**ESI** **The Erwin Schrödinger International Institute for Mathematical Physics** Boltzmanngasse 9 A-1090 Wien, Austria

# Conformally Symplectic Dynamics and Symmetry of the Lyapunov Spectrum

**Maciej P. Wojtkowski**
**Carlangelo Liverani**

Vienna, Preprint ESI 427 (1997) March 5, 1997

Supported by Federal Ministry of Science and Research, Austria
Available via http://www.esi.ac.at

# CONFORMALLY SYMPLECTIC DYNAMICS AND SYMMETRY OF THE LYAPUNOV SPECTRUM

Maciej P. Wojtkowski and Carlangelo Liverani

University of Arizona, University of Rome *Tor Vergata*

February 10, 1997

Abstract. A generalization of the Hamiltonian formalism is studied and the symmetry of the Lyapunov spectrum established for the resulting systems. The formalism is applied to the Gausssian isokinetic dynamics of interacting particles with hard core collisions and other systems.

## §0. Introduction.

We study the symmetry of the Lyapunov spectrum in systems more general than Hamiltonian but closely related to the symplectic formalism. We call these systems conformally Hamiltonian. They are determined by a non-degenerate 2-form $\Theta$ on the phase space and a function $H$, called again a Hamiltonian. The form $\Theta$ is not assumed to be closed but it satisfies the following basic condition $d\Theta = \gamma \wedge \Theta$, for some closed 1-form $\gamma$. This condition guarantees that, at least locally, the form $\Theta$ can be multiplied by a nonzero function to give a bona fide symplectic structure. The skew-orthogonality of tangent vectors is preserved under multiplication of the form by any nonzero function, hence the name *conformally symplectic structure*. These ideas were known to geometers for a long time, see for example the paper of Vaisman [V].

The conformally Hamiltonian (with respect to the form $\Theta$) vector field $\nabla_\Theta H$ is defined by the usual relation

$$\Theta\left(\cdot,\, \nabla_\Theta H\right) = dH(\cdot).$$

The Hamiltonian function $H$ is again a first integral of the system. In Section 2 we prove that for any conformally Hamiltonian system restricted to a smooth level set

**We thank Dmitri Alexeevski, Federico Bonetto and Philippe Choquard for many enlightening discussions during our stay at the ESI in December of 1996. In addition, we thank David Ruelle for valuable comments. We are also grateful for the opportunities provided by the hospitality of the Erwin Schrödinger Institute in Vienna where this work was done. M.P.Wojtkowski has been partially supported by NSF Grant DMS-9404420.**

Typeset by $\mathcal{A}_\mathcal{M}\mathcal{S}$-TEX

of the Hamiltonian the Lyapunov spectrum is symmetric, with symmetric exponents adding up to a constant. More precisely the direction of the flow has to be factored out. In Section 3 we extend this formalism to flows with collisions.

In Section 1 we give an independent proof of the fact that for any conformally symplectic cocycle we have the symmetry of the Lyapunov spectrum. The first proof of this fact in the symplectic case goes back to Benettin et al. [B-G-G-S]. Our proof is based on an alternative description of Lyapunov exponents and it borrows an idea from [W1] (Lemma 1.2).

In Section 4 we present examples which were recently the subject of several papers. We show that the Gaussian isokinetic dynamics can be viewed as a conformally Hamiltonian system, by which we immediately recover the results of Dettmann and Morriss [D-M 1], [D-M 2], on the symmetry of the Lyapunov spectrum and the Hamiltonian character of the dynamics. We extend these results to systems with collisions, taking advantage of the fact that our formalism works equally well for collisions as it does for flows. Such systems were studied in the paper by Dellago, Posch and Hoover, [D-P-H], and the symmetry of the Lyapunov spectra was demonstrated numerically. Chernov et al, [Ch-E-L-S] studied rigorously the Lorentz gas of periodic scatterers with an electric external field in dimension 2. Latz, van Beijeren and Dorfman, [L-B-D], considered thermostated random Lorentz gas in 3-dimensions and found the symmetry there. Let us note that we prove the symmetry of the Lyapunov spectrum for any invariant (ergodic) measure and not only for the SRB measure, which is the easiest to access numerically.

We also show that the Gaussian isokinetic dynamics on a Riemannian manifold can be given the same treatment.

The last application is to Nosé–Hoover dynamics. We show that the Hoover equations can be naturally viewed as a conformally Hamiltonian system, thus giving another proof of the symmetry of Lyapunov spectra for this system. It was originally proven by Dettmann and Morriss, [D-M 3].

Finally let us note that our approach is one of several possible. Recently Choquard, [Ch], showed that the isokinetic and Nosé-Hoover dynamics can be considered as Lagrangian systems. Even in the conformally symplectic framework one can keep the Hamiltonian unchanged and modify the form $\Theta$, or keep the form unchanged and modify the Hamiltonian, or keep both the form and the Hamiltonian unchanged, but change time on a level set of the Hamiltonian. We elaborate on that in Remark 2.1. We believe that our approach sheds new light on these issues.

## §1. Conformally symplectic group.

Let $\omega = \sum_{i=1}^n dp_i \wedge dq_i$ be the standard linear symplectic form in $\mathbb{R}^n \times \mathbb{R}^n$.

**Proposition 1.1.** *For an invertible linear mapping $S$ acting on $\mathbb{R}^{2n} = \mathbb{R}^n \times \mathbb{R}^n$ the following are equivalent*

(a) $\omega(Su, Sv) = \beta\omega(u, v)$ *for some scalar* $\beta$ *and all* $u, v \in \mathbb{R}^{2n}$;
(b) $\omega(Su, Sv) = 0$ *if and only if* $\omega(u, v) = 0$,
*i.e., $S$ preserves skew-orthogonality of vectors;*
(c) *$S$ takes Lagrangian subspaces of $\mathbb{R}^{2n}$ into Lagrangian subspaces.*

*Proof.* It is apparent that (a) implies (b) and that (b) is equivalent to (c). It remains to prove that (c) implies (a). By composing $S$ with an appropriate linear symplectic map we can assume without loss of generality that $S$ preserves the Lagrangian subspaces $\mathbb{R}^n \times \{0\}$ and $\{0\} \times \mathbb{R}^n$, i.e., $S$ is block diagonal. Moreover, again by multiplying by an appropriate linear symplectic map, we can assume that $S$ is equal to identity on $\mathbb{R}^n \times \{0\}$. By (b) we conclude that $S$ is diagonal on $\{0\} \times \mathbb{R}^n$. A simple calculation shows that to satisfy (b) this diagonal matrix must be a multiple of identity, which gives us (a). □

We will call a linear map from $GL(\mathbb{R}^{2n})$ conformally symplectic if it satisfies one of the properties in the Proposition 1.1. The group of all conformally symplectic maps will be denoted by $CSp(\mathbb{R}^{2n})$.

Let $X$ be a measurable space with probabilistic measure $\mu$ and let $T : X \to X$ be an ergodic map. Let further $A : X \to GL(\mathbb{R}^{2n})$ be a measurable map such that

$$\int_X \log_+ \|A(x)\| d\mu(x) < +\infty. \tag{1.1}$$

We define the matrix valued cocycle

$$A^m(x) = A(T^{m-1}x) \dots A(x).$$

By the Oseledets Multiplicative Ergodic Theorem, [O], which in this generality was first proven by Ruelle, [R], there are numbers $\lambda_1 < \dots < \lambda_s$, called the Lyapunov exponents of the measurable cocycle $A(x)$, $x \in X$, and for almost all $x$ a flag of subspaces

$$\{0\} = \mathbb{V}_0 \subset \mathbb{V}_1(x) \subset \dots \subset \mathbb{V}_{s-1}(x) \subset \mathbb{V}_s = \mathbb{R}^{2n},$$

such that for all vectors $v \in \mathbb{V}_k(x) \setminus \mathbb{V}_{k-1}(x)$

$$\lim_{m\to+\infty} \frac{1}{m} \log \|A^m(x)v\| := \lambda(v) = \lambda_k.$$

In addition, denoting by $d_k$ the difference between the dimensions of $\mathbb{V}_k$ and $\mathbb{V}_{k-1}$ ($d_k$ is called the multiplicity of the $k$-th Lyapunov exponent), the following holds:

$$\sum_{k=1}^{s} d_k \lambda_k = \int_X \log |\det(A(x))| d\mu(x), \tag{1.2}$$

i.e., the sum of all Lyapunov exponents is equal to the average exponential rate of volume growth.

Given a measurable cocycle $A(x)$, $x \in X$, satisfying (1.1) and with values in the conformally symplectic group $CSp(\mathbb{R}^{2n})$ we obtain a measurable function $\beta = \beta(x)$ such that

$$\omega(A(x)u, A(x)v) = \beta(x)\omega(u, v), \tag{1.3}$$

for all vectors $u, v \in \mathbb{R}^{2n}$. Let us define

$$b := \int_X \log |\beta(x)| d\mu(x). \tag{1.4}$$

**Lemma 1.1.** *If a measurable cocycle $A(x)$, $x \in X$, satisfies (1.1) and it has values in the conformally symplectic group $CSp(\mathbb{R}^{2n})$, then*

$$\sum_{k=1}^{s} d_k \lambda_k = nb.$$

*Proof.* Since $\omega^n$ is the volume form it follows from (1.3) that the determinant of $A(x)$ is

$$\det A(x) = \beta(x)^n.$$

The lemma follows by applying (1.2). □

**Lemma 1.2.** *Given a measurable cocycle $A(x)$, $x \in X$, satisfying (1.1) and with values in the conformally symplectic group $CSp(\mathbb{R}^{2n})$, for each two non skew-orthogonal vectors $u, v \in \mathbb{R}^{2n}$, i.e., $\omega(u,v) \neq 0$, we have*

$$\lambda(u) + \lambda(v) \geq b.$$

*Proof.* For the standard Euclidean norm $\|\cdot\|$ we have $|\omega(u,v)| \leq \|u\|\|v\|$.

From (1.3) we obtain

$$\omega(A^m(x)u, A^m(x)v) = \omega(u,v)\prod_{i=0}^{m}\beta(T^i x).$$

Therefore,

$$\log|\omega(A^m(x)u, A^m(x)v)| = \log|\omega(u,v)| + \sum_{i=0}^{m}\log|\beta(T^i x)|,$$

and

$$\frac{1}{m}\log|\omega(A^m(x)u, A^m(x)v)| \leq \frac{1}{m}\log\|A^m(x)u\| + \frac{1}{m}\log\|A^m(x)v\|.$$

Putting these relations together and using the Birkhoff Ergodic Theorem we conclude that

$$b = \int_X \log|\beta(x)| d\mu(x) \leq \lambda(u) + \lambda(v).$$

□

The following Lemma is obvious. We formulate it to streamline the proof of Theorem 1.4 where it is used twice.

For a linear subspace $X \subset \mathbb{R}^{2n}$ we denote by $X^{\angle}$ the skew-orthogonal complement of $X$, i.e, $X^{\angle} \subset \mathbb{R}^{2n}$ is the linear subspace containing all vectors $v$ such that $\omega(u, v) = 0$ for all $u \in X$. Since $\omega$ is assumed to be nondegenerate we have

$$\dim X + \dim X^{\angle} = 2n \quad \text{and} \quad (X^{\angle})^{\angle} = X.$$

**Lemma 1.3.** *Let $U, V \subset \mathbb{R}^{2n}$ be two linear subspaces. If $\omega(u, v) = 0$ for all $u \in U$ and $v \in V$, then*

$$U \subset V^{\angle}, \quad V \subset U^{\angle} \qquad \text{and} \qquad \dim U + \dim V \leq 2n.$$

□

**Theorem 1.4.** *If a measurable cocycle $A(x)$, $x \in X$, satisfies (1.1) and it has values in the conformally symplectic group $CSp(\mathbb{R}^{2n})$ then we have the following symmetry of the Lyapunov spectrum:*

$$\lambda_k + \lambda_{s-k+1} = b,$$

*where $b$ is given by (1.4), and the multiplicities of $\lambda_k$ and $\lambda_{s-k+1}$ are equal, for $k = 1, 2, \dots, s$. Moreover the subspace $\mathbb{V}_{s-k}$ is the skew-orthogonal complement of $\mathbb{V}_k$.*

*Proof.* Let $\mu_1 \leq \mu_2 \leq \cdots \leq \mu_{2n}$ be the Lyapunov exponents taken with repetitions according to their multiplicities. By Lemma 1.1 $\mu_1 + \mu_2 + \cdots + \mu_{2n} = nb$.

We can choose a flag of subspaces

$$\{0\} = \mathbb{W}_0 \subset \mathbb{W}_1(x) \subset \cdots \subset \mathbb{W}_{2n-1}(x) \subset \mathbb{W}_{2n} = \mathbb{R}^{2n},$$

such that $\dim \mathbb{W}_l = l$ and for all vectors $v \in \mathbb{W}_l(x) \setminus \mathbb{W}_{l-1}(x)$ the Lyapunov exponent $\lambda(v) = \mu_l$, for $l = 1, 2, \dots, 2n$. (Note that except in the case of all multiplicities equal to 1 there is a continuum of such flags.)

Since for any $l \leq n$, $\dim \mathbb{W}_l + \dim \mathbb{W}_{2n-l+1} = 2n + 1$, by Lemma 1.3 there are vectors $u \in \mathbb{W}_l$ and $v \in \mathbb{W}_{2n-l+1}$ such that $\omega(u, v) \neq 0$. By continuity there must be also vectors $\tilde{u} \in \mathbb{W}_l \setminus \mathbb{W}_{l-1}$ and $\tilde{v} \in \mathbb{W}_{2n-l+1} \setminus \mathbb{W}_{2n-l}$ such that $\omega(\tilde{u}, \tilde{v}) \neq 0$. It follows from Lemma 1.2 that

$$\mu_l + \mu_{2n-l+1} \geq b,$$

for $l = 1, 2, \dots, n$. Adding these inequalities together, we get

$$nb = \sum_{l=1}^{n} (\mu_l + \mu_{2n-l+1}) \geq nb,$$

which shows that all the inequalities must be actually equalities. It follows immediately that for any $k = 1, \dots, s$, the multiplicities of $\lambda_k$ and $\lambda_{s-k+1}$ are equal and $\lambda_k + \lambda_{s-k+1} = b$.

To show that the subspace $\mathbb{V}_{s-k}$ is the skew-orthogonal complement of the subspace $\mathbb{V}_k$ we observe that $\omega(u, v) = 0$ for any $u \in \mathbb{V}_k$ and $v \in \mathbb{V}_{s-k}$. Indeed, if this is not the case we could use Lemma 1.2 to claim that $\lambda_k + \lambda_{s-k} \geq b$, which leads to the contradiction

$$b = \lambda_k + \lambda_{s-k+1} > \lambda_k + \lambda_{s-k} \geq b.$$

We can now apply Lemma 1.3 and we obtain $\mathbb{V}_{s-k} \subset \mathbb{V}_k^{\angle}$. Since the dimensions of these subspaces are equal we must have $\mathbb{V}_{s-k} = \mathbb{V}_k^{\angle}$. □

## §2. Conformally symplectic manifolds and conformal Hamiltonian flows.

Let $M$ be a smooth manifold of even dimension. A conformally symplectic structure on $M$ is a differentiable 2-form $\Theta$ which is non-degenerate and has the following basic property

$$d\Theta = \gamma \wedge \Theta, \tag{2.1}$$

for some closed 1-form $\gamma$. A manifold with such a form $\Theta$ is called conformally symplectic. The origin of this name becomes clear when one observes that locally $\gamma = dU$ for some smooth function $U$ and

$$d(e^{-U}\Theta) = 0,$$

i.e. $e^{-U}\Theta$ defines a bona fide symplectic structure.

For a given function $H : M \to \mathbb{R}$, called a Hamiltonian, let us consider a vector field $\nabla_\Theta H$ defined by the usual relation

$$\Theta(\cdot, \nabla_\Theta H) = dH. \tag{2.2}$$

We will call it the conformally Hamiltonian vector field, or conformally symplectic, or simply a Hamiltonian vector field when the conformally symplectic structure is clearly chosen. Note that our definition does not coincide with the definition of a Hamiltonian vector field from [V].

Let $\Phi^t$ denote the flow defined by the vector field $F = \nabla_\Theta H$. The Hamiltonian function $H$ is a first integral of the system. Indeed we have

$$\frac{d}{dt}H = dH(\nabla_\Theta H) = \Theta(\nabla_\Theta H, \nabla_\Theta H) = 0.$$

Let us consider the Lie derivative of the form $\Theta$ in the direction of vector field $F$, i.e.,

$$(L_F\Theta)(\xi, \eta) := \frac{d}{du}\Theta(D\Phi^u\xi, D\Phi^u\eta)_{|u=0}.$$

**Theorem 2.1.** *For a Hamiltonian vector field $F = \nabla_\Theta H$ we have*

$$L_F\Theta = \gamma(F)\Theta + \gamma \wedge dH. \tag{2.3}$$

*Proof.* We will use the Cartan formula ([A-M-R])

$$L_F = i_F d + d i_F,$$

where $i_F$ is the interior and $d$ the exterior derivative. (For a differential $m$-form $\zeta$ the interior derivative $i_F\zeta$ is the differential $(m-1)$-form obtained by substituting $F$ as the first vector argument of $\zeta$.) We have $i_F\Theta = -dH$ and we get immediately

$$L_F\Theta = i_F d\Theta - d^2H = i_F(\gamma \wedge \Theta) = \gamma(F)\Theta + \gamma \wedge dH.$$

□

Let us restrict the flow $\Phi^t$ to one smooth level set of the Hamiltonian, $M^c = \{z \in M | H(z) = c\}$. In particular we assume that on $M^c$ the differential $dH$ and the vector field $F$ do not vanish. For two vectors $\xi, \eta$ from the tangent space $T_z M^c$, we introduce

$$w(t) = \Theta(D_z\Phi^t\xi, D_z\Phi^t\eta).$$

By (2.3) we get

$$\frac{d}{dt}w(t) = \gamma(F(\Phi^t z))w(t),$$

since $dH$ vanishes on the tangent space $T_z M^c$. We conclude that

$$\Theta(D_z\Phi^t\xi, D_z\Phi^t\eta) = \beta(t)\Theta(\xi,\eta), \tag{2.4}$$

for every $\xi, \eta$, from $T_z M^c$ and

$$\beta(t) = exp\left(\int_0^t \gamma(F(\Phi^u z))du\right). \tag{2.5}$$

**Remark 2.1**

Let us note that under a non-degenerate time change a conformally Hamiltonian vector field is still conformally Hamiltonian with the same Hamiltonian function but with respect to a modified conformally symplectic form. More precisely if $F = \nabla_\Theta H$ is a Hamiltonian vector field, then if the new time $\tau$ is related to the original time $t$ by

$$\frac{d\tau}{dt} = f,$$

for some function $f$ of the phase point, we get that the vector field $\frac{1}{f}F$ is conformally symplectic with respect to the form $\widetilde{\Theta} = f\Theta$. Indeed

$$d\widetilde{\Theta} = (d\ln f + \gamma)\wedge\widetilde{\Theta} \quad \text{and} \quad \widetilde{\Theta}(\cdot, \frac{1}{f}F) = dH.$$

Alternatively we can keep the same conformally symplectic form and modify the Hamiltonian separately on each level set. Indeed we have

$$\Theta(\cdot, \frac{1}{f}F) = \frac{1}{f}dH = d\left(\frac{1}{f}(H-c)\right),$$

where the last equality is valid only on the level set $\{H = c\}$.

Finally, let us consider the symplectic form $e^{-U}\Theta$. On the level set $\{H = c\}$ we have

$$d(e^{-U}(H-c)) = e^{-U}dH.$$

It follows that on this level set

$$e^{-U}\Theta(\cdot, F) = d\left(e^{-U}(H-c)\right),$$

and, as a result, the vector field $F$ coincides locally with the Hamiltonian vector field given by the Hamiltonian $e^{-U}(H-c)$ (with respect to the symplectic form $e^{-U}\Theta$). This observation provides an alternate way to derive (2.4) by using the preservation of the symplectic form $e^{-U}\Theta$ by any Hamiltonian flow (with respect to this symplectic form).

For a fixed level set $M^c$ we introduce the quotient of the tangent bundle $TM^c$ of $M^c$ by the vector field $F = \nabla_\Theta H$, i.e., by the one dimensional subspace spanned by $F$. Let us denote the quotient bundle by $\widehat{T}M^c$. The form $\Theta$ factors naturally from $TM^c$ to $\widehat{T}M^c$, in view of (2.2). The factor form defines in each of the quotient tangent spaces $\widehat{T}_z M^c, z \in M^c$, a linear symplectic form.

The derivative of the flow preserves the vector field $F$, i.e.,

$$D_z\Phi^t(F(z)) = F(\Phi^t z).$$

As a result the derivative can be also factored on the quotient bundle and we call it the transversal derivative cocyle and denote it by

$$A^t(z) : \widehat{T}_z M^c \to \widehat{T}_{\Phi^t z} M^c.$$

It follows immediately from (2.4) that the transversal derivative cocycle is conformally symplectic with respect to $\Theta$ (or more precisely the linear symplectic form it defines in the quotient tangent spaces). We fix an invariant probability measure $\mu_c$ on $M^c$ and assume that

$$\int_{M^c} \|D_z\Phi^t\| d\mu_c(z) < +\infty.$$

Under this assumption the derivative cocycle has well defined Lyapunov exponents, cf. [O],[R]. Then the transversal derivative cocycle has also well defined Lyapunov exponents which coincide with the former except that one zero Lyapunov exponent is skipped. We can immediately apply Theorem 1.4 to the transversal derivative cocycle and we get the following.

**Theorem 2.2.** *For a Hamiltonian flow $\Phi^t$, defined by the vector field $F = \nabla_\Theta H$, restricted to one level set $M^c$ we have the following symmetry of the Lyapunov spectrum of the transversal derivative cocyle with respect to an invariant ergodic probability measure $\mu$. Let*

$$\{0\} \subset \mathbb{V}_0(z) \subset \mathbb{V}_1(z) \subset \cdots \subset \mathbb{V}_{s-1}(z) \subset \mathbb{V}_s = \widehat{T}_z M^c$$

*be the flag of subspaces at $z$ associated with the Lyapunov spectrum $\lambda_1 < \lambda_2 < \cdots < \lambda_{s-1} < \lambda_s$ of the transversal derivative cocycle $A^t(z)$, $z \in M^c$. Then the multiplicities of $\lambda_k$ and $\lambda_{s-k+1}$ are equal and*

$$\lambda_k + \lambda_{s-k+1} = a, \quad \text{for} \quad k = 1, 2, \ldots, s,$$

*where $a = \int_{M^c} \gamma(F(z)) d\mu_c(z)$. Moreover the subspace $\mathbb{V}_{s-k}$ is the skew-orthogonal complement of the subspace $\mathbb{V}_k$.*

□

Note that the invariant measure $\mu_c$ can be supported on a single periodic orbit, so that Theorem 2.2 applies as well to the real parts of the Floquet exponents.

To apply Theorem 1.4 it is enough to have linear symplectic forms in each of the quotient tangent spaces (to the level set), not necessarily coming from a conformally symplectic structure on the phase space. But then one needs to check directly how the transversal derivative cocycle acts on these forms, because we do not have the advantage of Theorem 2.1. This is essentially the line of argument in [D-M 1] and [D-M 3].

## §3. Conformally symplectic flows with collisions.

Let $M$ be a smooth manifold with piecewise smooth boundary $\partial M$. We assume that the manifold $M$ is equipped with a conformally symplectic structure $\Theta$, as defined in Section 2. Given a smooth function $H$ on $M$ with *non vanishing* differential we obtain the non vanishing conformally Hamiltonian vector field $F = \nabla_\Theta H$ on $M$. The vector field $F$ is tangent to the level sets of the Hamiltonian $M^c = \{z \in M | H(z) = c\}$.

We distinguish in the boundary $\partial M$ the regular part, $\partial M_r$, consisting of the points which do not belong to more than one smooth piece of the boundary and where the vector field $F$ is transversal to the boundary. The regular part of the boundary is further split into "outgoing" part, $\partial M_-$, where the vector field $F$ points outside the manifold $M$ and the "incoming" part, $\partial M_+$, where the vector field is directed inside the manifold. Suppose that additionally we have a piecewise smooth mapping $\Gamma : \partial M_- \to \partial M_+$, called the collision map. We assume that the mapping $\Gamma$ preserves the Hamiltonian, $H \circ \Gamma = H$, and so it can be restricted to each level set of the Hamiltonian.

We assume that all the integral curves of the vector field $F$ that end (or begin) in the singular part of the boundary lie in a codimension 1 submanifold of $M$.

We can now define a flow $\Psi^t : M \to M$, called a flow with collisions, which is a concatenation of the continuous time dynamics $\Phi^t$ given by the vector field $F$, and the collision map $\Gamma$. More precisely a trajectory of the flow with collisions, $\Psi^t(x)$, $x \in M$, coincides with the trajectory of the flow $\Phi^t$ until it gets to the boundary of $M$ at time $t_c(x)$, the collision time. If the point on the boundary lies in the singular part then the flow is not defined for times $t > t_c(x)$ (the trajectory "dies" there). Otherwise the trajectory is continued at the point $\Gamma(\Psi^{t_c}x)$ until the next collision time, i.e., for $0 \leq t \leq t_c\left(\Gamma(\Psi^{t_c(x)}x)\right)$

$$\Psi^{t_c+t}x = \Phi^t\Gamma\Psi^{t_c}x.$$

We define a flow with collisions to be conformally symplectic, if for the collision map $\Gamma$ restricted to any level set $M^c$ of the Hamiltonian we have

$$\Gamma^*\Theta = \beta\Theta, \tag{3.1}$$

for some non vanishing function $\beta$ defined on the boundary. More explicitly we assume that for every vectors $\xi$ and $\eta$ from the tangent space $T_z\partial M^c$ to the boundary of the level set $M^c$ we have

$$\Theta(D_z\Gamma\xi, D_z\Gamma\eta) = \beta\Theta(\xi, \eta).$$

We restrict the flow with collisions to one level set $M^c$ of the Hamiltonian and we denote the resulting flow by $\Psi^t_c$. This flow is very likely to be badly discontinuous but we can expect that for a fixed time $t$ the mapping $\Psi^t_c$ is piecewise smooth, so that the derivative $D\Psi^t_c$ is well defined except for a finite union of codimension one submanifolds of $M^c$. We will consider only such cases. We choose an invariant measure in our system which satisfies the condition that all the trajectories that begin (or end) in the singular part of the boundary have measure zero. Usually there are many natural invariant measures satisfying this property. For instance we get one by taking a Lebesgue measure $\nu$ in $M^c$ and averaging it over increasing time intervals ($\frac{1}{T}\int_0^T \Psi^t_{c*}\nu dt$ as $T \to +\infty$). Let us denote the chosen invariant measure by $\mu_c$. This measure $\mu_c$ defines the measure $\mu_{cb}$ on the boundary $\partial M^c$, which is an invariant measure for the section of the flow (Poincaré map of the flow). With respect to the measure $\mu_c$ the flow $\Psi^t_c$ is a measurable flow in the sense of the Ergodic Theory and we obtain a measurable derivative cocycle $D\Psi^t_c : T_xM^c \to T_{\Psi^t_c x}M^c$. We can define Lyapunov exponents of the flow $\Psi^t_c$ with respect to the measure $\mu_c$, if we assume that

$$\int_{M^c} \log_+ ||D_x\Psi^t_c|| d\mu_c(x) < +\infty \quad \text{and} \quad \int_{\partial M^c_-} \log_+ ||D_y\Gamma|| d\mu_{cb}(y) < +\infty$$

(cf.[O],[R]).

The derivative of the flow with collisions can be also naturally factored onto the quotient of the tangent bundle $TM^c$ of $M^c$ by the vector field $F$, which we denote by $\widehat{T}M^c$. Note that for a point $z \in \partial M^c$ the tangent to the boundary at $z$ can be naturally identified with the quotient space.

We will again denote the factor of the derivative cocycle by

$$A^t(x) : \widehat{T}_xM^c \to \widehat{T}_{\Psi^t_c x}M^c.$$

We will call it the transversal derivative cocycle. If the derivative cocycle has well defined Lyapunov exponents then the transversal derivative cocycle has also well defined Lyapunov exponents which coincide with the former ones except that one zero Lyapunov exponent is skipped.

For a conformally symplectic flow with collisions the factor $A^t(x)$ of the derivative cocycle on one level set changes the form $\Theta$ by a scalar, (2.3) and (3.1), so that we can immediately apply Theorem 1.4 and we get

**Theorem 3.1.** *For a conformally symplectic flow with collisions $\Psi^t_c$ we have the following symmetry of the Lyapunov exponents for a given ergodic invariant probability measure $\mu_c$. Let $\{0\} \subset \mathbb{V}_0 \subset \mathbb{V}_1 \subset \dots \mathbb{V}_{s-1} \subset \mathbb{V}_s = \widehat{T}_xM^c$ be the flag of subspaces at $x$ associated with the Lyapunov spectrum $\lambda_1(x) < \lambda_2(x) < \dots < \lambda_{s-1}(x) < \lambda_s(x)$ of the transversal derivative cocycle $A^t(x)$, $x \in M^c$. Then the multiplicities of $\lambda_k$ and $\lambda_{s-k+1}$ are equal and*

$$\lambda_k + \lambda_{s-k+1} = a + b, \quad \text{for} \quad k = 1, 2, \dots, s,$$

*where $a = \int_{M^c} \gamma(F) d\mu_c$ and $b = \frac{1}{\tau}\int_{\partial M^c_-} \log|\beta(y)| d\mu_{cb}(y)$. $\tau = \int_{\partial M^c_-} t_c(y) d\mu_{cb}(y)$ is the average collision time on the section of the flow. Moreover the subspace $\mathbb{V}_{s-k}$ is the skew-orthogonal complement to $\mathbb{V}_k$.*

## §4. Applications.

**A. Gaussian isokinetic dynamics.**

The equations of the system are (cf. [D-M 1])

$$\begin{aligned} &\dot{q} = p, \\ (4.1) \qquad & \\ &\dot{p} = E - \alpha p, \quad \text{where} \quad \alpha = \frac{\langle E, p\rangle}{\langle p, p\rangle}. \end{aligned}$$

In these equations $q$ describes a point in the multidimensional configuration space $\mathbb{R}^N$, $p$ is the momentum (velocity) also in $\mathbb{R}^N$ and $\langle\cdot,\cdot\rangle$ is the arithmetic scalar product in $\mathbb{R}^N$. The field of force $E = E(q)$ is assumed to be irrotational, i.e., it has locally a potential function $U = U(q)$, $E = -\frac{\partial U}{\partial q}$.

Let us denote by $\kappa = \sum p dq$ the 1-form which defines the standard symplectic structure $\omega = d\kappa = \sum dp \wedge dq$. We introduce the following 2-form

$$\Theta = \omega + \frac{\langle E, dq\rangle}{\langle p, p\rangle} \wedge \kappa.$$

We choose the Hamiltonian to be $H = \frac{1}{2}\langle p, p\rangle$ and we denote the vector field defined by (4.1) by $F$. We have

$$\Theta(\cdot, F) = dH, \tag{4.2}$$

but the form $\Theta$ does not give us a conformally symplectic structure because the relation (2.1) fails. To correct this setback we fix one level set of the Hamiltonian $M^c = \{H = \frac{1}{2}\langle p, p\rangle = c\}$ and define another 2-form

$$\Theta_c = \omega + \frac{\langle E, dq\rangle}{2c} \wedge \kappa.$$

Now we get a conformally symplectic structure. Indeed

$$d\Theta_c = -\frac{\langle E, dq\rangle}{2c} \wedge \Theta_c, \quad \text{and locally} \quad \frac{\langle E, dq\rangle}{2c} = d\left(\frac{-U}{2c}\right).$$

Moreover on $M^c$ we still have $\Theta_c(\cdot, F) = dH$ so that the restriction of (4.1) to $M^c$ coincides with a conformally Hamiltonian system with respect to the 2-form $\Theta_c$ and with the Hamiltonian $H = \frac{1}{2}\langle p, p\rangle$. We can immediately apply Theorem 2.1 and we obtain that for any invariant ergodic probability measure $\mu_c$ on $M^c$ the Lyapunov exponents $\lambda_1 < \cdots < \lambda_s$ satisfy

$$\lambda_k + \lambda_{s-k+1} = -\frac{1}{2c}\int_{M^c} \langle E, p\rangle d\mu_c = \int_{M^c} \alpha d\mu_c.$$

Note that if the vector field of force has a global potential, $E = -\frac{\partial U}{\partial q}$, then by the Birkhoff Ergodic Theorem the integral $-\frac{1}{2c}\int_{M^c}\langle E, p\rangle d\mu_c = \frac{1}{2c}\int_{M^c} dU(F) d\mu_c$ is equal to the time average of $\frac{dU}{dt}$ and so it must vanish. Another way to see it is that $e^{-U}\Theta_c$ defines a global symplectic structure and on $M_c$ our flow is Hamiltonian with respect to this symplectic structure and a modified Hamiltonian

$$\widetilde{H} = e^{-U}(\frac{1}{2}\langle p, p\rangle - c).$$

Indeed as discussed in the Remark 2.1 on $M^c$ we have

$$e^{-U}\Theta_c(\cdot, F) = d\widetilde{H}.$$

For a Hamiltonian flow the symmetric Lyapunov exponents must add up to zero.

**B. Gaussian isokinetic dynamics on a Riemannian manifold.**

For a given Riemannian manifold $N$ with the metric tensor $ds^2 = \sum g_{ij}dq_i dq_j$ we can naturally generalize the form $\Theta$ to the cotangent bundle $T^*N$. Indeed the 1-form $\kappa = \sum pdq$ is independent of the coordinate system, cf. [A], and for a given closed 1-form $\gamma$ we put

$$\Theta_c = d\kappa - \frac{1}{c}\gamma \wedge \kappa.$$

We get $d\Theta_c = -\frac{1}{c}\gamma\wedge\Theta_c$. Taking $\gamma = dU$ for some potential function (single or multi-valued) and the Hamiltonian $H = \frac{1}{2}\sum g^{ij}p_i p_j$ we obtain the Gaussian isokinetic dynamics, [Ch], on the level set $H = c$ by the relation (4.2). We can repeat the discussion in part A and we conclude again that the Lyapunov exponents must be symmetric and they add up to zero, if the potential $U$ is single-valued.

**C. The Gaussian isokinetic dynamics with collisions.**

Let us consider $n$ spherical particles in a finite box $B$ contained in $\mathbb{R}^d$ or the torus $\mathbb{T}^d$. We assume that the particles interact with each other by the potential $V(q_1, q_2, \ldots, q_n)$ ($q_k \in B$, $k = 1, \ldots, n$ denote the positions of the particles) and that they are subjected to the external fields given by the potentials $V_k(q_k)$, $k = 1, \ldots, n$. Further we assume that the particles have the radii $r_1, \ldots, r_n$, the masses $m_1, \ldots, m_n$, and that they collide elastically with each other and the sides of the box, which can be flat or curved. The last element in the description of the system is the Gaussian isokinetic thermostat. As described in part A and B the Gaussian isokinetic thermostat gives rise to a conformally Hamiltonian flow with the Hamiltonian $H = \sum_{k=1}^{n} \frac{p_k^2}{2m_k}$ and an appropriate conformally symplectic structure. We will check below that the collisions in this system preserve the form $\Theta_c$ giving rise to a conformally symplectic flow with collisions. Theorem 3.1 can be thus applied to our system giving us the symmetry of the Lyapunov spectrum.

We introduce the canonical change of variables which bring the kinetic energy into the standard form,

$$x_k = \sqrt{m_k}q_k$$
$$v_k = \frac{p_k}{\sqrt{m_k}}.$$

The advantage of these coordinates is that although the collision manifolds in the configuration space become less natural, the collisions between particles (and the walls of the box) are given by the billiard rule in the configuration space. The equations of motions in the $(x, v)$ coordinates are

$$\dot{x} = v$$
(4.4)
$$\dot{v} = -\frac{\partial U}{\partial x} + \alpha v,$$

where $U = U(x) = V + \sum_{k=1}^{n} V_k$ is the total potential of the system and $\alpha = -\frac{dU(v)}{\langle v,v\rangle}$. We introduce the differential 2-form

$$\Theta_c = \sum dv \wedge dx - \frac{1}{2c} dU \wedge \langle v, dx\rangle$$

As in part A we conclude that the form satisfies (2.1) and the system (4.4) restricted to $M^c$ coincides with the conformally Hamiltonian system defined by this form and the Hamiltonian $H = \frac{1}{2}\langle v, v\rangle$.

**Proposition 4.1.** *The collision maps preserve the form $\Theta_c$.*

*Proof.* A collision manifold is locally given by an equation of the form $g(x) = 0$, where $g$ is some differentiable $\mathbb{R}^{nd}$ valued map. Note that the general form of the collision map is the same for collisions of particles and the collisions with the sides of the box. Let $n(x)$, for $x \in \{x \in \mathbb{R}^{nd} | g(x) = 0\}$, denote the unit normal vector to the collision manifold in the configuration space. The collision map is defined as

$$x^+ = x^-,$$
(4.4)
$$v^+ = v^- - 2\langle v^-, n(x^-)\rangle n(x^-).$$

where the index $^+$ corresponds to the values of $x$ and $v$ after the collision and the index $^-$ to the values before the collision. As a result of these formulas we get immediately that

$$\delta x^+ = \delta x^-. \tag{4.5}$$

It is well known, [W1],[W2], that in an elastic collision the symplectic form $\omega$ is preserved. It remains to show the preservation of the second term in $\Theta_c$. It follows immediately from (4.4) and (4.5), because

$$\langle v^+, \delta x^+\rangle = \langle v^-, \delta x^-\rangle - 2\langle v^-, n(x^-)\rangle\langle n(x^-), \delta x^-\rangle,$$

and the last term is zero since we only take the variations $(\delta x^-, \delta v^-)$ tangent to the collision manifold, i.e., $\delta x^-$ is orthogonal to $n(x)$.

The Proposition is proven.

□

It follows from Proposition 4.1 that also the form $e^{-U}\Theta_c$ is preserved under collisions. Hence, as remarked in parts A and B, if the potential $U$ is singlevalued then the system restricted to one energy level coincides with a globally Hamiltonian system (with collisions) with respect to the symplectic form $e^{-U}\Theta_c$ with the Hamiltonian function equal to $\widetilde{H} = e^{-U}(\frac{1}{2}\langle p, p\rangle - c)$. We conclude that the occurrence of dissipation in such systems is related to the topology of the configuration space (the multivaluedness of the potential $U$).

**D. Nosé-Hoover dynamics**

The Nosé Hamiltonian is, cf. [D-M 3]

$$H(q, s; \pi, p_s) = \sum_{i=1}^{N} \frac{\pi_i^2}{2m_i s^2} + \varphi(q) + \frac{p_s^2}{2} + C \ln s,$$

with a non-physical time denoted by $\lambda$ and some constant $C$. The symplectic form is $\omega = \sum d\pi \wedge dq + dp_s \wedge ds$. Changing the variables as $\pi = sp$ and $\sigma = \ln s$ the Hamiltonian becomes

$$H(q, s; p, p_s) = \sum_{i=1}^{N} \frac{p_i^2}{2m_i} + \varphi(q) + \frac{p_s^2}{2} + C\sigma, \tag{4.6}$$

and the symplectic form is $\omega = e^{\sigma}\left(\sum_i dp_i \wedge dq_i + dp_s \wedge d\sigma + d\sigma \wedge (\sum_i p_i dq_i)\right)$. Note that now in the Hamiltonian the thermostat $(\sigma, p_s)$ is decoupled from the system but the coupling is shifted to the symplectic form. We make finally the time change $\frac{d\lambda}{dt} = e^{\sigma}$. We choose not to change the Hamiltonian but rather to modify the 2-form,

$$e^{-\sigma}\omega(\cdot,\, e^{\sigma}\nabla_\omega H) = dH.$$

We end up with the Hamiltonian (4.6) and the conformally symplectic structure

$$\Theta = e^{-\sigma}\omega = \sum_i dp_i \wedge dq_i + dp_s \wedge d\sigma + d\sigma \wedge (\sum p_i dq_i + p_s d\sigma).$$

We have $d\Theta = d\sigma \wedge \Theta$. Note the similarity of $\Theta$ with the form used in the discussion of the isokinetic dynamics above. This form and the Hamiltonian give us the Hoover equations

$$\begin{aligned}
\dot{q}_i &= \frac{p_i}{m_i},\\
\dot{p}_i &= -\frac{\partial\varphi}{\partial q_i} - p_s p_i,\\
\dot{\sigma} &= p_s,\\
\dot{p}_s &= \sum_i \frac{p_i^2}{m_i} - C.
\end{aligned}$$

On any level set we can drop the equation for $\sigma$ since $\sigma$ can be trivially obtained from other variables using the constancy of the Hamiltonian.

By Theorem 2.1 we have the symmetry of the Lyapunov spectrum for this system reduced to one level of the Hamiltonian. Moreover the Lyapunov exponents add up to the time average $\dot{\sigma}$. This average must be zero, unless $p_s$ grows linearly, which is unlikely. Note that the Nosé–Hoover system is open in the sense that arbitrarily large fluctuations of $p_s$ cannot be ruled out.

## References


[A] V. I. Arnold, *Mathematical Methods in Classical Mechanics*, Springer Verlag, 1978.

[A-M-R] R. Abraham, J.E. Marsden, T. Ratiu, *Manifolds, tensor analysis and applications*, Springer Verlag, 1988.

[B-G-G-S] G. Benettin, I. Galgani, A. Giorgilli, J.-M. Strelcyn, *Lyapunov characteristic exponents for smooth dynamical systems and for Hamiltonian systems; a method for computing all of them*, Meccanica (1980), 9-20.

[Ch] Ph. Choquard, *Lagrangian formulation of Nosé–Hoover and of isokinetic dynamics*, ESI report (1996).

[Ch-E-L-S] N.I. Chernov, G.L. Eyink, J.L. Lebowitz, Ya.G. Sinai, *Steady-state electric conduction in the periodic Lorentz gas*, Commun. Math. Phys. **154** (1993), 569-601.

[D-P-H] C.P. Dellago, H.A. Posch, W.G. Hoover, *Lyapunov instability in a system of hard disks in equilibrium and nonequilibrium steady states*, Phys. Rev. E **53 n.2** (1996), 1485-1501.

[D-M 1] C.P. Dettmann, G.P. Morriss, *Proof of Lyapunov exponent pairing for systems at constant kinetic energy*, Phys. Rev. E **53** (1996), R5541.

[D-M 2] C.P. Dettmann, G.P. Morriss, *Hamiltonian formulation of the Gaussian isokinetic thermostat*, To appear in Phys. Rev. E.

[D-M 3] C.P. Dettmann, G.P. Morriss, *Hamiltonian reformulation and pairing of Lyapunov exponents for Nosé–Hoover dynamics*, To appear in Phys. Rev. E.

[L-B-D] A. Latz, H. van Beijeren, J.R. Dorfman, *Lyapunov spectrum and the conjugate pairing rule for a thermostatted random Lorentz gas: kinetic theory*, preprint (1996).

[O] V. I. Oseledets, *A multiplicative ergodic theorem: characteristic Lyapunov exponents of dynamical systems*, Trans. Moscow Math. Soc. **19** (1968), 197–231.

[R] D. Ruelle, *Ergodic theory of differentiable dynamical systems*, Publ. Math. IHES **50** (1979), 27 -58.

[V] I. Vaisman, *Locally conformal symplectic manifolds*, Int. J. Math.–Math.Sci. **8** (1985), 521–536.

[W1] M. P. Wojtkowski, *Measure Theoretic Entropy of the system of hard spheres*, Ergodic Theory and Dynamical Systems **8** (1988), 133–153.

[W2] M. P. Wojtkowski, *Systems of classical interacting particles with nonvanishing Lyapunov exponents*, Lyapunov Exponents, Proceedings, Oberwolfach 1990, L. Arnold, H. Crauel, J.-P. Eckmann (Eds.), Lecture Notes in Math., vol. 1486, 1991, pp. 243-262.



Liverani Carlangelo, Mathematics Department, University of Rome "Tor Vergata", 00133 Rome, Italy. Maciej P. Wojtkowski, Department of Mathematics, University of Arizona, Tuscon, Arizona 85 721 USA

*E-mail address*: liverani@mat.utovrm.it maciejw@math.arizona.edu