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Vienna, Preprint ESI 727 (1999) July 5, 1999

Supported by Federal Ministry of Science and Transport, Austria
Available via http://www.esi.ac.at

A SPECIAL CASE OF DE BRANGES' THEOREM ON MONODROMY MATRIX: ASSOCIATED RIEMANN SURFACE IS OF WIDOM TYPE WITH DIRECT CAUCHY THEOREM

Peter Yuditskii

Let $\mathcal{H}_0(t)$, $\mathcal{H}_1(t)$ be real 2×2 matrix–functions with entries from $L^1(0,1)$, $\mathcal{H}_1^*(t)=\mathcal{H}_1(t)$, $\mathcal{H}_0(t)\ge 0$. We associate with these data the solution of the Cauchy problem for the differential system

$$\frac{d\mathcal{A}(t,z)}{dt}=\mathcal{A}(t,z)\{z\mathcal{H}_0(t)+\mathcal{H}_1(t)\}J,\quad \mathcal{A}(0,z)=1_2,$$

where

$$J=\begin{bmatrix}0 & 1\\ -1 & 0\end{bmatrix}.$$

The matrix–function $\mathcal{A}(z)=\mathcal{A}(1,z)$ is called the monodromy matrix of the corresponding system [5]. More generally, let $\Sigma_0(t)$ be a continuous nondecreasing real 2×2 matrix–function of $t\in[0,1]$,

$$\operatorname{sp}\{\Sigma_0(1)-\Sigma_0(0)\}<\infty,$$

and $\Sigma_1(t)$ be a real symmetric 2×2 matrix–functions, whose entries are absolutely continuous functions with respect to the measure $\operatorname{sp}\{d\Sigma_0(t)\}$. In this case $\mathcal{A}(t,z)$ is defined as the solution of the matrix integral equation

$$\mathcal{A}(t,z)=1_2+\int_0^t\mathcal{A}(s,z)\{z\,d\Sigma_0(s)+d\Sigma_1(s)\}J, \tag{0.1}$$

and as before $\mathcal{A}(z)=\mathcal{A}(1,z)$.

How to restore the system on the monodromy matrix? When it could be done? Do we have a uniqueness theorem?

These problems were solved in the whole generality by L. de Branges [2]. His theorem states, that if $\mathcal{A}(z)$ is an entire 2×2 matrix–function, which satisfies the following properties:

$$\overline{\mathcal{A}(\bar z)}=\mathcal{A}(z) \tag{0.2.1}$$

$$\det\mathcal{A}(z)=1 \tag{0.2.2}$$

$$\frac{J-\mathcal{A}(z)J\mathcal{A}(z)^*}{z-\bar z}\ge 0, \tag{0.2.3}$$

This work was supported by the Austrian Founds zur Förderung der wissenschaftlichen Forschung, project–number P12985–TEC

then $\mathcal{A}(z)$ is the monodromy matrix of a system (0.1). Normalization $\Sigma_1(t) = 0$ defines $\Sigma_0(t)$ in unique way up to a continuous monotonic change of variable t, $[0,1] \to [0,1]$.

The main problem here is to prove that 1_2 and $\mathcal{A}(z)$ could be included in a monotonic continuous chain of entire matrix-functions $\mathcal{A}(\tau, z)$:

$$\frac{J}{z-\bar z} \ge \frac{\mathcal{A}(\tau,z)J\mathcal{A}(\tau,z)^*}{z-\bar z} \ge \frac{\mathcal{A}(z)J\mathcal{A}(z)^*}{z-\bar z}$$

and that this chain is complete i.e.: under some normalization any divisor $\mathcal{A}_1(z)$ of $\mathcal{A}(z)$,

$$\frac{J}{z-\bar z} \ge \frac{\mathcal{A}_1(z)J\mathcal{A}_1(z)^*}{z-\bar z} \ge \frac{\mathcal{A}(z)J\mathcal{A}(z)^*}{z-\bar z},$$

is present there ($\exists \tau_1 : \ \mathcal{A}_1(z) = \mathcal{A}(\tau_1, z)$).

Set

$$\mathcal{R} = \{(z,\lambda) : \ \det[\mathcal{A}(z) - \lambda] = 0\}.$$

Except some very special cases, when $\mathcal{A}(z)$ is a linear polynomial, or it is the monodromy matrix of a system with constant coefficients, $\mathcal{R}$ is two–sheeted Riemann surface (see section 1) and we define

$$\mathcal{R}_+ = \{(z,\lambda) : \ \det[\mathcal{A}(z) - \lambda] = 0, \ |\lambda| < 1\}.$$

According to the definition λ is an inner function on $\mathcal{R}_+$. If $\mathcal{A}(z)$ is a transcendental matrix function, λ remind the exponent: it is an inner function with only one or two singular points on the boundary. Our goal is to prove that any divisor of $\mathcal{A}(z)$ corresponds to inner devisor of λ on $\mathcal{R}_+$. It is really so, if the character–automorphic counterpart of the Beurling–Helson theorem on invariant subspaces of the Hardy space holds on $\mathcal{R}_+$. Such surfaces are called of Widom type with Direct Cauchy Theorem (for exact definition, see section 2). As a result, in this framework, we prove the following theorem.

Theorem. *Let $\mathcal{A}(z)$ be an entire transcendental matrix function satisfying (0.2). Assume that the surface*

$$\mathcal{R}_+ = \{(z,\lambda) : \ \det[\mathcal{A}(z) - \lambda] = 0, \ |\lambda| < 1\}$$

is of Widom type with Direct Cauchy Theorem. Accept the following normalization condition: $(0, \lambda_0) \in \mathcal{R}_+$, and moreover

$$\mathcal{A}(z) = \begin{bmatrix} \lambda_0 + \dots & (1/\lambda_0 - \lambda_0)z + \dots \\ 0 + \dots & 1/\lambda_0 + \dots \end{bmatrix}, \quad \lambda_0 > 0.$$

Then for any $\tau \in [\lambda_0, 1]$ there is unique entire matrix–function $\mathcal{A}(\tau, z)$ such that

1) $\overline{\mathcal{A}(\tau, \bar z)} = \mathcal{A}(\tau, z)$
2) $\det \mathcal{A}(\tau, z) = 1$
3) $\dfrac{J}{z-\bar z} \ge \dfrac{\mathcal{A}(\tau,z)J\mathcal{A}(\tau,z)^*}{z-\bar z} \ge \dfrac{\mathcal{A}(z)J\mathcal{A}(z)^*}{z-\bar z},$

with normalization

$$\mathcal{A}(\tau, z) = \begin{bmatrix} \tau + \dots & (1/\tau - \tau)z + \dots \\ 0 + \dots & 1/\tau + \dots \end{bmatrix}.$$

$\mathcal{A}(\tau, z)$ is a continuous matrix–function and $\frac{J - \mathcal{A}(\tau,z)J\mathcal{A}(\tau,z)^*}{z-\bar z}$ is a monotonic matrix–function of τ.

Corollary. *Under assertions of the previous theorem any divisor*

$$\mathcal{A}_1(z) = \begin{bmatrix} a_{11}^{(1)}(z) & a_{12}^{(1)}(z) \\ a_{21}^{(1)}(z) & a_{22}^{(1)}(z) \end{bmatrix}$$

of $\mathcal{A}(z)$ *is of the form*

$$\mathcal{A}_1(z) = \mathcal{A}(\tau, z) \begin{bmatrix} 1/\tau & 0 \\ 0 & \tau \end{bmatrix} \mathcal{A}_1(0),$$

where

$$1 - \tau^2 = a_{11}^{(1)}(0)(a_{12}^{(1)})'(0) - (a_{11}^{(1)})'(0)a_{12}^{(1)}(0) \le 1 - \lambda_0^2.$$

Note, that we use an internal point of $\mathcal{R}_+$ as a point of normalization (see section 3). In these terms de Branges' normalization corresponds to the case when we fix $\mathcal{A}(z)$ and its divisors in a boundary point, $0 \in \partial\mathcal{R}_+$.

A proof of the theorem is given in section 4.

Acknowledgment. The author thanks the International Erwin Schrödinger Institute for Mathematical Physics, where the paper is completed, for hospitality and personally Prof. James B. Cooper and Prof. Paul Müller for the kind invitation.

1. The Riemann surface associated with the monodromy matrix

First, we prove the following simple lemma.

Lemma 1. *Let* $\mathcal{A}(z)$ *possess properties (0.2). Assume that the* J*–form of the matrix* $\mathcal{A}(z)$ *is strictly positive in the upper half-plane,*

$$\frac{J - \mathcal{A}(z)J\mathcal{A}(z)^*}{z - \bar{z}} > 0.$$

Then the equation $\det[\mathcal{A}(z) - \lambda] = 0$ *has two different roots with respect to* λ *there, the module of one of them is strictly less then 1 (respectively, the module of the other one is strictly greater then 1).*

Proof. Let λ_1 be the eigenvalue of $\mathcal{A}(z_1)$, Im $z_1 > 0$, and f_1 be a corresponding eigenvector, $f_1\mathcal{A}(z_1) = \lambda_1 f_1$. Then

$$\frac{f_1 J f_1^*}{z_1 - \bar{z}_1}(1 - |\lambda_1|^2) > 0.$$

Therefore, $1 - |\lambda_1|^2 \neq 0$. But $\det\mathcal{A}(z_1) = 1$, and hence $\lambda_1\lambda_2 = 1$. So, if $|\lambda_1| < 1$, then $|\lambda_2| > 1$, and vice versa.

Now we prove, that except some very special cases J–form is strictly positive.

Lemma 2. *Let* j *be* 2×2 *matrix, such that* $j^2 = 1_2$*,* $j^* = j$*,* $j \neq 1_2$*. Let* $\mathcal{A}(z)$ *be a holomorphic in the upper half-plane* 2×2 *matrix-function, such that*

$$\Phi(z) = j - \mathcal{A}(z)j\mathcal{A}(z)^* \ge 0, \quad \text{Im } z > 0. \tag{1.1}$$

If the matrix-function $\Phi(z)$ is degenerated at least at one point z_0 ($\exists f_0 \in \mathbb{C}^2$: $f_0\Phi(z_0) = 0$, $f_0 \neq 0$), then, up to constant j–unitary matrices, $\mathcal{A}(z)$ has one of the following three forms:

$$\mathcal{A}(z) = \begin{bmatrix} s(z) & 0 \\ 0 & 1 \end{bmatrix}, \quad 1 - |s(z)|^2 \leq 0, \quad j = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}; \tag{1.2.1}$$

$$\mathcal{A}(z) = \begin{bmatrix} 1 & p(z) \\ 0 & 1 \end{bmatrix}, \quad p(z) + \overline{p(z)} \leq 0, \quad j = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}; \tag{1.2.2}$$

$$\mathcal{A}(z) = \begin{bmatrix} s(z) & 0 \\ 0 & 1 \end{bmatrix}, \quad 1 - |s(z)|^2 \geq 0, \quad j = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}. \tag{1.2.3}$$

Proof. We use the following essential property of j–contractive matrix–functions: inequality (1.1) implies positivity of the kernel [3]

$$\Phi(z_1, z_2) = i\frac{j - \mathcal{A}(z_1)j\mathcal{A}(z_2)^*}{z_1 - \bar{z}_2}, \quad \text{Im } z_1 > 0, \ \text{Im } z_2 > 0.$$

In particular,

$$\begin{bmatrix} i\frac{j-\mathcal{A}(z_0)j\mathcal{A}(z_0)^*}{z_0-\bar{z}_0} & \frac{\mathcal{A}(z)-\mathcal{A}(z_0)}{z-z_0} \\ \frac{\mathcal{A}(z)^*-\mathcal{A}(z_0)^*}{\bar{z}-\bar{z}_0} & i\frac{j-\mathcal{A}(z)^*j\mathcal{A}(z)}{z-\bar{z}} \end{bmatrix} \geq 0. \tag{1.3}$$

Let $f_0 \neq 0$ be a vector, such that $f_0\Phi(z_0) = 0$. Then, as it follows from (1.3),

$$\begin{bmatrix} 0 & f_0\frac{\mathcal{A}(z)-\mathcal{A}(z_0)}{z-z_0}f^* \\ f\frac{\mathcal{A}(z)^*-\mathcal{A}(z_0)^*}{\bar{z}-\bar{z}_0}f_0^* & if\frac{j-\mathcal{A}(z)^*j\mathcal{A}(z)}{z-\bar{z}}f^* \end{bmatrix} \geq 0,$$

for any $f \in \mathbb{C}^2$. It implies, $f_0\frac{\mathcal{A}(z)-\mathcal{A}(z_0)}{z-z_0}f^* = 0$, $\forall f \in \mathbb{C}^2$, $\forall z$, Im $z > 0$. Hence,

$$f_0\mathcal{A}(z) = f_0\mathcal{A}(z_0), \quad \text{Im } z > 0.$$

We consider three cases:

$$1) f_0jf_0^* > 0, \quad 2) f_0jf_0^* = 0, \quad 3) f_0jf_0^* < 0.$$

In the first case, it is convenient to take j in the form:

$$j = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}.$$

Since $f_0jf_0^* = f_0\mathcal{A}(z_0)j\mathcal{A}(z_0)^*f_0^* > 0$, one can find j–unitary matrices U_1 and U_2, in such a way, that

$$f_0U_1 = [0 \quad 1], \quad f_0\mathcal{A}(z_0)U_2 = [0 \quad 1]. \tag{1.4}$$

So, up to substitution $U_1^{-1}\mathcal{A}(z)U_2 \to \mathcal{A}(z)$, we have

$$[0 \quad 1]\,\mathcal{A}(z) = [0 \quad 1],$$

or

$$\mathcal{A}(z) = \begin{bmatrix} a_{11}(z) & a_{12}(z) \\ 0 & 1 \end{bmatrix}. \tag{1.5}$$

And now, we write explicitly the condition $j - \mathcal{A}(z)j\mathcal{A}(z)^* \geq 0$:

$$\begin{aligned} j - \mathcal{A}(z)j\mathcal{A}(z)^* &= \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} a_{11}(z) & a_{12}(z) \\ 0 & 1 \end{bmatrix} \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \overline{a_{11}(z)} & 0 \\ \overline{a_{12}(z)} & 1 \end{bmatrix} \\ &= \begin{bmatrix} -1 + |a_{11}(z)|^2 - |a_{12}(z)|^2 & -a_{12}(z) \\ -\overline{a_{12}(z)} & 0 \end{bmatrix} \geq 0. \end{aligned}$$

It implies, $|a_{12}(z)| = 0$ and $1 - |a_{11}(z)|^2 \leq 0$. Putting $s(z) = a_{11}(z)$, we get (1.2.1).

In the second case we take j in the form

$$j = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

Then we find matrices U_1 and U_2, such that (1.4) holds. After substitution $U_1^{-1}\mathcal{A}(z)U_2 \to \mathcal{A}(z)$, we have $\mathcal{A}(z)$ in the form (1.5), but positivity condition in this case means:

$$\begin{aligned} j - \mathcal{A}(z)j\mathcal{A}(z)^* &= \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} - \begin{bmatrix} a_{11}(z) & a_{12}(z) \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} \overline{a_{11}(z)} & 0 \\ \overline{a_{12}(z)} & 1 \end{bmatrix} \\ &= \begin{bmatrix} -\overline{a_{11}(z)}a_{12}(z) - \overline{a_{12}(z)}a_{11}(z) & 1 - a_{11}(z) \\ 1 - \overline{a_{11}(z)} & 0 \end{bmatrix} \geq 0. \end{aligned}$$

Therefore, $a_{11} = 1$, and $a_{12}(z) + \overline{a_{12}(z)} \leq 0$. Putting $p(z) = a_{12}(z)$, we get (1.2.2).

In the third case we take j in the form:

$$j = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}.$$

The next steps are completely the same, so we omit the proof.

Proposition 1. *Let $\mathcal{A}(z)$ be an entire 2×2 matrix-function, $\overline{\mathcal{A}(\bar{z})} = \mathcal{A}(z)$, $\det\mathcal{A}(z) = 1$, and such that*

$$\frac{J - \mathcal{A}(z)J\mathcal{A}(z)^*}{z - \bar{z}} \geq 0.$$

If $\mathcal{A}(z)$ is not a linear polynomial, then in the upper half plane $\mathcal{A}(z)$ has two different eigenvalues, module of one of them is strictly less then 1 (respectively, module of another one is strictly grater then 1).

Proof. Assume, that J–form is degenerated in some point z_0, $\operatorname{Im} z_0 > 0$. According to lemma 2, up to constant J–unitary factors, $\mathcal{A}(z)$ has one of the present there forms. But $\det\mathcal{A}(z) = 1$, so, if $\mathcal{A}(z) \neq \text{const}$, then $\mathcal{A}(z)$ should be of the form

$$\mathcal{A}(z) = \begin{bmatrix} 1 & p(z) \\ 0 & 1 \end{bmatrix}, \quad p(z) + \overline{p(z)} \leq 0,$$

where $p(z)$ is an entire function. This means, that $p(z)$ is a linear polynomial. Otherwise, J–form is not degenerated, and according to lemma 1, $\mathcal{A}(z)$ has two different eigenvalues.

In what follows we use the following notation. We denote by $\lambda(z)$ the eigenvalue of $\mathcal{A}(z)$ for which $|\lambda(z)| < 1$, $\operatorname{Im} z \neq 0$. We put

$$E = \{x \in \mathbb{R} : \ |\lambda(x + i0)| = 1\}.$$

Up to trivial case $\lambda = e^{az+b}$, $E \neq \mathbb{R}$ and we accept the normalization condition $0 \in \mathbb{C} \setminus E$. The set E is bounded if and only if $\mathcal{A}(z)$ is a polynomial. We will consider only the case when $\mathcal{A}(z)$ is a transcendental matrix–function. Note, that $\lambda(z) \neq 0$, $z \in \mathbb{C} \setminus E$ in this case.

Therefore, one can consider $\mathcal{R}$ as two–sheeted Riemann surface, which consists of two copies of the domain $\mathbb{C} \setminus E$ glued along the system of intervals E, and on the upper sheet

$$\mathcal{R}_+ = \{(z, \lambda) : \ \det[A(z) - \lambda] = 0, \ |\lambda| < 1\} = \{(z, \lambda(z)) : \ z \in \mathbb{C} \setminus E\}.$$

2. Hardy spaces on $\mathcal{R}_+$, Fuchsian groups of Widom type and Direct Cauchy Theorem

Let $\mathbb{D}$ denote the unit disk and $\mathbb{T}$ denote the unit circle

$$\mathbb{D} = \{\zeta : \ |\zeta| < 1\}, \quad \mathbb{T} = \{\zeta : \ |\zeta| = 1\}.$$

We use a standard terminology and notations of the theory of functions of bounded characteristic in $\mathbb{D}$ [4]. In particular, H^p denotes the standard Hardy space. We remind that an analytic in $\mathbb{D}$ function is said to be of Smirnov class if it can be presented in the form $f = f_1/f_2$, where $f_1, f_2 \in H^\infty$ and f_2 is an outer function.

Let ? be a discrete subgroup of $SU(1,1)$ consisting of elements of the form

$$\gamma = \begin{bmatrix} \gamma_{11} & \gamma_{12} \\ \gamma_{21} & \gamma_{22} \end{bmatrix}, \ \gamma_{11} = \overline{\gamma_{22}}, \ \gamma_{12} = \overline{\gamma_{21}}, \ \det \gamma = 1.$$

For $\gamma \in$?, put $\gamma(\zeta) = (\gamma_{11}\zeta + \gamma_{12})/(\gamma_{21}\zeta + \gamma_{22})$; as it well known, γ maps $\mathbb{D}$ and $\mathbb{T}$ onto themselves.

A character of ? is a complex–valued function $\alpha :$? $\to \mathbb{T}$, satisfying

$$\alpha(\gamma_1\gamma_2) = \alpha(\gamma_1)\alpha(\gamma_2) \quad (\gamma_1, \gamma_2 \in ?).$$

The characters form Abelian compact group, we denote it by $?^*$.

By the uniformization theorem, the domain $\mathbb{C} \setminus E$ is conformal equivalent the quotient of the unit disk $\mathbb{D}$ by the action of a group ? = ?(E). In other words, there exist an analytic function $z : \ \mathbb{D} \to \mathbb{C} \setminus E$ and a discrete group ? with the following properties:

z is automorphic with respect to ?, $z(\gamma(\zeta)) = z(\zeta), \ \forall \gamma \in ?$;

z maps $\mathbb{D}$ onto the domain $\mathbb{C} \setminus E$,

$$\forall z_0 \in \mathbb{C} \setminus E \ \exists \zeta_0 \in \mathbb{D} : \ z(\zeta_0) = \zeta_0,$$

in such a way that any two preimages of z_0 are ?–equivalent

$$z(\zeta_1) = z(\zeta_2) \ \Rightarrow \ \exists \gamma \in ? : \ \zeta_1 = \gamma(\zeta_2).$$

In this case, two classes of functions are equivalent:

$$\{\text{meromorphic functions } f(\zeta) \text{ in } \mathbb{D} \text{ such that } f(\gamma(\zeta)) = f(\zeta),\ \forall\gamma \in ?\,\}$$
$$\equiv \{\text{meromorphic functions } F(z) \text{ in } \mathbb{C} \setminus E\}. \blacksquare$$

This equivalence yields by the identity $F(z(\zeta)) = f(\zeta)$.

By the definition,

$$H^\infty(?\,) = \{f \in H^\infty :\ f \circ \gamma = f,\ \forall\gamma \in ?\,\}.$$

Let us note that if the space $H^\infty(?\,)$ is not trivial,

$$\exists f \in H^\infty(?\,) :\ f(\zeta) \not\equiv f(\zeta_0),$$

then the trajectory $\{\gamma(\zeta_0)\}_{\gamma\in\Gamma}$ satisfies the Blaschke condition. The Blaschke product

$$b(\zeta, \zeta_0) = b(\zeta, \zeta_0; ?\,) = \prod_{\gamma\in\Gamma} \frac{\gamma(\zeta_0) - \zeta}{1 - \overline{\gamma(\zeta_0)}\zeta} \frac{|\gamma(\zeta_0)|}{\gamma(\zeta_0)}$$

is called the *Green function* of ? with respect to ζ_0. It is a character–automorphic function, that is there exists $\mu_{\zeta_0} \in ?^*$ such that $b(\gamma(\zeta), \zeta_0) = \mu_{\zeta_0}(\gamma) b(\zeta, \zeta_0)$. To simplify notation we put $b(\zeta) = b(\zeta, 0)$ and $\mu = \mu_0$.

We will consider also spaces of character–automorphic functions: for $\alpha \in ?^*$

$$H^\infty(?\,, \alpha) = \{f \in H^\infty :\ f \circ \gamma = \alpha(\gamma) f,\ \forall\gamma \in ?\,\}.$$

The group ? is said to be of *Widom type* if for any $\alpha \in ?^*$ the space $H^\infty(?\,, \alpha)$ is not trivial, i.e. $H^\infty(?\,, \alpha) \neq \{\text{const}\}$ [12, 11]. ? is of Widom type if and only if the derivative $b'(\zeta)$ is of bounded characteristic. In this case, ? acts dissipative on $\mathbb{T}$ with respect to Lebesgue measure dm, that is there exists a measurable (fundamental) set $\mathbb{E} \subset \mathbb{T}$, such that

$$1)\mathbb{E} \bigcap \gamma(\mathbb{E}) = \emptyset, \quad \text{for all } \gamma \neq 1_2,$$
$$2) m(\bigcup_{\gamma\in\Gamma} \gamma(\mathbb{E})) = m(\mathbb{T}).$$

For an analytic function in $\mathbb{D}$, $\gamma \in ?$ and $k \in \mathbb{N}$ we write

$$f|[\gamma]_k = \frac{f(\gamma(\zeta))}{(\gamma_{21}\zeta + \gamma_{22})^k}$$

Then it easily verified that

$$f|[\gamma_1\gamma_2]_k = (f|[\gamma_1]_k)|[\gamma_2]_k.$$

Notice that $f|[\gamma]_2 = f,\ \forall\gamma \in ?$ means that the form $f(\zeta)d\zeta$ is invariant with respect to substitutions $\zeta \to \gamma(\zeta)$ ($f(\zeta)d\zeta$ is an Abelian integral on $\mathbb{D}/?\,$).

Definition. *Let ? be a group of Widom type and $\mathbb{E} \subset \mathbb{T}$ be its fundamental set. For $k = 1, 2$ and $\alpha \in ?^*$ the space $A_k^{2/k}(?, \alpha)$ is formed by the analytic functions f on $\mathbb{D}$ that satisfy the following three conditions*

$$1) f \text{ is of Smirnov class}$$
$$2) f|[\gamma]_k = \alpha(\gamma) f \ \forall \gamma \in ?$$
$$3) \int_{\mathbb{E}} |f|^{2/k}\, dm < \infty.$$

$A_1^2(?, \alpha)$ is a Hilbert space with the reproducing kernel $k^\alpha(\zeta, \zeta_0)$ (the point evaluation functional is bounded):

$$\langle f(\zeta), k^\alpha(\zeta, \zeta_0)\rangle = f(\zeta_0), \quad \zeta_0 \in \mathbb{D},\ f \in A_1^2(?, \alpha).$$

Put $k^\alpha(\zeta) = k^\alpha(\zeta, 0)$.

For a group of Widom type the following conditions are equivalent [7]:

• *Direct Cauchy Theorem* holds:

$$\int_{\mathbb{E}} \frac{f}{b}(\zeta)\, \frac{d\zeta}{2\pi i} = \frac{f}{b'}(0), \quad \forall f \in A_2^1(?, \mu). \tag{DCT}$$

• Let $L^2_{dm|\mathbb{E}}$ be the space of square–integrable functions on $\mathbb{E}$ with respect to dm. Then

$$L^2_{dm|\mathbb{E}} = \overline{\zeta A_1^2(?, \alpha^{-1})} \oplus A_1^2(?, \alpha) \quad \forall \alpha \in ?^*,$$

where $\overline{\zeta A_1^2(?, \alpha^{-1})} = \{g(\zeta) = \overline{\zeta f(\zeta)} :\ f \in A_1^2(?, \alpha^{-1})\}$.

• Every invariant subspace $M \subset A_1^2(?, \alpha)$ ($fM \subset M\ \forall f \in H^\infty(?)$) is of the form

$$M = s A_1^2(?, \sigma^{-1}\alpha)$$

for some character–automorphic inner function $s \in H^\infty(\sigma)$.

• The function $k^\alpha(0)$ is continuous on $?^*$.

• For all $\alpha \in ?^*$

$$b(\zeta)\overline{k^\alpha(\zeta)} = \zeta \frac{k^{\mu\alpha^{-1}}(\zeta)}{k^{\mu\alpha^{-1}}(0)} b'(0), \quad \zeta \in \mathbb{T}. \tag{2.1}$$

In particular (2.1) implies

$$k^\alpha(0) k^{\mu\alpha^{-1}}(0) = \{b'(0)\}^2 \tag{2.2}$$

Here, we just compere norm of the vectors in the right–hand and the left–hand sides.

We note that the origin is not a special point here, and

$$\int_{\mathbb{E}} \frac{f(\zeta)}{b(\zeta, \zeta_0)}\, \frac{d\zeta}{2\pi i} = \frac{f(\zeta_0)}{b'(\zeta_0, \zeta_0)}, \quad \forall f \in A_2^1(?, \mu_{\zeta_0}),$$

holds for all $\zeta_0 \in \mathbb{D}$ as well as

$$b(\zeta,\zeta_0)\overline{k^{\alpha}(\zeta,\zeta_0)} = \zeta \frac{k^{\mu_{\zeta_0}\alpha^{-1}}(\zeta,\zeta_0)}{k^{\mu_{\zeta_0}\alpha^{-1}}(\zeta_0,\zeta_0)} b'(\zeta_0,\zeta_0), \quad \zeta \in \mathbb{T}, \tag{2.3}$$

and

$$k^{\alpha}(\zeta_0,\zeta_0)k^{\mu_{\zeta_0}\alpha^{-1}}(\zeta_0,\zeta_0) = \{b'(\zeta_0,\zeta_0)\}^2. \tag{2.4}$$

Following Carleson, Jones and Marshall [6] considered the corona problem for the surface $\mathbb{C}\setminus E \equiv \mathbb{D}/?$, where E is a homogeneous set. Recall that a set E is homogeneous if there is $\eta > 0$ such that

$$|(x-\rho, x+\rho)\cap E| \ge \eta\rho \quad \text{for all } \rho > 0 \quad \text{and all } x \in E.$$

They showed that it would be enough to solve the corona problem in critical points of Green's function, i.e. $\{\zeta : \ b'(\zeta) = 0\}$, and then showed that for a homogeneous set critical points do form an interpolating sequence for H^∞. From the last property it follows immediately that *if E is a homogeneous set, then ? is of Widom type and the Direct Cauchy Theorem holds* [10].

3. Reproducing kernels and Weyl functions

For $\mathbb{D}/? = \mathbb{C}\setminus E$ we have a special representation for the kernel $k^{\alpha}(\zeta,\zeta_0)$.

Lemma 3. *Let ? be a group of Widom type, such that*

$$z : \ \mathbb{D}/? \equiv \mathbb{C}\setminus E,$$

with normalization $z(0) = 0$, $(z/\zeta)(0) < 0$. Then

$$k^{\alpha}(\zeta,\zeta_0) = -\left(\frac{b}{z}\right)(0)\frac{\overline{k^{\alpha}(\zeta_0)}\frac{k^{\alpha\mu}(\zeta)}{b(\zeta)k^{\alpha\mu}(0)} - \overline{\frac{k^{\alpha\mu}(\zeta_0)}{b(\zeta_0)k^{\alpha\mu}(0)}}k^{\alpha}(\zeta)}{z(\zeta)-\overline{z(\zeta_0)}} z(\zeta)\overline{z(\zeta_0)}. \tag{3.1}$$

Proof. Consider the function $b(\zeta)(1/z(\zeta) - 1/\overline{z(\zeta_0)})k^{\alpha}(\zeta,\zeta_0)$. Since $b(\zeta)(1/z(\zeta) - 1/\overline{z(\zeta_0)}) \in H^\infty(?,\mu)$, it belongs to $A^2_1(?,\alpha\mu)$. Besides this,

$$\begin{aligned}\langle b(\zeta)(1/z(\zeta) - 1/\overline{z(\zeta_0)})k^{\alpha}(\zeta,\zeta_0), b^2 f\rangle =& \langle k^{\alpha}(\zeta,\zeta_0), b(\zeta)(1/z(\zeta)-1/z(\zeta_0))f\rangle \\ =& \overline{b(\zeta_0)(1/z(\zeta_0)-1/z(\zeta_0))f(\zeta_0)} = 0,\end{aligned}$$

for all $f \in A^2_1(?,\alpha\mu^{-1})$. It is easy to see that orthogonal compliment

$$A^2_1(?,\alpha\mu)\ominus b^2 A^2_1(?,\alpha\mu^{-1})$$

is spanned by two functions $k^{\alpha\mu}(\zeta)$ and $b(\zeta)k^{\alpha}(\zeta)$. Therefore,

$$b(\zeta)(1/z(\zeta) - 1/\overline{z(\zeta_0)})k^{\alpha}(\zeta,\zeta_0) = C_1(\zeta_0)k^{\alpha\mu}(\zeta) + C_2(\zeta_0)b(\zeta)k^{\alpha}(\zeta).$$

To find $C_1(\zeta_0)$, we use identity

$$\left(\frac{b}{z}\right)(0)\overline{k^\alpha(\zeta_0)} = \langle b(\zeta)(1/z(\zeta) - 1/\overline{z(\zeta_0)})k^\alpha(\zeta,\zeta_0), k^{\alpha\mu}\rangle = C_1(\zeta_0)k^{\alpha\mu}(0).$$

Hence,

$$(1/z(\zeta) - 1/\overline{z(\zeta_0)})k^\alpha(\zeta,\zeta_0) = \left(\frac{b}{z}\right)(0)\overline{k^\alpha(\zeta_0)}\frac{k^{\alpha\mu}(\zeta)}{b(\zeta)k^{\alpha\mu}(0)} + C_2(\zeta_0)k^\alpha(\zeta).$$

And since the right–hand side is antisymmetric we get (3.1).

Define a meromorphic in $\mathbb{C}\setminus E$ function $r_+(z,\alpha)$ by the relation

$$r_+(z(\zeta),\alpha) = -\left(\frac{b}{z}\right)(0)\frac{1}{b(\zeta)}\frac{k^{\alpha\mu}(\zeta)}{k^{\alpha\mu}(0)}\frac{k^\alpha(0)}{k^\alpha(\zeta)}. \tag{3.2}$$

Lemma 3 implies that

$$\frac{r_+(z,\alpha) - \overline{r_+(z,\alpha)}}{z-\bar z} \ge 0.$$

Now we show that $r_+(z,\alpha)$ has generalized analytic continuation across the set E.

Lemma 4. *Let $\mathbb{C}\setminus E$ be a domain of Widom type with (DCT). Then*

$$\overline{r_+(z,\alpha)} = r_-(z,\alpha), \quad z\in E,$$

where $r_-(z,\alpha)$ is a meromorphic in $\mathbb{C}\setminus E$ function, such that

$$\frac{r_-(z,\alpha) - \overline{r_-(z,\alpha)}}{z-\bar z} \le 0. \tag{3.3}$$

Proof. According to (2.1)

$$\begin{aligned}\overline{r_+(z(\zeta),\alpha)} &= -\left(\frac{b}{z}\right)(0)b(\zeta)\frac{\overline{k^{\alpha\mu}(\zeta)}}{k^{\alpha\mu}(0)}\frac{b(\zeta)}{b(\zeta)}\frac{\overline{k^\alpha(0)}}{k^\alpha(\zeta)}\\ &= -\left(\frac{b}{z}\right)(0)b(\zeta)\frac{k^{\alpha^{-1}}(\zeta)}{k^{\mu\alpha^{-1}}(\zeta)},\end{aligned}$$

or

$$\overline{r_+(z(\zeta),\alpha)} = \mathcal{P}(\alpha)\left\{-\left(\frac{b}{z}\right)(0)\frac{1}{b(\zeta)}\frac{k^{\alpha^{-1}\mu}(\zeta)}{k^{\alpha^{-1}\mu}(0)}\frac{k^{\alpha^{-1}}(0)}{k^{\alpha^{-1}}(\zeta)}\right\}^{-1} = \mathcal{P}(\alpha)r_+^{-1}(z(\zeta),\alpha^{-1}),$$

where $\mathcal{P}(\alpha) = \left(\frac{b}{z}\right)^2(0)\frac{k^{\alpha^{-1}}(0)}{k^{\mu\alpha^{-1}}(0)}$. Since $\mathcal{P}(\alpha)$ is positive the meromorphic function

$$r_-(z,\alpha) = \mathcal{P}(\alpha)r_+^{-1}(z,\alpha^{-1})$$

satisfies (3.3).

The following lemma collects the properties of the pair of functions $r_\pm(z,\alpha)$ which, in fact, characterize functions of the form (3.2).

Lemma 5. *Let $\mathbb{C}\setminus E$ be a domain of Widom type with (DCT). Then the pair of meromorphic in $\mathbb{C}\setminus E$ functions $r_\pm(z,\alpha)$ possesses the following properties*

1)$\overline{r_+(z,\alpha)} = r_-(z,\alpha)$, $z\in E$,

2)$\pm\frac{r_\pm(z,\alpha)-\overline{r_\pm(z,\alpha)}}{z-\bar z}\ge 0$,

3) both functions $\{r_+^{\pm 1}(z,\alpha)-r_-^{\pm 1}(z,\alpha)\}^{-1}$ are holomorphic in $\mathbb{C}\setminus E$,

4)$r_-(z,\alpha)$ has zero and $r_+(z,\alpha)$ has pole in the origin, moreover

$$(zr_+(z,\alpha))_{z=0} = -1.$$

Proof. We have to prove the third property, since all other already have been proved. Note, that due to identity (2.2), the function $\mathcal{P}(\alpha)$ is symmetric,

$$\mathcal{P}(\alpha^{-1}) = \left(\frac{b}{z}\right)^2(0)\frac{k^\alpha(0)}{k^{\mu\alpha}(0)} = \left(\frac{b}{z}\right)^2(0)\frac{k^{\alpha^{-1}}(0)}{k^{\mu\alpha^{-1}}(0)} = \mathcal{P}(\alpha).$$

Therefore, $r_+(z,\alpha) = \mathcal{P}(\alpha^{-1})r_-^{-1}(z,\alpha^{-1}) = \mathcal{P}(\alpha)r_-^{-1}(z,\alpha^{-1})$, and

$$r_+^{-1}(z,\alpha) - r_-^{-1}(z,\alpha) = \frac{r_-(z,\alpha^{-1})-r_+(z,\alpha^{-1})}{\mathcal{P}(\alpha)}.$$

So, we will check only that $[r_+(z,\alpha)-r_-(z,\alpha)]^{-1}$ is holomorphic.

We start with the following nice identity:

$$\left(\frac{b}{z}\right)(0)\left(\frac{b'(0)}{z'(\zeta)}\right)\left(\frac{z}{b}\right)^2(\zeta)\left\{\frac{k^{\alpha\mu}(\zeta)}{k^{\alpha\mu}(0)}\frac{k^{\alpha^{-1}\mu}(\zeta)}{k^{\alpha^{-1}\mu}(0)} - \left(\frac{b(\zeta)}{b'(0)}\right)^2 k^{\alpha^{-1}}(\zeta)k^\alpha(\zeta)\right\} = 1. \tag{3.4}$$

To prove (3.4) we substitute (3.1) in (2.3)

$$\begin{aligned}\frac{k^{\mu_{\zeta_0}\alpha^{-1}}(\zeta,\zeta_0)}{k^{\mu_{\zeta_0}\alpha^{-1}}(\zeta_0,\zeta_0)}b'(\zeta_0,\zeta_0) &= \frac{b(\zeta,\zeta_0)}{\zeta}\overline{k^\alpha(\zeta,\zeta_0)}\\ &= -\left(\frac{b}{z}\right)(0)\frac{b(\zeta,\zeta_0)}{\zeta}\frac{k^\alpha(\zeta_0)b(\zeta)\overline{\frac{k^{\alpha\mu}(\zeta)}{k^{\alpha\mu}(0)}} - \frac{k^{\alpha\mu}(\zeta_0)}{b(\zeta_0)k^{\alpha\mu}(0)}\frac{b(\zeta)\overline{k^\alpha(\zeta)}}{b(\zeta)}}{z(\zeta)-z(\zeta_0)}z(\zeta)z(\zeta_0).\end{aligned}$$

Using (2.1) we get

$$\begin{aligned}&\frac{k^{\mu_{\zeta_0}\alpha^{-1}}(\zeta,\zeta_0)}{k^{\mu_{\zeta_0}\alpha^{-1}}(\zeta_0,\zeta_0)}b'(\zeta_0,\zeta_0)\\ &\quad= -\left(\frac{b}{z}\right)(0)b(\zeta,\zeta_0)\frac{k^\alpha(\zeta_0)k^{\alpha^{-1}}(\zeta)b'(0) - \frac{b'(0)}{b(\zeta_0)b(\zeta)}\frac{k^{\alpha\mu}(\zeta_0)}{k^{\alpha\mu}(0)}\frac{k^{\alpha^{-1}\mu}(\zeta)}{k^{\alpha^{-1}\mu}(0)}}{z(\zeta)-z(\zeta_0)}z(\zeta)z(\zeta_0).\ \blacksquare\end{aligned}$$

Putting $\zeta=\zeta_0$, we get

$$1 = -\left(\frac{b}{z}\right)(0)\frac{k^\alpha(\zeta)k^{\alpha^{-1}}(\zeta)b'(0) - \frac{b'(0)}{b^2(\zeta)}\frac{k^{\alpha\mu}(\zeta)}{k^{\alpha\mu}(0)}\frac{k^{\alpha^{-1}\mu}(\zeta)}{k^{\alpha^{-1}\mu}(0)}}{z'(\zeta)}z^2(\zeta).$$

So, (3.4) is proved, and, now, we transform the difference

$$r_+(z(\zeta),\alpha)-r_-(z(\zeta),\alpha)=-\left(\frac{b}{z}\right)(0)\left\{\frac{1}{b(\zeta)}\frac{k^{\alpha\mu}(\zeta)}{k^{\alpha\mu}(0)}\frac{k^{\alpha}(0)}{k^{\alpha}(\zeta)}-b(\zeta)\frac{k^{\alpha^{-1}}(\zeta)}{k^{\mu\alpha^{-1}}(\zeta)}\right\}$$
$$=\frac{-\left(\frac{b}{z}\right)(0)}{b(\zeta)k^{\alpha}(\zeta)k^{\mu\alpha^{-1}}(\zeta)}\left\{\frac{k^{\alpha\mu}(\zeta)}{k^{\alpha\mu}(0)}k^{\alpha}(0)k^{\alpha^{-1}\mu}(\zeta)-b^2(\zeta)k^{\alpha^{-1}}(\zeta)k^{\alpha}(\zeta)\right\}. \blacksquare$$

Since $k^{\alpha}(0)k^{\mu\alpha^{-1}}(0)=b'(0)^2$, we have

$$r_+(z(\zeta),\alpha)-r_-(z(\zeta),\alpha)=$$
$$\frac{-\left(\frac{b}{z}\right)(0)}{b(\zeta)k^{\alpha}(\zeta)k^{\mu\alpha^{-1}}(\zeta)}\left\{b'(0)^2\frac{k^{\alpha\mu}(\zeta)}{k^{\alpha\mu}(0)}\frac{k^{\alpha^{-1}\mu}(\zeta)}{k^{\alpha^{-1}\mu}(0)}-b^2(\zeta)k^{\alpha^{-1}}(\zeta)k^{\alpha}(\zeta)\right\}.$$

Using (3.4), we get

$$[r_+(z(\zeta),\alpha)-r_-(z(\zeta),\alpha)]^{-1}=-\frac{k^{\alpha^{-1}}(\zeta)k^{\alpha}(\zeta)z^2(\zeta)}{b'(0)b(\zeta)z'(\zeta)}.$$

The inverse statement was proved in [10].

Theorem. *Let $\mathbb{C}\setminus E$ be a domain of Widom type with (DCT). Let a pair of meromorphic in $\mathbb{C}\setminus E$ functions $r_\pm(z)$ possesses the following properties*

1) $\overline{r_+(z)}=r_-(z)$, $z\in E$,

2) $\pm\frac{r_\pm(z)-\overline{r_\pm(z)}}{z-\bar z}\geq 0$,

3) both functions $\{r_+^{\pm 1}(z)-r_-^{\pm 1}(z)\}^{-1}$ *are holomorphic in* $\mathbb{C}\setminus E$,

4) $r_-(z)$ *has zero and* $r_+(z)$ *has pole in the origin, moreover*

$$(zr_+(z))_{z=0}=-1.$$

Then there exists and unique $\alpha\in ?^*$ *such that* $r_+(z)=r_+(z,\alpha)$.

Let

$$\mathcal{A}(z)=\begin{bmatrix} a_{11}(z) & a_{12}(z)\\ a_{21}(z) & a_{22}(z)\end{bmatrix}$$

be an entire transcendental matrix–function, which satisfies (0.2). The associated Weyl functions is defined by the relations

$$[\,r_\pm(z)\quad 1\,]\,\mathcal{A}(z)=\lambda^{\pm 1}(z)\,[\,r_\pm(z)\quad 1\,],\quad z\in\mathbb{C}\setminus E.$$

Since

$$\frac{r_\pm(z)-\overline{r_\pm(z)}}{z-\bar z}(1-|\lambda^{\pm 1}(z)|^2)\geq 0,$$

we have

$$\pm\frac{r_\pm(z)-\overline{r_\pm(z)}}{z-\bar z}\geq 0.$$

Since $\lambda(z) = 1/\overline{\lambda(z)}$, when $z \in E$, we have

$$r_-(z) = \overline{r_+(z)}, \quad z \in E.$$

And since

$$r_\pm(z) = \frac{\lambda^{\pm 1}(z) - a_{22}(z)}{a_{12}(z)} = \frac{a_{21}(z)}{\lambda^{\pm 1}(z) - a_{11}(z)},$$

we have

$$\begin{aligned}[r_+(z) - r_-(z)]^{-1} &= \frac{a_{12}(z)\lambda(z)}{\lambda^2(z) - 1},\\ [r_+^{-1}(z) - r_-^{-1}(z)]^{-1} &= \frac{a_{21}(z)\lambda(z)}{\lambda^2(z) - 1}.\end{aligned}$$

These functions are clearly holomorphic in $\mathbb{C} \setminus E$, since $|\lambda(z)| < 1$ here.

To use previous theorem we have to add normalization condition 4). We note, that a substitution

$$\mathcal{A}(z) \to U\mathcal{A}(z)U^{-1}, \quad U \in SL_2(\mathbb{R}) \tag{3.5}$$

(U is real, $\det U = 1$) does not change the associated Riemann surface. Since $0 \in \mathbb{C} \setminus E$, we may assume that $\mathcal{A}(0)$ is diagonal, moreover

$$\mathcal{A}(0) = \begin{bmatrix} \lambda_0 & 0 \\ 0 & 1/\lambda_0 \end{bmatrix}, \quad \lambda_0^2 < 1.$$

In this case $r_+(z)$ has pole in the origin and $r_-(z)$ has zero. And then, with the help of substitution

$$\mathcal{A}(z) \to \begin{bmatrix} c & 0 \\ 0 & 1/c \end{bmatrix} \mathcal{A}(z) \begin{bmatrix} c & 0 \\ 0 & 1/c \end{bmatrix}^{-1},$$

we normalize $r_+(z)$ by the condition

$$(zr_+(z))_{z=0} = -1,$$

what is equivalent to $a_{12}'(0) = 1/\lambda_0 - \lambda_0$.

So, in fact, using substitution (3.5), we fixed the main terms of the decomposition $\mathcal{A}(z)$ in the origin in the form

$$\mathcal{A}(z) = \begin{bmatrix} \lambda_0 + \dots & (1/\lambda_0 - \lambda_0)z + \dots \\ 0 + \dots & 1/\lambda_0 + \dots \end{bmatrix}.$$

If it need we can also change $\mathcal{A}(z) \to -\mathcal{A}(z)$ and $\lambda(z) \to -\lambda(z)$. So without lost of generality we can add condition $\lambda_0 > 0$.

It would be useful to note, that the normalization is multiplicative, i.e.:

$$\begin{aligned}&\begin{bmatrix} \lambda_1 + \dots & (1/\lambda_1 - \lambda_1)z + \dots \\ 0 + \dots & 1/\lambda_1 + \dots \end{bmatrix} \begin{bmatrix} \lambda_2 + \dots & (1/\lambda_2 - \lambda_2)z + \dots \\ 0 + \dots & 1/\lambda_2 + \dots \end{bmatrix}\\ &\qquad = \begin{bmatrix} (\lambda_1\lambda_2) + \dots & \{1/(\lambda_1\lambda_2) - (\lambda_1\lambda_2)\}z + \dots \\ 0 + \dots & 1/(\lambda_1\lambda_2) + \dots \end{bmatrix}.\end{aligned}$$

We summarize result of this section as follows.

Proposition 2. *Let $\mathcal{A}(z)$ be an entire transcendental 2×2 matrix–function, which satisfies (0.2) and the normalization condition*

$$\mathcal{A}(z)=\begin{bmatrix}\lambda_0+\dots & (1/\lambda_0-\lambda_0)z+\dots\\ 0+\dots & 1/\lambda_0+\dots\end{bmatrix},\quad 0<\lambda_0<1. \tag{3.6}$$

Assume that the associated Riemann surface

$$\mathcal{R}_+=\{(z,\lambda):\ \det[\mathcal{A}(z)-\lambda]=0,\ |\lambda|<1\}\equiv\mathbb{D}/? \tag{3.7}$$

is of Widom type with (DCT). Then there is unique $\alpha\in ?^*$, such that the vector–function

$$f^\alpha(\zeta)=\frac{z(\zeta)}{b(\zeta)}\left[\ \sqrt{\mathcal{P}(\alpha)}\frac{k^{\alpha\mu}(\zeta)}{\sqrt{k^{\alpha\mu}(0)}}\quad b(\zeta)\frac{k^\alpha(\zeta)}{\sqrt{k^\alpha(0)}}\right]$$

is an eigenvector of the matrix–function $\mathcal{A}(z(\zeta))$ with the eigenvalue $\lambda(z(\zeta))$,

$$f^\alpha(\zeta)\mathcal{A}(z(\zeta))=\lambda(z(\zeta))f^\alpha(\zeta). \tag{3.8}$$

In addition the reproducing kernel of $A^2_1(?,\alpha)$ is of the form

$$k^\alpha(\zeta,\zeta_0)=\frac{f^\alpha(\zeta)Jf^\alpha(\zeta_0)^*}{z(\zeta)-\overline{z(\zeta_0)}}. \tag{3.9}$$

4. Proof of the main theorem

We brake the proof into uniqueness and existence parts.

We need two lemmas concerning the reproducing kernels. As well as for lemma 5 proofs are based on (3.4).

Lemma 6. *Let $\mathbb{C}\backslash E$ be a domain of Widom type with (DCT), and $z:\mathbb{D}/?\equiv\mathbb{C}\backslash E$. Then for almost every $\zeta\in\mathbb{T}$ there exists the limit*

$$\lim_{\rho\to 1}[z(\rho\zeta)-\overline{z(\rho\zeta)}]k^\alpha(\rho\zeta,\rho\zeta)=-\zeta z'(\zeta). \tag{4.1}$$

We note that the limit does not depend on α.

Proof. By (3.1) we have

$$\lim_{\rho\to 1}[z(\rho\zeta)-\overline{z(\rho\zeta)}]k^\alpha(\rho\zeta,\rho\zeta)=-\left(\frac{b}{z}\right)(0)\left\{\overline{k^\alpha(\zeta)}\frac{k^{\alpha\mu}(\zeta)}{b(\zeta)k^{\alpha\mu}(0)}-\overline{\frac{k^{\alpha\mu}(\zeta)}{b(\zeta)k^{\alpha\mu}(0)}}k^\alpha(\zeta)\right\}z^2(\zeta).$$

But on the boundary

$$b(\zeta)\overline{k^\alpha(\zeta)}=\zeta\frac{k^{\alpha^{-1}\mu}(\zeta)}{k^{\alpha^{-1}\mu}(0)}b'(0).$$

So,

$$\begin{aligned}&\lim_{\rho\to 1}[z(\rho\zeta)-\overline{z(\rho\zeta)}]k^\alpha(\rho\zeta,\rho\zeta)\\ &=-\left(\frac{b}{z}\right)(0)\left\{b'(0)\zeta\frac{k^{\alpha^{-1}\mu}(\zeta)}{b(\zeta)k^{\alpha^{-1}\mu}(0)}\frac{k^{\alpha\mu}(\zeta)}{b(\zeta)k^{\alpha\mu}(0)}-\frac{\zeta}{b'(0)}k^{\alpha^{-1}}(\zeta)k^\alpha(\zeta)\right\}z^2(\zeta)\\ &=-\zeta b'(0)\left(\frac{b}{z}\right)(0)\left\{\frac{k^{\alpha^{-1}\mu}(\zeta)}{k^{\alpha^{-1}\mu}(0)}\frac{k^{\alpha\mu}(\zeta)}{k^{\alpha\mu}(0)}-\frac{b^2(\zeta)}{b'(0)^2}k^{\alpha^{-1}}(\zeta)k^\alpha(\zeta)\right\}\frac{z^2(\zeta)}{b^2(\zeta)}.\end{aligned}$$

Using (3.4) we get (4.1).

Lemma 7. *Under the assertions of the previous lemma the vectors* $f^\alpha(\zeta)$ *and* $f^\alpha_*(\zeta) = \overline{\zeta f^\alpha(\zeta)}$ $(\zeta \in \mathbb{T})$ *are linearly independent, moreover,*

$$\det F^\alpha(\zeta) = -z'(\zeta), \tag{4.2}$$

where

$$F^\alpha(\zeta) = \begin{bmatrix} f^\alpha(\zeta) \\ f^\alpha_*(\zeta) \end{bmatrix} = \frac{z(\zeta)}{b(\zeta)} \begin{bmatrix} \sqrt{\mathcal{P}(\alpha)}\frac{k^{\alpha\mu}(\zeta)}{\sqrt{k^{\alpha\mu}(0)}} & b(\zeta)\frac{k^{\alpha}(\zeta)}{\sqrt{k^{\alpha}(0)}} \\ \sqrt{\mathcal{P}(\alpha)}b(\zeta)\frac{k^{\alpha^{-1}}(\zeta)}{\sqrt{k^{\alpha^{-1}}(0)}} & \frac{k^{\mu\alpha^{-1}}(\zeta)}{\sqrt{k^{\mu\alpha^{-1}}(0)}} \end{bmatrix}. \tag{4.3}$$

Proof. Using (2.1), (2.2) we have

$$\overline{\zeta f^\alpha(\zeta)} = \frac{z(\zeta)}{b(\zeta)} \begin{bmatrix} \sqrt{\mathcal{P}(\alpha)}b(\zeta)\frac{k^{\alpha^{-1}}(\zeta)}{\sqrt{k^{\alpha^{-1}}(0)}} & \frac{k^{\mu\alpha^{-1}}(\zeta)}{\sqrt{k^{\mu\alpha^{-1}}(0)}} \end{bmatrix} = f^{\alpha^{-1}}(\zeta) \begin{bmatrix} 0 & 1/\sqrt{\mathcal{P}(\alpha)} \\ \sqrt{\mathcal{P}(\alpha)} & 0 \end{bmatrix}. \quad \blacksquare$$

And then, due to (3.4), we get (4.2).

Proof of the uniqueness theorem. Let $\mathcal{A}(z)$ be an entire transcendental 2×2 matrix–function, which satisfies (0.2), (3.6), and assume that the Riemann surface (3.7) is of Widom type with (DCT). Let $\mathcal{A}_1(z)$ be an entire 2×2 matrix–function, which satisfies conditions

$$\overline{\mathcal{A}_1(\bar z)} = \mathcal{A}_1(z) \tag{4.4.1}$$

$$\det \mathcal{A}_1(z) = 1 \tag{4.4.2}$$

$$\frac{J}{z-\bar z} \geq \frac{\mathcal{A}_1(z) J \mathcal{A}_1(z)^*}{z-\bar z} \geq \frac{\mathcal{A}(z) J \mathcal{A}(z)^*}{z-\bar z} \tag{4.4.3}$$

and the normalization

$$\mathcal{A}_1(z) = \begin{bmatrix} \tau + \dots & (1/\tau - \tau)z + \dots \\ 0 + \dots & 1/\tau + \dots \end{bmatrix}, \quad \lambda_0 \leq \tau \leq 1.$$

Define $\tilde{\mathcal{A}}(z) = \mathcal{A}_1^{-1}(z)\mathcal{A}(z)\mathcal{A}_1(z)$. Then $\tilde{\mathcal{A}}(z)$ is an entire transcendental 2×2 matrix–function, which satisfies (0.2), (3.6), and has the same associated Riemann surface as $\mathcal{A}(z)$. According to proposition 2 there exists $\beta \in ?^*$ such that

$$f^\beta(\zeta)\tilde{\mathcal{A}}(z(\zeta)) = f^\beta(\zeta)\mathcal{A}_1^{-1}(z(\zeta))\mathcal{A}(z(\zeta))\mathcal{A}_1(z(\zeta)) = \lambda(z(\zeta))f^\beta(\zeta),$$

or

$$f^\beta(\zeta)\mathcal{A}_1^{-1}(z(\zeta))A(z(\zeta)) = \lambda(z(\zeta))f^\beta(\zeta)\mathcal{A}_1^{-1}(z(\zeta)).$$

But, $A(z(\zeta))$ has the eigenvector (3.8) with the same eigenvalue. So,

$$f^\alpha(\zeta)\mathcal{A}_1(z(\zeta)) = \lambda_1(\zeta)f^\beta(\zeta).$$

We are going to prove that $\lambda_1(\zeta)$ is an inner divisor of $\lambda(z(\zeta))$.

According to inequalities (4.4.3) and identity (3.9),

$$k^\alpha(\zeta,\zeta) \geq \lambda_1(\zeta)\overline{\lambda_1(\zeta)}k^\beta(\zeta,\zeta) \geq \lambda(z(\zeta))\overline{\lambda(z(\zeta))}k^\alpha(\zeta,\zeta), \tag{4.5}$$

and the same inequalities are fulfilled in the sense of the positive definite kernels. Hence, due to well known properties of positive definite kernels, for any $\zeta_0 \in \mathbb{D}$ the function $\lambda_1(\zeta)\overline{\lambda_1(\zeta_0)}k^\beta(\zeta,\zeta_0)$ belongs to $A_1^2(?\,,\alpha)$ and moreover

$$\|\lambda_1(\zeta)\overline{\lambda_1(\zeta_0)}k^\beta(\zeta,\zeta_0)\|^2_{A_1^2(\Gamma,\alpha)} \le |\lambda_1(\zeta_0)|^2 k^\beta(\zeta_0,\zeta_0).$$

Now we pass to the limit in the inequalities

$$1 \ge |\lambda_1(\rho\zeta)|^2 \frac{k^\beta(\rho\zeta,\rho\zeta)}{k^\alpha(\rho\zeta,\rho\zeta)} \ge |\lambda(z(\rho\zeta))|^2, \quad \rho \to 1,\ \zeta \in \mathbb{T}.$$

Lemma 6 implies that boundary value of $\lambda_1(\zeta)$ are unimodular a.e. on $\mathbb{T}$. Therefore, in fact,

$$\|\lambda_1(\zeta)\overline{\lambda_1(\zeta_0)}k^\beta(\zeta,\zeta_0)\|^2_{A_1^2(\Gamma,\alpha)} = |\lambda_1(\zeta_0)|^2 k^\beta(\zeta_0,\zeta_0).$$

Since $\lambda_1(\zeta)\overline{\lambda_1(\zeta_0)}k^\beta(\zeta,\zeta_0) \in \lambda_1 A_1^2(?\,,\beta)$ and form a complete set in this space, we have $\lambda_1 A_1^2(?\,,\beta) \subset A_1^2(?\,,\alpha)$. According to a character–automorphic counterpart of the Beurling–Helson Theorem $\lambda_1(\zeta)$ is an inner character–automorphic function.

In the same way we can prove that $\frac{\lambda\circ z}{\lambda_1} A_1^2(?\,,\alpha) \subset A_1^2(?\,,\beta)$. Therefore, λ_1 is an inner character–automorphic divisor of $\lambda \circ z$.

Note one more essential property of $\lambda_1(\zeta)$: $\overline{\lambda_1(\bar\zeta)} = \lambda_1(\zeta)$. Due to result of [1] (see, also, [8, Theorem 3.2]) any divisor of $\lambda\circ z$ with such a property is of the form

$$\lambda_1(\zeta) = (\lambda(z(\zeta))^t, \quad 0 \le t \le 1.$$

Denote by δ_t the character of $(\lambda\circ z)^t$. Then we have $\beta = \alpha\delta_t^{-1}$, and

$$(\lambda(z(\zeta))^t f^{\alpha\delta_t^{-1}}(\zeta) = f^\alpha(\zeta)\mathcal{A}_1(z(\zeta)). \tag{4.6}$$

Put here $\zeta = 0$,

$$\lambda_0^t \begin{bmatrix} \sqrt{k^{\alpha\delta_t^{-1}}(0)} & 0 \end{bmatrix} = [\,\sqrt{k^\alpha(0)} \quad 0\,] \begin{bmatrix} \tau & 0 \\ 0 & 1/\tau \end{bmatrix}.$$

So,

$$\tau = \lambda_0^t \sqrt{\frac{k^{\alpha\delta_t^{-1}}(0)}{k^\alpha(0)}}. \tag{4.7}$$

We note, that $k^\alpha(\zeta) - (\lambda(z(\zeta))^t \lambda_0^t k^{\alpha\delta_t^{-1}}(\zeta) \ne 0$ (as an element of $A_1^2(?\,,\alpha)$, $\forall\alpha \in ?^*$), and hence

$$\|k^\alpha(\zeta) - (\lambda(z(\zeta))^t \lambda_0^t k^{\alpha\delta_t^{-1}}(\zeta)\|^2_{A_1^2(\Gamma,\alpha)} = k^\alpha(0) - \lambda_0^{2t} k^{\alpha\delta_t^{-1}}(0) > 0.$$

It means that $\lambda_0^t \sqrt{\frac{k^{\alpha\delta_t^{-1}}(0)}{k^\alpha(0)}}$ is strictly monotonic and is changed from 1 to λ_0 then t runs over $[0,1]$.

Therefore, for a fixed $\tau \in [\lambda_0, 1]$ there is unique t, such that (4.7) holds.

For a fixed t relation (4.6) defines $\mathcal{A}_1(z(\zeta))$ uniquely. Really, taking conjugation in (4.6) and using (4.4.1), we get

$$(\lambda(z(\zeta))^{-t} f_*^{\alpha\delta_t^{-1}}(\zeta) = f_*^{\alpha}(\zeta)\mathcal{A}_1(z(\zeta)).$$

Together with (4.6) it means

$$\begin{bmatrix} (\lambda(z(\zeta))^t & 0 \\ 0 & (\lambda(z(\zeta))^{-t} \end{bmatrix} F^{\alpha\delta_t^{-1}}(\zeta) = F^{\alpha}(\zeta)\mathcal{A}_1(z(\zeta)).$$

And since $F^\alpha(\zeta)$ is invertible we have

$$\mathcal{A}_1(z(\zeta)) = F^\alpha(\zeta)^{-1}\Lambda^t(\zeta)F^{\alpha\delta_t^{-1}}(\zeta), \tag{4.8}$$

where

$$\Lambda^t(\zeta) = \begin{bmatrix} (\lambda(z(\zeta))^t & 0 \\ 0 & (\lambda(z(\zeta))^{-t} \end{bmatrix}.$$

Note that to prove existence part of the main theorem we only have to show that the right–hand side in (4.8) is an entire matrix–function which satisfies (4.4.3).

For an inner function $s \in H^\infty(?, \sigma)$ and $\alpha \in ?^*$ set

$$K_s(\alpha) = A_1^2(?, \alpha) \ominus sA_1^2(?, \sigma^{-1}\alpha).$$

If $s = s_1 s_2$, where s_1, s_2 are inner functions, $s_l \in H^\infty(?, \sigma_l)$, then

$$K_s(\alpha) = K_{s_1}(\alpha) \oplus s_1 K_{s_2}(\sigma_1^{-1}\alpha).$$

For $w \in H^\infty(?, \beta)$, define $T_w(K_s(\alpha)) : K_s(\alpha) \to K_s(\alpha\beta)$ as

$$T_w(K_s(\alpha))g = P_{K_s(\alpha\beta)}(wg),$$

where $P_{K_s(\alpha\beta)}$ is orthogonal projection onto the space $K_s(\alpha\beta)$. In this case the conjugate operator is of the form

$$T_w^*(K_s(\alpha))g = P_+(\alpha)(wg), \quad g \in K_s(\alpha\beta),$$

where $P_+(\alpha)$ is the orthogonal projection from $L^2_{dm|\mathbb{E}}$ onto $A_1^2(?, \alpha)$.

Note that if there exist functions $h_1 \in H^\infty(?, \beta^{-1})$ and $h_2 \in H^\infty(?, \sigma^{-1})$ such that

$$wh_1 + sh_2 = 1,$$

then $T_w(K_s(\alpha))$ is invertible for any $\alpha \in ?^*$.

When the space of the form $K_s(\alpha)$ is fixed we will write T_w instead of $T_w(K_s(\alpha))$.

We denote by $e^\alpha(\zeta, \zeta_0)$ the reproducing kernel of $K_s(\alpha)$. Evidently,

$$e^\alpha(\zeta, \zeta_0) = k^\alpha(\zeta, \zeta_0) - s(\zeta)\overline{s(\zeta_0)}k^{\alpha\sigma^{-1}}(\zeta, \zeta_0).$$

Let $f \in K_s(\alpha)$, then $\bar{s}f - A_1^2(?, \sigma^{-1}\alpha)$ and, therefore, it is of the form $\overline{\zeta f_*}$, where $f_* \in A_1^2(?, \sigma\alpha^{-1})$. In fact, $f_* \in K_s(\sigma\alpha^{-1})$. For $\zeta_0 \in \mathbb{D}, s(\zeta_0) \neq 0$, $f \to \frac{f_*(\zeta_0)}{s(\zeta_0)}$ is the antilinear functional on $K_s(\alpha)$. We define $e_*^\alpha(\zeta, \zeta_0) \in K_s(\alpha)$ by

$$\langle e_*^\alpha(\zeta, \zeta_0), f(\zeta)\rangle = \frac{f_*(\zeta_0)}{s(\zeta_0)}, \quad f \in K_s(\alpha).$$

For more detailed presentation of operator theory in such spaces, see [9].

Proof of the existence theorem. First we show that for $z_0 \in \mathbb{C}$ and $g_1 \in K_{b^2}(\alpha\mu\delta_t^{-1})$ there exists unique $g_2 \in K_{b^2}(\alpha\mu)$ such that

$$(\lambda \circ z)^t g_1 = (b - z_0\frac{b}{z})g + g_2, \tag{4.9}$$

where $g \in K_{(\lambda\circ z)^t}(\alpha)$.

Multiplying (4.9) by $\bar{b}^2$ and taking the projection $P_+(\alpha\mu^{-1})$, we get

$$P_+(\alpha\mu^{-1})\bar{b}^2(\lambda \circ z)^t g_1 = P_+(\alpha\mu^{-1})\left[\bar{b} - z_0\overline{\left(\frac{b}{z}\right)}\right] g = T^*_{\left(b - \bar{z}_0\frac{b}{z}\right)} g.$$

Note that $P_+(\alpha\mu^{-1})\bar{b}^2(\lambda \circ z)^t g_1 \in K_{(\lambda\circ z)^t}(\alpha\mu^{-1})$.

For $z_0 \in \mathbb{C} \setminus E$, put

$$h_1(\zeta) = \frac{z}{b}(\zeta)\frac{1 - \lambda(z(\zeta))/\lambda(z_0)}{z(\zeta) - z_0}, \; h_2(\zeta) = \frac{\lambda(z(\zeta))/\lambda(z_0)}{(\lambda \circ z)^t(\zeta)}.$$

For $z_0 \in E$, put

$$\begin{aligned} h_1(\zeta) =& \frac{z}{b}(\zeta)\frac{(1 - \lambda(z(\zeta))/\lambda(z_0 + i0))(1 - \lambda(z(\zeta))/\lambda(z_0 - i0))}{z(\zeta) - z_0}, \\ h_2(\zeta) =& \frac{1 - (1 - \lambda(z(\zeta))/\lambda(z_0 + i0))(1 - \lambda(z(\zeta))/\lambda(z_0 - i0))}{(\lambda \circ z)^t(\zeta)}. \end{aligned}$$

Evidently, $h_1 \in H^\infty(?, \mu^{-1})$, $h_2 \in H^\infty(?, \delta_t^{-1})$ and

$$(b - \bar{z}_0 b/z))h_1 + (\lambda \circ z)^t h_2 = 1.$$

Therefore $T_{(b - \bar{z}_0 b/z))} = T_b - \bar{z}_0 T_{b/z}$ is invertible.

Consider the vector

$$(\lambda \circ z)^t g_1 - [b - z_0 b/z][T_b^* - z_0 T_{b/z}^*]^{-1} P_+(\alpha\mu^{-1})\bar{b}^2(\lambda \circ z)^t g_1.$$

¿From one hand it belongs to $K_{b^2(\lambda\circ z)^t}(\alpha\mu)$, from the other hand it is orthogonal to $b^2 K_{(\lambda\circ z)^t}(\alpha\mu^{-1})$. Hence, it belongs to $K_{b^2}(\alpha\mu)$. So,

$$g_2(\zeta) = (\lambda\circ z)^t(\zeta)g_1(\zeta) - [b(\zeta) - z_0(b/z)(\zeta)]\{[T_b^* - z_0 T_{b/z}^*]^{-1} P_+(\alpha\mu^{-1})\bar{b}^2(\lambda\circ z)^t g_1\}(\zeta).$$

Let us fix the bases in $K_{b^2}(\alpha\mu)$ and $K_{b^2}(\alpha\mu\delta^{-t})$, then

$$\begin{aligned}&\left[\sqrt{\mathcal{P}(\alpha)}\frac{k^{\alpha\mu}(\zeta)}{\sqrt{k^{\alpha\mu}(0)}} \quad b(\zeta)\frac{k^{\alpha}(\zeta)}{\sqrt{k^{\alpha}(0)}}\right]\mathcal{B}(z_0)\begin{bmatrix}c_1\\c_2\end{bmatrix}\\&=(\lambda\circ z)^t(\zeta)g_1(\zeta)-[b(\zeta)-z_0(b/z)(\zeta)]\{[T_b^*-z_0T_{b/z}^*]^{-1}P_+(\alpha\mu^{-1})\bar{b}^2(\lambda\circ z)^t g_1\}(\zeta),\end{aligned} \tag{4.10}$$

with

$$g_1(\zeta)=\left[\sqrt{\mathcal{P}(\alpha\delta_t^{-1})}\frac{k^{\alpha\mu\delta_t^{-1}}(\zeta)}{\sqrt{k^{\alpha\mu\delta_t^{-1}}(0)}} \quad b(\zeta)\frac{k^{\alpha\delta_t^{-1}}(\zeta)}{\sqrt{k^{\alpha\delta_t^{-1}}(0)}}\right]\begin{bmatrix}c_1\\c_2\end{bmatrix}.$$

So, $\mathcal{B}(z_0)$ is an entire matrix function of z_0. But if we put $\zeta=\zeta_0$ ($z(\zeta_0)=z_0$) in (4.10), we get

$$\begin{aligned}&\left[\sqrt{\mathcal{P}(\alpha)}\frac{k^{\alpha\mu}(\zeta_0)}{\sqrt{k^{\alpha\mu}(0)}} \quad b(\zeta_0)\frac{k^{\alpha}(\zeta_0)}{\sqrt{k^{\alpha}(0)}}\right]\mathcal{B}(z_0)\begin{bmatrix}c_1\\c_2\end{bmatrix}\\&=(\lambda\circ z)^t(\zeta_0)\left[\sqrt{\mathcal{P}(\alpha\delta_t^{-1})}\frac{k^{\alpha\mu\delta_t^{-1}}(\zeta_0)}{\sqrt{k^{\alpha\mu\delta_t^{-1}}(0)}} \quad b(\zeta_0)\frac{k^{\alpha\delta_t^{-1}}(\zeta_0)}{\sqrt{k^{\alpha\delta_t^{-1}}(0)}}\right]\begin{bmatrix}c_1\\c_2\end{bmatrix}.\end{aligned}$$

Therefore, (see (4.3) and (4.8))

$$\mathcal{B}(z_0)=F^{\alpha}(\zeta_0)^{-1}\Lambda^t(\zeta_0)F^{\alpha\delta_t^{-1}}(\zeta_0).$$

Now we prove the inequality

$$\Phi(\zeta)=\frac{F^{\alpha}(\zeta)JF^{\alpha}(\zeta)^*}{z(\zeta)-\overline{z(\zeta)}}-\Lambda^t(\zeta)\frac{F^{\alpha\delta_t^{-1}}(\zeta)JF^{\alpha\delta_t^{-1}}(\zeta)^*}{z(\zeta)-\overline{z(\zeta)}}\Lambda^t(\zeta)^*\geq 0, \tag{4.11}$$

what is equivalent to

$$\frac{J-\mathcal{B}(z)J\mathcal{B}^*(z)}{z-\bar{z}}\geq 0.$$

Note that to prove

$$\frac{\mathcal{B}(z)J\mathcal{B}^*(z)-\mathcal{A}(z)J\mathcal{A}^*(z)}{z-\bar{z}}\geq 0,$$

we only have to use (4.11) with $t:=1-t$ and (then) $\alpha:=\alpha\delta_t^{-1}$.

The vector

$$\begin{aligned}e^{\alpha}(\zeta,\zeta_0)=&k^{\alpha}(\zeta,\zeta_0)-(\lambda\circ z)^t(\zeta)\overline{(\lambda\circ z)^t(\zeta_0)}k^{\alpha\delta_t^{-1}}(\zeta,\zeta_0)\\=&\frac{f^{\alpha}(\zeta)Jf^{\alpha}(\zeta_0)^*}{z(\zeta)-\overline{z(\zeta_0)}}-(\lambda\circ z)^t(\zeta)\overline{(\lambda\circ z)^t(\zeta_0)}\frac{f^{\alpha\delta_t^{-1}}(\zeta)Jf^{\alpha\delta_t^{-1}}(\zeta_0)^*}{z(\zeta)-\overline{z(\zeta_0)}}\end{aligned}$$

is the reproducing kernel of $K_{(\lambda\circ z)^t}(\alpha)$. According to the definition

$$\langle e_*^{\alpha}(\zeta,\bar{\zeta}_0),e^{\alpha}(\zeta,\zeta_1)\rangle=\left(\frac{\{k^{\alpha}(\zeta,\zeta_1)-(\lambda\circ z)^t(\zeta)\overline{(\lambda\circ z)^t(\zeta_1)}k^{\alpha\delta_t^{-1}}(\zeta,\zeta_1)\}_*}{(\lambda\circ z)^t(\zeta)}\right)_{\zeta=\bar{\zeta}_0}.$$

Since

$$\overline{\zeta k^{\alpha}(\zeta,\zeta_1)} = \frac{f^{\alpha}(\zeta_1) J \begin{bmatrix} f^{\alpha}_{*1}(\zeta) \\ f^{\alpha}_{*2}(\zeta) \end{bmatrix}}{z(\zeta_1) - z(\zeta)},$$

and $f^{\alpha}_{*l}(\bar{\zeta}_0) = \overline{f^{\alpha}_{*l}(\zeta_0)}$, $l = 1, 2$ we get

$$\langle e^{\alpha}_{*}(\zeta,\bar{\zeta}_0), e^{\alpha}(\zeta,\zeta_1)\rangle = \frac{f^{\alpha}(\zeta_1) J f^{\alpha}_{*}(\zeta_0)^{*}}{z(\zeta_1) - \overline{z(\zeta_0)}} - \frac{(\lambda\circ z)^{t}(\zeta_1)}{\overline{(\lambda\circ z)^{t}(\zeta_0)}} \frac{f^{\alpha\delta_t^{-1}}(\zeta_1) J f^{\alpha\delta_t^{-1}}_{*}(\zeta_0)^{*}}{z(\zeta_1) - \overline{z(\zeta_0)}}.$$

Therefore,

$$\Phi(\zeta_0) = \begin{bmatrix} \langle e^{\alpha}(\zeta,\zeta_0), e^{\alpha}(\zeta,\zeta_0)\rangle & \langle e_{*}(\zeta,\bar{\zeta}_0), e^{\alpha}(\zeta,\zeta_0)\rangle \\ \langle e^{\alpha}(\zeta,\bar{\zeta}_0), e^{\alpha}_{*}(\zeta,\bar{\zeta}_0)\rangle & \langle e^{\alpha}_{*}(\zeta,\bar{\zeta}_0), e^{\alpha}_{*}(\zeta_0,\bar{\zeta}_0)\rangle \end{bmatrix} \geq 0.$$

The theorem is proved.

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