The Cappelli–Itzykson–Zuber A–D–E Classification

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Abstract. In 1986 Cappelli, Itzykson and Zuber classified all modular invariant partition functions for the conformal field theories associated to the affine A_1 algebra; they found they fall into an A-D-E pattern. Their proof was difficult and attempts to generalise it to the other affine algebras failed – in hindsight the reason is that their argument ignored most of the rich structure present. We give here the "modern" proof of their result; it is an order of magnitude simpler and shorter, and much of it has already been extended to all other affine algebras. We conclude with some remarks on the A-D-E pattern appearing in this and other RCFT classifications.

1. The problem

One of the more important results in conformal field theory is surely the classification due to Cappelli, Itzykson, and Zuber [3; see also 4] of the genus 1 partition functions for the theories associated to $A_1^{(1)}$ (which in turn implies the classification of the minimal models). Their list was curious: Kac noticed that their partition functions fall into the A-D-E pattern familiar from the finite subgroups of $\mathrm{SU}_2(\mathbb{C})$, simple singularities, simply-laced Lie algebras, subfactors with index < 4, etc. See e.g. [9].

  The problem can be phrased as follows. Fix any integer $n \geq 3$. Let $P_+ = \{1, 2, \ldots, n-1\}$, and let S and T be the $(n-1) \times (n-1)$ matrices with entries

$$S_{ab} = \sqrt{\frac{2}{n}} \sin(\pi \frac{ab}{n}) , \qquad T_{ab} = \exp[\pi \mathrm{i} \frac{a^2}{2n}] \delta_{a,b} .$$

Find all $(n-1) \times (n-1)$ matrices M such that
- M commutes with S and T: $MS = SM$ and $MT = TM$
- M has nonnegative integer entries: $M_{ab} \in \mathbb{Z}_+$ for all $a, b \in P_+$
- M is normalised so that $M_{11} = 1$.

Call any such M a *physical invariant*. Since most entries M_{ab} are usually zero, it is more convenient to formally express M as the coefficient matrix for the combination

$$\mathcal{Z} = \sum_{a,b=1}^{n-1} M_{ab}\,\chi_a\,\chi_b^* \ .$$

THEOREM [3]. *The complete list of physical invariants is (using $Ja \overset{\text{def}}{=} n-a$)*

$$\mathcal{A}_{n-1} = \sum_{a=1}^{n-1} |\chi_a|^2 \ , \qquad\qquad \forall n \geq 3$$

$$\mathcal{D}_{\frac{n}{2}+1} = \sum_{a=1}^{n-1} \chi_a\,\chi_{J^a a}^* \ , \qquad\qquad \text{whenever } \frac{n}{2} \text{ is even}$$

$$\mathcal{D}_{\frac{n}{2}+1} = |\chi_1 + \chi_{J1}|^2 + |\chi_3 + \chi_{J3}|^2 + \cdots + 2|\chi_{\frac{n}{2}}|^2 \ , \qquad \text{whenever } \frac{n}{2} \text{ is odd}$$

$$\mathcal{E}_6 = |\chi_1 + \chi_7|^2 + |\chi_4 + \chi_8|^2 + |\chi_5 + \chi_{11}|^2 \ , \qquad\qquad \text{for } n = 12$$

$$\mathcal{E}_7 = |\chi_1 + \chi_{17}|^2 + |\chi_5 + \chi_{13}|^2 + |\chi_7 + \chi_{11}|^2$$
$$+ \chi_9\,(\chi_3 + \chi_{15})^* + (\chi_3 + \chi_{15})\,\chi_9^* + |\chi_9|^2 \ , \qquad\qquad \text{for } n = 18$$

$$\mathcal{E}_8 = |\chi_1 + \chi_{11} + \chi_{19} + \chi_{29}|^2 + |\chi_7 + \chi_{13} + \chi_{17} + \chi_{23}|^2 \ , \qquad \text{for } n = 30 \ .$$

These realise the A-D-E pattern, in the following sense. The Coxeter number h of the name $\mathcal{X}_\ell$ equals the corresponding value of n, and the exponents of X_ℓ (i.e. the m_i in the eigenvalues $4\sin^2(\pi\,\frac{m_i}{2n})$ of its Cartan matrix) equal those $a \in P_+$ for which $M_{aa} \neq 0$.

Cappelli-Itzykson-Zuber proved this by first finding an explicit basis for the space of all matrices commuting with S and T. Unfortunately their proof of the theorem was long and formidable. Considering all of the structure implicit in the problem, we should anticipate a much more elementary argument. This is not merely of academic interest, because there is a natural generalisation of this problem to all other affine algebras. Several people had tried to extend the argument of [3] to these larger algebras, but with [1] it became clear that some other approach was necessary, or the generalisation would never be achieved. And of course another reason is that the more transparent the argument, the better the chance of understanding the connection with A-D-E.

In this paper we provide a considerably shorter proof of the theorem, bearing no resemblance to the older arguments. Our proof is an example of the "modern" approach to physical invariant classifications. See [6] for a summary of the current status of these classifications for the other affine algebras.

The argument which follows is completely elementary: no knowledge of e.g. CFT or Kac-Moody algebras is assumed. It is based on various talks I've given, most recently at the Schrödinger Institute in Vienna where I wrote up this paper and who I thank for generous hospitality.

2. The combinatorial background

In this section we include some of the basic tools belonging to any classification of the sort, and we give a flavour of their proofs. We will state them for the specific problem given above, but everything generalises without effort [5].

First note that commutation of M with T implies the selection rule

$$M_{ab} \neq 0 \quad \Longrightarrow \quad (a+1)^2 \equiv (b+1)^2 \quad (\text{mod } 4n) . \tag{2.1}$$

Next, let us write down some of the basic properties obeyed by S. S is symmetric and orthogonal, and

$$S_{1b} \geq S_{11} > 0 . \tag{2.2}$$

The permutation J of P_+, defined by $Ja = n - a$, corresponds to the order 2 symmetry of the extended Dynkin diagram of $A_1^{(1)}$; it satisfies

$$S_{Ja,b} = (-1)^{b+1} S_{ab} . \tag{2.3}$$

Note that the element $1 \in P_+$ is both physically and mathematically special; our strategy will be to find all possible first rows and columns of M, and then for each of these possibilities to find the remaining entries of M.

The easiest result follows by evaluating $MS = SM$ at $(1, a)$ for any $a \in P_+$:

$$S_{11} + \sum_{b=2}^{n-1} M_{1b} S_{ba} \geq 0 , \tag{2.4}$$

with equality iff the ath column of M is identically 0. Equation (2.4) has two uses: it severely constrains the values of M_{1b} (and dually M_{b1}), and it says precisely which columns (and rows) are nonzero.

Another simple observation is

$$1 = M_{11} = \sum_{a,b=1}^{n-1} S_{1a} M_{ab} S_{1b} \geq S_{11}^2 \sum_{a,b=1}^{n-1} M_{ab} .$$

This tells us that each entry M_{ab} is bounded above by $\frac{1}{S_{11}^2}$ (we will use this below). In particular, there can only be finitely many physical invariants for each n. (This same calculation shows more generally that there will only be finitely many physical invariants for a given affine algebra $X_r^{(1)}$ and level k.)

Next, let's apply the triangle inequality to sums involving (2.3). Choose any $i, j \in \{0, 1\}$. Then

$$M_{J^i 1, J^j 1} = \sum_{a,b=1}^{n-1} (-1)^{(a+1)i} S_{1a} M_{ab} (-1)^{(b+1)j} S_{1b} .$$

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Taking absolute values, we obtain

$$M_{J^i 1, J^j 1} \leq \sum_{a,b=1}^{n-1} S_{1a}\, M_{ab}\, S_{1b} = M_{11} = 1 \ .$$

Thus $M_{J^i 1, J^j 1}$ can equal only 0 or 1. If it equals 1, then we obtain the selection rule: $(a+1)i \equiv (b+1)j \pmod{2}$ whenever $M_{ab} \neq 0$. This implies the symmetry $M_{J^i a, J^j b} = M_{ab}$ for all $a, b \in P_+$.

Whenever you have nonnegative matrices in your problem, and it makes sense to multiply those matrices, then you should seriously consider using Perron-Frobenius theory – a collection of results concerning the eigenvalues and eigenvectors of nonnegative matrices. Our M is nonnegative, and although multiplying M's may not give us back a physical invariant, at least it will give us a matrix commuting with S and T. In other words, the commutant is much more than merely a vector space, it is in fact an algebra.

Important applications of this thought are the following two lemmas.

LEMMA 1.   *Let M be a physical invariant, and suppose $M_{a1} = \delta_{a,1}$ – i.e. the first column of M is all zeros except for $M_{11} = 1$. Then M is a permutation matrix – i.e. there is some permutation π of P_+ such that $M_{ab} = \delta_{b,\pi a}$, and $S_{\pi a, \pi b} = S_{ab}$.*

This is proved by first showing that also $M_{1a} = \delta_{1,a}$ (evaluate $MS = SM$ at (1,1)), and then studying the powers $(M\, M^T)^L$ as L goes to infinity: its diagonal entries will grow exponentially with L, unless there is at most one nonzero entry on each row of M, and it equals 1. (Recall that the entries of $(M\, M^T)^L$ must be bounded above.) Lemma 1 was found independently by Schellekens and Gannon.

That argument is elementary enough that it required no knowledge of Perron-Frobenius. But that knowledge is needed to generalise it. In this fancier language, what the preceding argument shows is: write M as the direct sum of indecomposable submatrices; then the largest eigenvalue of the submatrix containing (1,1) bounds above the one for each other submatrix. Arguing with a little more sophistication, we obtain much more. The special case we need is:

LEMMA 2 [5].   *Let M be a physical invariant, and suppose $M_{a1} \neq 0$ only for $a = 1$ and $a = J1$, and similarly for M_{1a} – i.e. the first row and column of M are all zeros except for $M_{J^i 1, J^j 1} = 1$. Then the ath row (or column) of M will be identically 0 iff a is even. Moreover, let $a, b \in P_+$, both different from $\frac{n}{2}$, and suppose $M_{ab} \neq 0$. Then*

$$M_{ac} = \begin{cases} 1 & \text{if } c = b \text{ or } c = Jb \\ 0 & \text{otherwise} \end{cases}$$

and a similar formula holds for M_{cb}.

This lemma says that the indecomposable submatrices of M which don't involve $\frac{n}{2}$ (the fixed-point of J) will either be trivial (0) (for even places on the diagonal), or involve blocks $\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$. You can check this for the $\mathcal{D}_{even}$ and $\mathcal{E}_7$ partition functions.

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Our final ingredient is a Galois symmetry obeyed by S, and its consequence for M. Again, see e.g. [5] for a proof. Let L be the set of all ℓ coprime to $2n$. For each $\ell \in L$, there is a permutation $a \mapsto [\ell a]$ of P_+, and a choice of signs $\epsilon_\ell : P_+ \to \{\pm 1\}$, such that

$$M_{ab} = \epsilon_\ell(a)\,\epsilon_\ell(b)\,M_{[\ell a],[\ell b]}\;, \qquad (2.5)$$

for all $a, b \in P_+$. In particular, write $\{x\}$ for the number congruent to $x \pmod{2n}$ satisfying $0 \le \{x\} < 2n$. Then if $\{\ell a\} < n$, put $[\ell a] = \{\ell a\}$ and $\epsilon_\ell(a) = +1$, while if $\{\ell a\} > n$, put $[\ell a] = 2n - \{\ell a\}$ and $\epsilon_\ell(a) = -1$. This 'Galois symmetry' (2.5) comes from hitting $M = SMS$ with the ℓth 'Galois automorphism'. Any polynomial over $\mathbb{Q}$ with a $2n$th root of unity ζ as a zero – and $M = SMS$ can be interpreted in that way – also has ζ^ℓ as a zero. We then use $\sin(\pi\,\ell\frac{ab}{n}) = \epsilon_\ell(a)\,\sin(\pi\,\frac{[\ell a]b}{n})$. From (2.5) and the positivity of M, we get for all $\ell \in L$ the Galois selection rule

$$M_{ab} \ne 0 \quad \Longrightarrow \quad \epsilon_\ell(a) = \epsilon_\ell(b)\;. \qquad (2.6)$$

(2.5) and (2.6), valid for any affine algebras, were first found independently by Gannon and Ruelle-Thiran-Weyers. The Galois interpretation, and extension to all RCFT, is due to Coste-Gannon.

3. The "modern" proof of the $A_1^{(1)}$ classification

The last section reviewed the basic tools shared by all modular invariant partition function classifications. In this section we specialise to $A_1^{(1)}$.

The first step will be to find all possible values of a such that $M_{1a} \ne 0$ or $M_{a1} \ne 0$. These a are severely constrained. We know two generic possibilities: $a = 1$ (good for all n), and $a = J1$ (good when $\frac{n}{2}$ is odd). We now ask the question, what other possibilites for a are there? Our goal is to prove (3.4). Assume $a \ne 1, J1$.

There are only two constraints on a which we will need. One is (2.1):

$$(a-1)(a+1) \equiv 1 \pmod{4n}\;. \qquad (3.1)$$

More useful is the Galois selection rule (2.6), which we can write as $\sin(\pi\ell\frac{a}{n})\sin(\pi\ell\frac{1}{n}) > 0$, for all $\ell \in L$. But a product of sines can be rewritten as a difference of cosines, so we get

$$\cos(\pi\,\ell\,\frac{a-1}{n}) > \cos(\pi\,\ell\,\frac{a+1}{n})\;. \qquad (3.2)$$

Since ℓ obeys (3.2) iff $\ell + n$ does, we can take ℓ in (3.2) to be coprime merely to n instead of $2n$. Call L' the set of these ℓ. (3.2) is strong and easy to solve; here is my argument.

Define $d = \gcd(a-1, 2n)$, $d' = \gcd(a+1, 2n)$. Note from (3.1) that $\gcd(d, d') = 2$ and $dd' = 4n$, so $d, d' \ge 6$. We can choose $\ell_0, \ell' \in L'$ so that $\ell'(a+1) \equiv d' \pmod{2n}$ and

$$\ell_0\,(a-1) \equiv \begin{cases} n - d & \text{if } \frac{d}{2} \text{ is odd and } \frac{n}{2} \text{ is even} \\ n - 2d & \text{if } \frac{d}{2} \text{ is odd and } \frac{n}{2} \text{ is odd} \\ n - \frac{d}{2} & \text{otherwise, i.e. if } \frac{d'}{2} \text{ is odd} \end{cases} \pmod{2n}\;.$$

Now define $\ell_i = \frac{2ni}{d} + \ell_0$. Then $\ell_i(a-1) \equiv \ell_0(a-1) \pmod{2n}$ for all i, and for $0 \le i < \frac{d}{2}$ the numbers $\ell_i(a+1)$ will all be distinct $\pmod{2n}$. For those i, precisely $\phi(\frac{d}{2})$ of the ℓ_i will be in L', where $\phi(x)$ is the Euler totient, i.e. the number of positive integers less than x coprime to x.

Now, the numbers $\ell_i(a+1)$ are all multiples of d'. So (3.2) with $\ell = \ell_i$ gives us

$$(\phi(\frac{d}{2}) - 1)\,d' < \begin{cases} 2d & \text{if } \frac{d}{2} \text{ is odd and } \frac{n}{2} \text{ is even} \\ 4d & \text{if } \frac{d}{2} \text{ is odd and } \frac{n}{2} \text{ is odd} \\ d & \text{otherwise} \end{cases} \qquad . \qquad (3.3)$$

Also, (3.2) with $\ell = \ell'$ requires $d < d'$. Combining this with (3.3), we get $\phi(\frac{d}{2}) - 1 < 2$, 4, or 1, which has the solutions $d = 6$ (for n some multiple of 4), and $d = 6$ or 10 (for n an odd multiple of 2). (3.3) now gives us exactly 3 possibilities: $d = 6$, $d' = 8$, $n = 12$ (which yields $\mathcal{E}_6$ as we will see below); $d = 6$, $d' = 20$, $n = 30$, and $d = 10$, $d' = 12$, $n = 30$ (both which correspond to $\mathcal{E}_8$).

So what we have shown is that, provided $n \ne 12, 30$, M obeys the strong condition

$$M_{a1} \ne 0 \text{ or } M_{1a} \ne 0 \qquad \Longrightarrow \qquad a \in \{1, J1\} . \qquad (3.4)$$

Consider first **case 1**: $M_{a1} = \delta_{a,1}$. This is the condition in Lemma 1, and so we know $M_{ab} = \delta_{b,\pi a}$ for some permutation π of P_+ obeying $S_{ab} = S_{\pi a, \pi b}$. We know $\pi 1 = 1$, so put $m := \pi 2$. Then $\sin(\pi \frac{2}{n}) = \sin(\pi \frac{m}{n})$, and so we get either $m = 2$ or $m = J2$. By T-invariance (2.1), the second possibility can only occur if $4 \equiv (n-2)^2 \pmod{4n}$, i.e. 4 divides n. Then $\mathcal{D}_{\frac{n}{2}+1}$ is also a permutation matrix. Thus replacing M if necessary with the matrix product $M\,\mathcal{D}_{\frac{n}{2}+1}$, we can require $m = 2$, i.e. $\pi 2 = 2$.

Now take any $a \in P_+$ and write $b = \pi a$: we have both $\sin(\pi \frac{a}{n}) = \sin(\pi \frac{b}{n})$ and $\sin(\pi \frac{2a}{n}) = \sin(\pi \frac{2b}{n})$. Dividing these gives $\cos(\pi \frac{a}{n}) = \cos(\pi \frac{b}{n})$, and we read off that $b = a$, i.e. that M is the identity matrix $\mathcal{A}_{n-1}$.

The other possibility, **case 2**, is that both $M_{1,J1} \ne 0$ and $M_{J1,1} \ne 0$. Then Lemma 2 applies. (2.1) says $1 \equiv (n-1)^2 \pmod{4n}$, i.e. $\frac{n}{2}$ is odd. $n = 6$ is trivial (the only unknown entry, $M_{3,3}$, is fixed by $MS = SM$ at (1,3)), so consider $n \ge 10$. The role of '2' in **case 1** will be played here by '3'. The only difference is the complication caused by the fixed-point $\frac{n}{2}$. Can $M_{3,\frac{n}{2}} \ne 0$? If so, then Lemma 2 would imply $M_{3,a} = 0$ for all $a \ne \frac{n}{2}$. Evaluating $MS = SM$ at (3,1), we obtain $M_{3,\frac{n}{2}} = 2\sin(\pi \frac{3}{n})$, i.e. $n = 18$, which corresponds to $\mathcal{E}_7$ as we show later.

Thus we can assume for now that both $M_{3,\frac{n}{2}} = M_{\frac{n}{2},3} = 0$, and so by Lemma 2 there will be a unique $m < \frac{n}{2}$ for which $M_{3,m} \ne 0$. $MS = SM$ at (3,1) now gives $m = 3$. For any odd $a \in P_+$, $a \ne \frac{n}{2}$, can we have $M_{\frac{n}{2},a} \ne 0$? If so then $MS = SM$ at (1,a) and (3,a) would give us $2\sin(\pi \frac{a}{n}) = M_{\frac{n}{2},a} = 2\sin(\pi \frac{3a}{n})$, which is impossible for $\frac{n}{2}$ odd. Therefore Lemma 2 again applies, and we get a unique $b < \frac{n}{2}$ for which $M_{ba} \ne 0$. The usual argument forces $b = a$, and we obtain the desired result: $M = \mathcal{D}_{\frac{n}{2}+1}$.

3.1. The exceptional at $n = 12$

We know $M_{1a} \geq 1$ for some $a \in P_+$ with $\gcd(a+1, 24) = 8$ – i.e. $a = 7$. ¿From (2.4) at $a = 2$, we get $\sin(\frac{\pi}{6}) - M_{1,7} \sin(\frac{\pi}{6}) \geq 0$. Thus $M_{1,7} = 1$. Applying the Galois symmetry (2.5) for $\ell = 5, 7, 11$, we obtain the terms $|\chi_1 + \chi_7|^2 + |\chi_5 + \chi_{11}|^2$ in $\mathcal{E}_6$. Now use (2.4) to show that among the remaining entries of M, only the 4th and 8th rows and columns will be nonzero. $M_{J1,J1} = 1$ tells us $M_{44} = M_{88}$ and $M_{84} = M_{48}$. These must be equal, by evaluating $MS = SM$ at (4,2), and then either Perron-Frobenius or $MS = SM$ at $(1,4)$ forces that common value to be 1. We thus obtain $M = \mathcal{E}_6$.

3.2. The exceptional at $n = 18$

We know $M_{3,9} = 1$ and that $M_{3,a} = 0$ for all other $a \neq 9$. T-invariance (2.1) and Lemma 2 applied to the other odd $a < 9$, force $M_{aa} = 1$. The only remaining entry is $M_{9,9}$, which is fixed by $MS = SM$ at (9,1). We get $M = \mathcal{E}_7$.

3.3. The exceptional at $n = 30$

We know either $M_{1,11}$ or $M_{1,19}$ is nonzero; the only other (potentially) nonzero M_{1a} are at $a = 1, J1$. Suppose first that $M_{1,J1} = 1$, so $M_{1,11} = M_{1,19}$. Then (2.4) at $a = 3$ forces $M_{1,11} = 1$; Galois (2.5) for $\ell = 7, 11, 13, 17, 19, 23, 29$ gives us all the nonzero terms in $\mathcal{E}_8$, and (2.4) tells us all other entries of M must vanish.

If instead $M_{1,J1} = 0$, then (2.4) at $a = 3, 2, 4$ gives our contradiction.

4. Closing remarks

There are two reasons to be optimistic about the possibilities of a classification of all modular invariant partition functions (= physical invariants) for all simple X_r. One is the main general result in the problem [5], which gives the analogue of the $\mathcal{A}_\star$, $\mathcal{D}_\star$, and $\mathcal{E}_7$ physical invariants for any X_r. See [6] for a discussion. The other cause for optimism is the shortness and simplicity of the above proof for $A_1^{(1)}$.

The reader should be warned though that $A_1^{(1)}$ is an exceptionally simple case – the proof quickly reduces essentially to combinatorics. Our argument here is a projection of the general argument onto this special case, and this loses most of the structure present in the general proofs. The general arguments are necessarily more subtle and sophisticated. Nevertheless this paper should help the interested reader understand the further literature on this fascinating problem, and make the proof of the important classification of Cappelli-Itzykson-Zuber more accessible.

A big question is, does this new proof shed any light on the main mystery here: the A-D-E pattern to our Theorem? It does not appear to. But it should be remarked that it is entirely without foundation to argue that this $A_1^{(1)}$ classification is 'equivalent' to any other A-D-E one. There is a connection with the other A-D-E classifications which should be explained, and which has not yet been satisfactorily explained. But what we should look for is some critical combinatorial part of a proof which can be identified with critical parts in other A-D-E classifications. There has been some progress elsewhere at understanding this A-D-E. Nahm [10] constructed the invariant $\mathcal{X}_\ell$ in terms of the compact simply-connected Lie group of type X_ℓ, and in this way could interpret the $n = h$ and

$M_{m_i m_i} \neq 0$ coincidences. A very general explanation for A-D-E has been suggested by Ocneanu [11] using his theory of path algebras on graphs, but unfortunately it has never been published. Related to this is the work by Zuber and Petkova on fusion graphs (see e.g. [12]). Nevertheless, the A-D-E in CFT remains almost as mysterious now as it did a dozen years ago...

Incidently, there is a nice little curiousity contained within many modular invariants: another A-D-E! This A-D-E applies to *any* physical invariant (i.e. for any RCFT, not necessarily related to $A_1^{(1)}$) which looks like $\mathcal{Z} = |\chi_1 + \chi_{1'}|^2 + \text{stuff}$. The label $1'$ can be anything in P_+, and 'stuff' can be any sesquilinear combination of χ_i's, provided it doesn't contain χ_1 (the vacuum) or $\chi_{1'}$. In other words, the indecomposable submatrix of M containing $(1,1)$ is required to be $\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$, but otherwise M is unconstrained. Then to M we can associate several extended Dynkin diagrams of A-D-E type, as follows.

Put a node on the left of the page for each $a \in P_+$ whose row $M_{a\star}$ is nonzero, and put a node on the right of the page for each $b \in P_+$ whose column $M_{\star b}$ is nonzero. Connect a (on the left) and b (on the right) with precisely M_{ab} edges. The result will be a set of extended Dynkin diagrams of A-D-E type! (For these purposes we will identify two nodes connected with 2 lines as the extended A_1 diagram.)

For example, let's apply this to our $A_1^{(1)}$ classification. Any partition function $\mathcal{D}_{2\ell}$ is of this kind, and its corresponding graph will consist of $\ell - 1$ diagrams of (extended) A_2 type, and one of A_1 type. The exceptional $\mathcal{E}_6$ consists of three A_2's, and the exceptional $\mathcal{E}_7$ consists of three A_2's and one D_5. Again, this fact (proved in [5]) is not restricted to the $A_1^{(1)}$ physical invariants.

This little curiousity is not as deep or mysterious as the Cappelli-Itzykson-Zuber A-D-E pattern, and has to do with the $\mathbb{Z}_+$-matrices with largest eigenvalue 2.

There are 4 other claims for A-D-E classifications of families of RCFT physical invariants, and all of them inherit their (approximate) A-D-E pattern from the more fundamental $A_1^{(1)}$ one. The two rigourously established ones are the $c < 1$ minimal models, also proven in [3], and the $N = 1$ superconformal minimal models, proved in [2]. In both cases the physical invariants are parametrised by pairs of A-D-E diagrams. The list of known $c = 1$ RCFTs [8] also looks like A-D-E (two series parametrised by $\mathbb{Q}_+$, and three exceptionals), but the completeness of that list has never been successfully proved.

The fourth classification often quoted as A-D-E, is the $N = 2$ superconformal minimal models. The only rigourous classification of these is accomplished in [7], assuming the generally believed but still unproven coset realisation $(SU(2)_k \times U(1)_4)/U(1)_{2k+4}$. The connection here with A-D-E turns out to be rather weak: e.g. 20, 30, and 24 distinct invariants would have an equal right to be called $\mathcal{E}_6$, $\mathcal{E}_7$, and $\mathcal{E}_8$ respectively. It appears to this author that the frequent claims that the $N = 2$ minimal models fall into an A-D-E pattern are without serious foundation, or at least require major reinterpretation.

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