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A. Cappelli
L. S. Georgiev
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A. Cappelli *

I.N.F.N. and Dipartimento di Fisica,

Largo E. Fermi 2, I-50125 Firenze, Italy

L. S. Georgiev † I. T. Todorov ‡

E. Schrödinger Inst. for Mathematical Physics,

Boltzmanngasse 9, A-1090 Wien, Austria

and

Institute for Nuclear Research and Nuclear Energy,

Tsarigradsko Chaussee 72, BG-1784 Sofia, Bulgaria

Abstract

The wave functions of the Haldane-Rezayi paired Hall state have been previously described by a non-unitary conformal field theory with central charge $c = -2$. Moreover, a relation with the $c = 1$ unitary Weyl fermion has been suggested. We construct the complete unitary theory and show that it consistently describes the edge excitations of the Haldane-Rezayi state. Actually, we show that the unitary ($c = 1$) and non-unitary ($c = -2$) theories are related by a local map between the two sets of fields and by a suitable change of conjugation. The unitary theory of the Haldane-Rezayi state is found to be the same as that of the 331 paired Hall state. Furthermore, the analysis of modular invariant partition functions shows that no alternative unitary descriptions are possible for the Haldane-Rezayi state within the class of rational conformal field theories with abelian current algebra. Finally, the known $c = 3/2$ conformal theory of the Pfaffian state is also obtained from the 331 theory by a reduction of degrees of freedom which can be physically realized in the double-layer Hall systems.

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*E-mail: andrea.cappelli@fi.infn.it

†E-mail: lgeorg@inrne.bas.bg

‡E-mail: todorov@inrne.bas.bg, itodorov@esi.ac.at

Contents

1 Introduction

The so-called 331 [1], Pfaffian [2] and Haldane-Rezayi (HR) [3] $\nu = 1/2$ $(5/2)$ quantum Hall (QH) states [4] have been analysed extensively in the recent literature. They are called *paired* Hall states [5] because they contain two kinds of electrons, carrying spin or layer index, which first bind in pairs and then form incompressible fluids [6].

One would like to identify the Conformal Field Theories (CFT) [7] corresponding to these states, which describe their low-energy edge excitations [8]. This requires some guesswork and ingenuity for reconstructing the complete Hilbert space from the knowledge of the ground-state wave function and possibly some quasi-particle states. There are well-established procedures which have been used for the spin-polarized single-layer Hall states [8, 4], but they do not seem good enough for the paired states. In particular, the CFT proposed for the HR state is puzzling for the lack of unitarity [9, 10, 11], or locality [12, 13].

In this paper, we present a unified description of the paired Hall states which uses the same conformal fields (or a subset of them) in all three cases. We show that an unitary description of the HR state is possible and that this is given by the same CFT as that of the 331 state; furthermore, we interpret the Pfaffian as a projection of the 331 state, which can be obtained in the limit of low potential barrier between the two layers [5].

This common CFT description is rather useful for the physical interpretation; moreover, it allows the discussion of the $W_{1+\infty}$ symmetry of the paired states. This symmetry characterizes the incompressible Hall fluids [14], and is a definitive building criterion for the CFTs of the hierarchical single-layer states [15]. We show that the $W_{1+\infty}$ symmetry also characterizes the 331 and HR double-layer states, and that it is broken at the quantum level in the Pfaffian state.

In view of the controversial literature on this subject, it is important to state the hypotheses made in this work: we consider rational conformal field theories (RCFT), whose completeness can be checked by computing their modular invariant partition functions [16]; we require the unitarity of the theories, because they describe physical excitations propagating at the edge. Moreover, we consider, whenever possible, theories with a (multi-component) abelian current algebra, which possess the $W_{1+\infty}$ symmetry and can be extended to RCFTs [17] (henceforth called *lattice* RCFTs).

Therefore, in this paper we specifically prove that there is only one $c = 2$ unitary lattice RCFT suitable for the HR state – that of the 331 state. In particular, the HR ground state appears as an excited state in the 331 CFT. This result is at variance with the common opinion that these Hall states identify two independent universality classes, with different numerical energy spectrum [3, 18] and topological order [3, 19]. Our result may imply that this is not completely correct, or, alternatively, that the HR state is not described by a lattice RCFT; in either case, the unitarity problem is cleared up. Finally, the Pfaffian state is consistently described in terms of the same 331 conformal fields.

1.1 Review of the Haldane-Rezayi State

Here we review the basic characteristics of the model and its available theoretical treatments. The wave function Ψ of a paired QH system of $2N$ electrons is written as a product of the usual Gaussian factor and an analytic function of the electron coordinates z_i and w_i of the first and the second layer, respectively (alternatively, of the up and down spin components):

$$\Psi(z_i, w_i; \bar{z}_i, \bar{w}_i) = \Phi(z_1, w_1, \ldots z_N, w_N) \exp\left(-\frac{1}{4} \sum_i (|z_i|^2 + |w_i|^2) \right). \quad (1.1)$$

In the framework developed in [2][9] (see also earlier work cited there), the analytic factor Φ is interpreted as the CFT correlation function: $\langle \Phi_N | \phi^1(z_1) \cdots \phi^1(z_N) \phi^2(w_1) \cdots \phi^2(w_N) | 0 \rangle$, where ϕ^i is a chiral conformal field of effective charge $\mathbf{q}^i$ $(i = 1, 2)$ representing the electrons of layer i, and $\langle \Phi_N |$ is the out state carrying a compensating charge $N(\mathbf{q}^1 + \mathbf{q}^2)$. The holomorphic wave function of the HR state [3] is written as the product,

$$\Phi_{HR}(z_i; w_i) = \Phi_m(z_i; w_i) \Phi_{ds}(z_i; w_i) \ , \quad (1.2)$$

of a Laughlin type wave-function [6],

$$\Phi_m(z_i; w_i) = \prod_{i<j} (z_{ij} w_{ij})^m \prod_{i,j} (z_i - w_j)^m, \quad m = 2, 4, \ldots, \ z_{ij} = z_i - z_j \ , \quad (1.3)$$

and a neutral *d-wave spin-singlet* part, Φ_{ds}, that is skew-symmetric in $(z_1, \ldots, z_N)$ and in $(w_1, \ldots, w_N)$, separately:

$$\Phi_{ds}(z_i; w_i) = (-1)^{\frac{N(N-1)}{2}} \det \left(\frac{1}{(z_i - w_j)^2} \right). \quad (1.4)$$

This expression can be viewed, following Ref.[9], as the $2N$-point vacuum expectation value of a pair of Fermi fields $\psi_\pm(z)$ ($\psi_{\pm\frac{1}{2}}(z)$, the subscript of ψ referring, alternatively to spin projection or layer):

$$\Phi_{ds}(z_i; w_i) = \langle 0 | \psi_+(z_1) \cdots \psi_+(z_N) \ \psi_-(w_1) \cdots \psi_-(w_N) | 0 \rangle \ . \quad (1.5)$$

Equation (1.4) would then follow (in a local field theory with energy bounded from below): (i) from the *"quasi-free"* anti-commutation relations [9, 10]:

$$[\psi_\rho(z), \psi_\sigma(w)]_+ = -\varepsilon_{\rho\sigma}\delta'(z-w) \ , \quad \varepsilon_{\rho\sigma} = -\varepsilon_{\rho\sigma} \ , \quad \varepsilon_{+-} = 1 \ , \tag{1.6}$$

where $\delta(z-w)$ is the Dirac delta function for holomorphic test functions; and (ii) from the knowledge of the 2-point functions (which restricts the choice of vacuum). We adopt (1.6) in what follows as a phenomenological input.

The knowledge of the $2N$-point function (1.5) allows to determine the operator content of (the vacuum sector of) the CFT generated by the pair $\psi_\pm(z)$. To do that we write the determinant in Eq. (1.4) in the form

$$\begin{aligned} \det\left(\frac{1}{(z_i-w_j)^2}\right) &= \det\left(\frac{1}{z_i-w_j}\right) \operatorname{perm}\left(\frac{1}{z_i-w_j}\right) , \\ \det\left(\frac{1}{z_i-w_j}\right) &= (-1)^{N(N-1)/2}\, \frac{\prod\limits_{i<j} z_{ij} w_{ij}}{\prod\limits_{i,j}(z_i-w_j)}; \end{aligned}$$

here the *permanent* is the symmetrized product of $(z_i-w_j)^{-1}$ which has a non-zero limit

$$\operatorname{perm}\left(\frac{1}{z_i-w_j}\right) = \sum_{\sigma\in S_N} \prod_{i=1}^{N} \frac{1}{z_i - w_{\sigma(i)}} \quad \rightarrow \quad \frac{N!}{(z-w)^N} \ ,$$

for $z_i \to z$, $w_j \to w$, $i,j = 1,\ldots,N$. These properties of the $2N$-point function (1.4) imply that products of $\psi_\pm(z)$ give rise to a sequence of composite fields $V_{\pm s}$ ($V_{\pm\frac{1}{2}}(z) = \psi_\pm(z)$) of dimension $\Delta(2s)$ determined inductively by the operator-product expansion (OPE):

$$\psi_\pm(z_1) V_{\pm s}(z_2) \sim z_{12}^{2s}\ V_{\pm s \pm \frac{1}{2}}(z_2) \ , \qquad (2s = 0,1,2,\ldots) \ , \tag{1.7}$$

implying,

$$\Delta(\pm 2s \pm 1) - \Delta(\pm 2s) - \Delta(\pm 1) = 2s \quad \Rightarrow \quad \Delta(\pm 2s) = s\,(2s + 2\Delta(\pm 1) - 1) \,. \tag{1.8}$$

Eq. (1.7) allows to express $V_{\pm s}$ as (normal) products of $\psi_\pm$ and their derivatives:

$$V_{\pm s}(z) = \prod_{i=1}^{2s} :\frac{1}{(2s-i)!}\partial^{2s-i}\psi_\pm(z): \ . \tag{1.9}$$

The values $\Delta(\pm 1)$ depend on the choice of the stress energy tensor; however, according to (1.4) their sum is fixed:

$$\Delta(1) + \Delta(-1) = 2 \ . \tag{1.10}$$

In Ref. [9], it was further assumed that $\psi_\pm$ are *primary conformal fields.* This implies that these Fermi fields have conformal dimension one in violation of the spin-statistics relation for a unitary CFT. Indeed, these fields can be found [9, 10, 11] in the non-unitary extension of the Virasoro minimal conformal models [7] with central charge $c_p = 1 - 6/(p(p+1))$, for the value $p = 1$, -i.e., $c = -2$. The non-unitary stress tensor $\mathcal{T}(z) = :\psi_-\psi_+:$ is then invariant under the SU(2)-spin group and so are the OPEs of any number of $\psi_\pm$ factors (see [9] as well as Section 3 and Appendix B below). It follows that for each $2s = 0, 1, 2, \ldots$, the composite operators $V_{\pm s}$ (1.9) are the lowest and highest spin projection components of a $(2s+1)$-dimensional multiplet of fields $\phi_{sm}(z)$, $m = -s, -s+1, \ldots, s$, of the same dimension (1.8) (for $\Delta(1) = 1$):

$$\Delta(\pm 2s) = s(2s+1) \qquad (2s = 0, 1, 2, \ldots). \tag{1.11}$$

This $c = -2$ CFT is known in the literature as the ξ-η "ghost system" [20], which is defined by the pair of canonical Fermi fields,

$$\begin{aligned} [\xi(z), \eta(w)]_+ &= \delta(z-w) \ , \\ [\xi(z), \xi(w)]_+ &= 0 \ = \ [\eta(z), \eta(w)]_+ \ , \\ \langle 0|\eta(z)\xi(w)|0\rangle &= \frac{1}{z-w} \ , \end{aligned} \tag{1.12}$$

of conformal dimensions $\Delta_\xi = 0$, $\Delta_\eta = 1$. Their normal product defines the current j :

$$\eta(z)\xi(w) = \frac{1}{z-w} + :\eta(z)\xi(w): \ , \tag{1.13}$$

$$j(z) = :\xi(z)\eta(z): = \sum_{n\in\mathbb{Z}} j_n z^{-n-1} \ . \tag{1.14}$$

We can set:

$$\psi_-(z) = \xi'(z) \ , \qquad \psi_+(z) = \eta(z) \ . \tag{1.15}$$

Note that the zero mode j_0 acts as (minus twice) the spin projection operator S_3, which counts the difference between spin up and spin down components $\psi_\pm$: $[\psi_\pm(z), j_0] \equiv [\psi_\pm(z), 2S_3] = \pm\psi_\pm(z)$, with $\vec{S} = (S_1, S_2, S_3)$ standing for the spin operator. Moreover, the stress tensor $\mathcal{T}(z) = :\xi'(z)\eta(z):$ assigns dimension one to $\psi_\pm$, and the central charge is $c = -2$.

According to the Kac determinant formula, the conformal dimensions of the $c = -2$ primary fields are given by [7]:

$$\Delta_l = \frac{l^2 - 1}{8} \geq -\frac{1}{8} \ , \qquad l = 0, 1, 2, \ldots \tag{1.16}$$

In particular, for odd l, $l = 4s+1$ $(2s = 0, 1, 2, \ldots)$, we recover the spin multiplets of integer dimensions given by (1.11). The state of lowest dimension in this theory is the *disorder state* [9] with $\Delta_0 = -1/8$, which is a spin singlet; it gives rise to a $\mathbb{Z}_2$ twisted sector in which the field ψ_ρ creates double-valued quasi-particles with dimensions $\Delta_{2n} = (n^2/2) - 1/8$. In fact, it follows from (1.16) that:

$$\psi_\rho(z)|\Delta_0\rangle \sim z^{-\frac{1}{2}}|\Delta_2\rangle \ , \qquad \left(\Delta_2 - \Delta_0 = \frac{1}{2}\right) \ . \tag{1.17}$$

The appearance of negative norm states, like $\mathcal{T}(0)|0\rangle$, and a negative conformal dimension ($\Delta_0 = -\frac{1}{8}$) is certainly untenable from a physical point of view. This non-unitarity problem has been recognized and various solutions have been proposed in Refs. [10, 11, 12, 13].

In this paper, we choose to relax the assumption that the fields ψ_ρ are *primary*, while keeping, at the same time, the property (1.17) for the quasi-particle excitations. We obtain a unitary theory of the HR model which is based on the correspondence between the Hilbert space and the fields of the $c = -2$ CFT and those of the $c = 1$ Weyl fermion theory (see also Refs.[11, 12, 13]); we keep the expression (1.2-1.4) of the HR wave function while preserving both *conformal invariance* and *modular invariance* [16, 21]. Moreover, we can define a *hermitean* stress tensor which, however, is not SU(2) invariant.

1.2 Outline of the Paper

In Section 2, we first introduce a $U(1) \times U(1)$ current algebra CFT, whose orthogonal lattice contains a Weyl fermion field and a Laughlin boson. We use it to describe the 331 CFT as a $\mathbb{Z}_2$ orbifold (in the sense of [17]), whose ($\mathbb{Z}_2$-even) states possess charge and fermion number coupled by the *parity ("projection") rule* defined in Ref.[10]. The modular invariant partition function for the 331 RCFT is also obtained in agreement with this rule.

In Section 3, we present the $c = -2$ to $c = 1$ correspondence for the HR theory: we first discuss the improvement of the $c = -2$ stress tensor which leads to a unitary theory and then map the fields and the characters of the current-algebra representations. Furthermore, we find that the $c = -2$ partition function of the HR model proposed in Ref. [10] is mapped into the 331 one; this shows that the CFT descriptions of the two models coincide, once unitarity is enforced; moreover, we find that the parity rule is the same in the two theories.

In Section 4, we classify all possible $U(1) \times U(1)$ lattice current algebras, which can be made with the excitations of the 331 and HR theories (assuming the standard charge-statistics relation for the observable electron-like excitations). We find that there is a unique modular invariant partition function: thus, there is only one possible unitary lattice RCFT which can describe the edge excitations of the HR state, which is the same as that of the 331 state.

In Section 5, we show that the Pfaffian state can be obtained from the 331 model by a *generalized gauge reduction*: namely, its $c = 3/2$ CFT of a Majorana fermion and a Laughlin boson [2] is reproduced by projecting out the imaginary part of the Weyl fermion of the 331 CFT. The corresponding reduction at the level of partition functions shows that the Pfaffian theory inherits the parity rule of the two other paired Hall states. This gauge reduction breaks the $W_{1+\infty}$ symmetry present in the 331 theory and gives rise to non-Abelian statistics for the quasi-particles. In Section 5.2, a similar reduction allows to relate the maximally-symmetric $SU(2) \times SU(2)$ $c = 3$ lattice RCFT of Ref.[22] with the 331 CFT.

In the Conclusions (Section 6), we discuss some problems left open by our analysis, most notably the possible ways to distinguish between the 331 and the HR states. The Appendices contain more technical discussions: Appendix A sums up some basic facts about charge lattices and orbifolds of finite cyclic groups needed in the text. In Appendix B we prove that the $SU(2)$ invariant OPE of $\psi_\rho(z)\psi_\sigma(w)$ is independent of the choice of the stress tensor and of the dimensions of these fields. Finally, Appendix C provides a complete list of modular invariants of the orthogonal lattice algebra (Section 2) underlying both the 331 and the HR states. This is used in Section 4 to show that there is a unique lattice RCFT for the 331 and HR states.

2 The 331 Model as a $\mathbb{Z}_2$ Orbifold of an Orthogonal Lattice Algebra

2.1 The $(m+1)(m+1)(m-1)$ Holomorphic Wave Function and the Associated Charge Lattice

We shall be dealing in this section with a natural generalization of the 331 model corresponding to the filling fraction $\nu = 1/m$, m even, and to the holomorphic wave function:

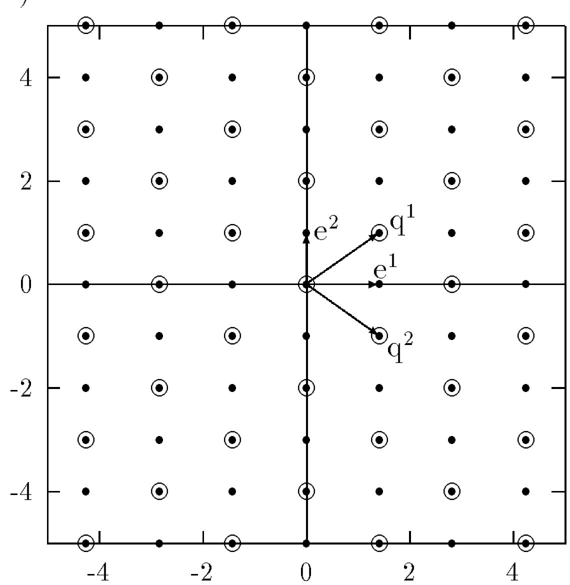
$$\begin{aligned}\Phi_{(m+1)(m+1)(m-1)}\left(z_i; w_i\right) &= \prod_{1\leq i<j\leq N} \left(z_{ij} w_{ij}\right)^{m+1} \prod_{i,j} \left(z_i - w_j\right)^{m-1} \\ &= (-1)^{N(N-1)/2}\, \Phi_m\left(z_i; w_i\right) \det\left(\frac{1}{z_i - w_j}\right) \quad (2.1)\end{aligned}$$

where Φ_m is the $U(1)$ factor (1.3). This ground state wave function is reproduced in a $c = 2$ RCFT whose chiral algebra $\mathcal{A}(L)$ is an extension of the $\widehat{u(1)} \times \widehat{u(1)}$ current algebra by two pairs of oppositely charged fields of charge vectors $\mathbf{q}^1$ and $\mathbf{q}^2$ spanning a two-dimensional lattice L. The Gram matrix of L is

$$G_L = \left((\mathbf{q}^i|\mathbf{q}^j)\right) = \begin{pmatrix} m+1 & m-1 \\ m-1 & m+1 \end{pmatrix} , \qquad (m = 2, 4, \ldots) . \tag{2.2}$$

The resulting RCFT, called the $(m+1)(m+1)(m-1)$ model, can be constructed from the following observation: we can embed the lattice L in a finer, orthogonal one, such that the corresponding conformal theory is the direct product of a Weyl fermion and a Laughlin anyon with $\nu = 1/m$ (Fig. 1). This basis will give the natural description for the quasi-particle excitations of all the paired Hall states, which will only differ in the treatment of the neutral fermionic factor.

Figure 1: The original lattice L (encircled dots) as a sub-lattice of the orthogonal $\Gamma_{m,1}$ (dots)



Proposition 2.1 *(a) The lattice L is a sub-lattice of index two of the (integral) orthogonal lattice $\Gamma_{m,1} = \mathbb{Z}\mathbf{e}^1 \oplus \mathbb{Z}\mathbf{e}^2$ with Gram matrix:*

$$G_{m,1} = \left((\mathbf{e}^i|\mathbf{e}^j)\right) = \begin{pmatrix} m & 0 \\ 0 & 1 \end{pmatrix} . \tag{2.3}$$

(b) We associate with the lattice L a $\widehat{u(1)} \otimes \widehat{u(1)}$ *chiral current algebra* $\mathcal{A}_{m,1}$, *by letting the vectors* $\pm\mathbf{e}^1$ *correspond to a pair of oppositely charged Bose fields* $E^g(z)$, $g = \pm\sqrt{m}$, *of dimension* $m/2 \in \mathbb{N}$, *and* $\pm\mathbf{e}^2$ *to a pair of conjugate Weyl fermions* $(\psi(z), \psi^*(z))$ *of dimension* $1/2$. *Then, the algebra* $\mathcal{A}_{m,1}$ *admits an involutive inner automorphism* α *defined on the generating fields by:*

$$\alpha\left[E^g(z)\right] = -E^g(z) \ , \qquad \alpha\left[\psi^{(*)}(z)\right] = -\psi^{(*)}(z) \ , \qquad g = \pm\sqrt{m} \ , \quad (2.4)$$

while $\mathcal{A}(L)$ *appears as the sub-algebra of* $\mathbb{Z}_2$ *invariant elements of* $\mathcal{A}_{m,1}$ *(the fixed points of* α*). It is generated by the products of commuting fields:*

$$\psi^1(z) = \psi^*(z)E^{-\sqrt{m}}(z) \ , \qquad \psi^2(z) = \psi(z)E^{-\sqrt{m}}(z) \ . \qquad (2.5)$$

Proof. (a) The lattice $L = \mathbb{Z}\mathbf{q}^1 + \mathbb{Z}\mathbf{q}^2$ is identified as a sub-lattice of $\Gamma_{m,1}$ by setting,

$$\mathbf{q}^1 = \mathbf{e}^1 + \mathbf{e}^2, \ \ \mathbf{q}^2 = \mathbf{e}^1 - \mathbf{e}^2 \ . \qquad (2.6)$$

Note that $\det G_L = 4m = 2^2 \det G_{m,1}$. (This is a necessary condition for the lattice L to be a sub-lattice of index two of $\Gamma_{m,1}$.)

(b) The vertex operator $E^g(z)$ is constructed in the standard fashion from the electric current $J(z)$ (see Appendix A) and satisfies the OPE:

$$J(z_1)E^g(z_2) \sim \frac{g}{z_{12}}E^g(z_2) \quad \Leftrightarrow \quad [J(z_1), E^g(z_2)] = g\delta(z_{12})E^g(z_2) \ . \quad (2.7)$$

The second $\widehat{u(1)}$ current (commuting with J) is the counterpart of the "spin current" (1.14) in the unitary $c = 2$ theory:

$$j(z) = :\psi^*(z)\psi(z): \ , \qquad [\psi^{(*)}(z_1), E^g(z_2)] = 0 = [j(z_1), J(z_2)] \ , \qquad (2.8)$$

with both j and J normalized by $\langle 0|j(z_1)j(z_2)|0\rangle = \langle 0|J(z_1)J(z_2)|0\rangle = z_{12}^{-2}$.

The automorphism $\alpha[A]$, $A \in \mathcal{A}_{m,1}$ is defined by:

$$\alpha\left[A\right] = \mathrm{e}^{i\pi J_0^1} A \mathrm{e}^{-i\pi J_0^1} \ , \qquad J_0^1 = \frac{1}{\sqrt{m}}J_0 + j_0 \ , \qquad (2.9)$$

(note that $\alpha[\alpha[A]] = A$ for all $A \in \mathcal{A}_{m,1}$). The property (2.4) is implied by (2.9) (in view of (2.7) and (2.8)). The invariance of (2.5) is then obvious.

Remark 2.1 This provides a simple example of the $\mathbb{Z}_2$ orbifold construction [17, 23]; in the present case, the orbifold actually corresponds to a manifold (with no singular points) because both $\widehat{u(1)}$ currents are invariant under the parity operation.

2.2 Superselection Sectors: Spectrum of Charges and Dimensions; Partition Function

We proceed to studying the *positive-energy representations* of $\mathcal{A}(L)$ which define the *superselection sectors* for our RCFT and are equipped with the fusion rules corresponding to the addition of charges.

Proposition 2.2 *(a) The superselection sectors $\mathcal{H}_\lambda$ of the $\mathcal{A}(L)$ theory are labelled by the elements of the cyclic group,*

$$L^*/L \simeq \mathbb{Z}_{4m} \ , \qquad (|L^*/L| = \det G_L = 4m) \ , \tag{2.10}$$

where L^* is the lattice dual to L:

$$L^* = \mathbb{Z}\mathbf{q}_1^* + \mathbb{Z}\mathbf{q}_2^*, \ (\mathbf{q}^i|\mathbf{q}_j^*) = \delta_j^i \ , \quad G_{L^*} = \frac{1}{4m}\begin{pmatrix} m+1 & 1-m \\ 1-m & m+1 \end{pmatrix} \ , \tag{2.11}$$

while $L^*/L = \{\lambda\mathbf{q}_1^* + L; \lambda \text{ mod } 4m\}$.

*(b) The (*visible-*in the terminology of ref. [22]) electric charge vector $\mathbf{Q}$, whose square gives the filling fraction of the model, belongs to the orthogonal sub-lattice $\Gamma_{m,1}^* = \Gamma_m^* \oplus \Gamma_1^*$ of L^*:*

$$\begin{aligned} \mathbf{Q} \ &= \ \mathbf{q}_1^* + \mathbf{q}_2^* = \mathbf{e}_1^*, \ \ (\mathbf{e}^i|\mathbf{e}_j^*) = \delta_j^i \\ &\Rightarrow (\mathbf{Q}|\mathbf{q}^i) = 1 \ , \quad |\mathbf{Q}|^2 = \nu = \frac{1}{m} \ . \end{aligned} \tag{2.12}$$

The cyclic group (2.10) is generated by either of the four cosets $\pm\mathbf{q}_i^* + L$.

(c) The characters $\chi_\lambda(\tau,\zeta;m)$ of the coset (λ) is expressed in terms of sums of products of $c=1$ lattice characters:

$$\chi_\lambda(\tau,\zeta;m) = \mathrm{e}^{-\frac{\pi}{m}\frac{(\mathrm{Im}\,\zeta)^2}{\mathrm{Im}\,\tau}}\mathrm{ch}_L^\lambda(\tau,\zeta), \tag{2.13}$$

$$\begin{aligned} \mathrm{ch}_L^\lambda(\tau,\zeta) \ &\equiv \ \mathrm{tr}_{\mathcal{H}_\lambda}\left(\mathrm{e}^{2\pi i\left[\tau(L_0-1/12)+\zeta J_0/\sqrt{m}\right]}\right) \\ &= \ K_\lambda(\tau;4)K_\lambda(\tau,2\zeta;4m) + K_{\lambda+2}(\tau;4)K_{\lambda+2m}(\tau,2\zeta;4m) \end{aligned} \tag{2.14}$$

where $K_l(\tau,\zeta;M)$ is given by,

$$K_l(\tau,\zeta;M) = \frac{1}{\eta(\tau)}\sum_{n\in\mathbb{Z}} q^{\frac{M}{2}(n+\frac{l}{M})^2}\mathrm{e}^{2\pi i\zeta(n+\frac{l}{M})} \ , \tag{2.15a}$$

$$q = \mathrm{e}^{2\pi i\tau}, \qquad \eta(\tau) = q^{\frac{1}{24}}\prod_{n=1}^{\infty}(1-q^n), \tag{2.15b}$$

and we used the notation $K_l(\tau;M) = K_l(\tau,0;M)$.

Remark 2.2. The non-analytic prefactor multiplying the character ch_L^λ corresponds to a constant term added to the Hamiltonian and ensures the invariance under the "spectral flow" [16, 21] of the resulting partition function (see Eq. (2.21b) below).

Sketch of the proof. Statement (a) is known (see Appendix A for a brief review of background material). (b) One can choose in each class a representative $\mathbf{q}^*$ such that the absolute value of the electric charge $|(\mathbf{Q}|\mathbf{q}^*)|$ and its conformal dimension $\Delta = |\mathbf{q}^*|^2/2$ are minimal. In Table 1, we list the representatives of each coset $(\lambda) = \lambda \mathbf{q}_1^* + L$ along with the corresponding electric charge and conformal dimension.

The periodicity condition:

$$K_{\lambda+M}(\tau, \zeta; M) = K_\lambda(\tau, \zeta; M) \ , \tag{2.16}$$

confirms that there are precisely $4m$ different characters χ_λ (2.14). We also note the symmetry property:

$$K_{-\lambda}(\tau, -\zeta; M) = K_\lambda(\tau, \zeta; M) \ , \tag{2.17}$$

which clarifies, in particular, the observation that the cosets $(\pm\lambda)$ in Table 1 give rise to the same conformal dimension.

Table 1: $(m+1)(m+1)(m-1)$ **superselection sectors**

coset $(\lambda) = \lambda \mathbf{q}_1^* + L$	**representative** $\mathbf{q}^*$	**charge** $(\mathbf{Q}\vert\mathbf{q}^*)$	**dimension** $\Delta = \frac{1}{2}\vert\mathbf{q}^*\vert^2$
(0)	$\mathbf{0}$	0	0
(± 1)	$\pm \mathbf{q}_1^*$	$\pm \frac{1}{2m}$	$\frac{m+1}{8m}$
$\pm(2m-1)$	$\pm \mathbf{q}_2^*$	$\pm \frac{1}{2m}$	$\frac{m+1}{8m}$
$\cdots$	$\cdots$	$\cdots$	$\cdots$
$(\pm m)$	$\pm(-1)^{m/2}\frac{1}{2}\mathbf{e}^1$	$\pm(-1)^{m/2}\frac{1}{2}$	$\frac{m}{8}$
$(m \pm 1)$	$\frac{1}{2}\left\{\pm (m-1)\, \mathbf{Q} - \pm(-1)^{m/2}\frac{1}{2}\mathbf{e}^2\right\}$	$\pm \frac{m-1}{4}$	$\frac{m^2-m+1}{8m}$
$\cdots$	$\cdots$	$\cdots$	$\cdots$
$(2m)$	$\mathbf{e}^2$	0	$\frac{1}{2}$

The partition function of any RCFT is given by the quadratic form [7]:

$$Z = \sum_{\lambda, \bar{\lambda}} \mathcal{N}_{\lambda\bar{\lambda}}\ \chi_\lambda\ \overline{\chi}_{\bar{\lambda}} \ ,$$

where the chiral (resp. antichiral) factors $\chi(\bar{\chi})$ pertain to the outer (inner) edge of the annular quantum Hall sample [16]. The integer coefficients $\mathcal{N}_{\lambda\bar{\lambda}}$ are obtained by the conditions of modular invariance of Z which are specific for the quantum Hall systems. They have been formulated in Ref.[16] and analyzed for lattice theories in Ref.[21]; Z should be invariant under the transformations S and T^2 which generate a subgroup of $SL(2,\mathbb{Z})$, as well as under the U and V transformations specified below.

The SL(2,$\mathbb{Z}$) transformation properties of K_λ,

$$\begin{aligned} T \ : \ K_\lambda(\tau+1,\zeta;M) &= \mathrm{e}^{i\pi(\frac{\lambda^2}{M}-\frac{1}{12})} K_\lambda(\tau,\zeta;M) \ , \\ S \ : \ K_\lambda(-\frac{1}{\tau},\frac{\zeta}{\tau};M) &= \frac{1}{\sqrt{M}} \sum_{\mu \,\mathrm{mod}\, M} \mathrm{e}^{-2\pi i \frac{\lambda\mu}{M}} K_\mu(\tau,\zeta;M) \ , \end{aligned} \tag{2.18}$$

imply the following transformations for χ_λ:

$$T^2 : \chi_\lambda(\tau+2,\zeta) = \mathrm{e}^{i\pi(\lambda^2\frac{m+1}{2m})-\frac{1}{3}} \chi_\lambda(\tau,\zeta) \ , \tag{2.19a}$$

$$S \ : \ \chi_\lambda(-\frac{1}{\tau},\frac{\zeta}{\tau}) = \mathrm{e}^{i\frac{\pi\zeta^2}{m\tau}} \sum_{\mu \,\mathrm{mod}\, M} S_{\lambda\mu}\chi_\mu(\tau,\zeta) \ ,$$

$$S_{\lambda\mu} = \frac{1}{2\sqrt{m}} \mathrm{e}^{-i\pi\frac{m+1}{2m}\lambda\mu} \ . \tag{2.19b}$$

The diagonal partition function:

$$Z_{(m+1)(m+1)(m-1)} = \sum_{\lambda=1-2m}^{2m} |\chi_\lambda(\tau,\zeta)|^2 \tag{2.20}$$

is then invariant under (2.19) as well as under the ζ-shift U and the spectral flow V:

$$U \ : \ \chi_\lambda(\tau,\zeta) \ \to \ \chi_\lambda(\tau,\zeta+1) = \mathrm{e}^{i\pi\frac{\lambda}{m}} \chi_\lambda(\tau,\zeta) \ , \tag{2.21a}$$

$$V \ : \ \chi_\lambda(\tau,\zeta) \ \to \ \chi_\lambda(\tau,\zeta+\tau) = \mathrm{e}^{\frac{-2\pi i}{m} Re(\zeta+\frac{\tau}{2})} \chi_{\lambda+2m+2}(\tau,\zeta). \tag{2.21b}$$

This diagonal partition function has the standard form for a lattice RCFT discussed at length in Refs.[16][21].

Let us briefly recall here the physical meaning of these modular invariance conditions [16]: the T^2 (resp. U) conditions require that the observable electrons have half-integer spin (resp. integer charge); S (resp. V) are self-consistency conditions for the completeness of the excitations under the change of temperature (resp. electric potential).

In the following, we shall consider the modular invariance of the partition function as part of the building criteria; this leads to the following postulate:

P1. *The partition function of the RCFT describing a fractional QH system should be invariant under (2.19) and (2.21).*

In the following, we are mostly interested in the 331 theory, corresponding to $m = 2$; in Section 4 and Appendix C, we prove that the partition function (2.20) is the unique solution to the modular invariance conditions P1.

An explicit expression of the 331 partition function (2.20), $m = 2$, is useful for the following discussion:

$$\begin{aligned} Z_{331} = \sum_{r=0}^{1} \Big\{ & |K_0(\tau;4)K_{2r}(\tau,2\zeta;8) + K_2(\tau;4)K_{2r+4}(\tau,2\zeta;8)|^2 \\ & + |K_2(\tau;4)K_{2r}(\tau,2\zeta;8) + K_0(\tau;4)K_{2r+4}(\tau,2\zeta;8)|^2 \\ & + |K_1(\tau;4)K_{2r+1}(\tau,2\zeta;8) + K_{-1}(\tau;4)K_{2r-3}(\tau,2\zeta;8)|^2 \\ & + |K_{-1}(\tau;4)K_{2r+1}(\tau,2\zeta;8) + K_1(\tau;4)K_{2r-3}(\tau,2\zeta;8)|^2 \Big\} \end{aligned} \quad (2.22)$$

(we used the periodicity condition (2.16)). This expression coincides, term by term, with that obtained in Ref. [10], for $\zeta = 0$ (see their Eq. (4.20)), by taking into account the symmetry (2.17) for the last two terms.

3 SU(2) Invariance versus Unitarity in the Haldane-Rezayi Model: the Mapping from $c = -2$ to $c = 1$

3.1 SU(2) Covariant OPE of $\psi_\rho(z)\psi_\sigma(w)$

The four-point function of the fermionic field ψ_ρ, $\rho = \pm 1/2$, appearing in the HR wave function (1.4) and (1.5) can be written in a manifestly SU(2) invariant form:

$$\langle 0|\psi_{\rho_1}(z_1)\psi_{\rho_2}(z_2)\psi_{\rho_3}(z_3)\psi_{\rho_4}(z_4)|0\rangle = \frac{\varepsilon_{\rho_1\rho_2}\varepsilon_{\rho_3\rho_4}}{z_{12}^2 z_{34}^2} + \frac{\varepsilon_{\rho_1\rho_4}\varepsilon_{\rho_2\rho_3}}{z_{14}^2 z_{23}^2} - \frac{\varepsilon_{\rho_1\rho_3}\varepsilon_{\rho_2\rho_4}}{z_{13}^2 z_{24}^2} . \quad (3.1)$$

(Note that it is only non-vanishing for $\rho_1+\rho_2+\rho_3+\rho_4 = 0$ and then the right hand side of (3.1) involves two non-zero terms.) It follows that the OPE of two ψ fields can also be written in an SU(2) covariant form (see Appendix B). We stress that this OPE just follows from the expression for the $2N$-point correlation function: it does not use the Virasoro properties of the fields; in fact, it does not require the knowledge of the stress energy tensor (and admits different CFT implementations). Keeping the first three terms in the small distance expansion we can write (using (B.2) and (B.14)):

$$\begin{aligned}\psi_\rho(z)\psi_\sigma(w) &= \varepsilon_{\rho\sigma}\left\{\frac{1}{(z-w)^2} - \mathcal{T}(w) - \frac{z-w}{2}\mathcal{T}'(w)\right\} \\ &- (z-w)V_{\rho+\sigma}(w) + O((z-w)^2)\ ,\end{aligned} \tag{3.2}$$

where $\mathcal{T}$ and V are composite fields of ψ_ρ,

$$\begin{aligned}\mathcal{T}(z) &= :\psi_-(z)\psi_+(z):\ , \\ V_{\rho+\sigma}(z) &= \frac{1}{2}:\{\psi_\rho(z)\partial\psi_\sigma(z) - (\partial\psi_\rho(z))\psi_\sigma(z)\}:,\end{aligned} \tag{3.3}$$

(cf. (1.9)). They satisfy:

$$\langle 0|\psi_\rho(z_1)\psi_\sigma(z_2)\mathcal{T}(z_3)|0\rangle = \frac{\varepsilon_{\rho\sigma}}{z_{13}^2 z_{23}^2}\ , \qquad \langle 0|\mathcal{T}(z_1)\mathcal{T}(z_2)|0\rangle = -\frac{1}{z_{12}^4}, \tag{3.4}$$

$$\langle 0|\psi_\rho(z_1)\psi_\sigma(z_2)V_a(z_3)|0\rangle = -\frac{z_{12}}{z_{13}^3 z_{23}^3}G_{\rho+\sigma,a}\ , \quad \langle 0|V_a(z_1)V_b(z_2)|0\rangle = \frac{G_{ab}^0}{z_{12}^6}, \tag{3.5}$$

where the non-zero elements of G_{ab} (for $a,b = 0,\pm 1$) are $G_{00} = -1$ and $G_{1-1} = G_{-11} = 2$. A salient feature of this OPE is the absence of the current j (1.14) from the right hand side of (3.2); however, it will reenter our discussion when we impose unitarity.

3.2 Other Choices of the Stress Tensor

If we now assume that $\psi_\rho(z)$ are primary fields (of dimension 1), we should identify $\mathcal{T}$ with the stress tensor of the theory; then, we arrive, in agreement with [9], at a non-unitary conformal model with $c = -2$ (indeed, one identifies the coefficient -1 in front of $\mathcal{T}$ in the right hand side of (3.2) with $2\Delta/c$, yielding $c = -2$ for $\Delta = \Delta(\pm 1) = 1$ [7]).

One can, however, preserve conformal invariance without allowing for negative norm squares and negative dimensions. We remark that the canonical anticommutation relations (1.12) and the two-point function (1.13,1.14) are sufficient for determining the HR wave function (1.4). These relations give room to a family of stress tensors [20]:

$$T_\kappa(z) = (1-\kappa):\xi'(z)\eta(z):-\kappa:\xi(z)\eta'(z): = \frac{1}{2}\left[:j^2(z): + (1-2\kappa)j'(z)\right]\ , \tag{3.6}$$

$$j(z) = :\xi(z)\eta(z):\ , \tag{3.7}$$

where $T_0 \equiv \mathcal{T}$. The dimensions of $\xi(z) = \xi(z,\kappa)$ and $\eta(z) = \eta(z, 1-\kappa)$ become κ and $(1-\kappa)$, respectively. The current anomaly and the Virasoro central charge depend on κ:

$$T_\kappa(z_1) j(z_2) \sim \frac{\partial}{\partial z_2}[z_{12}^{-1} j(z_2)] + (2\kappa - 1) z_{12}^{-3} \ , \tag{3.8a}$$

$$c_\kappa = 1 - 3(2\kappa - 1)^2 \ . \tag{3.8b}$$

Clearly, $\psi_- = \xi'$ (1.15) is only primary for $\kappa = 0$, i.e., $c = -2$, which accounts for this choice in [9, 10, 11]).

Here we remark that there is a *unique unitary point*, $\kappa = 1/2$, $c = 1$, in which the current is primary, and we can construct a model for the HR state satisfying all desiderata listed in the Introduction. In this case, we can identify ξ-η with a pair of conjugate Weyl spinors:

$$\eta(z, \frac{1}{2}) \equiv \psi(z) \ , \qquad \xi(z, \frac{1}{2}) \equiv \psi^*(z) \ ; \tag{3.9a}$$

furthermore, the stress tensor (3.6) takes the canonical form,

$$T(z) = T_{\frac{1}{2}}(z) = \frac{1}{2} {:}\big(\psi^{*\prime}(z)\psi(z) - \psi^*(z)\psi'(z)\big){:}. \tag{3.9b}$$

The identification (3.9a) implies a change of hermitean conjugation, which is only possible at $\kappa = 1/2$, where ξ, η have the same dimension. For $\kappa \neq 1/2$, we had instead $\eta^* = \eta$ and $\xi^* = \xi$. Note that the current $j(z)$ (3.7) is only hermitean with respect to the new conjugation rule (3.9a).

The SU(2) invariant tensor $\mathcal{T}(z)$ appearing in the OPE (3.2) becomes now non-hermitean: in terms of modes, setting $\mathcal{T}(z) = \sum_{n\in\mathbb{Z}} \mathcal{L}_n z^{-n-2}$ and $T(z) = \sum_{n\in\mathbb{Z}} L_n z^{-n-2}$, we have:

$$\mathcal{L}_n = L_n - \frac{n+1}{2} j_n, \qquad \mathcal{L}_n^* = \mathcal{L}_{-n} - n j_{-n} \ , \quad L_n^* = L_{-n} \ , \quad j_n^* = j_{-n} \ . \tag{3.10}$$

The tensor $\mathcal{T}(z)$ is an $SU(2)$ singlet, while the unitary one, $T(z)$, is not, because $j_0 \propto S_3$ as discussed in Section 1.2.

Remark 3.1. The inner products of $\widehat{u(1)}$ ground states $|c, \sigma\rangle$, where σ is the eigenvalue of j_0, and the behaviour of j_n under conjugation depend on c. Indeed, the properties of the vacuum ket $|-2, 0\rangle$ and of its bra counterpart are dictated by the expression (1.12) for the ξ-η two-point function and the mode expansions:

$$\begin{aligned} \xi(z) &= \sum_n \xi_n z^{-n} \ , \quad \eta(w) = \sum_n \eta_n w^{-n-1} \ , \\ &\Rightarrow \ \eta_0 \xi_0 |-2, 0\rangle = |-2, 0\rangle \ , \qquad \eta_0 |-2, 0\rangle = 0 \ . \end{aligned} \tag{3.11}$$

Furthermore, for $c=-2$, the independent hermiticity of ξ and η implies the following conjugation properties of the current:

$$j_n = \sum_{m\geq -n} \xi_{-m}\eta_{m+n} - \sum_{m\geq 1} \eta_{-m}\xi_{m+n} \Rightarrow j_n^* = \delta_{n,0} - j_{-n} \ , \qquad (c=-2). \quad (3.12)$$

As a corollary, there are different bra-ket couplings in the two cases:

$$\langle -2,\sigma| -2,\tau\rangle = \delta^1_{\sigma+\tau} \ , \qquad \langle 1,\sigma|1,\tau\rangle = \delta_{\sigma,\tau}. \quad (3.13)$$

In particular, the bra counterpart of the ket vacuum $|-2,0\rangle$ is $\langle -2,1|$ both being annihilated by the $c=-2$ Virasoro energy

$$\mathcal{L}_0 = \frac{1}{2} j_0(j_0-1) + \sum_{n=1}^{\infty} j_{-n}j_n \quad (= \mathcal{L}_0^*) \, . \quad (3.14)$$

It is the presence of a second conjugation for $c=1$ (which, incidently, does not change the hermiticity of $\mathcal{L}_0$) which renders the $c=1$ model unitary. It is remarkable that this c dependence of inner products and conjugation does not affect the correlation functions of charged fields.

Remark 3.2. Lee and Wen [12] (building on a development in [10]) propose a different way out of the unitarity problem. They abandon the CFT calculations and give instead an explicit construction of the Fermi fields in (1.5):

$$\begin{aligned} \psi_\uparrow(z) &= \sum_{n=1}^{\infty} \sqrt{n}(c_{n,\uparrow} z^{-n-1} + c^*_{n,\downarrow} z^{n-1}) \ , \quad (3.15) \\ \psi_\downarrow(z) &= \sum_{n=1}^{\infty} \sqrt{n}(c_{n,\downarrow} z^{-n-1} - c^*_{n,\uparrow} z^{n-1}) \ , \end{aligned}$$

where $c_{n\rho}$ satisfy canonical anticommutation relations, which imply (1.6) (for $\psi_\pm = \psi_{\uparrow\downarrow}$). The Hamiltonian of the system is defined by the manifestly SU(2) invariant expression:

$$H = \sum_{n=1}^{\infty} n \left(c^*_{n,\uparrow} c_{n,\uparrow} + c^*_{n,\downarrow} c_{n,\downarrow} \right) \ , \quad (3.16)$$

(but no attempt is made to introduce a hermitean stress energy tensor). H and ψ_ρ are found to satisfy the standard commutation relations for a dimension 1 field: $[H,\psi_\rho(z)] = (z\frac{\mathrm{d}}{\mathrm{d}z}+1)\psi_\rho(z)$. (Incidently, the expression (3.15) for a pair of Fermi fields of integer dimension is a special case of a procedure proposed in Ref. [24] to circumvent the spin-statistics theorem in any number of space-time dimensions.)

The fields ψ_ρ (3.15) - much like ours, (1.15)(3.9)- are not conjugate to each other. The SU(2) symmetry, however, has a more fundamental status in [12]: the Hamiltonian (3.16) is SU(2) invariant, the fields $\psi_\uparrow$ and $\psi_\downarrow$ (3.15) have the same dimension 1. What we regard, on the other hand, as a shortcoming of this model displayed is its *non-locality whenever the fields hermitean conjugate* to (3.15) are included; indeed, while:

$$[\psi_\uparrow(z_1), \psi_\downarrow(z_2)]_+ = \sum_{n=1}^{\infty} n(z_1^{n-1} z_2^{-n-1} - z_1^{-n-1} z_2^{n-1}) = \delta'(z_{12}) \ ,$$

one has:

$$[\psi_\uparrow(z_1), \psi_\uparrow^*(z_2)]_+ = \sum_{n=1}^{\infty} n(z_1^{n-1} z_2^{-n-1} + z_1^{-n-1} z_2^{n-1}) = \frac{1}{z_{12}^2} + \frac{1}{z_{21}^2} \ .$$

3.3 Operator Correspondence between the $c = -2$ and $c = 1$ Theories

We relate the $c = -2$ and the $c = 1$ models by identifying their $\widehat{u(1)}$ currents. Noting that Eq. (3.6) expresses, for $\kappa = 0$ and $1/2$, the corresponding stress energy tensors $\mathcal{T}$ and T in terms of j, we consider the $\widehat{u(1)}$ extension of the $c = -2$ Virasoro algebra Vir(-2) and compare it with the corresponding known $\widehat{u(1)}$ extension at $c = 1$. Next, the Vir(-2) primary field $\psi_-(z) = \xi'(z)$ satisfies $[j(z), \xi'(w)] = \xi(w)\frac{\partial}{\partial w}\delta(z-w)$; hence, it isn't primary with respect to the current algebra. As a result, it is not primary with respect to Vir(1) either. On the other hand, the $\widehat{u(1)}$- primary fields for $c = -2$ are also $\widehat{u(1)}$ primary for $c = 1$.

Proposition 3.1 *Let ϕ_σ be a Vir(-2) primary field of j_0-"charge" σ and dimension $\Delta_\sigma^{(-2)}$. Then it is $\widehat{u(1)}$- and, by implication, Vir(1)-primary iff the dimension and charge are related by:*

$$\Delta_\sigma^{(-2)} = \frac{1}{2}\sigma(\sigma - 1) \ ; \qquad (3.17)$$

moreover, ϕ_σ satisfies the Knizhnik–Zamolodchikov equation [7],

$$\frac{d}{dz}\phi_\sigma = :j\phi_\sigma: \equiv \sum_{n=0}^{\infty}(j_{-n-1}z^n\phi_\sigma + \phi_\sigma j_n z^{-n-1}) \ . \qquad (3.18)$$

The Vir(1) dimension, on the other hand, is symmetric under charge conjugation:

$$\Delta_\sigma^{(1)} = \frac{1}{2}\sigma^2 \ . \tag{3.19}$$

The **proof** uses standard current algebra techniques (see, e.g., [7]).

We remark that the dimension (3.17) of the $Vir(-2)$ fields reproduces the Kac formula Δ_l (1.16) for $l = 2\sigma - 1$; however, we find that only a subset of the degenerate $Vir(-2)$ fields (forming $\widehat{SU(2)}$ multiplets) satisfy the charge-dimension relation. A complete set of $\widehat{u(1)}$-primary fields (which are single-valued in the vacuum sector) is given by:

$$\begin{aligned} \phi_\sigma(z) &= \frac{\xi(z)}{0!}\frac{\xi'(z)}{1!}\cdots\frac{\xi^{(\sigma-1)}(z)}{(\sigma-1)!} \ , \\ \phi_{-\sigma}(z) &= \frac{\eta(z)}{0!}\frac{\eta'(z)}{1!}\cdots\frac{\eta^{(\sigma-1)}(z)}{(\sigma-1)!} \ , \end{aligned} \tag{3.20}$$

Each $\widehat{u(1)}$ primary $\phi_{-\sigma}$ can be identified with the lowest spin-projection member $\phi_{s,-s}$ of the SU(2) spin multiplet ϕ_{sm} , $2s = \sigma$, which was discussed in the Introduction (see the text preceding Eq.(1.11)). All other fields ϕ_{sm}, $m \neq -s$, are still Vir(-2) primary fields but are not $\widehat{u(1)}$-primary (and not even quasi-primary with respect to Vir(1)).

In order to construct an RCFT involving the series (3.20) we consider the extended chiral algebra $\mathcal{A}(c, \sigma^2)$ for both $c = -2$ and $c = 1$, which is generated by the $\sigma = \pm 2$ local Bose fields:

$$\phi_2(z) = \xi(z)\xi'(z) \ , \qquad \phi_{-2}(z) = \eta(z)\eta'(z). \tag{3.21}$$

It is the even part of the superalgebra $\mathcal{S}(c)$ of the ξ-η pair. The bosonic algebra $\mathcal{A}(c,4)$ has 4 irreducible representations (with energy bounded below) whose state spaces $\mathcal{H}_\sigma^{(c)}$ split into infinite direct sums of irreducible $\widehat{u(1)}$ modules $\mathcal{V}_\sigma$:

$$\mathcal{H}_\sigma^{(c)} = \bigoplus_{n\in\mathbb{Z}} \mathcal{V}_{\sigma+2n}, \qquad \sigma = 0, \pm\frac{1}{2}, 1. \tag{3.22}$$

The integer "charges", $\sigma = 0, 1$, correspond to the Neveu-Schwarz sector of $\mathcal{S}(c)$; $\sigma = \pm\frac{1}{2}$ label the Ramond sector.

Eqs. (3.22) and (3.10) for $n = 0$ allow to compute the $c = -2$ characters in terms of the standard $c = 1$ lattice characters,

$$\chi_\sigma^{(1)}(\tau,\zeta) = \mathrm{tr}_{\mathcal{H}_\sigma^{(1)}} \mathrm{e}^{2\pi i\{\tau(L_0 - \frac{1}{24}) + \frac{1}{2}\zeta j_0\}} = K_{2\sigma}(\tau,\zeta;4) \ , \tag{3.23}$$

where $K_\lambda(\tau,\xi;M)$ is defined in (2.15).

Indeed, we have, in view of (3.10),

$$\begin{aligned}
\chi_\sigma^{(-2)}(\tau,\zeta) &= \mathrm{tr}_{\mathcal{H}_\sigma^{(-2)}} q^{\mathcal{L}_0+\frac{1}{12}} \mathrm{e}^{2\pi i\zeta\frac{1}{2}j_0} = q^{\frac{1}{8}}\mathrm{tr}_{\mathcal{H}_\sigma^{(1)}} q^{L_0-\frac{1}{24}} \mathrm{e}^{2\pi i(\zeta-\tau)\frac{1}{2}j_0} \\
&= q^{\frac{1}{8}} K_{2\sigma}(\tau,\zeta-\tau;4) = K_{2\sigma-1}(\tau,\zeta;4)\ . \qquad (3.24)
\end{aligned}$$

This correspondence between the $c=-2$ and $c=1$ dimensions $\Delta_\sigma^{(c)}$ and characters $\chi_\sigma^{(c)}$ is summarized in Table 2; the "index shift" for the characters $(K_l(\tau;4) \to K_{l+1}(\tau;4))$ is graphically represented on Fig. 2.

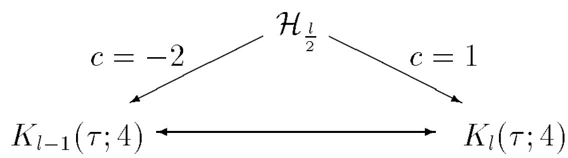


Figure 2: Shift of the characters in the mapping between the $c=-2$ and $c=1$ theories.

Furthermore, in Ref. [10], a partition function for the $c=-2$ CFT has been proposed (see their Eq. (4.16)). This can be written in our notation as follows:

$$\begin{aligned}
Z_{HR} \;=\; \sum_{r=0}^{1}\Big\{ & |K_0(\tau;4)K_{2r+1}(\tau,2\zeta;8)+K_2(\tau;4)K_{2r-3}(\tau,2\zeta;8)|^2 \\
& +|K_2(\tau;4)K_{2r+1}(\tau,2\zeta;8)+K_0(\tau;4)K_{2r-3}(\tau,2\zeta;8)|^2 \\
& +|K_1(\tau;4)K_{2r}(\tau,2\zeta;8)+K_{-1}(\tau;4)K_{2r+4}(\tau,2\zeta;8)|^2 \\
& +|K_{-1}(\tau;4)K_{2r}(\tau,2\zeta;8)+K_1(\tau;4)K_{2r+4}(\tau,2\zeta;8)|^2 \Big\}\ . \qquad (3.25)
\end{aligned}$$

This expression is *not* modular invariant; however, under the $c=-2\to 1$ mapping this remarkably becomes the 331 partition function (2.22) found in Section 2:

$$Z_{HR} \to Z_{331}\ , \qquad \text{for}\quad c\to 1\ . \qquad (3.26)$$

This shows that the CFT for the HR state proposed in Ref. [10], once made unitary, becomes the same as the 331 CFT. Furthermore, the parity rules coupling neutral and charged excitations is the same in the two Hall fluids. Actually, the above mapping amounts to a redefinition of the energy and of the scalar product on the same set of states (see also Refs.[12, 13, 25]).

Table 2: **The $c=-2 \Longleftrightarrow c=1$ correspondence**

l	$\sigma = l/2$	$\Delta_\sigma^{(c=-2)} = \sigma(\sigma-1)/2$	$\chi_\sigma^{(-2)}(\tau)$	$\Delta_\sigma^{(c=1)} = \sigma^2/2$	$\chi_\sigma^{(1)}(\tau)$
-1	$-1/2$	$3/8$	$K_2(\tau;4)$	$1/8$	$K_{-1}(\tau;4)$
0	0	0	$K_{-1}(\tau;4)$	0	$K_0(\tau;4)$
1	$1/2$	$-1/8$	$K_0(\tau;4)$	$1/8$	$K_1(\tau;4)$
2	1	0	$K_1(\tau;4)$	$1/2$	$K_2(\tau;4)$

The correspondence between the dimensions in the two models is given by:

$$\Delta^{(-2)}_{\sigma+\frac{1}{2}} + \frac{1}{12} = \Delta^{(1)}_{\sigma} - \frac{1}{24} \ , \tag{3.27}$$

(the additive constant on each side being just $-c/24$). The fields (3.20), for integer σ, satisfy (for both values of c) the *charge–statistics relation*:

$$\phi_\sigma(z)\phi_\tau(w) = (-1)^{\sigma\tau}\phi_\tau(w)\phi_\sigma(z). \tag{3.28}$$

4 Admissible $c=2$ Descriptions of the $\nu=\frac{1}{2}$ Double Layer States

In the previous section we have shown that the 331 model and the (unitarized) HR model of Ref. [10] share the same partition function and therefore have the same RCFT description. Nevertheless, we have not so far excluded the possibility of other modular invariants made by the same basis of states which could correspond to an alternative RCFT for the HR theory. The construction of the 331 partition function in Section 2 shows that its chiral building blocks (2.14) are expressed as sums of products of $\mathcal{A}(\Gamma_8)$- and $\mathcal{A}(\Gamma_4)$-characters corresponding to the orthogonal lattice

$$\Gamma_{8,4} = \Gamma_8 \oplus \Gamma_4 \subset L \subset \Gamma_{2,1} \subset \Gamma^*_{2,1} \subset L^* \subset \Gamma^*_{8,4}. \tag{4.1}$$

This orthogonal basis of states allows, in principle, for other RCFTs with different couplings between neutral and charged sectors, i.e. with different parity rules. We have the following result.

Theorem 4.1 *There are 7 distinguishable RCFT models with ($c=2$ extensions of the) chiral algebra $\mathcal{A}(\Gamma_{8,4})$ that have modular invariant partition* *functions (see P1 in Section 2.2). Six of them factorize into products of neutral and charged parts*

$$
\begin{aligned}
Z^{(l)}_{8,4}(\tau,\zeta) &= Z^{(l)}_{8}(\tau,\zeta) Z_4(\tau,0), \quad l=1,-3, &(4.2)\\
Z^{(l)}_{8,1}(\tau,\zeta) &= Z^{(l)}_{8}(\tau,\zeta) Z_1(\tau,0), \quad l=1,-3, &(4.3)\\
Z_{2,4}(\tau,\zeta) &= Z_2(\tau,\zeta) Z_4(\tau,0)\ , &(4.4)\\
Z_{2,1}(\tau,\zeta) &= Z_2(\tau,\zeta) Z_1(\tau,0)\ , &(4.5)
\end{aligned}
$$

where

$$
Z^{(l)}_M(\tau,\zeta) = \mathrm{e}^{-\frac{4\pi}{M}\frac{(\mathrm{Im}\,\zeta)^2}{\mathrm{Im}\,\tau}} \sum_{\lambda \,\mathrm{mod}\, M} K_\lambda(\tau,2\zeta;M)\overline{K_{l\lambda}}(\tau,2\zeta;M)\ , \qquad (4.6)
$$

and $Z_M(\tau;\zeta) \equiv Z^{(1)}(\tau,\zeta)$*. The only non-factorizable one is* Z_{331} *(2.20).*

Remark 4.1. Here and in what follows we deal exclusively with *physical* partition functions in which all characters enter with non-negative integer multiplicities and the vacuum character appears with multiplicity one (cf. Refs.[16, 21]).

A proof of Theorem 4.1, based on ref. [21], is given in Appendix C.

We shall analyze the seven RCFT singled out by the above theorem in order to select those which can describe the $\nu = 1/2$ state. To begin with, we note that all 7 partition functions contain a pair of fields with the properties of the electrons in the two layers,

$$
\psi^1(z) = \psi^*(z) E^{-\sqrt{2}}(z)\ , \qquad \psi^2(w) = \psi(w) E^{-\sqrt{2}}(w)\ , \qquad (4.7)
$$

of charges $\mathbf{q}^i$ with Gram matrix and electric charge:

$$
\begin{pmatrix} (\mathbf{q}^1|\mathbf{q}^1) & (\mathbf{q}^1|\mathbf{q}^2) \\ (\mathbf{q}^2|\mathbf{q}^1) & (\mathbf{q}^2|\mathbf{q}^2) \end{pmatrix} = \begin{pmatrix} 3 & 1 \\ 1 & 3 \end{pmatrix}, \quad (\mathbf{Q}|\mathbf{q}^i) = 1, \quad i=1,2, \quad |\mathbf{Q}|^2 = \nu = \frac{1}{2} \qquad (4.8)
$$

(thus $\psi^i(z)$ can be identified with $E^{\mathbf{q}^i}(z)$ in the notation of Appendix A). This has been made explicit in Eq. (2.5) for the $\mathbb{Z}_2$ orbifold model of partition function Z_{331} (2.22). Such fermion fields also belong to the chiral algebra of the $Z_{2,1}$ model (Eq. (4.5)) and to suitable extensions $\mathcal{A} = \mathcal{A}(\Gamma_{\mathrm{phys}})$ of the lattice chiral algebras $\mathcal{A}(\Gamma)$, $\Gamma = \Gamma_{2.4^\mu,4^\nu}$, $\mu,\nu = 0,1$, corresponding to the other six cases. Here Γ_{phys} is an *integral extension* of Γ satisfying [22]:

$$
\begin{aligned}
\Gamma &\subset \Gamma_{\mathrm{phys}} \subset \Gamma^*, \\
\exists \mathbf{q}^i &\in \Gamma_{\mathrm{phys}},\ i=1,2, \ \text{ so that } \ (\mathbf{Q}|\mathbf{q}^i) = 1 = |\mathbf{q}^i|^2 \mod 2. \qquad (4.9)
\end{aligned}
$$

In fact, if we exclude the case $\Gamma_{\rm phys} = L \supset \Gamma_{8,4}$ which naturally leads to Z_{331} for all 3 lattices ($\Gamma = \Gamma_{8,4}, \Gamma_{8,1}, \Gamma_{2,4}$), $\Gamma_{\rm phys}$ coincides with the maximal integral extension $\Gamma_{2,1}$ of Γ; this contains two charges $\mathbf{e}^1$ and $\mathbf{e}^2$ of lengths 2 and 1, respectively (corresponding to the fields $E^{\pm\sqrt{2}}(z)$ and $\psi^{(*)}$ of (4.7)) and hence their sum and difference $\mathbf{q}^{1,2} = \mathbf{e}^1 \pm \mathbf{e}^2$ satisfy (4.8).

There are, however, important distinctions between the Z_{331} model and the models with factorizable partition functions (4.2-4.5). The common factorizable extensions $\Gamma_{2,1}$ of the three Γ involves a pair of Fermi fields $\psi^{(*)}$ of dimension 1/2 - smaller than the dimension 3/2 of the basic electron fields ψ^i. Moreover, all factorizable models contradict a natural postulate about the charge–statistics relation [15][22].

P2. *If* $\mathbf{Q}$ *is the electric charge vector (cf. (2.12)), then there exists a* $\mathbf{q}' \in \Gamma_{\rm phys}$ *such that* $(\mathbf{Q}|\mathbf{q}') = 1$ *and for any* $\mathbf{q} \in \Gamma_{\rm phys}$ *we must have:*

$$(\mathbf{Q}|\mathbf{q}) = |\mathbf{q}|^2 \mod 2 \qquad \left(\mathbf{Q} \in \Gamma^*_{\rm phys}\right) \ , \tag{4.10}$$

(i.e., $\mathbf{Q}$ *is an odd vector with respect to* $\Gamma_{\rm phys}$ *in the terminology of ref.* [22]*).*

This postulate states that there exists an electron excitation in the spectrum and that all observable charged excitations are made out of electrons.

Thus the two postulates P1 and P2 together with the requirement:

$$\Gamma_{8,4} \subset \Gamma_{\rm phys} \subset \Gamma_{2,1} = \{\mathbb{Z}\mathbf{e}^1 + \mathbb{Z}\mathbf{e}^2\} \ , \quad \mathbf{Q} = \mathbf{e}^*_1 \ , \tag{4.11}$$

yield $\Gamma_{\rm phys} = L$ (2.6), i.e., a unique lattice CFT with partition function Z_{331}.

5 Gauge Reductions

5.1 The Pfaffian State as a Projection of the 331 State in the Low-Barrier Limit

The $c = 3/2$ *Pfaffian state* [2] could describe a double-layer sample in the particular limit in which the tunnelling amplitude between the layers is large and the two species of electrons become indistinguishable [5]. This limit can be described in CFT by performing the following (generalized) gauge reduction on the 331 CFT of distinguishable electrons. Splitting the Weyl spinor ψ into real and imaginary Majorana components $\varphi_{1,2}$ ($\varphi^*_j(z) = \varphi_j(z)$):

$$\psi^*(z) = \frac{1}{\sqrt{2}}\left(\varphi_1(z) + i\varphi_2(z)\right) \ , \qquad \psi(z) = \frac{1}{\sqrt{2}}\left(\varphi_1(z) - i\varphi_2(z)\right) \ , \tag{5.1}$$

we observe that the two layers just differ by the sign of φ_2; thus, in the limit of low potential barrier between the two layers, the imaginary part φ_2 of ψ

should be "gauged away". This is achieved by the usual coset construction [26], which consists first in decomposing the stress tensor in two parts, as follows:

$$T_L(z) = T_{Pf}(z) + \frac{1}{2}{:}\varphi_2(z)\varphi_2'(z){:}. \tag{5.2}$$

The last term is then projected out, leading to a $c = 3/2$ CFT. The resulting theory can once more be viewed as a $\mathbb{Z}_2$ orbifold - this time the chiral algebra $\mathcal{A}_{Pf}$ appears as the $\mathbb{Z}_2$ invariant part of the tensor product algebra $\mathcal{A}(\Gamma_2) \otimes \mathcal{A}_{Ising}(\varphi_1)$. The Pfaffian ground state wave function is fully antisymmetric [2]:

$$\begin{aligned} \Phi_{Pf}(z_1, \ldots, z_{2N}) &= \langle 0|\varphi_1(z_1)\varphi_1(z_2)\cdots\varphi_1(z_{2N})|0\rangle \Phi_m(z_1, \ldots, z_{2N}) \\ &= Pf\left(\frac{1}{z_{ij}}\right)\Phi_m(z_1, \ldots, z_{2N}) \end{aligned} \tag{5.3}$$

where Φ_m is the usual Laughlin factor (1.3) and the Pfaffian is,

$$Pf\left(\frac{1}{z_{ij}}\right) = \frac{1}{2^N N!}\sum_{\sigma\in S_{2N}} \mathrm{sgn}(\sigma)\prod_{k=1}^{N}\frac{1}{z_{\sigma(2k-1)} - z_{\sigma(2k)}}. \tag{5.4}$$

The resulting CFT exhibits some interesting features and deserves a special attention. The total antisymmetry of the wave function signals the fact that the electrons of the two layers are indeed indistinguishable. The neutral part (5.3) of the wave function appears as the vacuum expectation value of a product of free Majorana fermions which generate the Neveu-Schwarz sector of the $c = 1/2$ Ising model [7]. The Pfaffian model has topological order 6. The resulting representation spaces $\mathcal{H}_\lambda$ ($\lambda = 0, \pm 1, \pm 2, 4$) have lowest conformal weights:

$$\Delta_0 = 0\ , \quad \Delta_{\pm 1} = \frac{1}{8}\ , \quad \Delta_{\pm 2} = \frac{1}{4}\ , \quad \Delta_4 = \frac{1}{2}\ , \tag{5.5}$$

and the same charges as the 331 model - see Table 1.

The characters ch_λ are expressed, in parallel with (2.14), in terms of products of the Ising characters χ_ν, $\nu = 0, 1, 2$ and the characters $K_l(\tau, 2\zeta; 8)$ of the 331 CFT:

$$\begin{aligned} \mathrm{ch}_{2n}(\tau,\zeta) &= \chi_0(\tau)K_{2n}(\tau,2\zeta;8) + \chi_2(\tau)K_{2n+4}(\tau,2\zeta;8)\ , \quad n = 0, \pm 1, 2, \quad \text{(5.6a)} \\ & \qquad \chi_0(\tau) \pm \chi_2 = q^{-1/48}\prod_{n=1}^{\infty}\left(1 \pm q^{n-\frac{1}{2}}\right)\ ; \\ \mathrm{ch}_{\pm 1}(\tau,\zeta) &= \chi_1(\tau)\left(K_{\pm 1}(\tau,2\zeta;8) + K_{\mp 3}(\tau,2\zeta;8)\right)\ , \quad \text{(5.6b)} \\ & \qquad \chi_1(\tau) = q^{1/24}\prod_{n=1}^{\infty}(1+q^n)\,. \end{aligned}$$

The modular invariant partition function of the Pfaffian state is obtained by the diagonal quadratic form in the basis (5.6):

$$\begin{aligned} Z_{Pf} &= \sum_{r=0}^{1} \Big\{ |\chi_0(\tau) K_{2r}(\tau,2\zeta;8) + \chi_2(\tau) K_{2r+4}(\tau,2\zeta;8)|^2 \\ &\quad + |\chi_2(\tau) K_{2r}(\tau,2\zeta;8) + \chi_0(\tau) K_{2r+4}(\tau,2\zeta;8)|^2 \\ &\quad + |\chi_1(\tau) \left(K_{2r+1}(\tau,2\zeta;8) + K_{2r-3}(\tau,2\zeta;8)\right) |^2 \Big\}. \end{aligned} \tag{5.7}$$

This coincides with the expression proposed in Ref. [10] (Eq.(4.13)); we have verified that it satisfies the modular conditions (postulate **P1**); in particular, the Pfaffian S matrix,

$$S_{mn} = \frac{1}{2} \sin\left(\frac{\pi}{4}\sigma_m\sigma_n\right) \mathrm{e}^{-i\frac{\pi}{4}mn}, \qquad \sigma_n = \frac{3-(-1)^n}{2}, \qquad m,n = 0,\pm1,\pm2,4. \tag{5.8}$$

has dimension 6×6; this matches the topological order 6, in agreement with the general arguments of Ref.[16].

It is instructive to see the gauge reduction from the 331 CFT to the Pfaffian in the partition functions. Using the Jacobi identity, the neutral characters $K_l(\tau;4)$ in the 331 partition function (2.22) can be written in fermionic Fock-space form:

$$\begin{aligned} K_{0,2}(\tau;4) &= \frac{q^{-1/24}}{2}\left(\prod_{n=1}^{\infty}\left(1+q^{n-\frac{1}{2}}\right)^2 \pm \prod_{n=1}^{\infty}\left(1-q^{n-\frac{1}{2}}\right)^2\right) , \\ K_{\pm1}(\tau;4) &= q^{1/12}\prod_{n=1}^{\infty}\left(1+q^n\right)^2 . \end{aligned} \tag{5.9}$$

The projection of one Majorana fermion can be realized by eliminating the squares in these characters, i.e.,

$$K_0(\tau;4) \ \to\ \chi_0(\tau) , \quad K_2(\tau;4) \ \to\ \chi_2(\tau) , \quad K_{\pm1}(\tau;4) \ \to\ \chi_1(\tau) . \tag{5.10}$$

Moreover, only one for each pair of twisted sectors of the 331 CFT survives in (5.6). After these transformations the 331 partition function (2.22) is seen to become the Pfaffian one (5.7). It follows, in particular, that the Pfaffian theory inherits the parity rule of the 331 state.

The quantum dimension [27] of the representations (±1) (of characters (5.6b)) is $\sqrt{2}$ signalling the presence of a non-abelian (braid group) statistics for quasi-holes [2, 28, 18]. The corresponding fusion rules [7] read:

$$(\pm\mathbf{1})\times(\pm\mathbf{1}) = (\mathbf{2}) + (-\mathbf{2}) , \qquad (\mathbf{1})\times(-\mathbf{1}) = (\mathbf{0}) + (\mathbf{4}). \tag{5.11}$$

One may speculate that we observe here a manifestation of a more general phenomenon: the non-abelian statistics comes from a gauge reduction (in this case the projection $\varphi_2 \to 0$) of a lattice abelian model. The $\mathrm{U}(1)^n$ lattice models possess the characteristic $W_{1+\infty}$ symmetry of the incompressible quantum Hall fluids under area preserving diffeomorphisms [14]. This has been found to be a crucial feature of the CFT describing the single-layer hierarchical Hall states [15]. Therefore, it seems natural to expect it to be present in the paired Hall states as well. However, the Pfaffian state is not described by a $W_{1+\infty}$ CFT; this is signalled by the fact that its central charge $c = 3/2$ is not an integer [15]. We thus see that the $W_{1+\infty}$ symmetry is broken by the gauge reduction, which is a quantum effect occuring in the low potential barrier limit of a two-layer system.

It is important to remark that the $W_{1+\infty}$ symmetry is still present at the semiclassical level, which corresponds to the limit $N \to \infty$ [14]; actually, in this limit, the Pfaffian wave function (5.4) is dominated by the Laughlin factor (angular momentum of order $O(N^2)$), while the Pfaffian is a subleading $O(1/N)$ relative correction. The dominant term corresponds to the lattice RCFT which is $W_{1+\infty}$ symmetric.

5.2 Maximally Symmetric $c = 3$ Description of Paired Hall States

Until now we restricted our attention to rank 2 charge lattices which correspond to central charge $c = 2$. It seems natural to assume that the charge lattice associated with a QH state should have minimal possible rank. From this point of view it may appear superfluous, once we have a satisfactory description of the 331 state (2.1) in terms of the $c = 2$ RCFT of Section 2, to search for higher rank lattices in connection with this state. It has been argued, however, by Fröhlich et al. (see Sections 5 and 7 and Table (B.2) in Appendix B of the second of Ref. [22]), that a rank 3 lattice provides a "maximally symmetric" RCFT for this $\nu = 1/2$ state. It involves an $SU(2) \times SU(2)$ current algebra symmetry corresponding to spin rotation and layer; it is realized by a pair of Weyl fermions which are intertwined with the Laughlin boson.

We shall also demonstrate that the $c = 2$ CFT considered in the Sections 2 appears as a U(1) gauge reduction of this one, by projecting out the difference of the two Weyl currents. This identifies layer and spin quantum numbers of excitations, yielding the 331 CFT. In Ref.[22], the chiral QH lattice, - i.e., the pair $(\Gamma, \mathbf{Q})$, is defined in a *normal basis* as follows:

$$\Gamma = \Gamma_{\text{phys}} \quad = \quad \{\mathbb{Z}\mathbf{q} + \mathbb{Z}\underline{\alpha}^1 + \mathbb{Z}\underline{\alpha}^2\} \ ,$$

$$G_\Gamma = \begin{pmatrix} 3 & 1 & 1 \\ 1 & 2 & 0 \\ 1 & 0 & 2 \end{pmatrix}, \qquad \mathbf{Q} = \mathbf{q}^* = \frac{\mathbf{q}}{2} - \frac{\underline{\alpha}^1 + \underline{\alpha}^2}{4}\,. \quad (5.12)$$

The structure of superselection sectors and their fusion rules is similar to that of the 331 model. We have:

$$\Gamma^*/\Gamma = \mathbb{Z}_8 \quad (\Rightarrow |\Gamma| = \det G_\Gamma = 8)\ ; \qquad (5.13)$$

this shows [16] that the topological order is again 8. In Table 3, we display representatives of the non zero-elements of the additive group Γ^*/Γ (in parallel to the basis of L^*/L given in Table 1).

Table 3: **Superselection sectors for $\mathcal{A}(\Gamma)$**

coset $(\lambda) = \lambda\underline{\alpha}_1^* + \Gamma$	**representative** $\mathbf{q}^*$	**charge** $(\mathbf{Q}\vert\mathbf{q}^*)$	**dimension** $\Delta = \frac{1}{2}\vert\mathbf{q}^*\vert^2$
(0)	$\mathbf{0}$	0	0
(± 1)	$\pm\underline{\alpha}_1^*$	$\mp\frac{1}{4}$	$\frac{5}{16}$
(± 2)	$\pm(2\underline{\alpha}_1^* - \underline{\alpha}^1)$	$\mp\frac{1}{2}$	$\frac{1}{4}$
(± 3)	$\pm(3\underline{\alpha}_1^* - 2\underline{\alpha}^1 + \mathbf{q})$	$\pm\frac{1}{4}$	$\frac{5}{16}$
(4)	$\frac{1}{2}(\underline{\alpha}^1 + \underline{\alpha}^2)$	0	$\frac{1}{2}$

We note that the dimensions Δ_{2n} corresponding to the untwisted sector of even charge cosets ($\Delta_0 = 0,\ \Delta_{\pm 2} = 1/4,\ \Delta_4 = 1/2$) coincide precisely with those of the $\mathbb{Z}_4$ subgroup of L^*/L given in Table 1 for $m = 2$ and $\lambda = 0, \pm m, 2m$. The same is true for the minimal (absolute) values of the inner products of $\mathbf{Q}$ with vectors in each such coset, listed in the third column of Table 3. In both cases, *the smallest positive* (fractional) *electric charge of a quasi-particle is* $1/4$. The lattice Γ (5.12) obeys the postulate P2 and the electric charge satisfies (4.8) ($|\mathbf{Q}|^2 = \nu = 1/2$).

In order to verify this statement we pass from the normal basis (5.12) of Γ to its *symmetric basis* [22]:

$$\mathbf{q}^1 = \mathbf{q} - \underline{\alpha}^1,\ \mathbf{q}^2 = \mathbf{q} - \underline{\alpha}^2,\ \mathbf{q}^3 = \mathbf{q},\ (\Rightarrow\ \underline{\alpha}^i = \mathbf{q}^3 - \mathbf{q}^i,\ i = 1, 2), \quad (5.14a)$$

characterized by,

$$G_\Gamma(\mathbf{q}^i) = \left((\mathbf{q}^i|\mathbf{q}^j)_{i,j=1,2,3}\right) = \begin{pmatrix} 3 & 1 & 2 \\ 1 & 3 & 2 \\ 2 & 2 & 3 \end{pmatrix},\ \ (\mathbf{Q}|\mathbf{q}^i) = 1. \qquad (5.14b)$$

The resulting RCFT with chiral algebra $\mathcal{A}(\Gamma)$ can again be viewed as a $\mathbb{Z}_2$ orbifold of a tensor product algebra corresponding to the orthogonal integral lattice $\Gamma_{2,1,1}$ (that extends Γ):

$$\begin{aligned}(\Gamma\subset)\Gamma_{2,1,1} &= \{\mathbb{Z}\mathbf{e}^0\oplus\mathbb{Z}\mathbf{e}^1\oplus\mathbb{Z}\mathbf{e}^2\},\quad \mathbf{e}^0=2\mathbf{Q}=\mathbf{q}-\mathbf{e}^1,\ e^1=\frac{1}{2}(\underline{\alpha}^1+\underline{\alpha}^2),\\ & e^2=\frac{1}{2}(\underline{\alpha}^1-\underline{\alpha}^2)\ (\mathbf{e}^\mu|\mathbf{e}^\nu)=|\mathbf{e}^\mu|^2\delta_{\mu\nu},\ \mu,\nu=0,1,2,\\ & |\mathbf{e}^0|^2=2,\ \ |\mathbf{e}^1|^2=|\mathbf{e}^2|^2=1. \end{aligned} \tag{5.15}$$

Indeed, if we introduce the current:

$$J(z)=J^{\mathbf{Q}}(z)+J^{\underline{\alpha}^1}(z)\ ,\quad \underline{\alpha}^1=\mathbf{e}^1+\mathbf{e}^2\ ,\quad J^{\mathbf{Q}}(z)=\frac{1}{2}J^{\mathrm{e}^0}(z)\ , \tag{5.16}$$

then the inner automorphism α of the extended chiral algebra given by

$$\alpha(A)=\mathrm{e}^{i\pi J_0}\ A\ \mathrm{e}^{-i\pi J_0},\quad \text{for}\ \ A\in\mathcal{A}_{2,1,1}\ , \tag{5.17}$$

leaves invariant each element of the physical subalgebra $\mathcal{A}(\Gamma)$ while changing the sign of the basic charge shift operators $E^{\pm\mathbf{e}^\nu}$ and the associated Bose (for $\nu=0$) and Fermi (for $\nu=1,2$) local charged fields (see Eqs. (A.8-A.10) of Appendix A).

The characters $\chi_\lambda^\Gamma(\tau,\zeta)$ of the irreducible representations of the chiral algebra $\mathcal{A}(\Gamma)$ are given by the following counterpart of (2.13) and (2.14)

$$\chi_\lambda^\Gamma(\tau,\zeta)=\mathrm{e}^{-\frac{\pi}{2}\frac{(\mathrm{Im}\,\zeta)^2}{\mathrm{Im}\,\tau}}\mathrm{ch}_\Gamma^\lambda(\tau,\zeta)\ , \tag{5.18a}$$

$$\mathrm{ch}_\Gamma^\lambda(\tau,\zeta)=K_0(\tau;2)K_\lambda(\tau;2)K_{-\lambda}(\tau,2\zeta;8)+K_1(\tau;2)K_{\lambda+1}(\tau;2)K_{4-\lambda}(\tau,2\zeta;8) \tag{5.18b}$$

($\lambda\,\mathrm{mod}\,8$) where the K-functions are again given by (2.15). The proof of (5.18) is completely analogous to that of (2.15) given in Appendix A.

We end up once more with a diagonal modular invariant partition function of type (2.20),

$$Z_\Gamma(\tau,\zeta)=\sum_{\lambda=-3}^{4}|\chi_\lambda^\Gamma(\tau,\zeta)|^2. \tag{5.19}$$

In addition, there is an $\widehat{su(2)}\otimes\widehat{su(2)}$ current subalgebra of $\mathcal{A}(\Gamma)$ generated by the charged currents $E^{\pm\underline{\alpha}^i}(z)$ satisfying:

$$[E^{\underline{\alpha}^i}(z_1),E^{-\underline{\alpha}^j}(z_2)]=\left(J^{\underline{\alpha}^i}(z_2)\delta(z_{12})-\delta'(z_{12})\right)\delta_{ij}\ , \tag{5.20}$$

where $J^{\underline{\alpha}}$ are the corresponding Cartan currents which commute with the electromagnetic (u(1) -) current $J^{\mathbf{Q}}(z)$ (defined in (5.16)). It provides a local realization of the $\mathrm{SU}(2)_{\rm spin} \times \mathrm{SU}(2)_{\rm layer}$ symmetry of the model (justifying the term "maximally symmetric" of [22]).

The rank 2 ($\mathcal{A}(L)$) realization of the $\nu = \frac{1}{2}$ state, given in the preceding sections, is recovered if we gauge the u(1) current:

$$I(z)(\equiv J^{\mathrm{e}^2}(z)) = \frac{1}{2}\left(J^{\underline{\alpha}^1}(z) - J^{\underline{\alpha}^2}(z)\right) \ , \tag{5.21}$$

generated by the pair of conjugate Weyl spinors,

$$\varphi(z) = E^{-\mathrm{e}^2}(z) \ , \quad \varphi^*(z) = E^{\mathrm{e}^2}(z) \ \Rightarrow \ I(z) = :\varphi^*(z)\varphi(z): \ . \tag{5.22}$$

In particular, the new stress tensor is:

$$T_L(z) = \frac{1}{2}:\left\{J_1(z)J^1(z) + J_2(z)J^2(z)\right\}: \tag{5.23}$$

where $\{J^i\}$ and $\{J_i\}$ are dual bases in the Cartan subalgebra of $\mathcal{A}(L)$ (cf. (A.12)). $T_L(z)$ is obtained from the stress tensor $T_\Gamma(z)$ of $\mathcal{A}(\Gamma)$ by subtracting the Sugawara contribution of $I(z)$:

$$T_L(z) = T_\Gamma(z) - \frac{1}{2}:I^2(z):. \tag{5.24}$$

In other words, $\mathcal{A}(L)$ is obtained from $\mathcal{A}(\Gamma)$ by gauging out the second factor in the extended algebra,

$$\mathcal{A}(\Gamma_{2,1,1}) = \mathcal{A}(\Gamma_{2,1}) \otimes \mathcal{A}(\Gamma_1) \ , \tag{5.25}$$

corresponding to the lattice (5.15) and then taking the $\mathbb{Z}_2$ orbifold.

Let us finally remark that the choice between this $c = 3$ models and its 331 reduction should be decided by experiment.

6 Conclusions

We presented a unified description of the double-layer $\nu = 1/2$ Hall states which was based on two assumptions: (P1) the use of lattice $c = 2$ CFTs with modular invariant partition functions; and (P2) the usual charge – statistics relation for the observed electron-like excitations.

Our analysis has singled out a unique $c = 2$ RCFT which corresponds to the already proposed model for the 331 state. We have shown that the $c = -2$ CFT previously proposed for the HR state is in fact equivalent to the 331 CFT, which is the unitary description of the same set of states. The result has been obtained by a one-to-one mapping between the $c = -2$ and

$c = 1$ CFTs (Section 3). Furthermore, in Section 4 we proved that there are no alternative lattice RCFTs for the HR state which are made of the same chiral building blocks (e.g., there are no alternative parity rules for combining the fermion number and the charge).

The Pfaffian state has also been related to this unique CFT by a gauge reduction which has physical interpretation in the double-layer geometry. This projection eliminates the fermionic U(1) current and breaks the $W_{1+\infty}$ symmetry at the quantum level; the parity rule for the excitations remains the same.

In our description, the d-wave spin-singlet part of the HR wave function (1.4) is represented in terms of the $c = 1$ Weyl fermion as follows:

$$\begin{aligned}\Psi_{ds} &= \langle 0|\partial\psi^*(z_1)\cdots\partial\psi^*(z_N)\psi(w_1)\cdots\psi(w_N)|0\rangle \\ &= \frac{\partial^N}{\partial z_1\cdots\partial z_N}\det\left(\frac{1}{z_i - w_j}\right) = (-1)^N \det\left(\frac{1}{(z_i - w_j)^2}\right) \ . \qquad (6.1)\end{aligned}$$

The physical electron fields can also be written:

$$\begin{aligned}:j(z)\psi^1(z): &= :j(z)\psi^*(z):E^{-\sqrt{2}}(z) \ , \\ \psi^2(z) &= \psi(z)E^{-\sqrt{2}}(z) \ ; \qquad (6.2)\end{aligned}$$

here, we have used the K-Z equation [7] to substitute the derivative of ψ^* by its product with the current j, and ψ^1 and ψ^2 are the electron fields of the 331 state in Section 2.

Equations (6.1,6.2) show that in our unitary description the HR ground state is an excited state of the 331 CFT. Moreover, it has the same topological order 8 (for $m = 2$, i.e., $\nu = 1/2$) and its excitations obey abelian statistics as in any lattice RCFT. These results are conclusive in the framework of lattice RCFTs.

On the other hand, there are arguments in the literature in favour of distinguishing the HR from the 331 state:

- There is a wide spread opinion that the HR quasi-particles obey non-abelian statistics [9, 10, 12, 18].
- The wave functions for the excitations of $2n$ quasi-holes on the sphere are represented by symmetric polynomials spanning a 2^{2n-1} dimensional space for the 331 model and a 2^{2n-3} dimensional one for the HR state [18].
- The topological order of the HR state has been found to be 10 by constructing its ground-state wave functions on the torus [3, 19, 18] and by numerically diagonalizing the energy spectrum [3].

Finally, the unitary description of the HR state in this paper does not explain the stability of the HR ground state, which can naively decay in the 331 ground state. Possible solutions to these problems might be found by describing the HR with other, non-lattice conformal theories; for example, orbifold theories obtained from the 331 RCFT, which give rise to non-abelian statistics. Furthermore, other quantizations of the original ξ-η system might be possible based on a careful analysis of the zero modes, as suggested in Refs.[20, 11]. These issues are left for future investigations.

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Appendix A Charge Lattices, Cyclic Groups and Orbifolds

We briefly recall some background material about integral lattices (concise, physicist oriented surveys can be found in [22, 21]). An *Euclidean integral lattice* Γ *of rank* r is an r-dimensional $\mathbb{Z}$-module (free abelian group) with a positive definite integer valued bilinear form:

$$\Gamma = \mathbb{Z}\mathbf{e}^1 + \cdots \mathbb{Z}\mathbf{e}^r, \quad (\mathbf{e}^i|\mathbf{e}^j) = (\mathbf{e}^j|\mathbf{e}^i) \in \mathbb{Z}, \tag{A.1}$$

$$\Gamma \ni \mathbf{v} \Rightarrow (\mathbf{v}|\mathbf{v}) \in \mathbb{Z}_+, (\mathbf{v}|\mathbf{v}) = 0 \Leftrightarrow \mathbf{v} = 0. \tag{A.2}$$

The basis $\{\mathbf{e}^1, \ldots, \mathbf{e}^r\}$ of Γ is determined up to GL(n,$\mathbb{Z}$) transformations,

$$GL(n,\mathbb{Z}) \ni A = (a^i_j) : \mathbf{e}^i \to a^i_j \mathbf{e}^j, \quad a^i_j \in \mathbb{Z}, \;\; i,j = 1,\ldots,r, \;\; \det A = \pm 1. \tag{A.3}$$

Hence, the *determinant* $|\Gamma|$ of the *Gram matrix* G_Γ is an invariant of the lattice Γ. The *dual lattice* Γ^* is defined as the set of all vectors $\mathbf{u} \in \mathbb{R}^r$ such that $(\mathbf{u}|\mathbf{v}) \in \mathbb{Z}$ for any $\mathbf{v} \in \Gamma$. The basis $\{\mathbf{e}^*_i\} \subset \Gamma^*$ is dual to $\{\mathbf{e}^i\} \subset \Gamma$ if $(\mathbf{e}^i|\mathbf{e}^*_j) = \delta^i_j$. Clearly, $\Gamma \subset \Gamma^*$ is an (invariant) subgroup of the (abelian) group Γ^*. The quotient Γ^*/Γ is a finite abelian group (a product of cyclic groups) of order $|\Gamma|$:

$$|\Gamma^*/\Gamma| = \left(\frac{\det G_\Gamma}{\det G_{\Gamma^*}}\right)^{\frac{1}{2}} = \det G_\Gamma = |\Gamma| \quad (G_{\Gamma^*} = G_\Gamma^{-1}). \tag{A.4}$$

Let L be a sub-lattice of the integral lattice Γ. Then $\Gamma^* \subset L^*$ and the finite abelian groups Γ/L and L^*/Γ^* are isomorphic:

$$L \subset \Gamma \subset \Gamma^* \subset L^*, \quad L^*/\Gamma^* \simeq \Gamma/L. \tag{A.5}$$

To each integral lattice Γ there corresponds a chiral vertex algebra $\mathcal{A}(\Gamma)$ (see, e.g., Section 1.2 of [17] and references therein). It involves, to begin with, r linearly independent u(1) currents $J^i(z) = \sum_n J^i_n z^{-n-1}$ (in the basis $\{\mathbf{e}^i,\ i = 1, \ldots, r\}$ of Γ) such that:

$$[J^i(z_1), J^j(z_2)] = -(\mathbf{e}^i|\mathbf{e}^j)\delta'(z_{12}) \quad \Leftrightarrow \quad [J^i_n, J^j_m] = (\mathbf{e}^i|\mathbf{e}^j)\delta^0_{n+m}. \tag{A.6}$$

It defines a free Bose field subalgebra $\mathcal{A}_r$ of $\mathcal{A}(\Gamma)$. We consider a (reducible) positive energy *vacuum representation* of $\mathcal{A}_r$ in a Hilbert space $\mathcal{H}_\Gamma$ that splits into an infinite direct sum of irreducible $\mathcal{A}_r$ modules $\mathcal{H}_\mathbf{v}$ with distinguished cyclic vectors $|\mathbf{v}\rangle$ ($\mathbf{v} \in \Gamma$):

$$\mathcal{H}_\Gamma = \bigoplus_{\mathbf{v}\in\Gamma} \mathcal{H}_\mathbf{v}, \quad |\mathbf{v}\rangle \in \mathcal{H}_\mathbf{v}, \quad J^i_n|\mathbf{v}\rangle = (\mathbf{e}^i|\mathbf{v})\delta^0_n|\mathbf{v}\rangle \ \text{ for } \ n \geq 0. \tag{A.7}$$

To each ($\mathbf{u} \in \Gamma$) we associate a unitary *shift operator* $E^\mathbf{u}$ acting in $\mathcal{H}_\Gamma$ such that:

$$E^\mathbf{u}|\mathbf{v}\rangle = \varepsilon(\mathbf{u},\mathbf{v})|\mathbf{u}+\mathbf{v}\rangle\ , \quad J^i_n E^\mathbf{u} = E^\mathbf{u}\left(J^i_n + (\mathbf{e}^i|\mathbf{u})\delta^0_n\right). \tag{A.8}$$

The factor $\varepsilon(\mathbf{u},\mathbf{v})$ takes values ± 1. It is a 2-cocycle, - i.e.,

$$\varepsilon(\mathbf{u}_1,\mathbf{u}_2)\varepsilon(\mathbf{u}_1+\mathbf{u}_2,\mathbf{u}_3) = \varepsilon(\mathbf{u}_1,\mathbf{u}_2+\mathbf{u}_3)\varepsilon(\mathbf{u}_2,\mathbf{u}_3),$$

and satisfies

$$\varepsilon(\mathbf{u},0) = \varepsilon(0,\mathbf{u}) = 1, \quad \varepsilon(\mathbf{u},\mathbf{v}) = (-1)^{|\mathbf{u}|^2|\mathbf{v}|^2-(\mathbf{u}|\mathbf{v})}\varepsilon(\mathbf{v},\mathbf{u}), \quad |\mathbf{u}|^2 = (\mathbf{u}|\mathbf{u}),$$

thus guaranteeing the normal spin-statistics relation for different fields.

To each vector $\mathbf{v} = v_i\mathbf{e}^i$ in the real r-dimensional space $\mathbb{R}^r$ we associate a u(1) current $J^\mathbf{v}$ by:

$$J^\mathbf{v}(z) = v_i J^i(z)\ , \qquad (J^i(z) = J^{\mathbf{e}^i}(z)). \tag{A.9}$$

If $\mathbf{v} \in \Gamma$ then there exists a *local (Bose or Fermi)* charged field $E^{\mathbf{v}}(z)$ such that:

$$E^{\mathbf{v}}(z)|0\rangle = \exp\left\{\sum_{n=1}^{\infty} J^{\mathbf{v}}_{-n}\frac{z^n}{n}\right\}|\mathbf{v}\rangle, \quad |\mathbf{v}\rangle = E^{\mathbf{v}}|0\rangle \ , \tag{A.10}$$

and

$$E^{\mathbf{u}}(z)E^{\mathbf{v}}(w) = (-1)^{|\mathbf{u}|^2|\mathbf{v}|^2}E^{\mathbf{v}}(w)E^{\mathbf{u}}(z) \quad \text{for } \ w \neq z. \tag{A.11}$$

Let $\{J_i(z) := J^{\mathbf{e}_i^*}(z)\}$ be a basis of u(1) currents dual to $\{J^i(z)\}$. The stress energy tensor T for each chiral algebra $\mathcal{A}(\Gamma)$ belongs to its subalgebra $\mathcal{A}_r$ and is given by the Sugawara formula:

$$T(z) = \frac{1}{2}{:}J_i(z)J^i(z){:} \ . \tag{A.12}$$

It implies that the energy of a ground state vector $\mathbf{v}$ of $\mathcal{A}_r$ is given by:

$$(L_0 - \frac{1}{2}|\mathbf{v}|^2)|\mathbf{v}\rangle = 0 \quad \text{if} \quad (J_{in} - \delta^0_n v_i)|\mathbf{v}\rangle = 0 \ \text{ for } \ n \geq 0, \ v_i = (\mathbf{v}|\mathbf{e}^*_i). \tag{A.13}$$

The irreducible positive energy representations of $\mathcal{A}(\Gamma)$ are labeled by the elements of the finite abelian group Γ^*/Γ. Let $\mathbf{v}^* \in \Gamma^*$ be a representative of the coset $\mathbf{v}^* + \Gamma$ satisfying:

$$|\mathbf{v}^*|^2 = \inf_{\mathbf{u}\in\Gamma} |\mathbf{v}^* + \mathbf{u}|^2. \tag{A.14}$$

Such a $\mathbf{v}^*$ is, in general, not unique in $\mathbf{v}^* + \Gamma$, however the representation space,

$$\mathcal{H}_{\mathbf{v}^*+\Gamma} = \bigoplus_{\mathbf{u}\in\Gamma} \mathcal{H}_{\mathbf{v}^*+\mathbf{u}} \ , \tag{A.15}$$

which generalizes the vacuum space (A.7) is clearly independent of the choice of $\mathbf{v}^*$. The minimal (ground state) energy in $\mathcal{H}_{\mathbf{v}^*}$ is $\frac{1}{2}|\mathbf{v}^*|^2$ (provided (A.14) takes place).

Returning to the chain of embedded lattices (A.5) we shall demonstrate that $\mathcal{A}(L) \subset \mathcal{A}(\Gamma)$ appears as an orbifold of Γ.

Proposition A.1 *The finite abelian group L^*/Γ^* acts by automorphisms on the chiral algebra $\mathcal{A}(\Gamma)$ leaving each element of its subalgebra $\mathcal{A}_r \subset \mathcal{A}(\Gamma)$ invariant.*

Proof. Indeed, for each $\mathbf{u}^* \in L^*$ we can define a *gauge operator* $U(= U_{\mathbf{u}^*}) = e^{2\pi i J_0^{\mathbf{u}^*}}$ satisfying:

$$\alpha_{\mathbf{u}^*}[E^{\mathbf{v}}] = UE^{\mathbf{v}}U^{-1} = \exp\{2\pi i(\mathbf{u}^*|\mathbf{v})\}E^{\mathbf{v}} \ , \qquad (\mathbf{v} \in \Gamma). \tag{A.16}$$

(Although the operator U depends on the choice of representative $\mathbf{u}^*$ in the coset $\mathbf{u}^* + \Gamma^*$ the automorphism $\alpha_{\mathbf{u}^*}$ of $\mathcal{A}(\Gamma)$ does not.) For $\mathbf{v} \in L(\subset \Gamma)$ $(\mathbf{u}^*|\mathbf{v}) \in \mathbb{Z}$ so that $E^{\mathbf{v}}$ is unaltered by the automorphism $\alpha_{\mathbf{u}^*}$ in (A.16).

We are now prepared to apply Proposition A1 to the pair $L \subset \Gamma_{m,1} \equiv \Gamma_m \oplus \Gamma_1$ of integral lattices studied in Section 2. In this case:

$$\frac{|L|}{|\Gamma_{m,1}|} = \frac{4m}{m} = 2^2 \quad \Rightarrow L^*/\Gamma^*_{m,1} \simeq \mathbb{Z}_2 \quad (\Gamma^*_{m,1} = \Gamma^*_m \oplus \Gamma^*_1)\ , \tag{A.17}$$

and the non-trivial automorphism α is given by (2.4); note that the coset $L^*/\Gamma^*_{m,1}$ can be represented by either of the four vectors $\pm\mathbf{q}^*_i$ and we have,

$$J^{\mathbf{q}^*_1}(z) = \frac{1}{2}J^1(z) = \frac{1}{2}\left\{\frac{1}{\sqrt{m}}J(z) + j(z)\right\}\ , \tag{A.18}$$

in the notation (2.7-2.9).

We shall now sketch a derivation of the formula (2.14) for the character of the representation $(\lambda) = \lambda\mathbf{q}^*_1 + L$ of the orbifold algebra $\mathcal{A}(L)$.

Proposition A.2 *The character* $\mathrm{ch}^\lambda_L(\tau,\zeta)$ (2.14) *is the chiral partition function of the conformal Hamiltonian* $L_0 - \frac{2}{24}$ *and the visible charge operator* $J^{\mathbf{Q}}_0$ *in the* $\mathcal{A}(L)$ *module* $\mathcal{H}_\lambda$ $(= \mathcal{H}_{\lambda\mathbf{q}^*_1+L})$*:*

$$\mathrm{ch}^\lambda_L(\tau,\zeta) = \mathrm{tr}_{\mathcal{H}_\lambda}\left\{q^{L_0-\frac{1}{12}}\mathrm{e}^{2\pi i\zeta J^{\mathbf{Q}}_0}\right\},\quad q = \mathrm{e}^{2\pi i\tau}\ . \tag{A.19}$$

Proof. Substitute $\mathcal{H}_\lambda$ by the direct sum (A.15),

$$\mathcal{H}_\lambda = \bigoplus_{n_1,n_2\in\mathbb{Z}} \mathcal{H}_{\lambda\mathbf{q}^*_1+n_1\mathbf{q}^1+n_2\mathbf{q}^2},$$
$$n_1\mathbf{q}^1 + n_2\mathbf{q}^2 = (n_1+n_2)\mathbf{e}^1 + (n_1-n_2)\mathbf{e}^2. \tag{A.20}$$

The contribution of each term to the trace (the sum over the currents' excitations in $\mathcal{H}_\lambda$) gives a factor $[\eta(\tau)]^{-2}\mathrm{e}^{2\pi i\{\frac{\tau}{2}(\mathbf{u}+\lambda\mathbf{q}^*_1)^2+\zeta(\mathbf{Q}|\mathbf{u}+\lambda\mathbf{q}^*_1)\}}$ where $\mathbf{Q}$ is given by (2.12). Noting that n_1+n_2 and n_1-n_2 have the same parity we split the resulting double sum in two terms: one with $n_1+n_2 = 2n$, $n_1-n_2 = 2n'$ and another with $n_1+n_2 = 2n+1$, $n_1-n_2 = 2n'+1$. The first of these terms gives the first summand in (2.14):

$$\frac{1}{\eta^2(\tau)}\sum_{n,n'} q^{2m(n+\frac{\lambda}{4m})^2+2(n'+\frac{\lambda}{4})^2}\mathrm{e}^{4\pi i\zeta(n+\frac{\lambda}{4m})} = K_\lambda(\tau;4)K_\lambda(\tau,2\zeta;4m).$$

Similarly, the second one reproduces the second term, $K_{\lambda+2}(\tau;4)K_{\lambda+2m}(\tau,2\zeta;4m)$, thus completing the proof of the proposition.

A different method of computing $\mathrm{ch}^\lambda_L(\tau,\zeta)$ (and hence of proving Proposition A2) using technique of orbifold theory [17] is provided in [29].

Appendix B Conformal OPE for the HR Anticommuting Fields

Standard (global) conformal OPE are written as series of integrals of quasi-primary fields with respect to a given stress energy tensor (see, e.g., Appendix A of [30], [31] and [29]). Here we shall write down such an expansion for the product of (free) fields $\psi_\rho(z)\psi_\sigma(w)$ without committing ourselves to a choice of the stress energy tensor (and the associated Virasoro algebra). All we shall need is the 4-point function (3.1) with typical element:

$$\begin{aligned}&\langle 0|\psi_+(z_1)\psi_-(w_1)\psi_+(z_2)\psi_-(w_2)|0\rangle = \\ &= (z_1-w_1)^{-2}(z_2-w_2)^{-2} - (z_1-w_2)^{-2}(w_1-z_2)^{-2}. \end{aligned} \tag{B.1}$$

We shall verify (using techniques developed in the above references) that it gives rise to an OPE of the form:

$$\begin{aligned}(z-w)^2\psi_\rho(z)\psi_\sigma(w) \;=\;& \varepsilon_{\rho\sigma}\left\{1-\int_w^z [6\frac{(z-\zeta)(\zeta-w)}{z-w}\mathcal{T}(\zeta)\right.\\ &\left.+70\frac{(z-\zeta)^3(\zeta-w)^3}{(z-w)^3}\Lambda(\zeta)]\mathrm{d}\zeta + (z-w)^6 S(z,w)\right\}\\ &-30\int_w^z \frac{(z-\zeta)^2(\zeta-w)^2}{(z-w)^2}V_{\rho+\sigma}(\zeta)\mathrm{d}\zeta\\ &+(z-w)^5 R_{\rho+\sigma}(z,w)\ , \qquad \rho,\sigma=\pm\frac{1}{2}\ .\end{aligned} \tag{B.2}$$

Here the $\mathcal{T}$, Λ and V_a are (translation invariant) local fields satisfying:

$$\langle 0|\psi_\rho(z)\psi_\sigma(w)\mathcal{T}(\zeta)|0\rangle = \frac{\varepsilon_{\rho\sigma}}{(z-\zeta)^2(w-\zeta)^2}\ , \tag{B.3a}$$

$$\mathcal{T}(z_1)\mathcal{T}(z_2) = -z_{12}^{-4} + 12 z_{12}^{-5}\int_{z_2}^{z_1}(z_1-\zeta)(\zeta-z_2)\mathcal{T}(\zeta)\mathrm{d}\zeta + :\mathcal{T}(z_1)\mathcal{T}(z_2):$$

$$\Lambda(z) = :\mathcal{T}(z)^2:, \quad \langle 0|\psi_\rho(z)\psi_\sigma(w)\Lambda(\zeta)|0\rangle = \frac{6}{5}\frac{\varepsilon_{\rho\sigma}(z-w)^2}{(z-\zeta)^4(w-\zeta)^4}\ ; \tag{B.3b}$$

$$\langle 0|\psi_\rho(z)\psi_\sigma(w)V_a(\zeta)|0\rangle = -G_{\rho+\sigma,a}\frac{z-w}{(z-\zeta)^3(w-\zeta)^3}\ , \tag{B.4a}$$

$$V_a(z_1)V_b(z_2) \;=\; G_{ab}\left\{z_{12}^{-6} + \frac{18}{z_{12}^7}\int_{z_2}^{z_1}(z_1-\zeta)(\zeta-z_2)\mathcal{T}(\zeta)\mathrm{d}\zeta + \cdots\right\}$$

$$+30\frac{C^c_{ab}}{z_{12}^8}\int_{z_2}^{z_1}(z_1-\zeta)^2(\zeta-z_2)^2V_c(\zeta)\mathrm{d}\zeta+\cdots\ , \qquad \text{(B.4b)}$$

where the non-zero elements of G_{ab} and C^c_{ab} are:

$$G_{1-1}=G_{-11}=2,\ \ G_{00}=-1,\quad C^0_{1-1}=-C^0_{-11}=-2,\ \ C^{\mp 1}_{\pm 10}=\pm 1. \quad \text{(B.4d)}$$

In fact, the expansion (B.2) and the conditions (B.3) (B.4) are implied by the following result.

Proposition B.1 *The 4-point function* (B.1) *is reproduced by the OPE,*

$$\psi_+(z)\psi_-(w)=\frac{1}{(z-w)^2}-\sum_{l=1}^{\infty}(z-w)^{l-1}\int_w^z P_l(z,w;\zeta)\Phi_{l+1}(\zeta)\mathrm{d}\zeta \qquad \text{(B.5)}$$

where the (normalized) weight function,

$$P_l(z,w;\zeta)=\frac{(2l+1)!}{(l!)^2}\frac{(z-\zeta)^l(\zeta-w)^l}{(z-w)^{2l+1}} \qquad \left(\int_{z_1}^{z_2}P_l(z_1,z_2;\zeta)\mathrm{d}\zeta=1\right), \qquad \text{(B.6)}$$

is determined by the condition,

$$\int_{z_1}^{z_2}P_l(z_1,z_2;\zeta)(\zeta-z_3)^{-2l-2}\mathrm{d}\zeta=z_{13}^{-l-1}z_{23}^{-l-1} \qquad (z_{ij}=z_i-z_j), \qquad \text{(B.7)}$$

while the fields Φ_n are mutually orthogonal satisfying,

$$\langle 0|\Phi_n(z_1)\Phi_m(z_2)|0\rangle = -C_n(z_{12})^{-2n}\delta_{nm}\ , \qquad \text{(B.8)}$$

$$\langle 0|\psi_+(z)\psi_-(w)\Phi_n(\zeta)|0\rangle = C_n\frac{(z-w)^{n-2}}{(z-\zeta)^n(w-\zeta)^n},\ n=2,3,\dots\ , \qquad \text{(B.9)}$$

and the constants C_n are given by,

$$C_{n+1}=\frac{n(n+1)}{\binom{2n}{n}},\quad n=1,2,\dots(C_2=1=C_3). \qquad \text{(B.10)}$$

Conversely, the fields Φ_n can be expressed from (B.5) *as composites of $\psi_\pm$:*

$$\begin{aligned}\Phi_{n+1}(z) &= -\frac{1}{(2n)!}\lim_{z_1,z_2\to z}\partial_1^n(-\partial_2)^n\{z_{12}^{n+1}{:}\psi_+(z_1)\psi_-(z_2){:}\}\\ &= -\frac{(n+1)!}{(2n)!}\sum_{k=0}^{n-1}(-1)^{n-k-1}\binom{n}{k}\binom{n}{k+1}{:}\partial^k\psi_+(z)\partial^{n-k-1}\psi_-(z){:} \qquad \text{(B.11)}\end{aligned}$$

The **proof** is based on the expansion:

$$(1-\eta)^{-2} = \sum_{n=2}^{\infty} C_n F(n,n;2n;\eta) \ , \tag{B.12}$$

for C_n satisfying the recurrence relation,

$$\sum_{l=1}^{n-1} C_{l+1} \frac{(2l+1)!(n-2)!(n-1)!}{(l!)^2(n-l-1)!(n+l)!} = 1 \ ,$$

which is solved by (A.9); here we use the following representation for the Hypergeometric function (cf. [31, 29]):

$$F(n,n;2n;\eta) = z_{12}^n w_{12}^n \int_{w_2}^{z_2} \frac{P_{n-1}(z_2,w_2;\zeta)}{(z_1-\zeta)^n(w_1-\zeta)^n} \mathrm{d}\zeta \ , \quad \eta = \frac{(z_1-w_1)(z_2-w_2)}{z_{12}w_{12}}. \tag{B.13}$$

It remains to set $\Phi_2 = \mathcal{T}$, $\Phi_3 = V_0$, $\Phi_4 = \frac{1}{2}\Lambda$ in order to recover (B.2). The OPE of composite fields (Eqs. (B.3b) (B.4b) etc.) follow from their expressions (B.11) in terms of the free fields $\psi_\pm$. Short distance expansions (like (3.2)) are obtained from here using:

$$\int_w^z P_l(z,w;\zeta)\Phi_{l+1}(\zeta)\mathrm{d}\zeta = \Phi_{l+1}(w) + \frac{z-w}{2}\Phi'_{l+1}(w) + O((z-w)^2). \tag{B.14}$$

Appendix C Modular Invariants Partition Functions Involving the Chiral Algebra $\mathcal{A}(\Gamma_{8,4})$

Let the lattice base vectors $\mathbf{e}^i$ and their dual $\mathbf{e}_i^*$, $i=1,2$ satisfy:

$$G_{2,1} = \left((\mathbf{e}^i|\mathbf{e}^j)\right) = \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}, \quad \mathbf{e}^1 = 2\mathbf{e}_1^*, \ \mathbf{e}^2 = \mathbf{e}_2^*. \tag{C.1}$$

The corresponding $\widehat{u(1)}$ currents (which define, for $r=2$, the stress energy tensor (A.11)) are:

$$J^1(z) \ (\equiv J^{\mathbf{e}^1}(z)) = 2J_1(z), \text{ and } J^2(z) = J_2(z).$$

We are going to classify the ($c=2$) RCFT whose chiral algebra includes the tensor product algebra $\mathcal{A}(\Gamma_{8,4})$ where the lattice $\Gamma_{8,4}$ is spanned by $2\mathbf{e}^1$, $2\mathbf{e}^2$.

We shall continue, however, to identify the electric charge $\mathbf{Q}$ with (2.12), - that is,

$$\mathbf{Q} = \mathbf{e}_1^* \ , \qquad \left(|\mathbf{Q}|^2 = \frac{1}{2} \right) . \tag{C.2}$$

Proposition C.1 *There are 5 S and T^2 invariant partition functions for the $c = 1$ chiral algebra $\mathcal{A}(\Gamma_8)$, 3 such invariants for $\mathcal{A}(\Gamma_4)$ and hence 15 factorizable partition functions for $\mathcal{A}(\Gamma_{8,4})$. The rank 2 chiral algebra $\mathcal{A}(\Gamma_{8,4})$ also admits 4 non-factorizable modular invariants.*

Proof. We first note that the choice (C.2) of the electric charge requires substituting ζ by 2ζ in the second argument of the characters $K_\lambda(.,.;8)$ (2.15a).

According to ref. [21] there are 4 S invariants made out of $K_\lambda(\tau, 2\zeta; 8)$ (and their conjugate) for which the chiral algebra is unextended and the number of superselection sectors is 8. They can be labeled by an integer $l \bmod 8$ satisfying $l^2 = 1 \bmod 8$ and are given by

$$Z_{l,8}(\tau, \zeta) = \sum_{\lambda=-3}^{4} K_\lambda(\tau, 2\zeta; 8) \overline{K_{l\lambda}(\tau, 2\zeta; 8)} \quad \text{for} \quad l = \pm 1, \pm 3. \tag{C.3}$$

Two of them, $Z_{1,8}$ and $Z_{-1,8}$ are SL(2,$\mathbb{Z}$) invariant; the two others,

$$Z_{3,8} = K_0 \overline{K_0} + K_4 \overline{K_4} + \left(K_1 \overline{K_3} + K_{-1} \overline{K_{-3}} + K_2 \overline{K_{-2}} + c.c. \right) \ ,$$

and $Z_{-3,8}$ are only S and T^2 invariant. There is one more invariant partition function corresponding to the $\mathcal{A}(\Gamma_2) \supset \mathcal{A}(\Gamma_8)$ extension of the chiral algebra:

$$\begin{aligned} Z_2(\tau, \zeta) \ (= Z(\Gamma_2 \supset \Gamma_8)) \ &= \ |K_0(\tau, 2\zeta; 8) + K_4(\tau, 2\zeta; 8)|^2 \\ &\quad + |K_2(\tau, 2\zeta; 8) + K_{-2}(\tau, 2\zeta; 8)|^2 . \end{aligned} \tag{C.4}$$

Due to the identity:

$$K_{2l}(\tau, 2\zeta; 4m) + K_{2l+2m}(\tau, 2\zeta; 4m) = K_l(\tau, \zeta; m) \ , \tag{C.5}$$

Eq.(C.4) is just the diagonal invariant of the level 1 su(2) current algebra.

Similarly, there are two modular invariants:

$$Z_{\pm 1,4}(\tau) = \sum_{\lambda=-1}^{2} K_\lambda(\tau; 4) \overline{K_{\pm\lambda}(\tau; 4)} \ , \tag{C.6}$$

of the $\mathcal{A}(\Gamma_4)$ RCFT and one S invariant,

$$Z_1(\tau) = |K_0(\tau; 4) + K_2(\tau; 4)|^2 \quad (= Z_1(\tau, +2)) \ , \tag{C.7}$$

corresponding to the $\mathcal{A}(\Gamma_1) \supset \mathcal{A}(\Gamma_4)$ fermionic extension of the bosonic chiral algebra $\mathcal{A}(\Gamma_4)$. The products of each of the 3 invariants (C.6) (C.7) with any of the 5 invariants (C.3) (C.4) give the 15 factorizable partition functions.

The 4 non-factorizable S invariants correspond to the $\mathcal{A}(L) \supset \mathcal{A}(\Gamma_{8,4})$ extension of the original chiral algebra where L is defined by (2.6) and (2.2) with $m = 2$. They are related to the partition function (2.20) (for $m = 2$) in the same way as the invariants (C.3) are related to the diagonal one for $\mathcal{A}(\Gamma_8)$:

$$\begin{aligned} Z_{l,L}(\tau,\zeta) \;=\; & \sum_{\lambda=-3}^{4} \left(K_\lambda(\tau;4)K_\lambda(\tau,2\zeta;8) + K_{\lambda+2}(\tau;4)K_{\lambda+4}(\tau,2\zeta;8)\right) \times \\ & \left(\overline{K_{l\lambda}}(\tau;4)\overline{K_{l\lambda}}(\tau,2\zeta;8) + \overline{K_{l\lambda+2}}(\tau;4)\overline{K_{l\lambda+4}}(\tau,2\zeta;8)\right) \;, \\ & \quad l = \pm 1, \pm 3. \end{aligned} \tag{C.8}$$

The argument of Gannon [21] then proves that there are no other S invariants.

Remark C1. Note that Gannon[1] [21] (see, in particular, Example 2 of Section 4) identifies rank 2 charge lattices obtained from each other by a rotation by π and hence views partition functions differing only in the sign of l as equivalent. In his count there are, therefore, 3×2 (rather than 5×3) factorizable and 2 (rather than 4) non-factorizable invariants. We shall see that some of the partition functions, viewed as equivalent in [21], actually differ in their $U - V$ transformation properties.

Proposition C.2 *U-invariance leaves us with 9 factorizable and 2 non-factorizable partition functions (among the $15+4$ ones given by Proposition C1). If modified by a suitable prefactor of the type appearing in (2.13) they become automatically also V-invariant.*

Proof. U (i.e., $\zeta \to \zeta + 1$) of $Z_{l,8}$ (C.3) requires:

$$\lambda - l\lambda = 0 \mod 4 \qquad \text{for } -3 \le \lambda \le 4 \;, \tag{C.9}$$

which is only satisfied for $l = 1, -3$. The same is true for the invariants (C.8).

Given that the electric charge vector (C.2) has no projection in the direction of $\mathbf{e}^2$ one can view a reflection of the $\mathbf{e}^2$ axis as a symmetry and regard as *indistinguishable theories* obtained from one another by such a reflection (they indeed have identical partition functions). (Thus we substitute the "equivalence" $(\mathbf{e}^1, \mathbf{e}^2) \to (-\mathbf{e}^1, -\mathbf{e}^2)$ used in [21] - see Remark C1 - by $(\mathbf{e}^1, \mathbf{e}^2) \to (\mathbf{e}^1, -\mathbf{e}^2)$.)

[1] We thank T. Gannon for a helpful correspondence on this point

Proposition C.3 *There are 7 distinct partition functions among the 11 modular invariant ones of Proposition C2. They are given by Theorem 4.1.*

Proof. Identifying the two modular invariants (C.6) we remain with 6 out of 9 factorizable partition functions. To verify that $Z_{1,L}(\tau,\zeta) = Z_{-3,L}(\tau,\zeta)$ we note that

$$K_{-3}(\tau;4)K_{-3}(\tau,2\zeta;8) + K_{-1}(\tau;4)K_1(\tau,2\zeta;8) =$$
$$= K_1(\tau;4)K_5(\tau,2\zeta;8) + K_{-1}(\tau;4)K_1(\tau,2\zeta;8)$$

differs from $K_1(\tau;4)K_1(\tau,2\zeta;8) + K_{-1}(\tau;4)K_5(\tau,2\zeta;8)$ by just a change of sign in l of the $K_l(\tau;4)$ factors.

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