced space can be identified with $S^3 \times S^3/S^1$ which, by [10], is diffeomorphic with $S^2 \times S^3$. (In this case, one can also avoid the topological arguments in [10] and identify the reduced space by observing that the following diffeomorphism of $S^3 \times S^3$: $(z_0, z_1, z_2, z_3) \mapsto (z_1z_4 + \overline{z_2z_3}, z_1z_3 - \overline{z_2z_4}, z_3, z_4)$ is equivariant with respect to the previous S^1 action which restricted to the second factor of the product is the usual action inducing the Hopf fibration; *mille grazie* to Rosa Gini and Maurizio Parton for letting us know it, [6]).

The reduced Sasakian structure obtained in this way on $S^2 \times S^3$ is easily checked to be Einstein and to project on the Kähler Einstein metric of $\mathbb{C}P^1 \times \mathbb{C}P^1$ making the fibre map be a Riemannian submersion. As by [10] such an Einstein metric is unique, our reduced Sasakian structure coincides with the Sasakian structure found in [8] viewing $S^2 \times S^3$ as minimal submanifold of S^7, total space of the pull-back over $\mathbb{C}P^1 \times \mathbb{C}P^1$ of the Hopf bundle $S^7 \to \mathbb{C}P^3$. The same Einstein-Sasakian metric on $S^2 \times S^3$ also appears in [9], constructed by a different approach.

Example 4.2. Consider again S^7 as starting Sasakian manifold, but let S^1 act by: $e^{it} \mapsto (e^{-kit}z_0, e^{it}z_1, e^{it}z_2, e^{it}z_3)$, $k \in \mathbb{Z}_+$. Now $\mu^{-1}(0) \cong S^1(\sqrt{\frac{k}{k+1}}) \times S^5(\sqrt{\frac{1}{k+1}})$. In order to identify the reduced space, consider the $k : 1$ mapping

$$S^1 \times S^5 \ni (z_0, z_1, z_2, z_3) \mapsto ((z_0)^{-k}, z_1, z_2, z_3) \in S^1 \times S^5.$$

It induces a $k:1$ map from $M = S^1 \times S^5/S^1$, where S^1 acts diagonally, to the reduced space $\mu^{-1}(0)/S^1$ with the action given above. As in [6], the map

$$(z_0, ..., z_3) \mapsto (z_0, \overline{z_0}z_1, \overline{z_0}z_2, \overline{z_0}z_3)$$

is an equivariant diffeomorphism of $S^1 \times S^5$, equivariant with respect to the diagonal action of S^1 and the action of s^1 on the first factor. Hence M is diffeomorphic to S^5 and the reduced Sasakian space is $S^5/\mathbb{Z}_k$.

Example 4.3. In general, consider the weighted action of S^1 on $S^{2n-1} \subset \mathbb{C}^n$ by:

$$(e^{it}, (z_0, ..., z_{n-1})) \mapsto (e^{\lambda_0 it}z_0, ..., e^{\lambda_n it}z_{n-1})$$

where $(\lambda_0, ..., \lambda_{n-1}) \in \mathbb{Z}^n$. The associated moment map

$$\mu(z) = \lambda_0|z_0|^2 + ... + \lambda_n|z_{n-1}|^2$$

is regular on $\mu^{-1}(0)$ for any $(\lambda_0, ..., \lambda_{n-1})$ such that $\lambda_0 ... \lambda_{n-1} \neq 0$, $(\lambda_0, ..., \lambda_{n-1}) = 1$ and at least two λ's have different signs (compare with the 3-Sasakian case where the weights obey to more restrictions, cf. [3]).

Now take $\lambda_0 = ... = \lambda_k = a$ and $\lambda_{k+1} = ... = \lambda_{n-1} = -b$, $a, b \in \mathbb{Z}_+$ relatively prime. Then $\mu^{-1}(0) \cong S^{2k+1}(\sqrt{\frac{a}{a+b}}) \times S^{2(n-k)-1}(\sqrt{\frac{b}{a+b}})$. Note that the induced metric on $\mu^{-1}(0)$ coincides with the product metric of the standard metrics of the two factors. We then see that the reduced space is diffeomorphic with an S^1 factor of the above product of spheres given by the following action:

$$(e^{it}, (x, y)) \mapsto (e^{iat}x, e^{-ibt}y).$$

One can now adapt the arguments of [10], Cor. 2.2 and prove that the reduced spaces are S^1 bundles over $\mathbb{C}P^k \times \mathbb{C}P^{n-k-1}$ and, for $1 \leq k$, $4 < n$, they are not homeomorphic to each other in general.

However, for $k = 1$, $n = 4$, the reduced space is always diffeomorphic with $S^2 \times S^3$. Hence, one obtains an infinite family of Sasakian structures on $S^2 \times S^3$.

Note also that if n is *even*, choosing like in the first example, the first half of the λ's to be -1, the rest of them 1, the reduced Sasakian metric is *Einstein*, again acoording to [10].

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