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A remark on a martingale inequality of J. Bourgain

by

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In this note we improve a maringale inequality of J. Bourgain and simplify its original proof.

We let $\mathcal{D}$ denote the collection of dyadic intervals in the interval $[0,1]$. For $I \in \mathcal{D}$ we let h_I be the L^∞-normalized Haar function supported on the interval I. The Haar function h_I takes the value $+1$ on the left half of the interval I, and the value -1 on the right half of I. For a function h with Haar expansion $h = \sum b_I h_I$, $b_I \in \mathbb{R}$, we say that h belongs to dyadic BMO space if

$$\sup_{I \in \mathcal{D}} \left(\frac{1}{|I|} \sum_{J \subseteq I} b_J^2 |J| \right)^{1/2} < \infty.$$

We write BMO(δ) for the dyadic BMO space and we denote the above supremum by $||h||_{\mathrm{BMO}(\delta)}$. For $f = \sum a_J h_J$ we have the dyadic square function given as

$$S(f)(x) = \left(\sum a_J^2 \mathbf{1}_J(x) \right)^{1/2}.$$

If g is a positive integrable function then the martingale inequality of J. Bourgain relates the above expressions as follows.

$$\int g S(h) \geq \delta \int f h dx - \delta^{-1} ||S(f) - g||_{L^1}^{1/2} ||S(f)||_{L^1}^{1/2} ||h||_{\mathrm{BMO}(\delta)},$$

where $\delta > 0$ is a universal constant. In [B] J. Bourgain uses this inequality to obtain estimates from below for

$$\int g S(h)$$

under the hypotheses that $\int fh = 1$ and that the error term $||S(f) - g||_{L^1}^{1/2}$ is small. It follows from the proof in [B] that improvements in the estimates for the error term translate into better estimates for the main theorem in [B]. It is the puprose of this note to improve this error term. We also obtain simple numerical constants from a simple straightforward proof.

Theorem 1 *Let $h \in BMO(\delta)$, $g \in L^1([0,1])$, $g \geq 0$, and let f be a function with $S(f) \in L^1([0,1])$. Then*

$$\int g S(h) dx \geq \frac{1}{2} \int f h dx - 2 ||S(f) - g||_{L^1} ||h||_{BMO}.$$

COMMENT. The proof we give is just the standard proof of H^1 – BMO duality as in [F-S] pp. 148-149. Clearly it is only the contrast to the original – quite delicate – argument of J. Bourgain [B] that justifies the presentation below.

PROOF. We let $f = \sum a_I h_I$ and $h = \sum b_I h_I$ be the respective Haar expansions of f respectively h. Then we write

$$\begin{aligned} S(f,m)(x) &= \left(\sum_{|I| \le 2^{-m}} a_I^2 \mathbf{1}_I(x) \right)^{1/2}, \\ h^{\#}(x) &= \sup_{I \ni x} \left(\frac{1}{|I|} \sum_{J \subseteq I} b_J^2 |J| \right)^{1/2}. \end{aligned}$$

Now we define the following stopping time

$$m(x) = \inf\{m : S(h,m)(x) < 2h^{\#}(x)\}.$$

We will use the following estimate which will be proved below

$$|\{x \in I : 2^{-m(x)} < |I|\}| \le |I|/2.$$

It follows from biorthogonality of the Haar functions, Fubini's theorem and the Cauchy Schwarz inequality that

$$\begin{aligned} \int f(x)h(x)dx &\le \int \sum_{I \in \mathcal{D}} |a_I||b_I|\mathbf{1}_I(x)dx \\ &\le 2 \int \sum_{\{I : |I| < 2^{-m(x)}\}} |a_I b_I| \mathbf{1}_I(x)dx \\ &\le 2 \int S(f)(x) S(h, m(x))(x)dx. \end{aligned}$$

We now add and subtract the function g, and we finish the proof using the definig property of the stopping time $m(x)$.

$$\begin{aligned} \int f(x)h(x)dx &\le 2 \int (S(f)(x) - g(x)) S(h,m(x))(x)dx + 2 \int g(x) S(h, m(x))dx \\ &\le 4 \int (S(f)(x) - g(x)) h^{\#}(x)dx + 2 \int g(x) S(h)(x)dx \\ &\le 4||S(f) - g||_1 ||h||_{\mathrm{BMO}(\delta)} + 2 \int g(x)S(h)(x)dx. \end{aligned}$$

We used the equality $||h^{\#}||_\infty = ||h||_{\mathrm{BMO}(\delta)}$ to obtain the last line.

∎

It remains to prove the estimate $|\{x \in I : 2^{-m(x)} < |I|\}| \le |I|/2$. We fix $I \in \mathcal{D}$ and write $A = \{x \in I : 2^{-m(x)} < |I|\}$. Then we choose $m \in \mathbb{N}$ such that

$$|I| = 2^{-m}$$

Note that for $x \in A$ we have the following pointwise estimate,

$$\begin{aligned} S^2(h, m(x)-1)(x) &\geq 4h^{\#2}(x) \\ &\geq 4\frac{1}{|\tilde{I}|}\int_{\tilde{I}} S^2(h, m-1)(t)dt, \end{aligned}$$

where $\tilde{I}$ is the dyadic interval satisfying $I \subseteq \tilde{I}$, $|\tilde{I}| = 2|I|$. We also have that

$$S^2(h, m-1)(x) \geq S^2(h, m(x)-1)(x), \text{ for } x \in A.$$

Hence

$$\begin{aligned} \frac{1}{|\tilde{I}|}\int_{\tilde{I}} S^2(h, m-1)(x)dx &\geq \frac{1}{|\tilde{I}|}\int_A S^2(h, m(x)-1)(x)dx \\ &\geq 4\frac{|A|}{|\tilde{I}|}\frac{1}{|\tilde{I}|}\int_{\tilde{I}} S^2(h, m-1)(x)dx. \end{aligned}$$

Cancelling the following factor from both sides of the above estimate

$$\frac{1}{|\tilde{I}|}\int_{\tilde{I}} S^2(h, m-1)(x)dx$$

gives

$$|A| \leq |I|/2$$

as claimed.

REMARKS.

1. Note that we actually proved more than we claimed. In fact we showed that the integral

$$\int g(x)\min\{S(h)(x), 2h^{\#}(x)\}dx$$

dominates the expression

$$\frac{1}{2}\int fhdx - 2||S(f) - g||_{L^1}||h||_{\mathrm{BMO}(\delta)}.$$

We should remark that this improvement of Bourgain's martingale inequality has further consequences. It allowes us to break the proof of the non-isomorphism theorem in [B] into two independent pieces, in such a way that the only place where one uses the notion of "order-inversion" is in Lemma 5 of [B]. In this way the content of the present paper helps to clearify somewhat the role played by the concept of "order-inversion" in the proof of the non-isomorphism between H^1 spaces.

2. The above proof uses only well known, and well understood tools developed to prove $H^1 - \mathrm{BMO}$ duality. Therefore it is clear that the validity of Theorem 1 is not limited to the case of dyadic martingales. Analogous versions can be obtained, e.g., for the case of H^1 spaces consisting of harmonic functions in the upper half space $\mathbb{R}^{n+1}_+$ which are defined as follows. For an integrable function $f : \mathbb{R}^n \to \mathbb{R}$ we denote by $F : \mathbb{R}^{n+1}_+ \to \mathbb{R}$ its harmonic extension to the upper half space

$$\mathbb{R}^{n+1}_+ = \{(y,t) : y \in \mathbb{R}^n, t > 0\}.$$

Then the square function is

$$S(f)(x) = \left(\int_{\Gamma(x)} |\nabla F(y,t)|^2 t^{1-n} dt dx \right)^{1/2},$$

where $\Gamma(x) = \{(y,t) \in \mathbb{R}^{n+1}_+ : |x-y| < t\}$. For $h : \mathbb{R}^n \to \mathbb{R}$, locally integrable, we let

$$||h||_{\mathrm{BMO}(\mathbb{R}^n)} = \left(\sup_Q \int_Q \left| h(x) - \int_Q h(y) \frac{dy}{|Q|} \right|^2 \frac{dx}{|Q|} \right)^{1/2},$$

where the supremum is extended over all cubes in $\mathbb{R}^n$. Finally we let $g \in L^1(\mathbb{R}^n)$ be a non negative integrable function. With essentially the same proof as above we can show that

$$\int_{\mathbb{R}^n} g(x) S(h)(x) dx \geq \delta \int_{\mathbb{R}^n} f(x) h(x) dx - \delta^{-1} ||S(f) - g||_{L^1(\mathbb{R}^n)} ||h||_{\mathrm{BMO}(\mathbb{R}^n)},$$

where $\delta > 0$ is a positive universal constant. (See [F-S] pp.148, 149.)

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