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Abstract

We consider control problems associated with nonlinear wave equations, in which the slope of the admissible trajectories can be made to approach infinity by choosing parameters in an appropriate form. Thus, the solutions near shock waves, and we try to control these.

A problem is first reformulated as one consisting of the minimization of an integral in a space of functions satisfying a set of integral equalities; this is then transfered to a nonstandard framework, in which Loeb measures take the place of the functions and a near-minimizer can always be found. This is mapped back to the standard world by means of the standard part map; its image is a minimizer, so that the optimization is *global*. The minimizer is shown to be the solution of an infinite dimensional linear program and by well-proven approximation procedures a finite dimensional linear program is found by means of which nearly-optimal curves can be constructed for the original problem. A numerical example is given.

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Key Words: Global Optimization, Control Theory, Discontinuous Solutions, Colombeau Algebras, Nonstandard Analysis, Loeb measures, Standard Part Map

1 Introduction

We have developed in many publications (see [Rubio, 1994, Rubio, 1986] and the references there) an approach to the study to the global optimization of nonlinear optimal control problems based on the consideration of measure spaces and related mathematical structures; this approach was suggested by the work of Young [Young, 1969] on the calculus of variations. In most of our previous work, all underlying sets—control sets being a special case—were taken to to be compact. In our recent work [Rubio, 1997] we have extended this approach, to variational problems which admit discontinuous solutions, and for which therefore the set of the slopes of the admissible curves is unbounded. A generalization to general finite dimensional control problems has been presented in [Rubio, 1998].

In this paper we deal with nonlinear wave equations and their solutions in the sense of Colombeau; see [Aragona and Biagioni, 1991], [Colombeau, 1984, Oberguggenberger, 1992b] and the references there. The solutions of these equations may exhibit discontinuities—shock waves—so that there is difficulty dealing with the derivatives of the main variables as well as with their products with the variables themselves; this is the main problem solved by Colombeau by having recourse to particular kinds of quotient algebras.

From the standpoint of the optimization problem, we need to consider unbounded sets for the slopes of the solutions, just as in our papers [Rubio, 1997, Rubio, 1998]. The actual discontinuities happen, if at all, at infinitesimal values $\lambda = [< \lambda_p >]$ of a parameter, λ, appearing in the equations chosen; we will consider the optimization problem for each of the values λ_p, and put the results together at the end. Thus the need to consider unbounded sets for the values taken by the slopes, the parameters λ_p tend to 0, and we will get closer and closer to the discontinuities, the shocks.

The problem still remains, how do we enlarge our spaces? It may appear natural to simply include 'delta functions', impulses; there are many ways of doing this, such as embedding the spaces into spaces of distributions, using nonstandard versions of this same construction, and so on. Alas, it is very difficult to work with impulses; in particular, it is very hard to define functions of impulses; see [Rubio, 1994], Chapter 6, for a discussion of this point. It this paper, our idealized elements lack the familiarity of impulses and such; they are *nonstandard elements*, not easily visualized but easily handled mathematically.

Thus, our path is as follows. The control problem will be written in a manner involving the solution of a set of integral equalities; these are mapped then into a nonstandard framework, in which the use of Loeb measures gives rise to an important result, that a near-minimizer for the nonstandard optimization problem always exists. The standard part map provided us with a *global minimizer* for the original problem, as well as with a measure-theoretical framework in the standard world in which a linear program is obtained with the minimizer as a solution.

Approximation tools developed in our previous work [Rubio, 1986] are then used to develop a finite dimensional approximation of the linear program, and construct nearly-optimal solutions of the variational problem. A numerical example is given.

2 The problem

Consider a system of partial differential equations:

$$\begin{aligned} -\lambda u_{xx} + u_t + u u_x &= F \\ -\lambda v_{xx} + v_t + u v_x &= G, \\ u(0,x) = u_0(x),\ v(0,x) &= v_0(x). \end{aligned} \tag{1}$$

where $u, v : [0, \infty) \times \mathbb{R} \to \mathbb{R}$, $u_0, v_0 : \mathbb{R} \to \mathbb{R}$ and $\lambda > 0$. The functions $F, G : (t, x) \to \mathbb{R}$ can be considered as *distributed control functions*; we shall assume that their support is in $J \times J$, with $J := [0, 1]$. Bounds will be put on some of these variables later on. Equations like these—without the control functions—have been studied much (see [Hopf, 1950, Biagioni and Oberguggenberger, 1992, J. J. Cauret and LeRoux, 1989]; in [Oberguggenberger, 1992a] we find a study of a (homogeneous) version of our equations. Shock waves may happen for *infinitesimal* λ, so that they will be approximated for small values of this parameter. Our program consists then in choosing an appropriate superstructure, not necessarily the same as in Section 3—$\aleph_0$-saturation would suffice here—and write

$$\lambda = [< \lambda_p >], \tag{2}$$

with $p \in \Pi$, an appropriate index set. We shall study below an optimization problem associated with (1) for standard, fixed values of λ, and we shall come back to our nonstandard setting later on, once this problem is solved.

Our interest now then is to study a control problem associated with (1) for small—as small as we wish—values of the *standard* parameter λ, so that it is convenient to consider as unbounded the sets in which the derivatives u_x, v_x take values. As we shall see below, the second derivative u_{xx} will play no role.

Our objective is to choose the control functions so as to improve the behaviour of the system, by for instance minimizing a performance criterion such as the the one defined in (3) below. However, it may happen—it most likely will happen—that no minimizer exists for this functional in the 'nice' class of control-trajectories quadruples $\mathcal{F}$ defined below. As indicated above, these equations may have solutions approximating shock waves arbitrarely;

they may need thus strange controls. According to our philosophy above, we shall start with the class of controls which are 'nice', well-behaved, as a starting point, a temporary device. Then this class will be enlarged, in a sense completed.

Let us consider then the class $\mathcal{F}$ of *admissible quadruples* (u, v, F, G), in which F, G are continuously differentiable and (u, v) are the corresponding *classical solutions* of (1). We assume that this class is nonempty and seek to minimize the functional $I : \mathcal{F} \to \mathbb{R}$

$$I(u, v, F, G) = \int_{J\times J} f_0(t, x, u, v, u_x, v_x, F, G)\, dt\, dx, \tag{3}$$

for $(u, v, F, G) \in \mathcal{F}$. Of course, we are looking for an optimal control function. Here f_0 is a continuous function defined on

$$\Omega := J \times J \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} = J \times J \times \mathbb{R}^6.$$

We will be interested in the solutions of (1) on the square $J \times J$; thus the supports of f_0 and F, G. Note we are putting no boundary conditions for u, v or their derivatives on the boundary of this set.

We develop some equalities. Let

$$D_0 := \{t = 0\} \times J,\ D_1 := \{x = 1\} \times J, ?_1 := J \times \{x = 0\} \cup J \times \{x = 1\},$$

Let $\mathcal{K}$ be the class of C_1-functions ϕ on $J \times J$ which are zero on D_1 and $?_1$. If we multiply pointwise the equations in (1) by ϕ and do some simple algebra, we obtain

$$\begin{aligned} &\int_{J\times J} [-u\phi_t + \lambda u_x \phi_x - \frac{1}{2}u^2\phi_x - F\phi]\, dtdx = \int_{D_0} u\phi\, dx \\ &\int_{J\times J} [-v\phi_t + +\lambda v_x\phi_x + uv_x\phi - G\phi]\, dtdx = \int_{D_0} v\phi\, dx, \forall \phi \in \mathcal{K}. \end{aligned} \tag{4}$$

It will be necessary to explicitely use integral relationships between u and u_x, v and v_x. Let $\mathcal{B}$ be the class of C_1 functions ψ on $J \times J$ which have zero normal derivative ψ_x on $?_1$. Then,

$$\begin{aligned} &\int_{J\times J} [-u\psi_{xx} + u_x\psi_x]\, dtdx = 0 \\ &\int_{J\times J} [-v\psi_{xx} + v_x\psi_x]\, dtdx = 0, \forall \psi \in \mathcal{B}. \end{aligned} \tag{5}$$

All integrals above can and will be considered as the action of an admissible quadruple (u, v, F, G) on a continuous function defined on Ω. It will be necessary to consider in the same light integrable functions $\xi : \Omega \to \mathbb{R}$ which depend only on (x, t). Then, trivially,

$$\int_{J\times J} \xi(t, x, u, v, u_x, v_x, F, G)\, dtdx = a_\xi, \tag{6}$$

with a_ξ the Lebesgue integral of ξ.

As explained in [Rubio, 1994, Rubio, 1997, Fakharzadeh, 1997] we shall take for each of these spaces countable sets of functions whose linear combinations are dense in the corresponding spaces in appropriate topologies, $\{\phi_i\}$ for $\mathcal{K}$, $\{\psi_j\}$ for $\mathcal{B}$, $\{\xi_h\}$ for the space associated with equation (6). Further, we shall consider a finite number of the resulting infinite number of equalities:

$$\begin{gathered}
\int_{J\times J}[-u\phi_{it} + \lambda u_x\phi_{ix} - \frac{1}{2}u^2\phi_{ix} - F\phi_i]\, dtdx = \int_{D_0} u\phi_i\, dx, i = 1, \ldots, M_1;\\
\int_{J\times J}[-v\phi_{jt} + \lambda v_x\phi_{jx} + uv_x\phi_j - G\phi_j]\, dtdx = \int_{D_0} v\phi_j\, dx, j = 1, \ldots, M_2;\\
\int_{J\times J}[-u\psi_{kxx} + u_x\psi_{kx}]\, dtdx = 0, k = 1, \ldots, M_3\\
\int_{J\times J}[-v\psi_{kxx} + v_x\psi_{kx}]\, dtdx = 0, k = 1, \ldots, M_3\\
\int_{J\times J}\xi_h(u, v, u_x, v_x, F, t, x)\, dtdx = a_{\xi_h}, h = 1, \ldots, M_4.
\end{gathered} \tag{7}$$

Eventually we shall take limits as $M_i \to \infty, i = 1, 2, 3, 4$.

3 The nonstandard way

We shall change our framework here—in a manner that appears minor. Let $\overline{\mathbb{R}}$ be the extended real line. Instead of assuming that the derivatives u_x, v_x corresponding to functions in admissible quadruples $(u, v, F, G) \in \mathcal{F}$ take values in $\mathbb{R}$, we shall take $\overline{\mathbb{R}}$ as a place of abode for these values. There will be no apparent change— the derivatives do take values in $\mathbb{R}$ and $\mathbb{R} \subset \overline{\mathbb{R}}$. But, as we shall see below, the introduction of $\overline{\mathbb{R}}$ is fundamental to our development.

In what follows we shall write equations (7) in a more economical way as in (9) below; we put $M := M_1 + M_2 + 2M_3 + M_4$. We consider therefore the problem of minimizing the functional

$$I(u, v, F, G) = \int_{J\times J} f_0(t, x, u(t, x), v(t, x), u_x(t, x), v_x(t, x), F(t, x), G(t, x))\, dt\, dx, \tag{8}$$

of the class $\mathcal{F}_M$ of quadruples (u, v, F, G) satisfying

$$\int_{J\times J} f_i(t, x, u(t, x), v(t, x), u_x(t, x), v_x(t, x), F(t, x), G(t, x))\, dtdx = b_i, i = 1, \dots, M, \tag{9}$$

where $u(\cdot), v(\cdot) \in C_1(J)$ and take values in closed bounded sets $A, B \subset \mathbb{R}$ respectively, and F, G take values in a closed bounded set $U \subset \mathbb{R}$. Here $f_0, f_i, i = 1, \dots, M$, are in $C(\Omega')$, with

$$\Omega' := J \times J \times A \times B \times \overline{\mathbb{R}} \times \overline{\mathbb{R}} \times U \times U.$$

The integer $M \geq 1$ is fixed, and so are the constant $b_i, i = 1, \dots, M$. We assume that the class $\mathcal{F}_M$ is nonempty. We shall develop in this section a procedure to enlarge the set $\mathcal{F}_M$, while at the same time extending the functional (8) to the whole of the new, larger set of admissible elements. This procedure will be based on nonstandard techniques.

In our quest for infinities, we shall start with the extended real line $\overline{\mathbb{R}}$. This will be part of our starting nonstandard construction, while also playing a major role when we return to the standard world. We will review briefly some of its properties; see [Berge, 1963, Monroe, 1953, Choquet, 1969].

- The extended real line $\overline{\mathbb{R}}$ is obtained by adding to the real line $\mathbb{R}$ two elements, ∞ and $-\infty$, so that $\overline{\mathbb{R}} := \mathbb{R} \cup \{\infty, -\infty\}$. These two elements satisfy a number of well-known postulates, such as
  - For every $x \in \mathbb{R}$, $-\infty < x < \infty$. This makes the extended real line into an ordered set.
  - The extended system will not be a field, but we can connect the new elements with the field operations by postulating that for every real number x we have:

    $$x/\pm\infty = 0;\ (\pm\infty)(\pm\infty) = \infty;\ \infty + \infty + x = \infty,$$

    etc.

- It is possible to put a topology on $\overline{\mathbb{R}}$ so that it is a *compact space*. Such a topology is generated by the following sets:
  - The open sets in $\mathbb{R}$.
  - The union of $\{\infty\}$ with an open set of $\mathbb{R}$ containing an interval of the form (λ, ∞).
  - The union of $\{-\infty\}$ with an open set of $\mathbb{R}$ containing an interval of the form $(-\infty, \lambda)$.

We proceed now with our nonstandard construction. For general treatments of this topic, see [Cutland, 1988, Rubio, 1994]. We will work in a nonstandard framework given by a superstructure $V(W)$, $\overline{\mathbb{R}} \subset W$. The superstructure $V({}^*V)$ is also an enlargement, and $\aleph_1$-saturated. We study integrals of the form (8,9), that is,

$$\int_{J\times J} f(t,x,u(t,x),v(t,x),u_x(t,x),v_x(t,x),F(t,x),G(t,x))\,dtdx, \qquad (10)$$

with $(u,v,F,G) \in \mathcal{F}_M$ and $f \in C(\Omega')$. Then,

$$(\forall (u,v,F,G) \in \mathcal{F}_M)$$
$$(\int_{J\times J} f(t,x,u(t,x),v(t,x),u_x(t,x),v_x(t,x),F(t,x),G(t,x))\,dtdx \in \overline{\mathbb{R}}, \qquad (11)$$

by transfer,

$$(\forall (u,v,F,G) \in^* \mathcal{F}_M)$$
$$({}^*\int_{{}^*J\times{}^*J} {}^*f(t,x,u(t,x),v(t,x),u_x(t,x),v_x(t,x),F(t,x),G(t,x))\,dtdx \in {}^*\overline{\mathbb{R}}, \qquad (12)$$

where here and below we write $u_x(\cdot)$ for $({}^*\frac{\partial}{\partial x}u(\cdot)$, and similarly for $v(\cdot)$. Thus, the nonstandard version of the optimization problem (8,9) consists in minimizing

$${}^*I(u,v,F,G)$$
$$=^* \int_{{}^*J\times{}^*J} {}^*f_0(t,x,u(t,x),v(t,x),u_x(t,x),v_x(t,x),F(t,x),G(t,x))\,dt\,dx, \qquad (13)$$

on the class ${}^*\mathcal{F}_M$ of quadruples (u, v, F, G) satisfying

$$
\begin{aligned}
&{}^*\int_{^*J\times {}^*J} {}^*f_i(t,x,u(t,x),v(t,x),u_x(t,x),v_x(t,x),F(t,x),G(t,x))\,dtdx \\
&= b_i, i = 1,\ldots,M,
\end{aligned}
\tag{14}
$$

For instance,

$$
{}^*f_i(t,x,u,v,F,G) := -u^*\phi_{it} + {}^*\lambda u_x + -(1/2)u_2{}^*\phi_{ix} - F^*\phi, i = 1,\ldots,M_1.
$$

Note that the standard λ becomes of course ${}^*\lambda$.

Consider now the map suggested by (11). If $(u, v, F, G) \in \mathcal{F}_M$ is fixed, the map

$$
\begin{aligned}
&\nu_{(u,v,F,G)} : f \to \\
&\int_J f(t,x,u(t,x),v(t,x),u_x(t,x),v_x(t,x),F(t,x),G(t,x))\,dxdt \in \overline{\mathbb{R}}, f \in C(\Omega')
\end{aligned}
\tag{15}
$$

is linear and positive. By Riesz' Theorem, there is a measure, to be denoted also by $\nu_{(u,v,F,G)}$, on the Borel sets $\mathcal{B}$ of Ω', that represents this map; remember that Ω' is compact. Then $({}^*\Omega', {}^*\mathcal{B}, {}^*\nu_{(u,v,F,G)})$ is a nonstandard measure space and then [Render, 1993],

Lemma 1 *There is a measure space* $({}^*\Omega', \mathcal{A}, \mu_L^{(u,v,F,G)})$ *so that* $\mu_L^{(u,v,F,G)}$ *is the Loeb measure associated with* $\nu_{(u,v,F,G)}$*; then,*

$$
\begin{aligned}
&{}^*\int_{^*J} f(t,x,u(t,x),v(t,x),u_x(t,x),v_x(t,x),F(t,x),G(t,x))\,dtdx = \\
&\mu_L^{(u,v,F,G)}(f) := \int_{^*\Omega'} f\,d\,\mu_L^{(u,v,F,G)}, f \in C({}^*\Omega').
\end{aligned}
\tag{16}
$$

The algebra $\mathcal{A}$ *is an extension of the algebra* ${}^*\mathcal{B}$.

Proof Follows directly from the reference given above. •

Thus, one can write the optimization problem (13,14) as the problem of minimizing

$$
J(\mu_L^{(u,v,F,G)}) := \mu_L^{(u,v,F,G)}({}^*f_0), \tag{17}
$$

over the set $\mathcal{M}_M^L$ of measures of the form $\mu_L^{(u,v,F,G)}$ defined by

$$
\mu_L^{(u,v,F,G)}({}^*f_i) = b_i, i = 1,\ldots,M. \tag{18}
$$

The following two propositions show that the solution of our problem is a *global optimizer.*

Proposition 1 *(i) The infima associated with the problems* $(13-14)$ *and* $(17-18)$ *are equal.*

(ii) For any positive infinitesimal $s \in {}^*\overline{\mathbb{R}}$*, we can find a* near-minimizer $\mu_s \in \mathcal{M}_M^L$ *for the functional* J *in (17) in the set* $\mathcal{M}_M^L$*, so that*

$$J(\mu_s) = \inf_{\mathcal{M}_M^L} J + s. \tag{19}$$

Proof It follows from Theorem 3.8 in [Rubio, 1994]. •

Let, then, s be a fixed positive infinitesimal in ${}^*\overline{\mathbb{R}}$, and μ_s the corresponding near-minimizer for J on $\mathcal{M}_M^L$. We can proceed to map back this measure to the standard world, by means of the *standard part map*, see [Henson, 1979, Aldaz, 1992, Render, 1993, Landers and Rogge, 1987].

Proposition 2 *There is a Baire measure* μ_{opt} *on* Ω' *so that:*

(i) If S *is a Baire set in* Ω'*,*

$$\mu_{opt}(S) = {}^\circ\mu_s(\mathrm{st}_{\Omega'}^{-1}(S)),$$

where $\mathrm{st}_{\Omega'}^{-1}(S)$ *is the union of the monads of the elements of* S*.*

(ii)

$$\mu_{opt}(f_0) := \int_{\Omega'} f_0 d\,\mu_{opt} \leq \inf_{\mathcal{F}_M} \int_J f_0(t, x, u, v, u_x, v_x, F, G)\, dt dx.$$

(iii) The measure μ_{opt} *is a solution of the following optimization problem. Minimize*

$$\mu(f_0) \tag{20}$$

over the set $\mathcal{M}_M^+(\Omega')$ *of positive Baire measures on* Ω' *satisfying*

$$\mu(f_i) = b_i, i = 1, \ldots, M. \tag{21}$$

(iv) If the support of μ_{opt} *contains subsets of* Ω' *in which at least one of the variables* $u_x, v_x \in \overline{\mathbb{R}}$ *is either* $-\infty$ *or* ∞*, the measure* μ_{opt} *is defined by a Baire measure on*

$$J \times J \times A \times B \times \mathbb{R} \times \mathbb{R} \times U \times U$$

plus atomic measures on those subsets.

Proof (i) See [Henson, 1979]. (ii), (iii). These statements follow from Proposition 1 and the fact that for all $f \in C(\Omega')$

$$\int_{\Omega'} f d\,\mu_{opt} = \int_{{}^*\Omega'} {}^\circ({}^*f) d\,\mu_s = {}^\circ \int_{{}^*\Omega'} {}^*f d\,\mu_s;$$

note that by continuity

$${}^\circ({}^*f(w)) = f(\mathrm{st}_{\Omega'}(w)) = f(y_w), w \in \Omega,'$$

where y_w is the (unique) element of Ω' so that w is in the monad of y.

(iv) We consider now the support of μ_{opt}. Consider a point $(t, x, u, v, u_x, \infty, F, G) \in S$, S being a Baire set in Ω'. Then

$$\mathrm{st}_{\Omega'}^{-1}(t, x, u, v, u_x, \infty, F, G) = M \times \bigcap_\lambda {}^*(\lambda, \infty] = M \times {}^*\{\infty\},$$

with M the monad of (t, x, u, v, u_x, F, G). Then, for $f \in C(\Omega')$, for some hyperreal α, there will be a contribution to the integral

$$\int_{{}^*\Omega'} {}^\circ({}^*f) d\,\mu_s$$

of

$${}^\circ[< \alpha_i f(t, x, u, v, u_x, \infty, F, G) >] = ({}^\circ\alpha) f(t, x, u, v, u_x, \infty, F, G),$$

which proves our contention; other cases, including u_x and maybe the element $-\infty$, can be treated similarly. •

Note that the infimum for the problem (20-21) can be strictly less than the classical infimum, as indicated in (ii) above; this is discussed in detail in [Rubio, 1994], Chapter 4-5.

In fact, values $\pm\infty$ do not actually happen for λ standard. Even so, in problems of interest, in which the function f_0 tends to infinity at infinity, and in which the infimum is finite, elements in Ω' with value ∞ or $-\infty$ do not really occur anyhow in the support of μ_{opt}; note that expressions such as $\infty - \infty$ are not defined for the extended real line. Thus,

Proposition 3 *Suppose that*

$$|f_0(t, x, u, v, u_x, v_x, F, G)| = \infty$$

whenever u_x or v_x are either ∞ or $-\infty$, and that the minimum associated with the linear program (20)-(21) is finite. Then such elements are not present in the support of μ_{opt}.

We are now in a strong position to solve our original problem— the optimization problem (8) and (9) in the standard world. Note that we have been able to construct an extension of the original space $\mathcal{F}_M$, made up of elements which are not quadruples; however, the action of μ_{opt}—a global optimizer—can be approximated by members of $\mathcal{F}_M$.

4 Existence and Approximation

From the results of the optimization problem (20-21)—which we can take as having been obtained for instance by Rudolph's method [Rudolph, 1987, Rudolph, 1990] we can build a near-optimal control pair (F, G); these are piecewise continuous functions on $\mathbb{R}^2$ with support in $J \times J$. We consider then the equations in $G_{s,g}(\mathbb{R} \times [0, \infty))$:

$$\begin{aligned} \lambda U_{xx} + U_t + UU_x &= \hat{F} \\ \lambda V_{xx} + V_t + UV_x &= \hat{G} \\ U(0, \cdot) = U_0, V(), \cdot) &= V_0. \end{aligned} \tag{22}$$

Here $U_0, V_0 \in G_{s,g}(\mathbb{R}), \hat{F}, \hat{G} \in G_{s,g}(\mathbb{R})$; of course, $\hat{F}, \hat{G}, U_0, V_0$ are sequences of smooth functions obtained by mollifying F, G, u_0, v_0 respectively.

It is not difficult to prove the existence of the solutions of (22)in $G_{s,g}(\mathbb{R} \times [0, \infty))$; see [Oberguggenberger, 1992b, Oberguggenberger, 1992a], [Biagioni and Oberguggenberger, 1992]. Then

Proposition 4 *Suppose that the minimum in the linear program (20)-(21) is finite, that the conditions of Proposition 3 are satisfied, and that the function f_0 is Lipschitz, that is, that there is a constant h so that*

$$\begin{aligned} &|f_0(t', x', u', v', z', r', F', G') - f_0(t, x, u, v, z, r, F, G)| \\ &\leq h(|t' - t| + |x' - x|| + |u' - u| + \\ &|v' - v| + |z' - z| + |r' - r| + |F' - F| + |G' - G|) \end{aligned} \tag{23}$$

for all $(t', x', u', v', z', r', F', G'), (t, x, u, v, z, r, F, G)$ in Ω. Then it is possible to construct a quadruple in $\mathcal{F}_M$ so that as $M_1, M_2, M_3, M_4 \to \infty$ the corresponding value of the performance index tends to

$$\inf_S \mu(f_0),$$

with $S := \cap_{(M_1,M_2,M_3,M_4 \in \mathbb{N})} S(M_1, M_2, M_3, M_4)$.

Proof (i) Let us fix M_1, M_2, M_3, M_4, and write

$$\zeta := 1/\max(M_1, M_2, M_3).$$

Let μ_{opt} be the minimizer for (20) over the set $S(M_1, M_2, M_3, M_4)$ defined by (21). Then, provided M_4 is sufficiently large, we can find piecewise continuous functions forming a quadruple $q := (u, v, F, G)$ so that

$$\begin{aligned} |\mu_q(f_0) - \mu_{opt}(f_0)| \leq \zeta, \\ |\mu_q(f_i) - b_i| \leq \zeta, 1 = 1, \dots, M. \end{aligned} \tag{24}$$

(ii) Take a solution of (1) in the algebra $G_{s,g}(\mathbb{R}\times[0,\infty))$as discussed above, corresponding to the piecewise-continuous controls (F, G). Then for any $\epsilon > 0$ we have an *admissible* quadruple $q_\epsilon = (u_\epsilon, v_e, F_\epsilon, G_e)$ associated with the solution in the algebra $G_{s,g}(\mathbb{R}\times[0,\infty))$; of course, F_ϵ, G_e are the mollified functions F, G; the initial solutions are also mollified, $u_{0\epsilon}, v_{0\epsilon}$. Then, for $i = 1, \dots, M_1$,

$$\mu_{q_\epsilon}(f_i) = \int_{D_0} u_{0\epsilon}(x)\phi_i(x)\, dx,$$

so that

$$|\mu_{opt} - \mu_{q_\epsilon})f_i| \leq |\int_{D_0} (u_{0\epsilon}(x) - u_0(x))\phi_i(x)\, dx| \leq C\epsilon,$$

with $C > 0$ a constant. This inequality is in fact true for all $i = 1, \dots, \hat{M}$, with $\hat{M} := M_1 + M_2 + 2M_3$. Thus

$$|(\mu_\epsilon - \mu_{q_\epsilon})f_i| \leq \zeta + C\epsilon, i = 1, \dots, \hat{M}. \tag{25}$$

(iii) Finally, we consider the approximation of the performance criterion. Since

$$|\mu_{q_\epsilon}(f_0) - \mu_{opt}(f_0)| \leq |\mu_{q_\epsilon}(f_0) - \mu_{(u,v,F,G)}(f_0)| + |\mu_{(u,v,F,G)}(f_0) - \mu_{opt}(f_0)|,$$

and since the inequality

$$|\mu_{q_\epsilon}(f_0) - \mu_{(u,v,F,G)}(f_0)| \leq h|u + v + u_x + v_x + F + G| \leq C_1\epsilon,$$

$C_1 > 0$, can be proved by means much like those used in proving a similar inequality in Theorem 1 in [Rubio, 1995], we have that

$$|\mu_{q_\epsilon}(f_0) - \mu_{opt}(f_0)| \leq C_1\epsilon + \zeta,$$

from which our contention follows. •

5 Further Approximation

We consider again the optimization problem (20)-(21). By means of a result of Rosenbloom in [Rosenbloom, 1952], and since Ω' is compact, we can state that the minimizer μ_{opt} for this problem has the form

$$\mu_{opt} = \sum_{\ell=1}^{M} \alpha_\ell \delta(w_\ell), \alpha_\ell \geq 0, w_\ell \in \Omega', \ell = 1, \ldots, M, \tag{26}$$

where $\delta(w)$ is the atomic measure with support $\{w\} \in \Omega'$. Thus, we wish to minimize

$$\sum_{\ell=1}^{M} \alpha_\ell f_0(w_\ell), \tag{27}$$

on the set defined by the elements

$$\alpha_\ell \geq 0, w_\ell \in \Omega', \ell = 1, \ldots, M,$$

which satisfy, further,

$$\sum_{\ell=1}^{M} \alpha_\ell f_i(w_\ell) = b_i, i = 1, \ldots, M, \tag{28}$$

A further concept must be introduced now; see [Rubio, 1986] . Note that we have in (27)-(28) a *nonlinear* optimization problem, in which the unknowns are the coefficients α_ℓ and supports $w_\ell, \ell = 1, \ldots, M$. In order to find a linear approximation to this problem, we consider ω, a countable dense subset of Ω'. Taking $N >> M$ elements from ω, including all elements of the form introduced in (iv) above in which some variables take values either $-\infty$ or ∞, we can write (27)-(28) as follows. We wish to minimize

$$\sum_{\ell=1}^{N} \alpha_\ell f_0(w_\ell), \tag{29}$$

on the set defined by the elements $\alpha_\ell \geq 0, \ell = 1, \ldots, M$, which satisfy, further,

$$\sum_{\ell=1}^{N} \alpha_\ell f_i(w_\ell) = b_i, i = 1, \ldots, M. \tag{30}$$

Here, then, the supports w_ℓ are fixed, in ω; the coefficients $\alpha_\ell, \ell = 1, \dots, M$, are the only unknowns; this is an $M \times N$ (finite dimensional) linear program. Of course as $N \to \infty$ the support of the optimal measure μ_{opt} in (27)-(28) can be approximated closer and closer by that of μ^N_{opt}, the solution of (29)-(30). Note, further, that at most M of the unknown α's are nonzero; we shall assume that the problem has essential regularity, and that exactly M of these α's are nonzero; see [Rubio, 1986], Chapters 3 and 4, for a discussion of this point.

Since no element in the support of μ^N_{opt} has values equal to either $-\infty$ or ∞, the approximation process has been studied in detail in [Rubio, 1986], and a quadruple in $\mathcal{F}_M$ can be constructed approximating the action of μ_{opt} on f_0. It is necessary to modify the set ω into a set ω_Q, in which the coordinates of the u_x, v_x directions take values in a portion of the dense set ω defined by a number Q; if Q is large enough, all the elements in the support of μ_{opt} will be approximated adequately. Then,

Proposition 5 *Suppose that the minimum in the linear program (20)-(21) is finite, that the conditions of Proposition 3 are satisfied, and that the function f_0 is Lipschitz. Then it is possible to construct suboptimal admissible control pairs (F, G) so that:*

(i) As $Q \to \infty$, the corresponding values of the perormance criterion tend to $\mu^N_{opt}(f_0)$.

(ii)As $N \to \infty$,

$$\mu^N_{opt}(f_0) \to \mu_{opt}(f_0).$$

The proof is much as that of Proposition 4 in [Rubio, 1997]. The actual construction is explained in detail in [Fakharzadeh, 1997] and [M. H. Farahi and Wilson, 1996]; see also the next Section.

6 An Example

We have carried out the numerical computations associated with the estimation of a nearly-optimal control for the system (1) introduced in the last two Sections. We took $F \equiv 0$, $v_0 \equiv 0$, while $u_0(x) = 0.5, x \in (0.5, 1]$, zero otherwise. The variables u, v, G are constrained to take values in the same set, $[0, 10]$; we took $\lambda = 0.01$ and aimed to approximate the high values of u_x, v_x by making their domain the set $[0, 1000]$, that is $Q = 1000$.

Our aim was to control the extent of the shocks in the variable v, so we took

$$f_0(t, x, u, v, u_x, v_x, F, G) := v^2 + v_x^2.$$

We set a total of 35 equations ($M_1 = M_2 = 6, M_3 = 3, M_4 = 17$). The domains corresponding to the variables (u, v, F, v_x, u_x) were divided into 4 subintervals, those corresponding to (x, y) into 5, so that we had a total of 36864 variables. The suboptimal control is shown in Figure 1, constructed by means explained in detail in [Fakharzadeh, 1997], [Farahi, 1996]; the variable v_x in Figure 2, constructed by the same means.

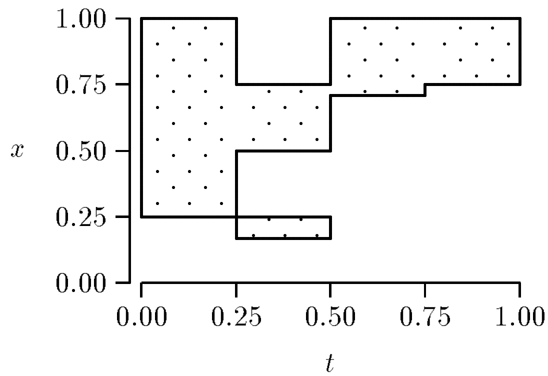

FIG. 1. *Graph of the suboptimal control, taking a value 10 at the shaded area of the (t, x)-plane, zero otherwise.*

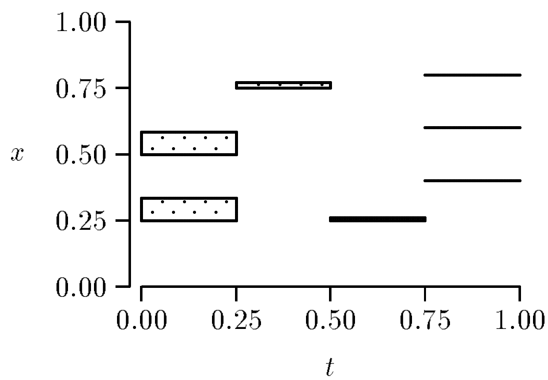

FIG. 2. *Graph of the variable v_x, taking a value 1000 at the shaded area of the (t, x)-plane, zero otherwise.*

Several points can be made concerning these results:

- In both figures we show the supports of the functions F and v_x in the (t, x)-plane, taking values 10 and 1000 respectively.
- Figure 2 shows therefore the regions in this plane where the function

v_x has large positive values, a precursor of the shocks most likely to be exhibited when λ is taken as an infinitesimal, as explained in the next section.

- Note then the action of the control in minimizing these areas: as times goes by, from 0 to 1, these 'shocks' get weaker and weaker, the initial ones caused by the initial conditions disappearing at the end into very weak regions; the last three 'shocks' are not to scale, their width being about 1/100 the width of the preceding one.
- These results are tentative; a thorough numerical investigation is needed, with much larger matrices and many values of λ; also, the actual numerical solution of (1) corresponding to the control F should be obtained for each value of λ.

7 A final step

Finally, we go back to our original nonstandard setting, associated with equation (2). If we solve the optimization problem for each value $\lambda_p, p \in \Pi$, obtaining for instance each time a suboptimal admissible control pair $(F, G)_p$, we can say that the nonstandard object

$$[< (F, g)_p >]$$

is a suboptimal pair for the infinitesimal λ defined in (2). Presumably, as explained above, the 'shocks' of the example would become actual shocks, with infinite slopes.

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