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Vienna, Preprint ESI 612 (1998) October 5, 1998

Supported by Federal Ministry of Science and Transport, Austria
Available via http://www.esi.ac.at

ENTROPY THEORY WITHOUT PAST

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February 4, 1998

Abstract. This paper treats the Pinsker algebra of a dynamical system in a way which avoids the use of an ordering on the acting group. This enables us to prove some of the classical results about entropy and the Pinsker algebra in the general setup of measure preserving dynamical systems, where the acting group is a discrete countable amenable group. We prove a basic disjointness theorem which asserts the relative disjointness in the sense of Furstenberg, of 0-entropy extensions from completely positive entropy (c.p.e.) extensions. This theorem is used to prove several classical results in the general setup. E.g. we show that the Pinsker factor of a product system is equal to the product of the Pinsker factors of the component systems. Another application is to obtain a generalization (as well as a simpler proof) of the quasifactor theorem for 0-entropy systems of [GW].

§0. Introduction

The classical theory of entropy was developed for $\mathbb{Z}$ actions and was based in part on the natural order on that group; i.e. the "time" order. For example a process $(\mathcal{P}, T)$ has zero entropy if and only if it is *deterministic* in the sense that $\mathcal{P}$ is contained in the "past" of the process defined as $\bigvee_{j=1}^{\infty} T^j\mathcal{P}$. The "remote past" defined by

$$\bigcap_{n=m}^{\infty} \bigvee_{j=m}^{\infty} T^j\mathcal{P},$$

played an essential role in proving basic theorems like the equivalence of the K-property, namely the triviality of the remote past, with the property of "complete positive entropy"; or more generally in proving the Rokhlin-Sinai theorem, which identifies the Pinsker algebra, $\Pi(T)$ as the remote past of a generating partition $\mathcal{P}$. Other important theorems then follow as easy consequences of this identification. E.g. for two measure preserving automorphisms S and T we get

$$\Pi(T \times S) = \Pi(T) \times \Pi(S),$$

(see for example, [P]).

The goal of this paper is to find a way of dealing with the Pinsker algebra of a dynamical system which avoids the use of an ordering on the acting group, and which enables us to prove some of the classical results about entropy and the Pinsker algebra, in the general setup of measure preserving dynamical systems,

1991 *Mathematics Subject Classification.* 28D05.

where the acting group is a discrete countable amenable group. (See [RW] for a related approach, where the classical equivalence of the K-property (i.e. complete positive entropy) with a uniform mixing property for $\mathbb{Z}$-actions, is generalized to amenable group actions.)

Thus, in this work G will be a discrete amenable group. The objects we work with are probability measure preserving G-systems $(X, \mathcal{X}, \mu, G)$, where $(X, \mathcal{X}, \mu)$ is a Lebesgue space. In some of the results we need to assume that G acts freely. This is a standing assumption in [OW,2]. For simplicity we will do the same in this work. Usually we omit the σ-algebra $\mathcal{X}$ and the group G from the notation of a system. Thus unless it is stated otherwise (X, μ), or even X, if the measure is clear, stands for $(X, \mathcal{X}, \mu, G)$. Often we confuse the space and the σ-algebra; thus we may sometimes say that a function f is X-measurable rather than $\mathcal{X}$-measurable. When $(X, \mathcal{X}, \mu, G) \overset{\pi}{\to} (Y, \mathcal{Y}, \nu, G)$ is a homomorphism of two such systems we say that (Y, ν) is a *factor* of (X, μ) or that (X, μ) is an *extension* of (Y, ν) and we consider $\mathcal{Y}$ as a subalgebra of $\mathcal{X}$. If $(Y, \mathcal{Y}, \nu, G)$and $(Z, \mathcal{Z}, \eta, G)$are two factors of $(X, \mathcal{X}, \mu, G)$we denote by $Y \vee Z$ the factor system of $(X, \mathcal{X}, \mu, G)$defined by, $\mathcal{Y} \vee \mathcal{Z}$, the G-invariant σ-subalgebra of $\mathcal{X}$ generated by $\mathcal{Y}$ and $\mathcal{Z}$. $E_\mu(f) = E(f)$ will denote the expectation of the function f with respect to μ; i.e. $E(f)$ is simply the integral $\int f d\mu$. And, when $(X, \mathcal{X}, \mu, G) \overset{\pi}{\to} (Y, \mathcal{Y}, \nu, G)$is a homomorphism, $E^Y(f)$ is the conditional expectation of f with respect to the σ-algebra $\mathcal{Y}$. Thus if

$$\mu = \int_Y \mu_y \, d\nu(y),$$

is the disintegration of μ over ν, then ν a.e.

$$E^Y(f)(y) = \int_X f d\mu_y.$$

Given a G-system $(X, \mathcal{X}, \mu, G)$, a measurable partition $\mathcal{P}$ of X and a subset F of G we let $\mathcal{P}^F = \bigvee_{g \in F} g^{-1}\mathcal{P}$. The entropy of the process $(\mathcal{P}, G)$ is given by

$$h(\mathcal{P}) = h(\mathcal{P}, G) = \lim_{n \to \infty} \frac{1}{|F_n|} H(\mathcal{P}^{F_n}),$$

where $\{F_n\}$ is a Følner sequence in the group G and H is the usual partition entropy. As is shown in [OW,2] (see also [Ki,1]) the limit exists and is independent of the Følner sequence. The *entropy* of the system $(X, \mathcal{X}, \mu, G)$is defined by

$$h(X, \mu) = \sup\{h(\mathcal{P}) : \mathcal{P} \text{ a finite partition of } X\}.$$

For the conditional entropy of $\mathcal{P}$ with respect to a G-invariant sub σ-algebra $\mathcal{Y} \subset \mathcal{X}$ we set:

$$h(\mathcal{P}|\mathcal{Y}) = \lim_{n \to \infty} \frac{1}{|F_n|} H(\mathcal{P}^{F_n}|\mathcal{Y}).$$

Once again one can deduce the existence of this limit and its independence of the sequence $\{F_n\}$ from the information provided in [OW,2]. This is worked out in [KR]. The approach in [Ki,1] was extended to the conditional case in [WaZ]. The conditional entropy of (X, μ) with respect to (Y, ν) is defined by

$$h(X|Y) = \sup\{h(\mathcal{P}|\mathcal{Y}) : \mathcal{P} \text{ a finite partition of } X\}.$$

An extension $(X,\mu) \to (Y,\nu)$ will be called a 0-*entropy extension* if $h(X|Y) = 0$. It will be called a *completely positive entropy extension* or *c.p.e. extension* if for any intermediate extension $(X,\mu) \to (Z,\eta) \to (Y,\nu)$ with $(Z,\eta) \to (Y,\nu)$ a proper extension, $h(Z|Y) > 0$.

The *Pinsker algebra* of the system $(X,\mathcal{X},\mu,G)$is defined as the σ-algebra of all subsets $A \in \mathcal{X}$ such that $h(\{A, A^c\}) = 0$, and the corresponding factor is called the *Pinsker factor* of the system (X,μ). The *relative Pinsker factor* for an extension $(X,\mathcal{X},\mu,G)\stackrel{\pi}{\to}(Y,\mathcal{Y},\nu,G)$, is defined similarly.

A *Bernoulli G-system* is a system $(X,\mathcal{X},\mu,G)$where an independent generating partition $\mathcal{P}$ exists.

In section 1 we present a basic formula (lemma 1.1 (1)) which governs entropy calculations, and deduce some corollaries. One of these is the definition of the Pinsker (relative Pinsker) factor of a system (an extension). In section 2 we use a lemma of J-P. Thouvenot (lemma 2.2, see [LPT]) to prove a disjointness theorem which asserts the relative disjointness in the sense of Furstenberg, of 0-entropy extensions from c.p.e. extensions. In section 3 we show how the c.p.e. property lifts through certain diagrams. This is used in section 4 to deal with the Pinsker factor of product systems. Next we generalize some theorems of K. Berg on maximal entropy and independence, [B]. Finally in the last section we use the relative disjointness theorem from section 2 to obtain a generalization (as well as a simpler proof) of the quasifactor theorem for 0-entropy systems of [GW].

An important part of the work on this paper was done during the special year in ergodic theory at the Institute for Advanced Studies of the Hebrew University in Jerusalem, 1996-7, where all three authors participated. We would like to thank the Institute for the very pleasant time we spent there.

§1. Background

Recall that the conditional entropy of a finite measurable partition $\mathcal{P}$ with respect to a G-invariant sub σ-algebra $\mathcal{Y} \subset \mathcal{X}$ is defined as:

$$h(\mathcal{P}|\mathcal{Y}) = \lim_{n\to\infty} \frac{1}{|F_n|} H(\mathcal{P}^{F_n}|\mathcal{Y}),$$

In case $G = \mathbb{Z}$, it is well known that

$$\lim_{n\to\infty} \frac{1}{n} H(\mathcal{P}^{[0,n)}) = \inf_n \frac{1}{n} H(\mathcal{P}^{[0,n)}).$$

This is a consequence of the fact that translates of the interval $[0,n)$ tile $\mathbb{Z}$ exactly. For amenable groups, where Følner sets that tile perfectly are not known to exist in general, we can substitute the ϵ-quasi-tiling developed in [OW,2]. That gives for any $\epsilon > 0$, a k and a rate of growth for the almost invariance of $F_{n_1}, F_{n_2}, \dots, F_{n_k}$, $(n_1 < n_2 < \cdots < n_k)$ such that

$$h(\mathcal{P}) \le \max_{1\le i\le k} \frac{1}{|F_{n_i}|} H(\mathcal{P}^{F_{n_i}}) + \epsilon H(\mathcal{P}),$$

for all finite partitions $\mathcal{P}$, and the same holds for the conditional entropy with respect to any invariant σ-algebra $\mathcal{C}$:

$$h(\mathcal{P}|\mathcal{C}) \le \max_{1\le i\le k} \frac{1}{|F_{n_i}|} H(\mathcal{P}^{F_{n_i}}|\mathcal{C}) + \epsilon H(\mathcal{P}). \tag{1}$$

Let now $\mathcal{R}_n$ be any sequence of finite partitions such that $\mathcal{R}_n^G$ refines to $\mathcal{Y}$. We then have

$$h(\mathcal{P}|\mathcal{Y}) = \lim_{n\to\infty} h(\mathcal{P}|\mathcal{R}_n^G). \tag{*}$$

In fact, denoting $h = h(\mathcal{P}|\mathcal{Y})$, we clearly have for all n, $h \le h(\mathcal{P}|\mathcal{R}_n^G)$. For the other direction, by definition of $h = h(\mathcal{P}|\mathcal{Y})$, for n_i sufficiently large

$$\frac{1}{|F_{n_i}|} H(\mathcal{P}^{F_{n_i}}|\mathcal{Y}) \le h + \epsilon. \tag{2}$$

We now fix n_i, $1 \le i \le k$ satisfying (1) and (2). If m is large enough then by the martingale convergence theorem, for all $n_i, 1 \le i \le k$,

$$\frac{1}{|F_{n_i}|} H(\mathcal{P}^{F_{n_i}}|\mathcal{R}_m^G) \le \frac{1}{|F_{n_i}|} H(\mathcal{P}^{F_{n_i}}|\mathcal{Y}) + \epsilon, \tag{3}$$

and by (1), (2) and (3) we get

$$\begin{aligned} h(\mathcal{P}|\mathcal{R}_m^G) &\overset{(1)}{\le} \max_{1\le i\le k} \frac{1}{|F_{n_i}|} H(\mathcal{P}^{F_{n_i}}|\mathcal{R}_m^G) + \epsilon \\ &\overset{(3)}{\le} \max_{1\le i\le k} \frac{1}{|F_{n_i}|} H(\mathcal{P}^{F_{n_i}}|\mathcal{Y}) + 2\epsilon \\ &\overset{(2)}{\le} (h + 2\epsilon) + \epsilon, \end{aligned}$$

so that (*) follows.

We let
$$h(X|Y) = \sup\{h(\mathcal{P}|\mathcal{Y}) : \mathcal{P} \text{ a finite partition of } X\}.$$

As usual the continuity of the entropy function enables one to replace the sup over *all* finite partitions by the sup over a dense set of partitions.

Lemma 1.1. *Let $(X, \mathcal{X}, \mu, G)$be a G-system with factor $(Z, \mathcal{Z}, \eta, G)$.*

(1) *For any two finite measurable partitions $\mathcal{P}$ and $\mathcal{Q}$:*

$$h(\mathcal{P} \vee \mathcal{Q}|\mathcal{Z}) = h(\mathcal{P}|\mathcal{Z}) + h(\mathcal{Q}|\mathcal{P}^G \vee \mathcal{Z}).$$

(2) *If $\mathcal{P}$ is a finite partition and $h(\mathcal{P}|\mathcal{Z}) = H(\mathcal{P})$ then the process $(\mathcal{P}, G)$ is Bernoulli and moreover $\mathcal{P}^G$ is independent of $\mathcal{Z}$.*

Proof. (1) For Z trivial this is theorem 4.4 of [WaZ]. If $\mathcal{Z}$ is generated by a finite partition $\mathcal{R}$ then we can write

$$h(\mathcal{P} \vee \mathcal{Q} \vee \mathcal{R}) = h(\mathcal{R}) + h(\mathcal{P} \vee \mathcal{Q}|\mathcal{R}^G) = h(\mathcal{R}) + h(\mathcal{P} \vee \mathcal{Q}|\mathcal{Z})$$

and then

$$\begin{aligned} h(\mathcal{P} \vee \mathcal{Q}|\mathcal{Z}) &= h(\mathcal{P} \vee \mathcal{Q} \vee \mathcal{R}) - h(\mathcal{R}) \\ &= h(\mathcal{P} \vee \mathcal{R}) + h(\mathcal{Q}|\mathcal{P}^G \vee \mathcal{Z}) - h(\mathcal{R}) \\ &= h(\mathcal{P} \vee \mathcal{R}) - h(\mathcal{R}) + h(\mathcal{Q}|\mathcal{P}^G \vee \mathcal{Z}) \\ &= h(\mathcal{P}|\mathcal{Z}) + h(\mathcal{Q}|\mathcal{P}^G \vee \mathcal{Z}). \end{aligned}$$

In the general case we let $\mathcal{R}_n$ be a sequence of finite partitions such that $\mathcal{R}_n^G$ refines to $\mathcal{Z}$. Then for each n

$$h(\mathcal{P} \vee \mathcal{Q}|\mathcal{R}_n^G) = h(\mathcal{P}|\mathcal{R}_n^G) + h(\mathcal{Q}|\mathcal{P}^G \vee \mathcal{R}_n^G),$$

and in the limit, by (*),

$$h(\mathcal{P} \vee \mathcal{Q}|\mathcal{Z}) = h(\mathcal{P}|\mathcal{Z}) + h(\mathcal{Q}|\mathcal{P}^G \vee \mathcal{Z}).$$

(2) The proof proceeds by showing that if for any finite set $F \subset G$ one would have

$$(**) \qquad H(\mathcal{P}^F|\mathcal{Z}) = |F| \cdot H(\mathcal{P}) - \delta$$

with $\delta > 0$, then $h(\mathcal{P}|\mathcal{Z}) < H(\mathcal{P})$. On the other hand, if for all finite F one has equality in $(**)$ with $\delta = 0$, then the conclusions that the process $(\mathcal{P}, G)$ is independent and that $\mathcal{P}^G$ is independent of Z follow immediately. To establish the first claim, let F_n be a Følner set that is almost F-invariant and let $\{Fg_i : 1 \leq i \leq L\}$ be a maximal set of disjoint translates of F that lie in F_n. If $a \in F_n$ and $Fa \subset F_n$ and one cannot add Fa to the collection $\{Fg_i\}$, it means that $a \in F^{-1}Fg_i$ for some g_i. It follows that $L \cdot |F| \geq c \cdot |F_n|$ for some positive constant c that depends only on F.

We can now estimate

$$H(\mathcal{P}^{F_n}|\mathcal{Z}) \leq \sum_{i=1}^{L} H(\mathcal{P}^{Fg_i}|\mathcal{Z}) + H(\mathcal{P}^E|\mathcal{Z}),$$

where $E = F_n \setminus \bigcup_{i=1}^L Fg_i$. By the invariance of the measure we conclude that

$$\begin{aligned} H(\mathcal{P}^{F_n}|\mathcal{Z}) &\leq L \cdot (|F| \cdot H(\mathcal{P}) - \delta) + |E| \cdot H(\mathcal{P}) \\ &= (L \cdot |F| + |E|) \cdot H(\mathcal{P}) - \delta L \\ &= |F_n| \cdot H(\mathcal{P}) - \delta L. \end{aligned}$$

from which it would follow that $h(\mathcal{P}|\mathcal{Z}) < H(\mathcal{P})$, contrary to our assumption. □

Lemma 1.2.

(1) *If* $(X, \mu) \xrightarrow{\pi} (Z, \eta)$ *and* $(Y, \nu) \xrightarrow{\phi} (Z, \eta)$ *are 0-entropy extensions and* λ *is a joining of* μ *and* ν *over* η *(i.e.* $\pi \times \phi(\lambda) = \eta_\Delta$*), then* $(X \times Y, \lambda) \to (Z, \eta)$ *is also a 0-entropy extension.*

(2) *If* $\{(Y_i, \nu_i) \xrightarrow{\phi_i} (Z, \eta)\}$ *is an inverse system of 0-entropy extensions then so is the inverse limit extension* $\bigvee(Y_i, \nu_i) \to (Z, \eta)$.

Proof. (1) This follows directly from lemma 1.1 by taking $\mathcal{P}$ and $\mathcal{Q}$ any finite partitions which are X and Y measurable respectively.

(2) Again the assertion follows from lemma 1.1 and the observation that the collection of finite partitions measurable with respect to some Y_i are dense in the set of all finite measurable partitions of $\bigvee(Y_i, \nu_i)$. □

Let $(X, \mathcal{X}, \mu, G) \xrightarrow{\pi} (Y, \mathcal{Y}, \nu, G)$be an extension of ergodic systems. Put

$$\Pi(X|Y) = \{A \in \mathcal{X} : h(\{A, A^c\}|\mathcal{Y}) = 0\}.$$

Another corollary of lemma 1.1 is the fact that $\Pi(X|Y)$ is a G-invariant sub-σ-algebra of $\mathcal{X}$ which of course contains $\mathcal{Y}$. We call this algebra the *Pinsker algebra* of $\mathcal{X}$ relative to $\mathcal{Y}$, and the corresponding factor the *relative Pinsker factor*. When $\mathcal{Y}$ is the trivial algebra we get the *Pinsker algebra* and *Pinsker factor* of X and we denote this factor by $\Pi(X)$.

§2. The relative disjointness theorem

We say that $(X,\mathcal{X},\mu,G)\overset{\pi}{\to}(Y,\mathcal{Y},\nu,G)$is an *ergodic* extension if every G-invariant $\mathcal{X}$-measurable function is $\mathcal{Y}$-measurable.

Lemma 2.1. *Let $(X,\mu,\mathrm{id})\overset{\pi}{\to}(Z,\eta,\mathrm{id})$ and $(Y,\nu,G)\overset{\phi}{\to}(Z,\eta,\mathrm{id})$ be extensions of G-actions, where* id *denotes the trivial action, and assume that the extension $Y\to Z$ is ergodic. Then (X,μ,id) and (Y,ν,G) are relatively disjoint over (Z,η,id) i.e. $\lambda=\mu\underset{\eta}{\times}\nu$ is the only joining of X and Y over Z.*

Proof. Let λ be any joining of the systems (X,μ,id) and (Y,ν,G) over their common factor (Z,η,id). Let

$$\mu=\int_Z \mu_z\ d\eta(z),$$

be the disintegration of μ over η, and

$$\lambda=\int_Y \lambda_y\times\delta_y\ d\nu(y).$$

the disintegration of λ over ν. Then for every $g\in G$

$$\begin{aligned}\lambda=(\mathrm{id}\times g)\lambda&=\int_Y \lambda_y\times\delta_{gy}\ d\nu(y)\\ &=\int_Y \lambda_{g^{-1}y}\times\delta_y\ d\nu(y).\end{aligned}$$

By uniqueness of disintegration we have $\lambda_y=\lambda_{g^{-1}y}$ ν-a.e., hence, by ergodicity of the extension $Y\overset{\phi}{\to}Z$, $\lambda_y=\lambda_{\phi(y)}=\mu_z$ ν-a.e. (the latter equality follows by projecting the disintegration of λ onto the X coordinate). Thus

$$\begin{aligned}\lambda&=\int_Y \mu_{\phi(y)}\times\delta_y\ d\nu(y)\\ &=\int_Z\int_Y \mu_z\times\delta_y\ d\nu_z(y)d\eta(z)\\ &=\int_Z \mu_z\times\Big(\int_Y \delta_y\ d\nu_z(y)\Big)\ d\eta(z)\\ &=\int_Z \mu_z\times\nu_z\ d\eta(z)\\ &=\mu\underset{\eta}{\times}\nu.\end{aligned}$$

□

Lemma 2.2. *Let (X,μ) and (Y,ν) be Lebesgue spaces (not necessarily G-systems), λ a joining of μ and ν. Let*

$$\lambda=\int_X \delta_x\times\lambda_x\ d\mu(x),$$

be the disintegration of λ over μ and define a probability measure λ_∞ on $X \times Y^{\mathbb{Z}}$ by:

$$\lambda_\infty = \int_X \delta_x \times (\cdots \times \lambda_x \times \lambda_x \cdots)\ d\mu(x).$$

Let $\mathcal{Z}$ denote the largest σ-algebra common to the algebras $\mathcal{X}$ and $\mathcal{Y}^{\mathbb{Z}}$ mod λ_∞ and let (Z, η) denote the corresponding factor Lebesgue space. Then $\mathcal{X}$ and $\mathcal{Y}^{\mathbb{Z}}$ are relatively independent over $\mathcal{Z}$ with respect to λ_∞.

Proof. Define a transformation $S : X \times Y^{\mathbb{Z}} \to X \times Y^{\mathbb{Z}}$ by $S(x, y) = (x, \sigma y)$ where $y = (\cdots, y_{-1}, y_0, y_1, \cdots) \in Y^\infty$ and σ is the left shift on $Y^{\mathbb{Z}}$. If $f(x, y)$ is an S-invariant function on $X \times Y^{\mathbb{Z}}$ then for every x the function $f_x(y) = f(x, y)$ is a σ-invariant function on $(Y^{\mathbb{Z}}, \lambda_x^{\mathbb{Z}})$, hence a constant; i.e. $f(x, y) = f(x)$, λ_∞ a.e..

Thus every S-invariant function is $\mathcal{X}$-measurable and in particular the extension $(Y^{\mathbb{Z}}, \nu_\infty, \sigma) \to (Z, \eta, \mathrm{id})$, where

$$\nu_\infty = \int_X (\cdots \times \lambda_x \times \lambda_x \cdots)\ d\mu(x),$$

is an ergodic extension. Now we apply lemma 2.1 to the diagram

$$\begin{array}{ccccc} & & (X \times Y^{\mathbb{Z}}, \lambda_\infty, S) & & \\ & \swarrow & & \searrow & \\ (X, \mu, \mathrm{id}) & & & & (Y^{\mathbb{Z}}, \nu_\infty, \sigma) \\ & \searrow & & \swarrow & \\ & & (Z, \eta) & & \end{array}$$

to deduce that $\mathcal{X}$ and $\mathcal{Y}^{\mathbb{Z}}$ are relatively independent over $\mathcal{Z}$ as claimed. $\square$

Theorem 1. *Let $(X, \mu) \overset{\pi}{\to} (Z, \eta)$ and $(Y, \nu) \overset{\sigma}{\to} (Z, \eta)$ be two ergodic systems extending the system (Z, η). Suppose π is a c.p.e. extension while σ is a zero-entropy extension. Then*

(1) *X and Y are disjoint over Z; i.e. the relatively independent joining $\mu \underset{\eta}{\times} \nu$ is the only joining of μ and ν over η. In particular (taking (Z, η) to be the trivial one point system) we have that every c.p.e. system is disjoint from every zero-entropy system.*

(2) *The extension $(X \underset{Z}{\times} Y, \mu \underset{\eta}{\times} \nu) \to (Y, \nu)$ is a c.p.e. extension; i.e. Y is the relative Pinsker factor of $X \underset{Z}{\times} Y$ over Z. In particular (taking Z to be trivial) when X is a c.p.e. system and Y a zero-entropy system then Y is the Pinsker factor of $X \times Y$; i.e. the extension $X \times Y \to Y$ is a c.p.e. extension.*

Proof. (1) Let λ be a joining of (X, μ) and (Y, ν) over (Z, η). As in lemma 2.2, let

$$\lambda = \int_X \delta_x \times \lambda_x\ d\mu(x),$$

be the disintegration of λ over μ and define a probability measures λ_∞ on $X \times Y^{\mathbb{Z}}$ and ν_∞ on $Y^{\mathbb{Z}}$ by:

$$\lambda_\infty = \int_X \delta_x \times (\cdots \times \lambda_x \times \lambda_x \cdots)\ d\mu(x)$$

and

$$\nu_\infty = \int_X (\cdots \times \lambda_x \times \lambda_x \cdots)\, d\mu(x).$$

Let $\tilde{\mathcal{Z}}$ denote the largest σ-algebra common to the algebras $\mathcal{X}$ and $\mathcal{Y}^{\mathbb{Z}}$ mod λ_∞ and let $(\tilde{Z}, \tilde{\eta})$ denote the corresponding factor system. Then clearly $\mathcal{Z} \subset \tilde{\mathcal{Z}}$ and by lemma 2.2, $\mathcal{X}$ and $\mathcal{Y}^{\mathbb{Z}}$ are relatively independent over $\tilde{\mathcal{Z}}$ with respect to λ_∞.

Now lemma 1.2 implies that the extension $(Y^{\mathbb{Z}}, \nu_\infty) \to (Z, \eta)$ is a 0-entropy extension and a fortiori $(\tilde{Z}, \tilde{\eta}) \to (Z, \eta)$ is a 0-entropy extension. On the other hand $(X, \mu) \to (Z, \eta)$ is a c.p.e. extension; hence so is the extension $(\tilde{Z}, \tilde{\eta}) \to (Z, \eta)$ and we conclude that $\tilde{Z} = Z$. This completes the proof of part (1).

(2) Denote $\lambda = \mu \underset{\eta}{\times} \nu$ and let (U, ζ) be the relative Pinsker factor of $X \underset{Z}{\times} Y$ over Y:

$$\begin{array}{ccc} (X \underset{Z}{\times} Y, \lambda) & \longrightarrow & (U, \zeta) \\ & \searrow \quad\quad \swarrow & \\ & (Y, \nu) & \end{array} .$$

If $\mathcal{U} \neq \mathcal{Y}$ then we let f be any bounded $L_2(\lambda)$ non-zero function which is $\mathcal{U}$ measurable but orthogonal to $L_2(\mathcal{Y})$, so that $E^Y(f) = 0$. Let g and h be any bounded $\mathcal{X}$ and $\mathcal{Y}$ measurable functions respectively. Then by part (1), as fh is $\mathcal{U}$-measurable and $U \to Z$ is a 0-entropy extension, $E^Z(fgh) = E^Z(fh)E^Z(g)$. Now $E^Z(fh) = E^Z(E^Y(fh)) = E^Z(hE^Y(f)) = 0$. Thus $E^Z(fgh) = 0$, whence $E(fgh) = 0$. Since linear combinations of functions of the form gh are dense in $L_2(\lambda)$, this leads to a contradiction and we conclude that $\mathcal{U} = \mathcal{Y}$. $\square$

The following theorem is a simple application of the disjointness theorem.

Theorem 2. *If the ergodic system (X, μ) is the inverse limit of the sequence (X_n, μ_n) and $\bar{X}$ and $\bar{X}_n$ are the corresponding Pinsker factors, then $\bar{X}$ is the inverse limit of the sequence $\bar{X}_n$.*

Proof. Let $\bar{X}$ be the Pinsker factor of X and for each n, $\bar{X}_n$ the Pinsker factor of X_n. Finally let Z be the inverse limit of the X_n. Since the extension $X_n \to \bar{X}_n$ is c.p.e. while the extension $\bar{X} \to \bar{X}_n$ is a zero-entropy extension we get by theorem 1, that $X_n \underset{\bar{X}_n}{\times} \bar{X}$ is the relatively independent joining over $\bar{X}_n$. Let f be an $\mathcal{X}_n$ measurable function and g an $\bar{\mathcal{X}}$ measurable function. Then, since f is also $\mathcal{X}_m$ measurable for $m \geq n$ we have for such m:

$$E^{\bar{X}_m}(fg) = E^{\bar{X}_m}(f) \cdot E^{\bar{X}_m}(g).$$

By the martingale convergence theorem we get

$$E^Z(fg) = E^Z(f) \cdot E^Z(g).$$

Since the union of the $\mathcal{X}_n$ measurable functions is dense, we conclude that $\bar{X}$ is relatively independent of X over Z, which is possible only when $\bar{X} = Z$. $\square$

§3. Lifting c.p.e. extensions

Our goal in this section is to obtain the assertion of theorem 1.(2) without the assumption that the extension $(Y,\nu) \stackrel{\sigma}{\to} (Z,\eta)$ is a 0-entropy extension (theorem 3, below). Our strategy will be to put together two special cases of the theorem. The first is the case when σ is assumed to be a 0-entropy extension; i.e. theorem 1.(2). The second will be the case when $Y = Z \times B$, where B is a Bernoulli system (see also [RW]). For this we need the following three lemmas. The proof of the first lemma is straightforward. Denote by $\mathrm{Aut}(X,\mu,G)$ the group of measure preserving transformations of (X,μ) that commute with the action of G.

Lemma 3.1. *Let (B,ν,G) be a G-Bernoulli system; i.e. $B = \{0,1,\dots,s-1\}^G$, ν a Bernoulli measure on B and the action is the canonical left action: $gx(h) = x(g^{-1}h)$, $x \in B, g,h \in G$. Define the right action of G on B by*

$$g \circ x(h) = x(hg), \qquad (x \in B, g,h \in G).$$

Then the left and right actions of G commute. In particular the right G-action defines a subgroup of $\mathrm{Aut}(B,\nu,G)$ which acts ergodically on B.

In preparation for the next lemma we make the observation that for a Lebesgue space $(X,\mathcal{X},\mu)$, the space of all sub σ-algebras of $\mathcal{X}$ (mod μ) can be viewed as a Polish space. One way to see this is to identify a sub σ-algebra $\mathcal{A} \subset \mathcal{X}$, with the corresponding conditional expectation operator $E^{\mathcal{A}} : L_2(\mathcal{X},\mu) \to L_2(\mathcal{A},\mu)$.

Lemma 3.2. *Let $(X,\mu) \stackrel{\pi}{\to} (Z,\eta)$ be a c.p.e. extension and (Y,ν) an ergodic G-system with the property that the group $\mathrm{Aut}(Y,\nu,G)$ acts ergodically on Y. Then the extension $(X \times Y, \mu \times \nu) \to (Z \times Y, \eta \times \nu)$ is a c.p.e. extension.*

Proof. Let $X \times Y \to U \to Z \times Y$ be the relative Pinsker factor. Fix Γ a countable dense subgroup of $\mathrm{Aut}(Y,\nu,G)$, and note that Γ also acts ergodically on (Y,ν). Choose a countably generated sub-σ-algebra $\mathcal{U}$ whose completion is the σ-algebra of U such that (i) $\mathcal{U}$ is invariant under G and $\mathrm{Id} \times \gamma$ for every $\gamma \in \Gamma$ and (ii) $\mathcal{U}$ contains a sub-σ-algebra whose completion is the σ-algebra of Z. Consider the map $y \mapsto \mathcal{U}_y$ of Y into the Polish space of sub σ-algebras of $\mathcal{X}$, where $\mathcal{U}_y$ is the σ-algebra defined as the trace of $\mathcal{U}$ on the fiber $X \times \{y\}$, and the latter is identified with X. With $\mathcal{U}_y$ we also take the conditional expectation operator defined on $L_2(\mathcal{X},\mu)$ to the closed subspace spanned by the $\mathcal{U}_y$-measurable functions, $E^{\mathcal{U}_y}$. It is now easy to check that the map $y \mapsto E^{\mathcal{U}_y}$ is measurable. Moreover, since clearly for each $\gamma \in \Gamma$, the factor U is $I \times \gamma$-invariant, it follows that $\mathcal{U}_{\gamma y} = \mathcal{U}_y$. Since Γ acts ergodically on Y we conclude that, ν a.e, $\mathcal{U}_y$ does not depend on $y \in Y$ and defines a G-invariant factor σ-algebra $\hat{\mathcal{U}}$ of $\mathcal{X}$ with $\mathcal{Z} \subset \hat{\mathcal{U}}$. If $\hat{\mathcal{U}} \neq \mathcal{Z}$ then—as $X \to Z$ is a c.p.e. extension—it follows that the entropy of $\hat{U}$ is strictly larger than that of Z. However since X and Y are independent and $\hat{U}$ is a factor of X it follows that also the entropy of $\hat{U} \times Y$ is strictly larger than that of Y. Finally since $\hat{U} \times Y$ is a factor of U we get a contradiction to the assumption that U is the relative Pinsker factor of $X \times Y \to Y$. Thus $\hat{U} = Z$ and we conclude that $U = Z \times Y$. □

Lemma 3.3. *If (Z,η) is a factor of (Y,ν) with strictly smaller entropy then there is a Bernoulli factor of Y, B independent of Z such that $Z \times B$ has full entropy in Y.*

For $\mathbb{Z}$ this result is essentially found in [O]. It was made explicit in [OW,1] and [T]. (For another treatment of the relative theory see [Ki,2]). For discrete amenable groups all of this carries over in a straight forward fashion using the basic machinery developed in [OW,2]. It was not done there explicitly because the extension to *continuous* amenable groups presents new difficulties (cf. the discussion there in Appendix C).

Theorem 3. *Let $(X,\mu) \overset{\pi}{\to} (Z,\eta)$ and $(Y,\nu) \overset{\sigma}{\to} (Z,\eta)$ be two ergodic systems extending the system (Z,η). Suppose π is a c.p.e. extension. Let $\lambda = \mu \underset{\eta}{\times} \nu$ be the relatively independent joining of μ and ν over η. Then the extension π_Y in the diagram*

$$\begin{array}{ccccc} & & (X \underset{Z}{\times} Y, \lambda) & & \\ & \swarrow & & \searrow \pi_Y & \\ (X,\mu) & & & & (Y,\nu) \\ & \pi \searrow & & \swarrow \sigma & \\ & & (Z,\eta) & & \end{array}$$

is a c.p.e. extension.

Proof. By lemma 3.3, there exists a Bernoulli factor $Y \to B$ independent of Z and such that $Y \to Z \times B$ is a zero-entropy extension. Notice that $X \underset{Z}{\times} (Z \times B) \cong X \times B$. By lemma 3.1, $\mathrm{Aut}(B,G)$ acts ergodically on B and we can apply lemma 3.2 to the diagram

$$\begin{array}{ccccc} & & X \times B & & \\ & \swarrow & & \searrow \pi_B & \\ X & & & & Z \times B \\ & \pi \searrow & & \swarrow \sigma & \\ & & Z & & \end{array}$$

to deduce that π_B is a c.p.e. extension. Next observe that $[X \underset{Z}{\times} (Z \times B)] \underset{Z \times B}{\times} Y \cong (X \times B) \underset{Z \times B}{\times} Y \cong X \underset{Z}{\times} Y$, then apply theorem 1 to the diagram

$$\begin{array}{ccccc} & & X \underset{Z}{\times} Y & & \\ & \swarrow & & \searrow \pi_Y & \\ X \times B & & & & Y \\ & \pi_B \searrow & & \swarrow \text{0-ent.} & \\ & & Z \times B & & \end{array}$$

to deduce that $\pi_Y : X \underset{Z}{\times} Y \to Y$ is a c.p.e. extension. $\square$

§4. The Pinsker factor of a product system

In this section our main goal is a generalization to discrete amenable groups of the well known fact that for $\mathbb{Z}$-actions the product of two K-systems is a K-system. For previous results in this direction see e.g. [Ka].

Theorem 4. *Let $(X,\mu) \overset{\pi}{\to} (Z,\eta)$ and $(Y,\nu) \overset{\sigma}{\to} (Z,\eta)$ be two ergodic systems extending the system (Z,η) with relative Pinsker factors $\bar{X}$ and $\bar{Y}$ respectively. Then $\bar{X} \underset{Z}{\times} \bar{Y}$ is the relative Pinsker factor of the relative product system $(X \underset{Z}{\times} Y, \mu \underset{\eta}{\times} \nu)$*

over (Z,η). In particular (for the case that Z is trivial) we have that the Pinsker factor of the product system is the product of the Pinsker factors and specializing once again, the product of two c.p.e. systems is a c.p.e. system.

Proof. Let $\lambda = \mu \underset{\eta}{\times} \nu$. The measure space we work with is $(X \underset{Z}{\times} Y, \lambda)$. Let U be the relative Pinsker factor of $X \underset{Z}{\times} Y$ over Z. First consider the diagram

$$\begin{array}{ccccc} & & X \underset{Z}{\times} Y & & \\ & \swarrow & & \searrow & \\ X & & & & \bar{X} \underset{Z}{\times} Y \\ & \searrow & & \swarrow & \\ & & \bar{X} & & \end{array} .$$

By theorem 3 the extension $X \underset{Z}{\times} Y \to \bar{X} \underset{Z}{\times} Y$ is a c.p.e. extension.

Now consider the systems $X \vee U$ and $Y \vee U$. Since the extension $U \vee Y \to \bar{X} \underset{Z}{\times} Y$ is clearly a zero-entropy extension, theorem 1 implies that $X \underset{Z}{\times} Y$ is independent of $U \vee Y$ over $\bar{X} \underset{Z}{\times} Y$. Since the algebra corresponding to $X \underset{Z}{\times} Y$ is the total σ-algebra, we deduce that $U \vee Y$, and in particular U, is $\bar{X} \underset{Z}{\times} Y$ measurable. Symmetrically we get that U is $X \underset{Z}{\times} \bar{Y}$ measurable, and therefore deduce that U is $\bar{X} \underset{Z}{\times} \bar{Y} = (X \underset{Z}{\times} \bar{Y}) \cap (\bar{X} \underset{Z}{\times} Y)$ measurable; i.e. $U = \bar{X} \underset{Z}{\times} \bar{Y}$. $\square$

§5. Maximal entropy and independence

Theorem 5. *Let $(X,\mu) \overset{\pi}{\to} (Z,\eta)$ and $(Y,\nu) \overset{\sigma}{\to} (Z,\eta)$ be two ergodic systems extending the system (Z,η) with finite entropy and such that π is a c.p.e. extension. Let λ be a joining of μ and ν over η (i.e. λ projects under $\pi \times \sigma$ onto the diagonal measure $\eta^{(2)}$ on $Z \times Z$). Suppose*

$$h_\lambda(X \underset{Z}{\times} Y|Z) = h(X|Z) + h(Y|Z)$$

then λ is the relatively independent joining of μ and ν over η.

Proof. The measure space we work with is $(X \underset{Z}{\times} Y, \lambda)$. Entropies are computed with respect to the measure λ. We let $\mathcal{R}, \mathcal{S}$ and $\mathcal{U}$ be finite generating partitions for X, Y and Z respectively (see [Ro]).

(1) Assume first that $Y = B \times Z$ where B is a Bernoulli factor of Y independent of Z. We let $\mathcal{P}$ be a finite Bernoulli generating partition for B. By lemma 1.1 (1) and our assumption

$$\begin{aligned} h_\lambda(X \underset{Z}{\times} Y|Z) &= h(\mathcal{R} \vee \mathcal{S}|Z) = h(\mathcal{R}|Z) + h(\mathcal{S}|\mathcal{R}^G \vee \mathcal{Z}) \\ &= h(X|Z) + h(Y|X \vee Z) = h(X|Z) + h(Y|Z), \end{aligned}$$

hence $h(Y|X \vee Z) = h(Y|X) = h(Y|Z)$. Now

$$h(Y|X) = h(\mathcal{P} \vee \mathcal{U}|X) = h(\mathcal{P}|X) + h(\mathcal{U}|\mathcal{P}^G \vee \mathcal{X}) = h(\mathcal{P}|X),$$

hence

$$h(\mathcal{P}|X) = h(\mathcal{P} \vee \mathcal{U}|X) = h(Y|X) = h(Y|Z) = h(\mathcal{P} \vee \mathcal{U}|Z) = H(\mathcal{P}),$$

and lemma 1.1 (2) implies that B is independent of X. ¿From this we get that $Y = B \times Z$ is relatively independent of X over Z as required.

(2) In the general case we observe first that when $h(\mathcal{Y}|Z) = 0$, the assertion follows from theorem 1. If $h(\mathcal{Y}|Z) > 0$, we apply lemma 3.3 to choose a Bernoulli factor B of Y such that $h(B) = h(Y|Z)$ and B is independent of Z. Now by (1), B is independent of X, and $B \times Z$ is relatively independent of X over Z. By theorem 3, in the diagram

$$\begin{array}{ccc} & X \times B & \\ \swarrow & & \searrow \\ X & & Z \times B \\ \searrow & & \swarrow \\ & Z & \end{array},$$

the extension $X \times B \to Z \times B$ is a c.p.e. extension. Now, since $Y \to Z \times B$ is a zero-entropy extension, we can apply theorem 1 to the diagram

$$\begin{array}{ccc} & (X \underset{Z}{\times} Y, \lambda) & \\ \swarrow & & \searrow \\ X \times B & & Y \\ \searrow & & \swarrow \\ & Z \times B & \end{array},$$

and conclude that $X \times B$ and Y are relatively independent over $Z \times B$. Now let f and g be bounded $\mathcal{X}$ and $\mathcal{Y}$ measurable functions, respectively. Then

$$\begin{aligned} E_\lambda^Z(fg) &= E^Z(E^{Z\times B}(fg)) = E^Z(E^{Z\times B}(f)E^{Z\times B}(g)) \\ &= E^Z(E^Z(f)E^{Z\times B}(g)) = E^Z(f)E^Z(g). \end{aligned}$$

Thus X and Y are relatively independent over Z and $\lambda = \mu \underset{\eta}{\times} \nu$ as required. □

Theorem 6. *Let $(X_i, \mu_i) \overset{\pi_i}{\to} (Z, \eta)$, $i = 1, 2$ be two ergodic systems extending the system (Z, η). Let λ be a joining of X_1 and X_2 over Z. Let $(Z_i, \eta_i) \to (Z, \eta)$ be the relative Pinsker factors, $i = 1, 2$, and assume further that*

$$h_\lambda(X_1 \vee X_2|Z) = h(X_1|Z) + h(X_2|Z).$$

Then λ is the relatively independent joining of the two systems $(X_i, \mu_i), i = 1, 2$ over the joining induced by λ on the σ-algebra generated by the two relative Pinsker factors $(Z_i, \eta_i), i = 1, 2$. In particular if with respect to λ, Z_1 and Z_2 are independent over Z, then also X_1 and X_2 are independent over Z; i.e. λ is the relatively independent joining of μ_1 and μ_2 over η.

Proof. Consider the diagram

$$\begin{array}{ccc} X_1 \vee Z_2 & & Z_1 \vee X_2 \\ p_1 \searrow & & \swarrow p_2 \\ & Z_1 \vee Z_2 & \end{array}$$

where for example $X_1 \vee Z_2$ denotes the factor of $(X_1 \times X_2, \lambda)$ generated by the factors X_1 and Z_2. By theorem 1.(2), p_1 is a c.p.e. extension while by using lemma 1.1, we have

$$h_\lambda(X_1 \vee X_2 | Z_1 \vee Z_2) = h(X_1 \vee Z_2 | Z_1 \vee Z_2) + h(Z_1 \vee X_2 | Z_1 \vee Z_2).$$

Thus theorem 5 implies that $X_1 \vee Z_2$ and $Z_1 \vee X_2$ are independent over $Z_1 \vee Z_2$. Let f_i be bounded functions on $X_1 \times X_2$ which are $\mathcal{X}_i$-measurable, $i = 1, 2$. Then

$$E^{Z_1 \vee Z_2}(f_1 f_2) = E^{Z_1 \vee Z_2}(f_1) \cdot E^{Z_1 \vee Z_2}(f_2).$$

This proves the first part of the theorem; for the second part we use the relative independence of Z_1 and Z_2 over Z to get:

$$\begin{aligned} E^Z(f_1 f_2) &= E^Z(E^{Z_1 \vee Z_2}(f_1 f_2)) \\ &= E^Z(E^{Z_1 \vee Z_2}(f_1) \cdot E^{Z_1 \vee Z_2}(f_2)) \\ &= E^Z(E^{Z_1}(f_1) \cdot E^{Z_2}(f_2)) = E^Z(f_1) \cdot E^Z(f_2). \end{aligned}$$

Thus X_1 and X_2 are relatively independent over Z as claimed. □

§6 Quasi-factors of zero-entropy extensions

Our last application of theorem 1 is to obtain a new proof of the fact that a quasifactor of a zero-entropy system has zero-entropy. In fact we get a generalization of this statement to zero-entropy extensions and moreover this new proof is simpler than both proofs in [GW]. Of course the main application of the quasifactor theorem in [GW] was to get a proof of the absolute version of theorem 1.

Let $(X, \mathcal{X}, \mu, G)$ be an ergodic system. Let $(M(X), G)$ be the Borel space of probability measures on X with the induced G-action. Recall that a quasifactor of $(X, \mathcal{X}, \mu, G)$ is any probability measure λ on $M(X)$, invariant under G, and with barycenter equal to μ:

$$\int_{M(X)} \theta \; d\lambda(\theta) = \mu.$$

Let $(X, \mathcal{X}, \mu, G) \xrightarrow{\pi} (Y, \mathcal{Y}, \nu, G)$ be a homomorphism of ergodic systems. Let $M_\pi(X)$ be the subspace of $M(X)$ consisting of measures θ for which $\pi(\theta) = \delta_y$ for some $y \in Y$. We let $Q(\mu)$ be the set of all quasifactors of $(X, \mathcal{Y}, \mu, G)$ and we denote by $Q_\pi(\mu)$ the subset of those quasifactors $\lambda \in Q(\mu)$ that are supported on $M_\pi(X)$ i.e. those $\lambda \in Q(\mu)$ for which for λ almost every $\theta \in M(X)$ there exists $y = y(\theta) \in Y$ with $\theta(\pi^{-1}(y)) = 1$.

The projection map $\pi : X \to Y$ induces a map from $M(X)$ onto $M(Y)$ which for convenience we also denote by π. Given an element $\lambda \in Q_\pi(\mu)$ we get this way (identifying the Dirac measure δ_y with the point $y \in Y$) a factor map $\pi : (M_\pi(X), \lambda, G) \to (Y, \nu, G)$. We now have the corresponding disintegration of λ over ν

$$\lambda = \int_Y \lambda_y \; d\nu(y)$$

with λ_y a probability measure on $M(Z)$.

Theorem 7. *Let* $(X, \mathcal{X}, \mu, T) \stackrel{\pi}{\to} (Y, \mathcal{Y}, \nu, T)$ *be a homomorphism of ergodic systems with* $h_\mu(X|Y) = 0$. *Then every quasifactor* $\lambda \in Q_\pi(\mu)$ *satisfies*

$$h_\lambda(M_\pi|Y) = 0.$$

Proof. Let $\lambda \in Q_\pi(\mu)$ be a quasifactor and let (U, ζ) be the relative Pinsker factor of (M_π, λ) over (Y, ν):

$$\begin{array}{ccc} (M_\pi, \lambda) & \xrightarrow{\phi} & (U, \zeta) \\ & \searrow \quad \swarrow & \\ & (Y, \nu) & \end{array}.$$

Form the joining κ, of λ and μ, defined by

$$\kappa = \int_{M_\pi} \delta_\theta \times \theta \ d\lambda(\theta),$$

and consider the diagram

$$\begin{array}{ccc} & (M_\pi \vee X, \kappa) & \\ \swarrow & & \searrow \phi \times \mathrm{id} \\ (M_\pi, \lambda) & & (U \vee X, \gamma) \\ \phi \searrow & & \swarrow p \\ & (U, \zeta) & \end{array}$$

where $\gamma = (\phi \times \mathrm{id})\kappa$. In this diagram, by definition, ϕ is a c.p.e. extension and by lemma 1.2, p is a 0-entropy extension. Thus, by theorem 1, $\kappa = \lambda \underset{\zeta}{\times} \gamma$. If $\lambda = \int_U \lambda_u \ d\zeta(u)$ and we let $\mu_u = \int_{M_\pi} \theta \ d\lambda_u(\theta)$, then

$$\begin{aligned} \gamma = (\phi \times \mathrm{id})\kappa &= \int \int \delta_u \times \theta \ d\lambda_u(\theta) d\zeta(u) \\ &= \int \delta_u \times \Big(\int \theta \ d\lambda_u(\theta)\Big) d\zeta(u) = \int \delta_u \times \mu_u \ d\zeta(u). \end{aligned}$$

Thus implementing the isomorphism

$$M_\pi \underset{U}{\times} (U \vee X) \cong M_\pi \vee X$$

(given by $(\theta, (\phi(\theta), x)) \mapsto (\theta, x)$) we get on one hand:

$$\begin{aligned} \kappa &= \int_U \lambda_u \times \delta_u \times \mu_u \ d\zeta(u) \cong \int_U \lambda_u \times \mu_u \ d\zeta(u) \\ &= \int_U \Big(\int_{M_\pi} \delta_\theta \ d\lambda_u(\theta)\Big) \times \mu_u \ d\zeta(u) = \int_U \int_{M_\pi} \delta_\theta \times \mu_u \ d\lambda_u(\theta) d\zeta(u) \\ &= \int_{M_\pi} \delta_\theta \times \mu_{\phi(\theta)} \ d\lambda(\theta), \end{aligned}$$

and on the other

$$\kappa = \int_{M_\pi} \delta_\theta \times \theta \ d\lambda(\theta).$$

By uniqueness of disintegration we conclude that for λ-a.e θ,

$$\mu_{\phi(\theta)} = \int_{M_\pi} \theta' \ d\lambda_{\phi(\theta')} = \theta.$$

This clearly implies that ϕ is an isomorphism, hence that $(M_\pi, \lambda) \to (Y, \nu)$ is a 0-entropy extension; i.e.

$$h_\lambda(M_\pi|Y) = 0,$$

as claimed. $\square$

References

[B] K.R. Berg, *Independence and additive entropy*, Proc. Am. Math. Soc. **51** (1975), 366–370.

[GW] E. Glasner and B. Weiss, *Quasi-factors of zero-entropy systems*, J. of Amer. Math. Soc. **8** (1995), 665–686.

[Ka] B. Kamiński, *Generators of perfect σ-algebras of $\mathbb{Z}^d$-actions*, Studia Math. **99** (1991), 1–10.

[KR] J. Kammeyer and D.J. Rudolph, *Restricted orbit equivalence for actions of discrete amenable groups*, Preprint.

[Ki,1] J.C. Kieffer, *A generalized Shanon-McMillan theorem for the action of an amenable group on a probability space*, Ann. Prob. **3** (1975), 1031–1037.

[Ki,2] J.C. Kieffer, *A simple development of the Thouvenot relative isomorphism theory*, Ann. Prob. **12** (1984), 204–211.

[LPT] M. Lemańczyk, F. Parreau and J.-P. Thouvenot, *Gaussian automorphisms whose ergodic self-joinings are Gaussian*, Preprint.

[O] D.S. Ornstein, *Two Bernoulli shifts with infinite entropy are isomorphic*, Adv. in Math. **5** (1970), 339–348.

[OW,1] D.S. Ornstein and B. Weiss, *Unilateral codings of Bernoulli systems*, Israel J. of Math. **21** (1975), 159–166.

[OW,2] D.S. Ornstein and B. Weiss, *Entropy and isomorphism theorems for the action of an amenable group actions*, J. d'Analyse Math. **48** (1987), 1–141.

[P] W. Parry, *Topics in ergodic theory*, Cambridge University Press, 1981.

[Ro] A. Rosenthal, *Finite uniform generators for ergodic finite entropy free actions of amenable groups*, Prob. Th. Rel. Fields **77** (1988), 147–166.

[RW] D.J. Rudolph and B. Weiss, *Entropy and mixing for amenable group actions*, Preprint.

[T] J.-P. Thouvenot, *Quelques propriétés des systèmes dynamiques qui se décomposent en un produit de deu systèmes dont l'un est un schéma de Bernoulli*, Israel J. of Math. **21** (1975), 177–207.

[WaZ] T. Ward and Q. Zhang, *The Abramov-Rokhlin entropy addition formula for amenable group actions*, Monat. für Math. **114** (1992), 317–329.

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