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Vienna, Preprint ESI 691 (1999) April 7, 1999

Supported by Federal Ministry of Science and Transport, Austria
Available via http://www.esi.ac.at

JOHN'S THEOREM FOR AN ARBITRARY PAIR OF CONVEX BODIES

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Abstract

We provide a generalization of John's representation of the identity for the maximal volume position of L inside K, where K and L are arbitrary smooth convex bodies in $\mathbb{R}^n$. ¿From this representation we obtain Banach-Mazur distance and volume ratio estimates.

1. Introduction.

The definition of the Banach-Mazur distance between symmetric convex bodies can be extended to the non-symmetric case as follows [Gr]: Let K and L be two convex bodies in $\mathbb{R}^n$. Their geometric distance is defined by

$$\tilde{d}(K,L)=\inf\{ab:(1/b)L\subseteq K\subseteq aL\}. \tag{1}$$

If $z_1,z_2\in\mathbb{R}^n$, we consider the translates $K-z_1$ and $L-z_2$ of K and L, and their distance with respect to z_1,z_2,

$$d_{z_1,z_2}(K,L)=\inf\{\tilde{d}(T(K-z_1),L-z_2)\}, \tag{2}$$

where the inf is taken over all invertible linear transformations T of $\mathbb{R}^n$. Finally, we let z_1,z_2 vary and define

$$d(K,L)=\inf\{d_{z_1,z_2}(K,L):z_1,z_2\in\mathbb{R}^n\}. \tag{3}$$

John's theorem [J] provides a first estimate for $d(K,L)$. If K is any convex body in $\mathbb{R}^n$ and E is its maximal or minimal volume ellipsoid, then $d_{z,z}(K,E)\leq n$, where z is the center of E. Actually, the distance between the simplex and the ball is equal to n, and the simplex is the only body with this property [P]. It follows that the distance between any two convex bodies is at most n^2. Rudelson [R] has recently proved that $d(K,L)\leq cn^{4/3}\log^{\beta}n$ for some absolute constants $c,\beta>0$ (see also recent work of Litvak and Tomczak-Jaegermann [LTJ]). A well-known theorem of Gluskin [Gl] shows that $d(K,L)$ can be of the order of n even for symmetric bodies K and L.

In this paper we study the *maximal volume position* of a body L inside K: we say that L *is of maximal volume in* K if $L\subseteq K$ and, for every $w\in\mathbb{R}^n$ and every volume preserving linear transformation T of $\mathbb{R}^n$, the affine image $w+T(L)$ of L

is not contained in the interior of K. A simple compactness argument shows that for every pair of convex bodies K and L there exists an affine image $\tilde{L}$ of L which is of maximal volume in K.

Our main result is the following:

Theorem. *Let L be of maximal volume in K. If $z \in \mathrm{int}(L)$, we can find contact points $v_1, \ldots, v_m$ of $K - z$ and $L - z$, contact points $u_1, \ldots, u_m$ of $(K-z)^\circ$ and $(L-z)^\circ$, and positive reals $\lambda_1, \ldots, \lambda_m$, such that: $\sum \lambda_j u_j = o$, $\langle u_j, v_j \rangle = 1$, and*

$$Id = \sum_{j=1}^{m} \lambda_j u_j \otimes v_j. \tag{4}$$

We shall prove the above fact under the assumption that both K and L are smooth enough. The theorem may be viewed as a generalization of John's representation of the identity even in the case where L is the Euclidean unit ball. This generalization was observed by V.D. Milman in the case where K and L are o-symmetric and $z = o$ (see [TJ], Theorem 14.5). Using the theorem, we give a direct proof of the fact that $d(K, L) \leq n$ when both K and L are symmetric, and we obtain the estimate $d(K, L) \leq 2n - 1$ when L is symmetric and K is any convex body (this was recently proved by Lassak [L]).

Using the maximal volume position of L inside K, one can naturally extend the notion of *volume ratio* to an arbitrary pair of convex bodies. We define

$$\mathrm{vr}(K, L) = \left(\frac{|K|}{|\tilde{L}|} \right)^{\frac{1}{n}}, \tag{5}$$

where $\tilde{L}$ is an affine image of L which is of maximal volume in K (by $|\cdot|$ we denote n-dimensional volume). In Section 3, we prove the following general estimate:

Theorem. *Let K and L be two convex bodies in $\mathbb{R}^n$. Then,*

$$\mathrm{vr}(K, L) \leq n. \tag{6}$$

The same estimate can be given through K. Ball's result on $\mathrm{vr}(K, D_n)$ and $\mathrm{vr}(D_n, K)$, where D_n is the Euclidean unit ball. Ball [Ba] proved that both $\mathrm{vr}(K, D_n)$ and $\mathrm{vr}(D_n, K)$ are maximal when K is the simplex S_n. It follows that

$$\mathrm{vr}(K, L) \leq \mathrm{vr}(K, D_n)\mathrm{vr}(D_n, L) \leq \mathrm{vr}(S_n, D_n)\mathrm{vr}(D_n, S_n) = n.$$

However, our proof is direct and might lead to a better estimate; it might be true that $\mathrm{vr}(K, L)$ is always bounded by $c\sqrt{n}$.

Acknowledgement. The first named author acknowledges the hospitality of the Erwin Schrödinger International Institute for Mathematical Physics in Vienna, where his contribution to this work has been completed.

2. The main theorem and distance estimates.

We assume that $\mathbb{R}^n$ is equipped with a Euclidean structure $\langle\cdot,\cdot\rangle$, and denote the corresponding Euclidean norm by $|\cdot|$. We write D_n for the Euclidean unit ball, and S^{n-1} for the unit sphere.

If W is a convex body in $\mathbb{R}^n$ and $z \in \mathrm{int}(W)$, we define the radial function $\rho_W(z,\cdot)$ of W with respect to z by

$$\rho_W(z,\theta) = \max\{\lambda > 0 : z + \lambda\theta \in W\} \tag{7}$$

for $\theta \in S^{n-1}$, and extend this definition to $\mathbb{R}^n \backslash \{z\}$ by

$$\rho_W(z,x) = \frac{1}{t}\rho_W(z,\theta), \tag{8}$$

where $x = z + t\theta$, $t > 0$ and $\theta \in S^{n-1}$. If $\theta \in S^{n-1}$, we will write $\rho_W(z,\theta)$ instead of $\rho_W(z, z+\theta)$ (this will cause no confusion).

The polar body W^z of W with respect to $z \in \mathrm{int}(W)$ is the body

$$W^z = (W - z)^\circ = \{y \in \mathbb{R}^n : \langle y, x - z\rangle \leq 1 \text{ for all } x \in W\}. \tag{9}$$

Let o denote the origin. Since $\rho_W(z,x) = \rho_{W-z}(o, x-z)$, the support function h_{W^z} of W^z satisfies

$$h_{W^z}(x - z) = \frac{1}{\rho_W(z,x)} \tag{10}$$

for all $x \in \mathbb{R}^n \backslash \{z\}$.

Recall that, if $o \in \mathrm{int}(W)$, W is strictly convex and h_W is continuously differentiable, then $\nabla h_W(\theta)$ is the unique point on the boundary of W at which the outer unit normal to W is θ, and $\nabla h_W(\lambda\theta) = \nabla h_W(\theta)$ for all $\lambda > 0$.

With these definitions, we have the following description of the maximal volume position of L in K:

Lemma 1. *Let K and L be two convex bodies in $\mathbb{R}^n$. Then, L is of maximal volume in K if and only if, for every $z \in \mathrm{int}(L)$, for every $w \in \mathbb{R}^n$ and every volume preserving T, there exists $\theta \in S^{n-1}$ such that*

$$\rho_K\Big(z, z + w + T(\rho_L(z,\theta)\theta)\Big) \leq 1. \quad \square \tag{11}$$

We assume that the bodies K and L are smooth enough: we ask that they are strictly convex and their support functions are twice continuously differentiable. Under this assumption, we have that h_{K^z} and h_{L^z} are twice continuously differentiable for every $z \in \mathrm{int}(L)$.

Lemma 2. *Let L be of maximal volume in K, and $z \in \mathrm{int}(L)$. Then, for every $w \in \mathbb{R}^n$ and every $S \in L(\mathbb{R}^n, \mathbb{R}^n)$ we can find $\theta \in S^{n-1}$ such that $\rho_L(z,\theta) = \rho_K(z,\theta)$ and*

$$h_{K^z}\Big(w + \rho_K(z,\theta)S(\theta)\Big) \geq \frac{\mathrm{tr}S}{n}. \tag{12}$$

Proof: We follow the argument of [GM]. Let $w \in \mathbb{R}^n$ and $S \in L(\mathbb{R}^n, \mathbb{R}^n)$. If $\varepsilon > 0$ is small enough, then $T_\varepsilon = (I + \varepsilon S)/[\det(I + \varepsilon S)]^{1/n}$ is volume preserving, hence, using (10) and Lemma 1 for T_ε and εw, we find $\theta_\varepsilon \in S^{n-1}$ such that

$$h_{K^z}\Big(\varepsilon w + T_\varepsilon(\rho_L(z, \theta_\varepsilon)\theta_\varepsilon)\Big) \geq 1. \tag{13}$$

Since $[\det(I + \varepsilon S)]^{1/n} = 1 + \varepsilon \frac{\mathrm{tr} S}{n} + O(\varepsilon^2)$, we get

$$h_{K^z}\Big(\rho_L(z, \theta_\varepsilon)\theta_\varepsilon + \varepsilon w + \varepsilon \rho_L(z, \theta_\varepsilon) S(\theta_\varepsilon)\Big) \geq 1 + \varepsilon \frac{\mathrm{tr} S}{n} + O(\varepsilon^2). \tag{14}$$

Since $L \subseteq K$, we have $h_{K^z}(\rho_L(z, \theta_\varepsilon)\theta_\varepsilon) = \rho_L(z, \theta_\varepsilon)/\rho_K(z, \theta_\varepsilon) \leq 1$, and the subadditivity of h_{K^z} gives

$$h_{K^z}\Big(w + \rho_L(z, \theta_\varepsilon) S(\theta_\varepsilon)\Big) \geq \frac{\mathrm{tr} S}{n} + O(\varepsilon). \tag{15}$$

By compactness, we can find $\varepsilon_m \to 0$ and $\theta \in S^{n-1}$ such that $\theta_{\varepsilon_m} \to \theta$. Then, taking limits in (15) we get

$$h_{K^z}\Big(w + \rho_L(z, \theta) S(\theta)\Big) \geq \frac{\mathrm{tr} S}{n}, \tag{16}$$

and taking limits in (13) we see that $h_{K^z}(\rho_L(z, \theta)\theta) \geq 1$, which forces $\rho_L(z, \theta) = \rho_K(z, \theta)$. □

Making one more step, we obtain the following condition:

Lemma 3. *Let L be of maximal volume in K, and $z \in \mathrm{int}(L)$. Then, for every $w \in \mathbb{R}^n$ and every $T \in L(\mathbb{R}^n, \mathbb{R}^n)$ we can find $\theta \in S^{n-1}$ such that $\rho_L(z, \theta) = \rho_K(z, \theta)$ and*

$$\langle \nabla h_{K^z}(\theta), w + \rho_K(z, \theta) T(\theta)\rangle \geq \frac{\mathrm{tr} T}{n}. \tag{17}$$

Proof: Let $T \in L(\mathbb{R}^n, \mathbb{R}^n)$, and define $S_\varepsilon = I + \varepsilon T$, $\varepsilon > 0$. By Lemma 2, we can find $\theta_\varepsilon \in S^{n-1}$ such that $\rho_K(z, \theta_\varepsilon) = \rho_L(z, \theta_\varepsilon)$ and

$$h_{K^z}(\varepsilon w + \rho_K(z, \theta_\varepsilon)\theta_\varepsilon + \varepsilon \rho_K(z, \theta_\varepsilon) T(\theta_\varepsilon)) \geq \frac{\mathrm{tr}(I + \varepsilon T)}{n} = 1 + \varepsilon \frac{\mathrm{tr} T}{n}. \tag{18}$$

The left hand side is equal to

$$\begin{aligned} & h_{K^z}(\rho_K(z, \theta_\varepsilon)\theta_\varepsilon) + \varepsilon \langle \nabla h_{K^z}(\theta_\varepsilon), w + \rho_K(z, \theta_\varepsilon) T(\theta_\varepsilon)\rangle + O(\varepsilon^2) \\ & = 1 + \varepsilon \langle \nabla h_{K^z}(\theta_\varepsilon), w + \rho_K(z, \theta_\varepsilon) T(\theta_\varepsilon)\rangle + O(\varepsilon^2). \end{aligned} \tag{19}$$

Therefore,

$$\langle \nabla h_{K^z}(\theta_\varepsilon), w + \rho_K(z, \theta_\varepsilon) T(\theta_\varepsilon)\rangle \geq \frac{\mathrm{tr} T}{n} + O(\varepsilon). \tag{20}$$

Choosing again $\varepsilon_m \to 0$ such that $\theta_{\varepsilon_m} \to \theta \in S^{n-1}$, we see that $\rho_K(z,\theta) = \rho_L(z,\theta)$ and θ satisfies (17). $\square$

Lemma 3 and a separation argument give us a generalization of John's representation of the identity:

Theorem 1. *Let L be of maximal volume in K, and $z \in \mathrm{int}(L)$. There exist $m \leq n^2+n+1$ vectors $\theta_1,\dots,\theta_m \in S^{n-1}$ such that $\rho_K(z,\theta_j) = \rho_L(z,\theta_j)$ and $\lambda_1,\dots,\lambda_m > 0$, such that:*

(i) $\sum \lambda_j \nabla h_{K^z}(\theta_j) = o$,

(ii) $Id = \sum \lambda_j [(\nabla h_{K^z}(\theta_j)) \otimes (\rho_K(z,\theta_j)\theta_j)]$.

Proof: We identify the affine transformations of $\mathbb{R}^n$ with points in $\mathbb{R}^{n^2+n}$, and consider the set

$$\mathcal{C} = \mathrm{co}\Big\{[\nabla h_{K^z}(\theta) \otimes \rho_K(z,\theta)\theta] + \nabla h_{K^z}(\theta) : \theta \in S^{n-1}, \rho_K(z,\theta) = \rho_L(z,\theta)\Big\}. \tag{21}$$

Then, $\mathcal{C}$ is a compact convex set with the Euclidean metric, and we claim that $Id/n \in \mathcal{C}$. If not, there exist $w \in \mathbb{R}^n$ and $T \in L(\mathbb{R}^n,\mathbb{R}^n)$ such that

$$\langle Id/n, T+w\rangle > \Big\langle [\nabla h_{K^z}(\theta) \otimes \rho_K(z,\theta)\theta] + \nabla h_{K^z}(\theta), T+w\Big\rangle \tag{22}$$

whenever $\rho_K(z,\theta) = \rho_L(z,\theta)$. But, (22) is equivalent to

$$\frac{\mathrm{tr}T}{n} > \langle \nabla h_{K^z}(\theta), w + \rho_K(z,\theta)T(\theta)\rangle, \tag{23}$$

and this contradicts Lemma 3.

Caratheodory's theorem shows that we can find $m \leq n^2+n+1$ and positive reals $\lambda_1,\dots,\lambda_m$ such that

$$Id = \sum_{j=1}^{m} \lambda_j \Big([\nabla h_{K^z}(\theta_j) \otimes \rho_K(z,\theta_j)\theta_j] + \nabla h_{K^z}(\theta_j)\Big), \tag{24}$$

for $\theta_1,\dots,\theta_m \in S^{n-1}$ with $\rho_K(z,\theta_j) = \rho_L(z,\theta_j)$. This completes the proof. $\square$

Remark. Let $\theta \in S^{n-1}$ be such that $\rho_K(z,\theta) = \rho_L(z,\theta)$. Observe that

$$\langle \nabla h_{K^z}(\theta), \rho_K(z,\theta)\theta\rangle = \rho_K(z,\theta) h_{K^z}(\theta) = 1. \tag{25}$$

Also, $x = \nabla h_{L^z}(\theta)$ is the unique point of L^z for which $\langle x,\theta\rangle = h_{L^z}(\theta) = h_{K^z}(\theta)$. Since $\langle \nabla h_{K^z}(\theta),\theta\rangle = h_{K^z}(\theta)$ and $\nabla h_{K^z}(\theta) \in K^z \subseteq L^z$, we must have

$$\nabla h_{K^z}(\theta) = \nabla h_{L^z}(\theta). \tag{26}$$

Hence, the theorem can be stated in the following form:

Theorem 2. *Let L be of maximal volume in K. For every $z \in \mathrm{int}(L)$, we can find contact points $v_1, \ldots, v_m$ of $K - z$ and $L - z$, contact points $u_1, \ldots, u_m$ of K^z and L^z, and positive reals $\lambda_1, \ldots, \lambda_m$, such that: $\sum \lambda_j u_j = o$, $\langle u_j, v_j \rangle = 1$, and*

$$Id = \sum_{j=1}^{m} \lambda_j u_j \otimes v_j. \quad \square \tag{27}$$

Remark. The analogue of the Dvoretzky-Rogers lemma [DR] in the context of Theorem 2 is the following: If F is a k-dimensional subspace of $\mathbb{R}^n$ and P_F denotes the orthogonal projection onto F, then there exists $j \in \{1, \ldots, m\}$ such that

$$\langle P_F(u_j), P_F(v_j) \rangle \geq \frac{k}{n}.$$

This can be easily checked, since

$$k = \mathrm{tr} P_F = \sum_{j=1}^{m} \lambda_j \langle P_F(u_j), P_F(v_j) \rangle,$$

and $\sum \lambda_j = n$.

As an application of Theorem 1, we give a direct proof of the fact that the diameter of the Banach-Mazur compactum is bounded by n:

Proposition 1. *Let K and L be symmetric convex bodies in $\mathbb{R}^n$. Then, $d(K, L) \leq n$.*

Proof: We may assume that K and L satisfy our smoothness hypotheses, and that K is symmetric about o. Let L_1 be an affine image of L which is of maximal volume in K.

Claim: L_1 is also symmetric about o.

[Let z be the center of L_1. Then $L_1 = 2z - L_1 \subseteq K$ and the symmetry of K shows that $L_1 - 2z \subseteq K$. It follows that

$$\tilde{L} = L_1 - z = \frac{L_1 + (L_1 - 2z)}{2} \subseteq K, \tag{28}$$

and $L_1 - z$ is o-symmetric. If $z \neq o$, we obtain a contradiction as follows: we define a linear map T which leaves z^{-} unchanged and sends z to $(1+\alpha)z$, where $0 < \alpha < |z|^2 / h_{L_1 - z}(z)$. One can easily check that $T(L_1 - z) \subseteq \mathrm{co}(L_1, L_1 - 2z) \subseteq K$ and $|T(L_1 - z)| = (1+\alpha)|L_1| > |L_1|$.]

We write L for L_1. Let $x \in \mathbb{R}^n$ and choose $z = w = o$ and $T(y) = \langle \nabla h_{L^\circ}(x), y \rangle x$ in Lemma 3. Then there exists $\theta \in S^{n-1}$ such that $\rho_K(o, \theta) = \rho_L(o, \theta)$ and

$$\Big\langle \nabla h_{K^\circ}(\theta), \langle \nabla h_{L^\circ}(x), \rho_L(o, \theta)\theta \rangle x \Big\rangle \geq \frac{h_{L^\circ}(x)}{n}. \tag{29}$$

But, $\nabla h_{L^\circ}(x) \in L^\circ$ and $\rho_L(o,\theta)\theta \in L$. Since L is o-symmetric, we have

$$|\langle \nabla h_{L^\circ}(x), \rho_L(o,\theta)\theta\rangle| \le 1. \tag{30}$$

Using now the o-symmetry of K and the fact that $\nabla h_{K^\circ}(\theta) \in K^\circ$, from (29) and (30) we get

$$h_{K^\circ}(x) \ge \frac{h_{L^\circ}(x)}{n}. \tag{31}$$

Therefore, $L^\circ \subseteq nK^\circ$, and this shows that $K \subseteq nL$. □

We now assume that L is symmetric and K is any convex body:

Proposition 2. *Let L be a symmetric convex body and K be any convex body in $\mathbb{R}^n$. Then, $d(K,L) \le 2n-1$.*

Proof: We may assume that L is of maximal volume in K and L is symmetric about o.

Let $d > 0$ be the smallest positive real for which $h_{L^\circ}(y) \le dh_{K^\circ}(y)$ for all $y \in \mathbb{R}^n$. Then, duality, the symmetry of L and the fact that $L \subseteq K$ show that $h_K(-x) \le dh_L(-x) = dh_L(x) \le dh_K(x)$ for every $x \in \mathbb{R}^n$.

We define $T(y) = \langle n\nabla h_{L^\circ}(x), y\rangle x$ and $w = \gamma x$, where $\gamma \in [0,n)$ is to be determined. ¿From Lemma 3, there exists $\theta \in S^{n-1}$ such that $\rho_K(o,\theta) = \rho_L(o,\theta)$ and

$$\Big\langle \nabla h_{K^\circ}(\theta), \gamma x + n\langle \nabla h_{L^\circ}(x), \rho_L(o,\theta)\theta\rangle x\Big\rangle \ge \frac{n\langle \nabla h_{L^\circ}(x), x\rangle}{n} = h_{L^\circ}(x). \tag{32}$$

Since $\nabla h_{L^\circ}(x) \in L^\circ$, $\rho_L(o,\theta)\theta \in L$ and L is o-symmetric, we have

$$|\langle \nabla h_{L^\circ}(x), \rho_L(o,\theta)\theta\rangle| \le 1,$$

therefore

$$\gamma - n \le \gamma + n\langle \nabla h_{L^\circ}(x), \rho_L(o,\theta)\theta\rangle \le \gamma + n. \tag{33}$$

Let $s = \langle \nabla h_{L^\circ}(x), \rho_L(o,\theta)\theta\rangle$. Since $\nabla h_{K^\circ}(x) \in K^\circ$, from (32) and (33) we get

$$h_{L^\circ}(x) \le (\gamma + n)h_{K^\circ}(x), \tag{34}$$

if $\gamma + ns \ge 0$, and

$$h_{L^\circ}(x) \le (n-\gamma)dh_{K^\circ}(x), \tag{35}$$

if $\gamma + ns < 0$. It follows that

$$h_{L^\circ}(x) \le \max\{\gamma + n, (n-\gamma)d\}h_{K^\circ}(x). \tag{36}$$

This shows that $d \le \max\{\gamma + n, (n-\gamma)d\}$, and choosing $\gamma = n(d-1)/(d+1)$ we get $d \le 2n-1$. Hence, $L^\circ \subseteq (2n-1)K^\circ$ and the result follows. □

3. Volume ratio.

In this Section we give an estimate for the volume ratio of two convex bodies:

Theorem 3. *Let L be of maximal volume in K. Then, $(|K|/|L|)^{1/n} \le n$.*

Proof: Without loss of generality we may assume that $o \in \mathrm{int}(L)$. Then, Theorem 2 with $z = o$ shows that

$$Id = \sum_{j=1}^{m} \lambda_j u_j \otimes v_j, \tag{37}$$

where $\lambda_j > 0$, $u_1, \ldots, u_m$ are contact points of K° and L°, $v_1, \ldots, v_m$ are contact points of K and L, and $\sum \lambda_j u_j = o$. This last condition shows that $m \ge n+1$.

Since $u_j \in K^\circ$, $j = 1, \ldots, m$, we have the inclusion

$$K \subseteq U := \{x : \langle x, u_j \rangle \le 1, j = 1, \ldots, m\}. \tag{38}$$

Observe that U is a convex body, because $\sum \lambda_j u_j = o$. On the other hand, $v_j \in L$, $j = 1, \ldots, m$. Therefore,

$$L \supseteq V := \mathrm{co}\{v_1, \ldots, v_m\}. \tag{39}$$

It follows that

$$\frac{|K|}{|L|} \le \frac{|U|}{|V|}. \tag{40}$$

We define $\tilde{v}_j \in \mathbb{R}^{n+1}$ by

$$\tilde{v}_j = \frac{n}{n+1}(-v_j, 1) \quad , \quad j = 1, \ldots, m. \tag{41}$$

Then, we can estimate $|V|$ using the reverse form of the Brascamp-Lieb inequality (see [Bar]):

Lemma 4. *Let*

$$D_{\tilde{v}} = \inf \left\{ \frac{\det\left(\sum_{j=1}^{m} \lambda_j \alpha_j v_j \otimes v_j\right)}{\prod_{j=1}^{m} \alpha_j^{\lambda_j}} \ : \ \alpha_j > 0, \ j = 1, 2, \ldots, m \right\}.$$

Then, the volume of V satisfies the inequality

$$|V| \ge \left(\frac{n+1}{n}\right)^{n+1} \frac{\sqrt{D_{\tilde{v}}}}{n!}. \tag{42}$$

Proof: Let

$$N_V(x) = \begin{cases} \inf\left\{\sum_{i=1}^{m} \alpha_i \ : \ \alpha_i \ge 0 \text{ and } x = \sum_{i=1}^{m} \alpha_i \tilde{v}_i\right\} & \text{, if such } \alpha_i \text{ exist} \\ +\infty & \text{, otherwise.} \end{cases}$$

Let also $C = \mathrm{co}\{-v_1, -v_2, \ldots, -v_m\}$.

Claim: If $x = (y, r)$ for some $y \in \mathbb{R}^n$ and $r \in \mathbb{R}$, then

$$e^{-N_V(x)} \leq \chi_{\{y \in rC\}} \chi_{\{r \geq 0\}} e^{-\frac{n+1}{n} r}. \tag{43}$$

[If $r < 0$ then $N_V(x) = +\infty$ and the inequality is true. Otherwise, let $\alpha_i \geq 0$ be such that $x = \sum_{i=1}^m \alpha_i \tilde{v}_i$ and $\sum_{i=1}^m \alpha_i = N_V(x)$. Then, it is immediate that $N_V(x) = \frac{n+1}{n} r \geq 0$ and $y = \frac{n}{n+1} \sum_{i=1}^m \alpha_i(-v_i) \in rC$. From this (43) follows.]

Integrating the inequality (43) we get

$$\int_{\mathbb{R}^{n+1}} e^{-N_V(x)}\, dx \leq n! \left(\frac{n}{n+1} \right)^{n+1} |V|.$$

We now set $d_j = \frac{n+1}{n} \lambda_j$ and apply the reverse form of the Brascamp-Lieb inequality to the left hand side integral:

$$\begin{aligned}
\int_{\mathbb{R}^{n+1}} e^{-N_V(x)}\, dx &= \int_{\mathbb{R}^{n+1}} \sup_{\substack{\alpha_j \geq 0 \\ x = \sum_{j=1}^m \alpha_j \tilde{v}_j}} \prod_{j=1}^m e^{-\alpha_j}\, dx \\
&= \int_{\mathbb{R}^{n+1}} \sup_{x = \sum_{j=1}^m \alpha_j \tilde{v}_j} \prod_{j=1}^m \left(e^{\alpha_j / d_j} \chi_{\{\alpha_j \geq 0\}} \right)^{d_j} \\
&\geq \sqrt{D_{\tilde{v}}} \prod_{j=1}^m \left(\int_0^\infty e^{-t}\, dt \right)^{d_j} = \sqrt{D_{\tilde{v}}}.
\end{aligned}$$

¿From this (42) follows. □

We now turn to find an upper bound for $|U|$: as above, let $d_j = \frac{n+1}{n} \lambda_j$ and set $\tilde{u}_j = \left(-u_j, \frac{1}{n}\right)$ for $j = 1, \ldots, m$.

Lemma 5. *The volume of U satisfies the inequality*

$$|U| \leq \frac{1}{\sqrt{D_{\tilde{u}}}} \frac{(n+1)^{n+1}}{n! n}, \tag{44}$$

where

$$D_{\tilde{u}} = \inf \left\{ \frac{\det(\sum d_j \alpha_j \tilde{u}_j \otimes \tilde{u}_j)}{\prod \alpha_j^{d_j}}; \alpha_j > 0 \right\}. \tag{45}$$

Proof: We apply the Brascamp-Lieb inequality [BL] (see also [Bar]) in the spirit of K. Ball's proof of the fact that among all convex bodies having the Euclidean unit ball as their ellipsoid of maximal volume, the regular simplex has maximal volume [Ba].

For each $j = 1, \ldots, m$, define $f_j : \mathbb{R} \to [0, \infty)$ by $f_j(t) = e^{-t}\chi_{[0,\infty)}(t)$, and set

$$F(x) = \prod_{j=1}^{m} f_j(\langle \tilde{u}_j, x\rangle)^{d_j} \quad , \quad x \in \mathbb{R}^{n+1}. \tag{46}$$

The Brascamp-Lieb inequality gives

$$\int_{\mathbb{R}^{n+1}} F(x)dx \leq \frac{1}{\sqrt{D_{\tilde{u}}}} \prod_{j=1}^{m} \left(\int_{\mathbb{R}} f_j \right)^{d_j} = \frac{1}{\sqrt{D_{\tilde{u}}}}. \tag{47}$$

As in [Ba], writing $x = (y, r) \in \mathbb{R}^n \times \mathbb{R}$, we see that $F(x) = 0$ if $r < 0$. When $r \geq 0$, we have $F(x) \neq 0$ precisely when $y \in (r/n)U$, and then, taking into account the facts that $\sum \lambda_j u_j = o$ and $\sum d_j = n + 1$, we see that F is independent of y and equal to

$$F(x) = \exp(-r(n+1)/n). \tag{48}$$

It follows from (47) that

$$\frac{1}{\sqrt{D_{\tilde{u}}}} \geq \int_0^\infty \exp(-r(n+1)/n) \left(\frac{r}{n}\right)^n |U| dr = |U| \frac{n!n}{(n+1)^{n+1}}. \quad \square \tag{49}$$

Combining the two lemmata, we get

$$\frac{|K|}{|L|} \leq \frac{n^n}{\sqrt{D_{\tilde{u}} D_{\tilde{v}}}}. \tag{50}$$

Observe that $\tilde{u}_j$, $\tilde{v}_j$ and d_j satisfy $\langle \tilde{u}_j, \tilde{v}_j \rangle = 1$, $j = 1, \ldots, m$, and

$$Id = \sum_{j=1}^{m} d_j \tilde{u}_j \otimes \tilde{v}_j.$$

Thus, in order to finish the proof of Theorem 3 it suffices to prove the following proposition.

Proposition 3. *Let $\lambda_1, \ldots, \lambda_m > 0$, $u_1, \ldots u_m$ $v_1, \ldots, v_m$ be vectors satisfying $\langle u_j, v_j \rangle = 1$ for all $j = 1, \ldots m$ and*

$$Id = \sum_{j=1}^{m} \lambda_j u_j \otimes v_j. \tag{51}$$

Then $D_u D_v \geq 1$.

Proof: For $I \subseteq \{1, 2, \ldots, m\}$ we use the notation $\lambda_I = \prod_{i \in I} \lambda_i$, $\alpha_I = \prod_{i \in I} \alpha_i$, and for I's with cardinality n, we write $U_I = \det(u_i : i \in I)$ and $V_I = \det(v_i : i \in I)$. Moreover, we write $(\lambda U)_I$ for $\det(\lambda_i u_i : i \in I)$.

Applying the Cauchy-Binet formula we have

$$\det\left(\sum_{j=1}^{m}\lambda_j\alpha_j u_j\otimes v_j\right)=\sum_{\substack{|I|=n\\ I\subseteq\{1,2,\ldots,m\}}}\alpha_I(\sqrt{\lambda}U)_I(\sqrt{\lambda}V)_I. \tag{52}$$

But $\sum(\sqrt{\lambda}U)_I(\sqrt{\lambda}V)_I=\det\left(\sum_{j=1}^{m}\lambda_j u_j\otimes u_j\right)=\det I=1$. Hence, applying the arithmetic-geometric means inequality to the right side of (52) we deduce that

$$\begin{aligned}\sum_{\substack{|I|=n\\ I\subseteq\{1,2,\ldots,m\}}}\alpha_I(\sqrt{\lambda}U)_I(\sqrt{\lambda}V)_I &\geq \prod_{\substack{|I|=n\\ I\subseteq\{1,2,\ldots,m\}}}\alpha_I^{(\sqrt{\lambda}U)_I(\sqrt{\lambda}V)_I}\\ &= \prod_{j=1}^{m}\alpha_j^{\sum_{j\in I,|I|=n}(\sqrt{\lambda}U)_I(\sqrt{\lambda}V)_I}.\end{aligned}$$

Observe now that the exponent of α_j in the above product equals λ_j:

$$\begin{aligned}\sum_{j\in I,\ |I|=n}(\sqrt{\lambda}U)_I(\sqrt{\lambda}V)_I &= \sum_{|I|=n}(\sqrt{\lambda}U)_I(\sqrt{\lambda}V)_I-\sum_{j\notin I,\ |I|=n}(\sqrt{\lambda}U)_I(\sqrt{\lambda}V)_I\\ &= \det\left(\sum_{j=1}^{m}\lambda_j u_j\otimes v_j\right)-\det(I-\lambda_j u_j\otimes v_j)\\ &= \lambda_j,\end{aligned}$$

since $\langle u_j,v_j\rangle=1$. Thus, we have shown that

$$\det\left(\sum_{j=1}^{m}\lambda_j\alpha_j u_j\otimes v_j\right)\geq\prod_{j=1}^{m}\alpha_j^{\lambda_j}. \tag{53}$$

Now, for any $\gamma_j,\ \delta_j>0$ we have

$$\begin{aligned}&\det\left(\sum_{j=1}^{m}\lambda_j\gamma_j u_j\otimes u_j\right)\det\left(\sum_{j=1}^{m}\lambda_j\delta_j v_j\otimes v_j\right)\\ &=\sum_{|I|=n}\gamma_I(\sqrt{\lambda}U)_I^2\sum_{|I|=n}\delta_I(\sqrt{\lambda}V)_I^2.\end{aligned}$$

By the Cauchy-Schwarz inequality this is greater than

$$\left(\sum_{|I|=n}\lambda_I\sqrt{\gamma_I\delta_I}U_IV_I\right)^2.$$

Apply now (53) to get

$$\frac{\det\left(\sum_{j=1}^m \lambda_j\gamma_j u_j\otimes u_j\right)}{\prod_{j=1}^m \gamma_j^{\lambda_j}}\frac{\det\left(\sum_{j=1}^m \lambda_j\delta_j v_j\otimes v_j\right)}{\prod_{j=1}^m \delta_j^{\lambda_j}}\geq 1,$$

completing the proof. □

Remark. A different argument shows that $\mathrm{vr}(K,S_n)\leq c\sqrt{n}$ for every convex body K in $\mathbb{R}^n$, where $c>0$ is an absolute constant.

Without loss of generality we may assume that K is of maximal volume in D_n. Then, John's theorem gives us $\lambda_1,\dots,\lambda_m>0$ and contact points $u_1,\dots,u_m$ of K and D_n such that

$$Id=\sum_{j=1}^m \lambda_j u_j\otimes u_j.$$

The Dvoretzky-Rogers lemma [DR] shows that we can choose $u_1,\dots,u_n$ among the u_j's so that

$$|P_{\mathrm{span}\{u_s:s<i\}^\perp}u_i|\geq\left(\frac{n-i+1}{n}\right)^{1/2}\quad ,\quad i=2,\dots,n.$$

Therefore, the simplex $S=\mathrm{co}\{o,u_1,\dots,u_n\}$ has volume

$$|S|\geq\frac{1}{n!}\prod_{i=2}^n\left(\frac{n-i+1}{n}\right)^{1/2}=\frac{1}{(n!n^n)^{1/2}},$$

and $S\subseteq K\subseteq D_n$. It follows that

$$\begin{aligned}\mathrm{vr}(K,S_n)&\leq\left(\frac{|D_n|}{|S|}\right)^{1/n}\leq\frac{(n!)^{1/2n}\sqrt{n}\sqrt{\pi}}{[\Gamma(\frac{n}{2}+1)]^{1/n}}\\&\leq c\sqrt{n}.\end{aligned}$$

This supports the question if $\mathrm{vr}(K,L)$ is always bounded by $c\sqrt{n}$.

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