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ISOTROPY REPRESENTATION OF FLAG MANIFOLDS

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Abstract. Flag manifolds of a classical compact Lie group G considered up to a diffeomorphism are described in terms of painted Dynkin diagrams. The explicit decomposition of the isotropy representation into irreducible components is given.

1. Flag manifolds and painted Dynkin graphs

We describe flag manifold of a compact semisimple Lie group G up to some equivalence relation in terms of painted Dynkin graph .

Definition.

(1) *Flag manifold of a compact semisimple Lie group G is a quotient $M = G/K$ of G over a subgroup K which is the centralizer of an one-parametric subgroup* $\exp th$ *of G, or, in other words, the homogeneous manifold G/K which is G-diffeomorphic to the adjoint orbit* $\mathrm{Ad}\,Gh$ *of an element h of the Lie algebra* $\mathfrak{g} = \mathrm{Lie}\,G$.

(2) *Two flag manifold $M = G/K$, $M' = G'/K'$ are called to be equivalent if there exists an automorphism ϕ of G such that $\phi(K) = K'$.*

Remark. The automorphism ϕ induces a diffeomorphism

$$\bar{\phi} : M \longrightarrow M',\; gK \mapsto \phi(g)K'.$$

However, if ϕ is an outer automorphism, then $\bar{\phi}$ is not a G-equivariant diffeomorphism, but only $Aut(G)$-equivariant diffeomorphism, where $Aut(G)$ is the group of all automorphisms of G.

Let $\mathfrak{g} = (\mathrm{Lie}\,G)^{\mathbb{C}}$ be the complex Lie algebra associated with a compact semisimple Lie group G and $\mathfrak{h}$ its Cartan subalgebra. We have the following root decomposition of $\mathfrak{g}$

$$\mathfrak{g} = \mathfrak{h} + \sum_{\alpha \in R} \mathbb{C}E_\alpha$$

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where $R \subset \mathfrak{h}^*$ is the root system and E_α is the root vector of a root α with the standard normalization , such that $B(E_\alpha, E_\alpha) = 2/B^{-1}(\alpha,\alpha)$, where B is the Killing form. Denote by $\mathfrak{h}(\mathbb{R}) = B^{-1}\,\mathrm{span}_{\mathbb{R}}\, R$ the real form of the Cartan subalgebra and by τ the canonical antiinvolution

$$\tau|\mathfrak{h} = -1, \qquad \tau E_\alpha = E_{-\alpha}.$$

Then

$$\mathfrak{g}^\tau = \mathrm{Lie}\, G = i\mathfrak{h}(\mathbb{R}) + \sum \mathbb{R}(E'_\alpha, E''_\alpha)$$

where $E'_\alpha = E_\alpha + E_{-\alpha}, \quad E''_\alpha = i(E_\alpha - E_{-\alpha})$ is the compact real form of $\mathfrak{g}$.

Any adjoint orbit $\mathrm{Ad}\, Gh$ has a non trivial intersection with $\mathfrak{h}^\tau = i\mathfrak{h}(\mathbb{R})$, since any element h of the compact Lie algebra $\mathfrak{g}^\tau$ belongs to some Cartan subalgebra and all Cartan subalgebras are conjugated. Hence, without loss of generality, we may assume that the stabilizer K of a flag manifold $M = G/K$ is the centralizer of an element $t \in \mathfrak{h}^\tau$. The corresponding complex Lie algebra $\mathfrak{k} = (\mathrm{Lie}\, K)^{\mathbb{C}}$has the root decomposition

$$\mathfrak{k} = \mathfrak{h} + \sum_{\alpha \in R_K} \mathbb{C}E_\alpha = \mathfrak{z} \oplus \mathfrak{g}(R_K) = \mathfrak{z} \oplus \mathfrak{k}'$$

where $R_K = \{\alpha \in R,\ \alpha(t) = 0\} \subset R$ is the root system of $\mathfrak{k}$, $\mathfrak{z}$ is the center of $\mathfrak{k}$ and $\mathfrak{g}(R_K)$ is the semisimple subalgebra of $\mathfrak{g}$ generated by the root vectors $E_\alpha, \alpha \in R_K$.

Now we fix a basis Π_K of the root system R_K. Denote by Π its extension to a basis of R. (It is always exist , but is not unique, in general). We associate with the pair (Π, Π_K) a painted Dynkin graph Γ as follows. It is the Dynkin graph associated with the basis Π whose vertices corresponding to Π_K are painted in black. We call it a graph and not a diagram, because we do not assume that the correspondence between vertices and simple roots is fixed. If such correspondence is fixed, we say that we have an equipment of painted Dynkin graph (by simple roots).

Definition. *We say that a painted Dynkin graph has type* $(\mathfrak{g}, \mathfrak{k})$ *if the underlining graph is the Dynkin graph of a semisimple Lie algebra* $\mathfrak{g}$ *and deleting the black vertices we get the Dynkin graph of the semisimple Lie algebra* $\mathfrak{k}$*.*

Remark. It the Lie algebra $\mathfrak{g}$ has roots of different length , and $\mathfrak{k}$ has a simple summand of the type A_k , one has to indicate also wheather it corresponds to long roots (then we write A^l_k) or short roots ($A_k = A^{sh}_k$).

We associate with a painted Dynkin graph Γ of type $(\mathfrak{g}, \mathfrak{k}')$ a flag manifold $F_\Gamma = G/K$ as follows. Choose the standard equipment of Γ by simple roots $\Pi = \{\alpha_1, \dots, \alpha_l\}$ of the Lie algebra $\mathfrak{g}$ and denote by Π_W, resp., $\Pi_B = \Pi \setminus \Pi_W$ the subset of Π which corresponds to white , reps., black, verticies. The set Π_W generate a root subsystem

$$R_K = [\Pi_W] = \mathrm{span}_{\mathbb{Z}}\, \Pi_W \cap R$$

of the root system R of $\mathfrak{g}$. Denote by $\bar{\alpha}_i$ the fundamental weight associated to a simple root $\alpha_i \in \Pi$, such that $(\bar{\alpha}_i, \alpha_j) = \delta_{ij}$.

Then

$$\mathfrak{k} = \sum_{\beta \in \Pi_B} \mathbb{C} B^{-1} \bar{\beta} + \mathfrak{g}([\Pi_W])$$

is a subalgebra of $\mathfrak{g}$ which is the centralizer of the element $t = \sum_{\beta \in \Pi_B} B^{-1} \bar{\beta}$. The flag manifold associated with Γ is defined as the flag manifold $F_\Gamma = G^\tau / K^\tau = \operatorname{Ad} G^\tau t$, where G^τ is the adjoint group of the compact Lie algebra $\mathfrak{g}^\tau$ and K^τ is the connected subgroup , generating by the compact subalgebra $\mathfrak{k}^\tau$.

Definition. *We say that a painted Dynkin graph of the type $(D_l, \mathfrak{k})$ has class B (resp., WWB, or WWW) if at least one of the two end right vertices is black (resp., both these vertices are white , but connected with them vertex is black, or all these last three vertices are white).*

Theorem.

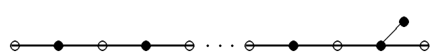
(1) *Any flag manifold $F = G/K$ of a compact semisimple Lie group G is equivalent to the flag manifold F_Γ associated to a painted Dynkin graph Γ of type $(\mathfrak{g}, \mathfrak{k}')$ where $\mathfrak{g} = \operatorname{Lie}(G)$ and $\mathfrak{k}'$ is the semisimple part of the reductive Lie algebra $\mathfrak{k} = \operatorname{Lie}(K)$.*

(2) *Two connected painted Dynkin graph Γ, Γ' define equivalent flag manifolds if and only if they have the same type $(\mathfrak{g}, \mathfrak{k})$ and , in the case $\mathfrak{g} = D_l$, the same class.*

Proof. This is an other version of the theorem from [Ale].

1.1. The standard painted Dynkin graph associated with a flag manifold of a classical group. To get 1-1 correspondence between equivalent classes of flag manifolds and painted Dynkin graphs, we choose a canonical representative of the set of painted Dynkin graphs of given type and class for classical Lie algebras $\mathfrak{g} = A_l, B_l, C_l, D_l$. We will consider painted Dynkin graph as oriented graph with the standard orientation for B_l, C_l, D_l.

Definition. *An oriented painted Dynkin graph Γ of a type $(\mathfrak{g}, \mathfrak{k})$, where $\mathfrak{g} = A_l, B_l, C_l, D_l$, different from the graph of the type (D_{2m}, mA_1) , class B, is called standard if all black vertices are isolated , with the exception , may be, one chain of black verticies, starting from the left vertex , and the length of white chains forms a non increasing sequence. The standard painted Dynkin graph of type (D_{2m}, mA_1) and class B is defined as*



Using simple combinatoric consideration of painted Dynkin graphs , we derive from the theorem the following corollary.

Corollary. *There exist a natural bijection between flag manifolds G/K of a classical compact Lie group G (up to equivalency) and standard painted Dynkin graphs of the type $(\mathfrak{g} = \operatorname{Lie} G, \mathfrak{k}')$ where $\mathfrak{k} = \operatorname{Lie} K = \mathfrak{z} \oplus \mathfrak{k}'$.*

2. T-roots and decomposition of the isotropy representation into irreducible components

Let

$$\mathfrak{g} = \mathfrak{h} + \sum_{\alpha \in R} \mathbb{C} E_\alpha$$

be the root decomposition of a complex semisimple Lie algebra $\mathfrak{g}$ and

$$\mathfrak{k} = \mathfrak{h} + \sum_{\alpha \in R_K} \mathbb{C} E_\alpha = \mathfrak{z} \oplus \mathfrak{g}(R_K) = \mathfrak{z} \oplus \mathfrak{k}'$$

the corresponding decomposition of the subalgebra $\mathfrak{k}$ which is the centralizer of an element $t \in \mathfrak{h}^\tau$, where $R_K = \{\alpha \in R,\, \alpha(t) = 0\} \subset R$ is the root system of $\mathfrak{k}$, $\mathfrak{z}$ is the centre of $\mathfrak{k}$ and $\mathfrak{k}' = \mathfrak{g}(R_K)$ is the semisimple part of $\mathfrak{k}$.

Denote by $\mathfrak{t} = \mathfrak{z}(\mathbb{R}) = \mathfrak{h}(\mathbb{R}) \cap \mathfrak{z}$ the real form of the center $\mathfrak{z}$.

Definition. *The restriction $\kappa : \alpha \mapsto \tilde{\alpha} = \alpha|\mathfrak{t}$ of a root $\alpha \in R_M = R \setminus R_K$ to the subspace $\mathfrak{t} \subset \mathfrak{h}(\mathbb{R})$ is called a T-root .*

We denote the set of T-roots by R_T. We fix a basis Π_K of the root system R_K and denote by Π its extension to a basis of R and by R_K^+, R^+ the corresponding systems of positive roots.

Definition. *A root $\alpha \in R' = R \setminus R_K$ is called K-simple, if $\alpha - \phi \notin R$ for any $\phi \in R_K^+$.*

Let $\mathfrak{g}^\tau = \mathfrak{k}^\tau + \mathfrak{m}^\tau$ be the reductive decomposition of $\mathfrak{g}^\tau$, where $\mathfrak{m}^\tau$ is the B-orthogonal complement to $\mathfrak{k}^\tau$ in $\mathfrak{g}^\tau$ and $\mathfrak{g} = \mathfrak{k} + \mathfrak{m}$ its complexification.

We identify $\mathfrak{m}^\tau$ with the tangent space of the corresponding flag manifold $F = G^\tau/K^\tau$ at the point $o = eK^\tau$ and $\mathfrak{m}$ with its complexification. Then the isotropy representation of the stabilizer K^τ and its Lie algebra $\mathfrak{k}^\tau$ is identified with the restriction of the adjoint representation to $\mathfrak{m}^\tau$ and the complexification of the isotropy representation of $\mathfrak{k}^\tau$ is identified with the restriction of the adjoint representation to $\mathfrak{m}$. We will consider $\mathfrak{m}$ as complex $\mathfrak{k}$-module.

Proposition.

(1) *There exist natural one-to-one correspondence between T-roots, irreducible $\mathfrak{k}$-submodules of the complexified tangent space $\mathfrak{m} = \sum_{\alpha \in R_M} \mathbb{C}E_\alpha = (\mathfrak{m}^\tau)^{\mathbb{C}}$ of the flag manifold G^τ/K^τ and the set of K-simple roots from $R_M = R \setminus R_K$, given by*

$$R_T \ni \tilde{\alpha} \Longleftrightarrow \mathfrak{m}(\tilde{\alpha}) = \sum_{\alpha \in R_M, \alpha|\mathfrak{t} = \tilde{\alpha}} \mathbb{C}E_\alpha \Longleftrightarrow (\tilde{\alpha})_-,$$

where $(\tilde{\alpha})_-$ is the lowest weight of irreducible $\mathfrak{k}$-module $\mathfrak{m}(\tilde{\alpha})$.

(2) *The irreducible $\mathfrak{k}$-modules $\mathfrak{m}(\tilde{\alpha})$ and $\mathfrak{m}(-\tilde{\alpha})$ are conjugated : $\mathfrak{m}(-\tilde{\alpha}) = \mathfrak{m}(\tilde{\alpha})^*$ and the real form $(\mathfrak{m}(\tilde{\alpha}) + \mathfrak{m}(-\tilde{\alpha}))^\tau$ of $\mathfrak{m}(\tilde{\alpha}) + \mathfrak{m}(-\tilde{\alpha})$ is an irreducible $\mathfrak{k}^\tau$-module.*

For proof see [Sieb], [Al-Per], [G-O-V].

Corollary. *The decomposition of the tangent space* $\mathfrak{m}^\tau$ *of a flag manifold* $F = G^\tau/K^\tau$ *into irreducible real* $\mathfrak{k}^\tau$ *submodules is given by*

$$\mathfrak{m}^\tau = \sum_{\tilde{\alpha}\in R_T^+} (\mathfrak{m}(\tilde{\alpha}) + \mathfrak{m}(-\tilde{\alpha}))^\tau,$$

where $R_T^+ = \kappa(R^+)$ *is the set of positive* T*-roots.*

3. Flag manifolds of the group $A_l = SU(l+1)$

Let $V = \mathbb{C}^{l+1}$ be the standard vector space of dimension $l+1$ with the standard basis e_i and dual basis e_i^* of V^* . The standard Cartan subalgebra $\mathfrak{h}$ of the Lie algebra $A_l = sl(V)$ consists of all diagonal endomorphisms

$$\mathfrak{h} = \{h = \sum \varepsilon_i e_i \otimes e_i^*, \sum \varepsilon_i = 0\}.$$

We will consider the coordinates $\varepsilon_i, i = 1, \ldots, l+1$ as linear forms on $\mathfrak{h}$.

A standard painted Dynkin diagram $\Gamma = \Gamma A(p, n^1, \cdots, n^{q+1})$ for A_l has the form

where p is the length of the chain of black vertices and $n^1, \cdots, n^{q+1}$ are lengths of chains of white vertices.

It is convenient to denote the vectors of the standard basis of V as follows

$$d^1, \cdots, d^p;\ e_1^1, \cdots, e_{n^1+1}^1;\ \cdots;\ e_1^{q+1}, \cdots, e_{n^{q+1}+1}^{q+1}.$$

Then any element of $\mathfrak{h}$ can be written as

$$h = \sum \delta^a d^a \otimes (d^a)^* + \sum \varepsilon_i^b e_i^b \otimes (e_i^b)^*.$$

In terms of 1-forms $\delta^a, \varepsilon_i^b \in \mathfrak{h}^*$ the root system of A_l is given by

$$R = \{\delta^{ab} = \delta^a - \delta^b;\ \pm(\delta^a - \varepsilon_i^b);\ \varepsilon_{ij}^{bc} = \varepsilon_i^b - \varepsilon_j^c\}.$$

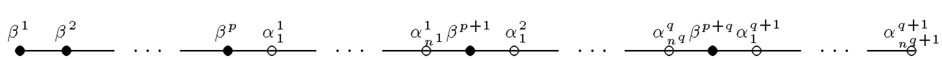
We choose the natural basis of the root system R as

$$\Pi = \{\beta^a = \delta^{a\,a+1},\ \beta^p = \delta^p - \varepsilon_1^1,\ \beta^{p+b} = \varepsilon_{n^b+1}^b - \varepsilon_1^{b+1},\ \alpha_i^c = \varepsilon_i^c - \varepsilon_{i+1}^c\}$$

where

$$a = 1, \ldots, p-1;\ b = 1, \ldots, q;\ c = 1, \ldots, q+1.$$

It defines the following standard equipment of Γ



with

$$\Pi_B = \{\beta^1, \cdots, \beta^{p+q}\} \qquad \Pi_W = \{\alpha_i^c\}.$$

Since the diagram Γ is determined by the type A_l and the numbers $p, n^1, \cdots, n^{q+1}$, we will write $\Gamma = \Gamma A(p; n^1, \cdot, n^{q+1})$ and we will denote by $F(\Gamma) = FA(p; n^1, \ldots, n^{q+1})$ the corresponding flag manifold.

The systems of positive roots R_K^+ and $R^+ = R_K^+ \cup R_M^+$ are given by

$$R_K^+ = \{\varepsilon_i^c - \varepsilon_j^c\}, \quad R_M^+ = \{\delta^{ab}, \delta^a - \varepsilon_i^c, \varepsilon_{ij}^{cd}\}.$$

The Weyl group W_K of $\mathfrak{k}$ acting on R_M^+ permutes low indices. Hence, dropping these indices, we get the set R_T^+ of positive T-roots :

$$R_T^+ = \{\delta^{ab}, \delta^a - \varepsilon^c, \varepsilon^{cd} = \varepsilon^c - \varepsilon^d, a < b; c < d\}.$$

Here $\varepsilon^c = \kappa(\varepsilon_i^c) = \varepsilon_i^c|\mathfrak{t}$ and we write $\kappa(\delta^{ab}) = \delta^{ab}$ since $\delta^{ab} \in \mathfrak{t}^*$.

The stability subalgebra of the flag manifold $F(\Gamma) = G^\tau/K^\tau$ is the compact real form $\mathfrak{k}^\tau$ of the complex subalgebra

$$\mathfrak{k} = \mathfrak{z} \oplus \mathfrak{a}^1(e_i^1 \otimes e_{-j}^1) \oplus \cdots \oplus \mathfrak{a}^{q+1}(e_i^{q+1} \otimes e_{-j}^{q+1})$$

of $\mathfrak{g}$, where

$$\mathfrak{z} = \mathrm{span}_{\mathbb{C}}\{H_{\beta^a} = [E_{\beta^a}, E_{-\beta^a}]\, , a = 1, \cdots, p+q\}$$

and

$$\mathfrak{a}^c(e_i^c \otimes e_{-j}^c) = \mathfrak{a}^c \simeq A_{n^c}$$

is a simple summand of $\mathfrak{k}$ with the root system $R^c = \{\varepsilon_i^c - \varepsilon_j^c\}$. From this description of T-roots and the proposition of section 2 we easily derive the following

Proposition. *The isotropy representation of the Lie algebra $\mathfrak{k}$ on the complexified tangent space $T_0^{\mathbb{C}} F_\Gamma = \mathfrak{m}$ is the direct sum of the following (non equivalent) irreducible $\mathfrak{k}$-modules:*

$$\mathfrak{m}(\delta^{ab}) = \mathbb{C}E_{\delta^{ab}}, \quad \mathfrak{m}(\varepsilon^{cd}) = \pi_1(\mathfrak{a}^c) \otimes \pi_1^*(\mathfrak{a}^d)$$

$$\mathfrak{m}(\delta^a - \varepsilon^c) = \pi_1^*(\mathfrak{a}^c), \quad \mathfrak{m}(\varepsilon^c - \delta^a) = \pi_1(\mathfrak{a}^c)$$

where $a, b \in \{1, \ldots, p\}$; $c, d \in \{1, \ldots, q+1\}$.

Corollary. *The decomposition of the isotropy $\mathfrak{k}^\tau$-module $\mathfrak{m}^\tau$ of the flag manifold into irreducible submodules is given by*

$$\mathfrak{m}^\tau = \sum_{a<b} \mathbb{R}(E'_{\delta^{ab}}, E''_{\delta^{ab}}) + \sum (\mathfrak{m}(\delta^a - \varepsilon^c) + \mathfrak{m}(\varepsilon^c - \delta^a))^\tau + \sum_{c<d} (\mathfrak{m}(\varepsilon^{cd}) + \mathfrak{m}(\varepsilon^{dc}))^\tau.$$

In particular there is $p(p-1)/2 + (q+1)(2p+q)/2$ irreducible (non equivalent) $\mathfrak{k}^\tau$-submodules in $\mathfrak{m}^\tau$.

4. Flag manifolds of the group B_l

We identify the Lie algebra $B_l = so(V) \subset gl(V)$, where $V = \mathbb{C}^{2l+1}$ is the Euclidean vector space, with the space $\wedge^2 V$ of bivectors by means of the Euclidean metric $< \cdot, \cdot >$.

We describe the stability subgroup and stability subalgebra of the flag manifold $F(\Gamma) = G^\tau / K^\tau$ associated to a standard painted Dynkin graph
$\Gamma = \Gamma B(p, n^1, \dots, n^q, n)$:

Here the length of the first black chain is p and the lengths of the white chains are $n^1, n^2, \cdots, n^q, n^{q+1} = n$ and

$$l = p + q + \sum n^b,\ n^1 \geq n^2, \cdots, \geq n^q > 0,\ n \geq 0.$$

Remark that if $n = 0$ the last root is black.

We choose a basis of V in the form

$$d^{\pm a},\ a = 1, \dots, p;\ e^b_{\pm i},\ i = 1, \dots, n^b + 1,\ b = 1, \dots, q;\ e_I,\ I = 0, \pm 1, \dots, \pm n$$

such that only non zero scalar products are

$$< d^a, d^{-a} >=< e^b_i, e^b_{-i} >=< e_0, e_0 >=< e_i, e_{-i} >= 1.$$

The natural Cartan subalgebra $\mathfrak{h}$ of B_l is defined by

$$\mathfrak{h} = \{h = \sum \delta^a d^a \wedge d^{-a} + \sum \varepsilon^b_i e^b_i \wedge e^b_{-i} + \sum \varepsilon_i e_i \wedge e_{-i}\}.$$

The coordinates $\delta^a, \varepsilon^b_i, \varepsilon_i$ of a vector $h \in \mathfrak{h}$ form a basis of the dual space $\mathfrak{h}^*$.

Denote by $\mathfrak{a}^b = \mathfrak{g}(e^b_i \wedge e^b_{-j})$ the Lie subalgebra of B_l generated by the root vectors $e^b_i \wedge e^b_{-j}$ with the roots $\varepsilon^b_i - \varepsilon^b_j$. It is isomorphic to A_{n^b}. Similar, the Lie subalgebra $\mathfrak{b} = \mathfrak{g}(e_I \wedge e_J)$ of $\mathfrak{g}$, generated by the root vectors with roots $\varepsilon_i \pm \varepsilon_j, \pm\varepsilon_i$ is isomorphic to B_n. Denote by $\mathfrak{z} = \mathrm{span}\{H_{ab}\}$ the subalgebra of $\mathfrak{h}$, spaned by elements

$$H_{ab} = [d^{-a} \wedge d^{-b}, d^a \wedge d^b] = d^a \wedge d^{-a} + d^b \wedge d^{-b}.$$

Then $\mathfrak{k} = \mathfrak{z} + \mathfrak{a}^1 + \cdots + \mathfrak{a}^q + \mathfrak{b}$ is a regular reductive subalgebra of B_l with the center $\mathfrak{z}$ which is the centerlizer of some element from $\mathfrak{h}$. Denote by

$$\mathfrak{n} = \mathrm{span}\{d^a \wedge d^{\pm b}, d^a \wedge e^b_{\pm i}, d^a \wedge e^b_{\pm i}, e^b_i \wedge e^b_j, e^b_i \wedge e^c_{\pm j}, e^b_i \wedge e_{\pm j}\}$$

the nilpotent subalgebra of B_l generated by indicated root vectors and by $\mathfrak{n}_-$ the subalgebra generated by opposite root vectors. Then

$$B_l = \mathfrak{n}_- + \mathfrak{k} + \mathfrak{n}$$

is the generalized Gauss decomposition of B_l and $\mathfrak{p} = \mathfrak{n}_- + \mathfrak{k}$ is a parabolic subalgebra.

The set

$$\Pi = \{\beta^a = \delta^a - \delta^{a+1};\ \beta^p = \delta^p - \varepsilon_1^1;\ \beta^{p+b} = \varepsilon^b_{n^b+1} - \varepsilon_1^{b+1};\ \alpha_i^b = \varepsilon_i^b - \varepsilon_{i+1}^b;$$
$$\alpha_i^b = \varepsilon_i^b - \varepsilon_{i+1}^b;\ \alpha_i = \varepsilon_j - \varepsilon_{j+1};\ \alpha_n = \varepsilon_n\}$$
$$a = 1, \dots, p;\ b = 1, \dots, q;\ i = 1, \dots, n^b;\ j = 1, \dots, n-1,$$

is a basis of the root system R of the Lie algebra B_l with respect to $\mathfrak{h}$. It defines the standard equipment of $\Gamma B(p, n^1, \dots, n^q, n)$ such that

$$\Pi_B = \{\beta^1, \cdots, \beta^{p+q}\}, \quad \Pi_W = \{\alpha_i^b, \alpha_i\}.$$

The positive roots R_K^+ of the root system $R_K = [\Pi_W]$ generated by Π_W is given by

$$R_K^+ = \{\varepsilon_{ij}^b = \varepsilon_i^b - \varepsilon_j^b,\ \varepsilon_{ij} = \varepsilon_i - \varepsilon_j;\ \varepsilon_{ij}^+ = \varepsilon_i + \varepsilon_j,\ (i < j),\ \varepsilon_i\}.$$

The root system R_K generates the semisimple Lie subalgebra

$$\mathfrak{k}' = \mathfrak{g}(R_K) = \mathfrak{a}^1(e_i^1 \wedge e_{-j}^1) + \cdots + \mathfrak{a}^q(e_i^q \wedge e_{-j}^q)) + \mathfrak{b}(e_I \wedge e_J) \approx$$
$$\approx A_{n^1} + \cdots + A_{n^q} + B_n$$

where $\mathfrak{a}^b(e_i^b \wedge e_{-j}^b) = \mathfrak{a}^b \approx A_{n^b}$ is the Lie subalgebra of $\mathfrak{g}$ generated by elements $e_i^b \vee e_{-j}^b$ and $\mathfrak{b}(e_I \wedge e_J) = \mathfrak{b} \approx B_n$ is the Lie subalgebra generated by elements $e_I \wedge e_J$, $I, J \in \{0, \pm 1, \dots, \pm n\}$.

The positive roots $R_M^+ = R^+ \setminus R_K^+$ are given by

$$R_M^+ = \{\delta^a;\ \delta^a \pm \delta^{a'};\ (a < a')\,\delta^a \pm \varepsilon_i^b; \delta^a \pm \varepsilon_i; \varepsilon_i^b; \varepsilon_i^b + \varepsilon_j^b; \varepsilon_i^b \pm \varepsilon_j^c\ (b < c), \varepsilon_i^b \pm \varepsilon_j\}.$$

Dropping the low indices, we get the system of positive T-roots :

$$R_T^+ = \{\delta^a; \delta^a \pm \delta^{a'}; \delta^a \pm \varepsilon^b; \varepsilon^b; 2\varepsilon^b; \varepsilon^b \pm \varepsilon^c; \varepsilon^b \pm \varepsilon\}.$$

As for the case A_l, we get the following proposition.

Proposition. *The complexified tangent space $\mathfrak{m}$ of the flag manifold $F = FB(p, n^1, \dots, n^q, n)$ of the group $B_l = SO_{2l+1}$ has the form*

$$\mathfrak{m} = \mathfrak{m}_+ + \mathfrak{m}_-, \qquad \mathfrak{m}_- \approx \mathfrak{m}_+^*,$$

where $\mathfrak{m}_+ = \sum_{\tilde\alpha \in R_T^+} \mathfrak{m}(\tilde\alpha)$ is the sum of the following irreducible $\mathfrak{k}'$-modules:

$$\mathfrak{m}(2\delta^a) = \mathbb{C}E_{\delta^a}, \quad \mathfrak{m}(\delta^a \pm \delta^b) = \mathbb{C}E_{\delta^a \pm \delta^b}$$
$$\mathfrak{m}(\delta^a + \varepsilon^b) = \pi_1(\mathfrak{a}^b), \quad \mathfrak{m}(\delta^a - \varepsilon^b) = \pi_1(\mathfrak{a}^b)^*$$
$$\mathfrak{m}(\delta^a + \varepsilon^b) = \pi_1(\mathfrak{a}^b), \quad \mathfrak{m}(\delta^a - \varepsilon^b) = \pi_1(\mathfrak{a}^b)^*$$
$$\mathfrak{m}(\delta^a + \varepsilon) = \pi_1(\mathfrak{b}), \quad \mathfrak{m}(\delta^a - \varepsilon) = \pi_1(\mathfrak{b})^* \approx \pi_1(\mathfrak{b})$$
$$\mathfrak{m}(\varepsilon^b) = \pi_1(\mathfrak{a}^b), \quad \mathfrak{m}(2\varepsilon^b) = \pi_1^2(\mathfrak{a}^b)$$
$$\mathfrak{m}(\varepsilon^b + \varepsilon^c) = \pi_1(\mathfrak{a}^b) \otimes \pi_1(\mathfrak{a}^c), \quad \mathfrak{m}(\varepsilon^b - \varepsilon^c) = \pi_1(\mathfrak{a}^b) \otimes \pi_1(\mathfrak{a}^c)^*$$
$$\mathfrak{m}(\varepsilon^b + \varepsilon) = \pi_1(\mathfrak{a}^b) \otimes \pi_1(\mathfrak{b}), \quad \mathfrak{m}(\varepsilon^b - \varepsilon) = \pi_1(\mathfrak{a}^b) \otimes \pi_1(\mathfrak{b})^*$$

Here $\pi_1^2(\mathfrak{a}^b)$ denotes the irreducible $\mathfrak{a}^b$-module with the highest weight equal twice the highest weight of $\pi_1(\mathfrak{a}^b)$ and $\pi_1(\mathfrak{b})$ is the tautological representation of $\mathfrak{b} = B_n$.

Corollary. *The isotropy representation of the flag manifold* $FB(p,n^1,\dots,n^q,n) = SO_{2l+1}/K$ *has* $p^2+2p(q+1)+q(q+3)$ *non equivalent irreducible components, between them* p^2 *2-dimensional , which are trivial as* $\mathfrak{k}'$*-modules.*

5. Flag manifolds of the group C_l

Let now $(V=\mathbb{C}^{2l},\omega)$ be the standard complex symplectic vector space. We identify the symplectic Lie algebra $C_l = sp(V)$ with the space S^2V of symmetric tensors by means of the symplectic form ω such that a tensor $x\vee y = x\otimes y + y\otimes x$ is identified with the endomorphism $x\otimes\omega(y,.)+y\otimes\omega(x,.)$. We describe the stability subgroup and stability subalgebra of the flag manifold $F(\Gamma) = FC(p,n^1,\dots,n^q,n) = G^\tau/K^\tau$ associated to a standard painted Dynkin graph $\Gamma=\Gamma(p,n^1,\dots,n^q,n)$:

Here the length of the first black chain is p and the lengths of the white chains are $n^1,n^2,\cdots,n^q,n^{q+1}=n$ and $l=p+q+\sum n^b, n^1\geq n^2,\cdots,\geq n^q>0, n\geq 0$. Remark that if $n=0$ the last root is black.

We choose a basis of V in the form

$$d^{\pm a},(a=1,\cdots,p);\quad e^b_{\pm i}(b=1,\cdots,q;\ i=1,\cdots,n^b+1);\quad e_I\ (I=\pm1,\cdots,\pm n)$$

such that the only non zero products are

$$\omega(d^a,d^{-a})=\omega(e^b_i,e^b_{-i})=\omega(e_i,e_{-i})=1.$$

Then a Cartan subalgebra of C_l can be choosen as

$$\mathfrak{h}=\{h=\sum\delta d^a\vee d^{-a}+\sum\varepsilon^b_i e^b_i\vee e^b_{-i}+\sum\varepsilon_i e_i\vee e_{-i}\}$$

where 1-forms $\delta^a,\varepsilon^b_{\pm i},\varepsilon_{\pm i}$ form a basis of $\mathfrak{h}^*$.

The set

$$\Pi=\{\beta^a=\delta^a-\delta^{a+1};\ \beta^p=\delta^p-\varepsilon^1_1;\ \beta^{p+b}=\varepsilon^b_{n^b+1}-\varepsilon^{b+1}_1;$$

$$\alpha^b_i=\varepsilon^b_i-\varepsilon^b_{i+1};\ \alpha_j=\varepsilon_j-\varepsilon_{j+1};\ \alpha_n=2\varepsilon_n\}$$

$$a=1,\dots,p-1;\ b=1,\dots,q;\ i=1,\dots,n^b;\ j=1,\dots,n-1,$$

is a basis of the root system R of the Lie algebra C_l with respect to $\mathfrak{h}$. It defines the standard equipment of Γ such that

$$\Pi_B=\{\beta^1,\cdots,\beta^{p+q}\},\qquad \Pi_W=\{\alpha^b_i,\alpha_i\}.$$

The positive roots R^+_K of the root system $R_K=[\Pi_W]$ generated by Π_W is given by

$$R_K^+ = \{\varepsilon_{ij}^b = \varepsilon_i^b - \varepsilon_j^b,\ \varepsilon_{ij} = \varepsilon_i - \varepsilon_j;\ \varepsilon_{ij}^+ = \varepsilon_i + \varepsilon_j,\ (i<j),\ 2\varepsilon_i\}.$$

The root system R_K generates the semisimple Lie subalgebra

$$\mathfrak{k}' = \mathfrak{g}(R_K) = \mathfrak{a}^1(e_i^1 \vee e^1{}_-j) + \cdots + \mathfrak{a}^q(e_i^q \vee e_{-j}^q)) + \mathfrak{c}(e_{\pm i} \vee e_{\pm j}) =$$

$$\approx A_{n^1} + \cdots + A_{n^q} + C_n$$

where $\mathfrak{a}^b(e_i^b \vee e_{-j}^b) = \mathfrak{a}^b \approx A_{n^b}$ is the Lie subalgebra of $\mathfrak{g}$ generated by elements $e_i^b \vee e_{-j}^b$ and similar for $\mathfrak{c}(e_{\pm i} \vee e_{\pm j}) = \mathfrak{c} \approx C_n$.

The positive roots $R_M^+ = R^+ \setminus R_K^+$ are given by

$$R_M^+ = \{2\delta^a;\ \delta^a \pm \delta^{a'};\ (a<a')\,\delta^a \pm \varepsilon_i^b;\ \delta^a \pm \varepsilon_i;\ 2\varepsilon_i^b;\ \varepsilon_i^b + \varepsilon_j^b;\ \varepsilon_i^b \pm \varepsilon_j^c\ (b<c), \varepsilon_i^b \pm \varepsilon_j\}.$$

Dropping the low indices, we get the system of positive T-roots :

$$R_T = \{2\delta^a; \delta^a \pm \delta^{a'}; \delta^a \pm \varepsilon^b; \delta^a \pm \varepsilon; 2\varepsilon^b; \varepsilon^b \pm \varepsilon^c; \varepsilon^b \pm \varepsilon\}.$$

We get the following proposition.

Proposition. *The complexified tangent space* $\mathfrak{m}$ *of the flag manifold* $F = FC(p, n^1, \dots, n^q, n)$ *of the group* $C_l = Sp_n$ *has the form*

$$\mathfrak{m} = \mathfrak{m}_+ + \mathfrak{m}_-, \qquad \mathfrak{m}_- \approx \mathfrak{m}_+^*,$$

where $\mathfrak{m}_+ = \sum_{\tilde\alpha \in R_T^+} \mathfrak{m}(\tilde\alpha)$ *is the sum of the following irreducible* $\mathfrak{k}'$*-modules*

$$\mathfrak{m}(2\delta^a) = \mathbb{C}E_{2\delta^a}, \quad \mathfrak{m}(\delta^a \pm \delta^b) = \mathbb{C}E_{\delta^a \pm \delta^b}$$

$$\mathfrak{m}(\delta^a + \varepsilon^b) = \pi_1(\mathfrak{a}^b), \quad \mathfrak{m}(\delta^a - \varepsilon^b) = \pi_1(\mathfrak{a}^b)^*$$

$$\mathfrak{m}(\delta^a + \varepsilon^b) = \pi_1(\mathfrak{a}^b), \quad \mathfrak{m}(\delta^a - \varepsilon^b) = \pi_1(\mathfrak{a}^b)^*$$

$$\mathfrak{m}(\delta^a + \varepsilon) = \pi_1(\mathfrak{c}), \quad \mathfrak{m}(\delta^a - \varepsilon) = \pi_1(\mathfrak{c})^* \approx \pi_1(\mathfrak{c})$$

$$\mathfrak{m}(\varepsilon^b + \varepsilon^c) = \pi_1(\mathfrak{a}^b) \otimes \pi_1(\mathfrak{a}^c), \quad \mathfrak{m}(\varepsilon^b - \varepsilon^c) = \pi_1(\mathfrak{a}^b) \otimes \pi_1(\mathfrak{a}^c)^*$$

$$\mathfrak{m}(\varepsilon^b + \varepsilon) = \pi_1(\mathfrak{a}^b) \otimes \pi_1(\mathfrak{c}), \quad \mathfrak{m}(\varepsilon^b - \varepsilon) = \pi_1(\mathfrak{a}^b) \otimes \pi_1(\mathfrak{c})^*$$

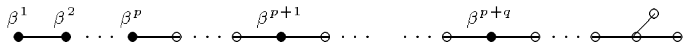
$$\mathfrak{m}(2\varepsilon^b) = \pi_1^2(\mathfrak{a}^b).$$

Corollary. *The isotropy representation of the flag manifold* $FC(p, n^1, \dots, n^q, n)$ *has* $p^2 + 2p(q+1) + q(q+2)$ *non equivalent irreducible components, between them* p^2 *2-dimensional , which are trivial as* $\mathfrak{k}'$*-modules.*

6. Flag manifolds of the group D_l

As for B_l we identify the Lie algebra $D_l = so(V)$, where $V = \mathbb{C}^{2l}$, with the space $\wedge^2 V$ of bivectors.

A standard painted Dynkin graph for D_l has one of the following forms $\Gamma = \Gamma(p; n^1, \cdots, n^q; n)$



or $\Gamma' = \Gamma'(p; n^1, \cdots, n^q; n)$

where $p + q \leq l - 2$, $p \geq 0$ is the length of the first black chain , $0 < n^1 \leq n^2 \leq \cdots \leq n^q$ are the lengths of the white chains and $n \geq 2$ for Γ and $n - 1 \geq 1$ for Γ' are the number of white vertices on the right from β^{p+q} .

As for B_l we denote the vectors of the standard basis of V by

$$d^{\pm a}, a = 1, \cdots, p;\ e^b_{\pm i}, i = 1, \cdots, n^b + 1, b = 1, \cdots, q;\ e_I, I = \pm 1, \cdots, \pm n.$$

The natural Cartan subalgebra $\mathfrak{h}$ of B_l is defined by

$$\mathfrak{h} = \{h = \sum \delta^a d^a \wedge d^{-a} + \sum \varepsilon^b_i e^b_i \wedge e^b_{-i} + \sum \varepsilon_i e_i \wedge e_{-i}\}.$$

The coordinates $\delta^a, \varepsilon^b_i, \varepsilon_i$ of a vector $h \in \mathfrak{h}$ form a basis of the dual space $\mathfrak{h}^*$.

We consider the standard equipment of painted Dynkin graph Γ and Γ' by the elements of the basis $\Pi = \Pi_B \cup \Pi_W$, where for Γ

$$\Pi_B = \Pi_B(\Gamma) = \{\beta^a = \delta^a - \delta^{a+1}, (a < p), \beta^p = \delta^p - \varepsilon^1_1, \beta^{p+b} = \varepsilon^b_{n^b+1} - \varepsilon^{b+1}_1, (b \leq q)\}$$

$$\Pi_W = \Pi_W(\Gamma) = \{\alpha^b_i = \varepsilon^b_i - \varepsilon^b_{i+1}, (b \leq q), \alpha_j = \varepsilon_j - \varepsilon_{j+1}, (j < n), \alpha_n = \varepsilon_{n-1} + \varepsilon_n\}$$

and

$$\Pi_B(\Gamma') = \Pi_B(\Gamma) \cup \{\alpha_n\} \quad \Pi_W(\Gamma') = \Pi_W(\Gamma) \setminus \{\alpha_n\}.$$

Then we have

$$R^+_K(\Gamma) = \{\varepsilon^b_{ij} = \varepsilon^b_i - \varepsilon^b_j,\ (b = 1, \ldots, q),\ \varepsilon_i \pm \varepsilon_j\}$$

for Γ and

$$R^+_K(\Gamma') = \{\varepsilon^b_{ij}, (b = 1, \ldots, q), \varepsilon_i - \varepsilon_j\}$$

for Γ'.

The semisimple part $\mathfrak{k}'$ of the stability subalgebra $\mathfrak{k}$, associated with the root system R_K is given for the case Γ by

$$\mathfrak{k}'(\Gamma) = \mathfrak{g}(R_K(\Gamma)) = \mathfrak{a}^1(e^1_i \wedge e^1_{-j}) + \cdots + \mathfrak{a}^q(e^q_i \wedge e^q_j) + \mathfrak{d}(e_I \wedge e_J)$$

where $\mathfrak{a}^a(e^a_i \wedge e^a_j) = \mathfrak{a}^a \approx A_{n^a}$ and $\mathfrak{d}(e_I \wedge e_J) = \mathfrak{d} \approx D_n$, and for Γ' by

$$\mathfrak{k}'(\Gamma') = \mathfrak{g}(R_K(\Gamma')) = \mathfrak{a}^1(e^1_i \wedge e^1_{-j}) + \cdots + \mathfrak{a}^q(e^q_i \wedge e^q_{-j}) + \mathfrak{a}(e_i \wedge e_{-j}) \approx$$

$$\approx A_{n^1} + \cdots + A_{n^q} + A_n.$$

We get the following proposition.

Proposition. *The complexified tangent space* $\mathfrak{m}$ *of the flag manifold* $FD(p, n^1, \dots, n^q, n)$ *of the group* $D_l = SO_{2n}$ *has the form*

$$\mathfrak{m} = \mathfrak{m}_+ + \mathfrak{m}_-, \quad \mathfrak{m}_- \approx \mathfrak{m}_+^*,$$

where $\mathfrak{m}_+ = \sum_{\tilde{\alpha} \in R_T^+} \mathfrak{m}(\tilde{\alpha})$ *is the sum of the following irreducible* $\mathfrak{k}'$*-modules*

$$\mathfrak{m}(\delta^a \pm \delta^b) = \mathbb{C}E_{\delta^a \pm \delta^b}$$

$$\mathfrak{m}(\delta^a + \varepsilon^b) = \pi_1(\mathfrak{a}^b), \quad \mathfrak{m}(\delta^a - \varepsilon^b) = \pi_1(\mathfrak{a}^b)^*$$
$$\mathfrak{m}(\delta^a + \varepsilon^b) = \pi_1(\mathfrak{a}^b), \quad \mathfrak{m}(\delta^a - \varepsilon^b) = \pi_1(\mathfrak{a}^b)^*$$
$$\mathfrak{m}(\delta^a + \varepsilon) = \pi_1(\mathfrak{d}) \quad \mathfrak{m}(\delta^a - \varepsilon) = \pi_1(\mathfrak{d})^* \approx \pi_1(\mathfrak{d})$$
$$\mathfrak{m}(\varepsilon^b + \varepsilon^c) = \pi_1(\mathfrak{a}^b) \otimes \pi_1(\mathfrak{a}^c), \quad \mathfrak{m}(\varepsilon^b - \varepsilon^c) = \pi_1(\mathfrak{a}^b) \otimes \pi_1(\mathfrak{a}^c)^*$$
$$\mathfrak{m}(\varepsilon^b + \varepsilon) = \pi_1(\mathfrak{a}^b) \otimes \pi_1(\mathfrak{d}), \quad \mathfrak{m}(\varepsilon^b - \varepsilon) = \pi_1(\mathfrak{a}^b) \otimes \pi_1(\mathfrak{d})^*$$

$$\mathfrak{m}(2\varepsilon^b) = \pi_1^2(\mathfrak{a}^b).$$

For the flag manifold $F' = F'D(p, n^1, \dots, n^q, n)$ *in all formulas the* $\mathfrak{d}$*-module* $\pi_1(\mathfrak{d})$ *have to change for* $\mathfrak{a}$*-module* $\pi_1(\mathfrak{a})$ *and one new irreducible module* $\mathfrak{m}(2\varepsilon) = \pi_1^2(\mathfrak{a})$ *appears.*

Corollary. *The isotropy representation of the flag manifold* $F = FD(p, n^1, \dots, n^q, n)$ *has* $p(p-1) + 2p(q+1) + q(q+2)$ *non equivalent irreducible components, between them* $p(p-1)$ *2-dimensional , which are trivial as* $\mathfrak{k}'$*-modules. For the manifold* $F' = F'D(p, n^1, \dots, n^q, n)$ *the number irreducible components is increased by one .*

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