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Some Aspects of Noncommutative Differential Geometry

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Abstract

We discuss in some generality aspects of noncommutative differential geometry associated with reality conditions and with differential calculi. We then describe the differential calculus based on derivations as generalization of vector fields, and we show its relations with quantum mechanics. Finally we formulate a general theory of connections in this framework.

1 Introduction

In [23], J.L. Koszul described a powerful algebraic version of differential geometry in terms of a commutative associative algebra \mathcal{C} , \mathcal{C} -modules and connections (“derivation laws”) on these modules. For the applications to differential geometry, \mathcal{C} is the algebra of smooth functions on a manifold and the \mathcal{C} -modules are modules of smooth sections of smooth vector bundles over the manifold. The fact that classical differential geometry admits such an algebraic formulation is at the very origin of the idea of noncommutative differential geometry. Historically, the motivation of noncommutative geometry was the development of quantum theory [12]. In noncommutative geometry, one replaces the commutative associative algebra \mathcal{C} by an associative algebra \mathcal{A} which is not assumed to be commutative. However this replacement raises several problems which will be discussed in this lecture.

First problem: what should replace the \mathcal{C} -modules? The problem arises because there are at least four inequivalent generalizations of the notion of a module over a commutative algebra when the algebra is replaced by a noncommutative algebra \mathcal{A} . There is the notion of right \mathcal{A} -module and the dual notion of left \mathcal{A} -module. If one recalls that a module over a commutative algebra is canonically a bimodule (of a specific kind), there is a notion of

bimodule over \mathcal{A} . Finally, since a commutative algebra coincides with its center, there is the notion of module over the center $Z(\mathcal{A})$ of \mathcal{A} . As will be explained later, there is also a duality between $Z(\mathcal{A})$ -modules and bimodules over \mathcal{A} .

Second problem: what should be the generalization of the classical notions of reality? For classical differential geometry one can use for \mathcal{C} either the real commutative algebra of smooth real-valued functions or the complex commutative $*$ -algebra of smooth complex-valued functions. More generally, if \mathcal{C} is a complex commutative $*$ -algebra then the set \mathcal{C}^h of its hermitian elements is a real commutative algebra and \mathcal{C} is the complexification of \mathcal{C}^h . Conversely if $\mathcal{C}_{\mathbb{R}}$ is a real commutative algebra, then its complexification \mathcal{C} is canonically a complex commutative $*$ -algebra and one has $\mathcal{C}_{\mathbb{R}} = \mathcal{C}^h$. In fact $\mathcal{C} \mapsto \mathcal{C}^h$ is an equivalence of the category of commutative associative $*$ -algebras over \mathbb{C} and $*$ -homomorphisms onto the category of commutative associative algebras over \mathbb{R} and homomorphisms of real algebras. The situation is quite different for noncommutative algebras. If \mathcal{A} is a complex associative $*$ -algebra, the set \mathcal{A}^h of its hermitian elements is generally not an associative algebra but a real Jordan algebra. This means that one has two choices for the generalization of the algebra of real-valued functions, either the real Jordan algebra \mathcal{A}^h of all hermitian elements of a complex associative $*$ -algebra \mathcal{A} , which plays the role of the algebra of complex-valued functions, or a real associative algebra. Here we take the first point of view. This choice, which is the standard one, is dictated by quantum theory and, more generally, by spectral theory. This reality problem is not independent of the first problem because if \mathcal{C} is a complex commutative associative $*$ -algebra there is again an obvious equivalence between the involutive \mathcal{C} -modules and

the \mathcal{C}^h -modules [18].

Third problem: which differential calculus should be used? In other words what should be the generalization of differential forms? Such a generalization is needed for instance to define connections. There is a minimal set of assumptions which must be satisfied and which will be described in the sequel but nevertheless the choice is not straightforward. We can make here the following remarks. In his pioneer work on the subject [7], A. Connes defined the cyclic cohomology of an algebra and showed that the correct generalization of the homology of a manifold is the reduced cyclic cohomology. This means that the generalization of the cohomology of a manifold in noncommutative geometry must be the reduced cyclic homology of the algebra \mathcal{A} which replaces the algebra of smooth functions. In classical differential geometry, the de Rham theorem states that the cohomology of a manifold, (a topological invariant), coincides with the cohomology of its differential forms. This does not mean that any cochain complex which has the reduced cyclic homology as cohomology is an acceptable generalization of differential forms, and this for at least two reasons. First, even in the classical situation, there are many ways to compute the cohomology of a manifold and, in particular, there are complexes which are not connected with the differential structure and which have this cohomology. Second, the de Rham theorem is not a tautological result but a deep theorem of differential topology which means that there may well be proper noncommutative generalizations of differential geometry for which the generalization of de Rham theorem fails to be true.

The problems quoted above will be discussed in the first part of this lecture. Then the differential calculus based on the derivations as generalization

of vector fields will be introduced [13]. This differential calculus is the direct generalization of the one used by J.L. Koszul in [23]; it is also connected with the differential calculus used by A. Connes for noncommutative dynamical systems in [8]. The noncommutative symplectic structures will be defined in this framework and the relation with quantum mechanics will be described. Finally we shall describe the theory of connections in this framework. Examples of such connections and applications to gauge field theory may be found in [14], [15], [16], [18], [24].

Let \mathcal{A} be an associative algebra. If M and N are right \mathcal{A} -modules, the space of all right \mathcal{A} -module homomorphisms of M into N will be denoted by $\text{Hom}^{\mathcal{A}}(M, N)$; if M and N are left \mathcal{A} -modules, the space of all left \mathcal{A} -module homomorphisms of M into N will be denoted by $\text{Hom}_{\mathcal{A}}(M, N)$. When \mathcal{A} is a commutative algebra \mathcal{C} , both notions coincide and the space of \mathcal{C} -module homomorphisms of M into N will be denoted by $\text{Hom}_{\mathcal{C}}(M, N)$. If \mathcal{B} is another associative algebra and if M and N are $(\mathcal{A}, \mathcal{B})$ -bimodules, $\text{Hom}_{\mathcal{A}}^{\mathcal{B}}(M, N)$ will denote the space of all bimodules homomorphisms of M into N . In the sequel, we shall often use the word algebra to mean associative algebra.

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2 Modules, bimodules and reality

In the following \mathcal{A} is a complex unital associative $*$ -algebra which is to be considered as a noncommutative generalization of an algebra of complex func-

tions. As a consequence what must replace the algebra of real-valued functions is generally not an associative real algebra but the Jordan algebra \mathcal{A}^h of hermitian elements of \mathcal{A} . Although quite familiar in quantum theory, this fact has non trivial consequences for the noncommutative generalization of classical reality conditions. In fact we are here interested in noncommutative differential geometry, which means that \mathcal{A} is to be considered as the generalization of the algebra of complex smooth functions on a manifold. The Jordan algebra \mathcal{A}^h replaces then the algebra of real smooth functions.

Let E be a smooth complex vector bundle of finite rank over a manifold V . Then the set $?(E)$ of its smooth sections is a finite projective module over the algebra $\mathcal{C}^\infty(V)$ of smooth complex functions on V . Furthermore the correspondence $E \mapsto ?(E)$ is an equivalence of the category of smooth complex vector bundles of finite rank over V onto the category of finite projective modules over $\mathcal{C}^\infty(V)$. Let now $E_{\mathbb{R}}$ be a smooth real vector bundle over V , its complexification E is a smooth complex vector bundle over V equipped with a canonical antilinear involution $\xi \mapsto \xi^*$ such that $\xi \in E_{\mathbb{R}}$ if and only if $\xi = \xi^*$. The module $?(E)$ is then a $*$ -module over the $*$ -algebra $\mathcal{C}^\infty(V)$ in the sense that it is equipped with an antilinear involution $\psi \mapsto \psi^*$ such that $(f\psi)^* = f^*\psi^*$, $\forall f \in \mathcal{C}^\infty(V)$ and $\forall \psi \in ?(E)$, where $f \mapsto f^*$ is the complex conjugation. A section of $E_{\mathbb{R}}$ is a section $\psi \in ?(E)$ such that $\psi = \psi^*$. Clearly, one can replace $E_{\mathbb{R}}$ by the $*$ -module $?(E)$. With this in mind, let us more generally consider the notion of module and the notion of $*$ -module over a commutative $*$ -algebra \mathcal{C} and investigate their generalizations when \mathcal{C} is replaced by the noncommutative $*$ -algebra \mathcal{A} .

As pointed out in the introduction a \mathcal{C} -module has several natural gener-

alizations: a right \mathcal{A} -module, a left \mathcal{A} -module, a module over the center $Z(\mathcal{A})$ of \mathcal{A} and a bimodule over \mathcal{A} . Right \mathcal{A} -modules and left \mathcal{A} -modules are dual in the sense that if M is a right \mathcal{A} -module, its dual $M^* = \text{Hom}^{\mathcal{A}}(M, \mathcal{A})$ is a left \mathcal{A} -module and if N is a left \mathcal{A} -module, its dual $N^* = \text{Hom}_{\mathcal{A}}(N, \mathcal{A})$ is a right \mathcal{A} -module; this duality generalizes the duality of \mathcal{C} -modules. Similarly, there is a natural duality between bimodules over \mathcal{A} and $Z(\mathcal{A})$ -modules [18]: if M is a bimodule over \mathcal{A} , its \mathcal{A} -dual $M^{*\mathcal{A}} = \text{Hom}_{\mathcal{A}}^{\mathcal{A}}(M, \mathcal{A})$ is canonically a $Z(\mathcal{A})$ -module and if N is a $Z(\mathcal{A})$ -module, its \mathcal{A} -dual $N^{*\mathcal{A}} = \text{Hom}_{Z(\mathcal{A})}(N, \mathcal{A})$ is canonically a bimodule over \mathcal{A} . This duality (\mathcal{A} -duality) also generalizes the duality of \mathcal{C} -modules when the bimodules over \mathcal{C} are the underlying bimodules of \mathcal{C} -modules.

Concerning the generalization of $*$ -modules over \mathcal{C} , (i.e. the generalization of the description of real vector bundles), one notices that one cannot use right or left \mathcal{A} -modules because, since the involution of \mathcal{A} reverses the order of the product in \mathcal{A} , there cannot be a notion of right or left $*$ -module over \mathcal{A} . In contrast, since $Z(\mathcal{A})$ is a commutative $*$ -algebra, the notion of $*$ -module over $Z(\mathcal{A})$ is perfectly defined and one can introduce a dual notion of $*$ -bimodule over \mathcal{A} : a bimodule M over \mathcal{A} is a $*$ -bimodule over \mathcal{A} if it is equipped with an antilinear involution $m \mapsto m^*$ such that one has $(xmy)^* = y^*m^*x^* \quad \forall x, y \in \mathcal{A} \text{ and } \forall m \in M$.

Thus, simple considerations of reality rule out right or left \mathcal{A} -modules for the description of a generalization of real vector bundles. This does not mean that one cannot use them for the generalization of complex vector bundles, this simply means that all the above generalizations of the notion of \mathcal{C} -module have to be considered when \mathcal{C} is replaced by the noncommutative

algebra \mathcal{A} . The fact that bimodule structures arise in connection with reality in noncommutative geometry has been also pointed out in [10] by A. Connes in the context of his spectral triples approach to noncommutative geometry [8],[9].

It must be stressed that not every bimodule over a commutative algebra \mathcal{C} is the underlying bimodule of a \mathcal{C} -module and therefore not every bimodule over \mathcal{A} can be considered as the generalization of a \mathcal{C} -module. One must select an appropriate class of bimodules, for instance the class of central bimodules [18],[19]. A bimodule M over \mathcal{A} is called a *central bimodule* if one has $zm = mz$, $\forall m \in M$ and $\forall z \in Z(\mathcal{A})$. A central bimodule over a commutative algebra \mathcal{C} is just a \mathcal{C} -module for its underlying bimodule structure. In [19], the more restrictive notion of diagonal bimodule was introduced. A bimodule M over \mathcal{A} is called a *diagonal bimodule* if it is isomorphic to a subbimodule of \mathcal{A}^I for some set I . A diagonal bimodule is central. A bimodule M over \mathcal{A} is diagonal if and only if the canonical mapping of M into its \mathcal{A} -bidual $M^{*\mathcal{A}*\mathcal{A}}$ is injective. In particular a diagonal bimodule over a commutative algebra \mathcal{C} is just a \mathcal{C} -module such that the canonical mapping in its bidual is injective; projective \mathcal{C} -modules are therefore diagonal bimodules. If N is a $Z(\mathcal{A})$ -module, its \mathcal{A} -dual $N^{*\mathcal{A}}$ is a diagonal bimodule over \mathcal{A} .

3 Differential calculus

In this section we wish to discuss some general features of the noncommutative versions of differential forms.

A *graded differential *-algebra* is a complex graded differential algebra

$\Omega = \bigoplus_{n \in \mathbb{N}} \Omega^n$ equipped with an antilinear involution $\omega \mapsto \omega^*$ which preserves the degree and satisfies $(\alpha\beta)^* = (-1)^{ab}\beta^*\alpha^*$ and $(d\omega)^* = d(\omega^*)$ for $\alpha \in \Omega^a, \beta \in \Omega^b$ and $\omega \in \Omega$, where d is the differential of Ω . Notice that then Ω^0 is a $*$ -algebra. Given (as before) the complex unital $*$ -algebra \mathcal{A} , a *differential calculus over \mathcal{A}* is a graded differential $*$ -algebra Ω with $\Omega^0 = \mathcal{A}$. Among the differential calculi over \mathcal{A} , there is a universal one [21], $\Omega_u(\mathcal{A})$, which we now review.

Let $\mu : \mathcal{A} \otimes \mathcal{A} \rightarrow \mathcal{A}$ be the product $\mu(x \otimes y) = xy$. The mapping μ is a bimodule homomorphism so its kernel $\Omega_u^1(\mathcal{A})$ is a bimodule over \mathcal{A} . One defines a derivation d_u of \mathcal{A} into $\Omega_u^1(\mathcal{A})$ by setting $d_u x = \mathbb{1} \otimes x - x \otimes \mathbb{1}$ for $x \in \mathcal{A}$. The pair $(\Omega_u^1(\mathcal{A}), d_u)$ is characterized uniquely (up to an isomorphism) by the following universal property [4], [1]: given a derivation $\delta : \mathcal{A} \rightarrow M$ of \mathcal{A} into a bimodule M over \mathcal{A} , there is a unique bimodule homomorphism $j_\delta : \Omega_u^1(\mathcal{A}) \rightarrow M$ such that $\delta = j_\delta \circ d_u$. Let $\Omega_u(\mathcal{A})$ be the tensor algebra over \mathcal{A} of the bimodule $\Omega_u^1(\mathcal{A})$ i.e. $\Omega_u^0(\mathcal{A}) = \mathcal{A}$ and $\Omega_u^n(\mathcal{A}) = \bigotimes_{\mathcal{A}}^n \Omega_u^1(\mathcal{A})$ for $n \geq 1$. The derivation d_u extends uniquely into a differential, again denoted by d_u , of the graded algebra $\Omega_u(\mathcal{A})$. Using the above universal property of $(\Omega_u^1(\mathcal{A}), d_u)$ and the universal property of the tensor product over \mathcal{A} , one sees that the graded differential algebra $\Omega_u(\mathcal{A})$ is characterized by the following universal property: given a graded differential algebra $\Omega = \bigotimes_n \Omega^n$ with $\Omega^0 = \mathcal{A}$, there is a unique homomorphism of graded differential algebra $\varphi : \Omega_u(\mathcal{A}) \rightarrow \Omega$ which induces the identity mapping of \mathcal{A} onto itself (i.e. $\varphi \upharpoonright \mathcal{A} = id_{\mathcal{A}}$). Furthermore, there is a unique antilinear involution $\omega \mapsto \omega^*$ on $\Omega_u(\mathcal{A})$ which extends the involution of \mathcal{A} and for which it is a graded differential $*$ -algebra [27]; this involution is induced on $\Omega_u^n(\mathcal{A}) (\subset \bigotimes^{n+1} \mathcal{A})$ by the involution of $\bigotimes^{n+1} \mathcal{A}$ defined by $(x_0 \otimes x_1 \otimes \dots \otimes x_n)^* = (-1)^{\frac{n(n+1)}{2}} x_n^* \otimes \dots \otimes x_1^* \otimes x_0^*$. Equipped with

this involution, $\Omega_u(\mathcal{A})$ is a differential calculus over \mathcal{A} which is universal in the sense that for any differential calculus Ω over \mathcal{A} there is a unique homomorphism of graded differential $*$ -algebra of $\Omega_u(\mathcal{A})$ into Ω which induces the identity mapping of \mathcal{A} onto itself.

One can expect, and it is our point of view here, that a noncommutative generalization of differential forms is a differential calculus over \mathcal{A} when \mathcal{A} replaces the algebra $\mathcal{C}^\infty(V)$ of smooth functions on a manifold V . However not every differential calculus over \mathcal{A} is appropriate. For instance the universal differential calculus is not a proper generalization of the algebra of differential forms. Indeed $\Omega_u(\mathcal{C}^\infty(V))$ does not coincide with the algebra $\Omega(V)$ of differential forms on V although, by the universal property, there is a homomorphism of graded differential algebra of $\Omega_u(\mathcal{C}^\infty(V))$ into $\Omega(V)$. More generally, if \mathcal{C} is a commutative algebra, the bimodule $\Omega_u^1(\mathcal{C})$, for instance, is not the underlying bimodule of a module since left and right multiplications by elements of \mathcal{C} do not coincide. In any case the choice of a differential calculus Ω over \mathcal{A} as generalization of the algebra of complex differential forms is not unique and depends on the applications one has in mind [7], [8], [11], [13], [18], [20], [22], [24], [27]. In the next section we will describe a choice for Ω based on derivations as generalization of vector fields. This choice, which is a direct generalization of [23], is natural in the sense that it only depends on the algebra \mathcal{A} (and not on additional structures).

Before leaving this section, two points are worth noticing. First some authors, e.g. G. Maltiniotis [25], consider that a proper generalization of differential geometry is given by a graded differential algebra which then replaces the algebra of differential forms; this point of view is more general

than the one, implicit here, where \mathcal{A} replaces the algebra of smooth functions. Second there are generalizations of the space of differential forms which are not differential algebras but merely differential complexes. For instance, it was shown in [21] that the subspace $[\Omega_u(\mathcal{A}), \Omega_u(\mathcal{A})]$ of graded commutators in $\Omega_u(\mathcal{A})$ is stable by d_u and that the cohomology of the complex $(\Omega_u(\mathcal{A})/[\Omega_u(\mathcal{A}), \Omega_u(\mathcal{A})], d_u)$, (which is not a differential algebra in general), is the reduced cyclic homology of \mathcal{A} which in many aspects is a good generalization of de Rham cohomology. This is why this complex is a natural generalization of the de Rham complex which is often called *the noncommutative de Rham complex*.

4 Derivations and differential calculus

In this section we explain our approach to the (noncommutative) differential calculus over \mathcal{A} , (a complex unital $*$ -algebra), based on the derivations of \mathcal{A} as generalization of vector fields [13], [14], [15], [16], [18], [19], [20]. This approach is a noncommutative generalization of the one of J.L. Koszul [23] which is based on the fact that a vector field on a manifold V , i.e. a smooth section of the tangent bundle over V , is the same thing as a derivation of the algebra $\mathcal{C}^\infty(V)$ of smooth functions on V . More generally, since the derivations are the infinitesimal algebra automorphisms, they are the natural right-hand sides of differential evolution equations. This is why the differential calculus based on derivations is the natural one for commutative and noncommutative dynamical systems i.e. for classical as well as for quantum mechanics.

Let $\text{Der}(\mathcal{A})$ denote the space of all derivations of \mathcal{A} , i.e. the space of

all linear mappings X of \mathcal{A} into itself satisfying the Leibniz rule $X(xy) = X(x)y + xX(y)$. The space $\text{Der}(\mathcal{A})$ is in a natural way a module over the center $Z(\mathcal{A})$ of \mathcal{A} and in fact a $*$ -module over $Z(\mathcal{A})$ when equipped with the involution $X \mapsto X^*$ defined by $X^*(x) = (X(x^*))^*$. The space $\text{Der}(\mathcal{A})$ is also a Lie algebra with Lie bracket $(X, Y) \mapsto [X, Y] = X \circ Y - Y \circ X$. This bracket satisfies the reality condition $[X, Y]^* = [X^*, Y^*]$. Furthermore, $Z(\mathcal{A})$ is stable under $\text{Der}(\mathcal{A})$ and one has $[X, zY] = X(z)Y + z[X, Y]$, for any $X, Y \in \text{Der}(\mathcal{A})$ and $z \in Z(\mathcal{A})$. This last equality ensures that, in the complex $C(\text{Der}(\mathcal{A}), \mathcal{A})$ of the \mathcal{A} -valued Lie-algebra cochains of $\text{Der}(\mathcal{A})$, the subspace $\underline{\Omega}_{\text{Der}}(\mathcal{A})$ of $Z(\mathcal{A})$ -multilinear cochains is stable under the differential, i.e. is a subcomplex.

More precisely, let $\underline{\Omega}_{\text{Der}}^n(\mathcal{A})$ be the space of $Z(\mathcal{A})$ -multilinear antisymmetric mappings of $(\text{Der}(\mathcal{A}))^n$ into \mathcal{A} , (i.e. $\underline{\Omega}_{\text{Der}}^n(\mathcal{A}) = \text{Hom}_{Z(\mathcal{A})}(\Lambda_{Z(\mathcal{A})}^n \text{Der}(\mathcal{A}), \mathcal{A})$). Then the graded space $\underline{\Omega}_{\text{Der}}(\mathcal{A}) = \bigoplus_n \underline{\Omega}_{\text{Der}}^n(\mathcal{A})$ is in a natural way a graded algebra (the product combining the product of \mathcal{A} with antisymmetrisation in the arguments). One verifies that one defines a differential d of $\underline{\Omega}_{\text{Der}}(\mathcal{A})$, i.e. an antiderivation of degree 1 satisfying $d^2 = 0$, by setting, for $\omega \in \underline{\Omega}_{\text{Der}}^n(\mathcal{A})$ and $X_i \in \text{Der}(\mathcal{A})$,

$$(d\omega)(X_0, \dots, X_n) = \sum_{k=0}^n (-1)^k X_k \omega(X_0, \overset{\mathbf{k}}{\underset{\vee}{\dots}}, X_n) \\ + \sum_{0 \leq r < s \leq n} (-1)^{r+s} \omega([X_r, X_s], X_0, \overset{\mathbf{r}}{\underset{\vee}{\dots}} \overset{\mathbf{s}}{\underset{\vee}{\dots}}, X_n)$$

where $\overset{\mathbf{i}}{\underset{\vee}{\dots}}$ means omission of X_i . Thus, equipped with this differential, $\underline{\Omega}_{\text{Der}}(\mathcal{A})$ is a graded differential algebra and the subalgebra $\underline{\Omega}_{\text{Der}}^0(\mathcal{A})$ coincides with \mathcal{A} . If one equips $\underline{\Omega}_{\text{Der}}(\mathcal{A})$ with the involution $\omega \mapsto \omega^*$ defined by $\omega^*(X_1, \dots, X_n) =$

$(\omega(X_1^*, \dots, X_n^*))^*$, it becomes a differential calculus over \mathcal{A} .

Let $\Omega_{\text{Der}}(\mathcal{A})$ be the smallest differential subalgebra of $\underline{\Omega}_{\text{Der}}(\mathcal{A})$ which contains \mathcal{A} . The differential algebra $\Omega_{\text{Der}}(\mathcal{A})$ is the canonical image of $\Omega_u(\mathcal{A})$ in $\underline{\Omega}_{\text{Der}}(\mathcal{A})$ and is stable by the involution; it consists of finite sums of elements of the form $x_0 dx_1 \dots dx_n$, $x_i \in \mathcal{A}$. The graded differential $*$ -algebra $\Omega_{\text{Der}}(\mathcal{A})$ is also a differential calculus over \mathcal{A} .

Both $\Omega_{\text{Der}}(\mathcal{A})$ and $\underline{\Omega}_{\text{Der}}(\mathcal{A})$ are generalizations of the algebra of complex differential forms. If V is a finite-dimensional paracompact manifold then $\Omega_{\text{Der}}(\mathcal{C}^\infty(V))$ and $\underline{\Omega}_{\text{Der}}(\mathcal{C}^\infty(V))$ both coincide with the graded differential $*$ -algebra $\Omega(V)$ of complex differential forms on V . In general the inclusion $\Omega_{\text{Der}}(\mathcal{A}) \subset \underline{\Omega}_{\text{Der}}(\mathcal{A})$ is a strict one: $\Omega_{\text{Der}}(\mathcal{A})$ is the minimal version of noncommutative differential forms based on derivations while $\underline{\Omega}_{\text{Der}}(\mathcal{A})$ is the maximal one. It is worth noticing here that even in the classical situation the above inclusion may be strict, e.g. if V is a manifold which does not admit a partition of unity then the inclusion $\Omega_{\text{Der}}(\mathcal{C}^\infty(V)) \subset \underline{\Omega}_{\text{Der}}(\mathcal{C}^\infty(V))$ is a strict one. There is however a density result of $\Omega_{\text{Der}}(\mathcal{A})$ in $\underline{\Omega}_{\text{Der}}(\mathcal{A})$ which we now describe at the level of one-forms [18].

By its very definition, the bimodule $\underline{\Omega}_{\text{Der}}^1(\mathcal{A})$ is the \mathcal{A} -dual of the $Z(\mathcal{A})$ -module $\text{Der}(\mathcal{A})$, i.e. $\underline{\Omega}_{\text{Der}}^1(\mathcal{A}) = (\text{Der}(\mathcal{A}))^{*\mathcal{A}} = \text{Hom}_{Z(\mathcal{A})}(\text{Der}(\mathcal{A}), \mathcal{A})$. On the other hand, by the universal property of $(\Omega_u^1(\mathcal{A}), d_u)$, $\text{Der}(\mathcal{A})$ can be identified with $\text{Hom}_{\mathcal{A}}^{\mathcal{A}}(\Omega_u^1(\mathcal{A}), \mathcal{A})$ through the canonical mapping $X \mapsto j_X$ (see in last section). However the intersection of the kernels of the bimodule homomorphisms of $\Omega_u^1(\mathcal{A})$ into \mathcal{A} , (which is the intersection of the kernels of the j_X when X runs over $\text{Der}(\mathcal{A})$), is just the kernel of the

canonical bimodule homomorphism of $\Omega_u^1(\mathcal{A})$ onto $\Omega_{\text{Der}}^1(\mathcal{A})$ [13] and therefore one has $\text{Hom}_{\mathcal{A}}^{\mathcal{A}}(\Omega_u^1(\mathcal{A}), \mathcal{A}) = \text{Hom}_{\mathcal{A}}^{\mathcal{A}}(\Omega_{\text{Der}}^1(\mathcal{A}), \mathcal{A})$. So one has finally $\text{Hom}_{\mathcal{A}}^{\mathcal{A}}(\Omega_{\text{Der}}^1(\mathcal{A}), \mathcal{A}) = \text{Der}(\mathcal{A})$ which means that the $Z(\mathcal{A})$ -module $\text{Der}(\mathcal{A})$ is the \mathcal{A} -dual of the bimodule $\Omega_{\text{Der}}^1(\mathcal{A}) : \text{Der}(\mathcal{A}) = (\Omega_{\text{Der}}^1(\mathcal{A}))^{*\mathcal{A}}$. Thus $\underline{\Omega}_{\text{Der}}^1(\mathcal{A})$ is the \mathcal{A} -bidual bimodule of $\Omega_{\text{Der}}^1(\mathcal{A})$, i.e. one has $\underline{\Omega}_{\text{Der}}^1(\mathcal{A}) = (\Omega_{\text{Der}}^1(\mathcal{A}))^{*\mathcal{A}*\mathcal{A}}$. This is an obvious density result which implies in particular that $\Omega_{\text{Der}}^1(\mathcal{A})$ is a diagonal bimodule; however this fact is obvious since $\underline{\Omega}_{\text{Der}}^1(\mathcal{A})$ is diagonal by definition ($\subset \mathcal{A}^{\text{Der}(\mathcal{A})}$).

Using $\text{Hom}_{\mathcal{A}}^{\mathcal{A}}(\Omega_u^1(\mathcal{A}), \mathcal{A}) = \text{Hom}_{\mathcal{A}}^{\mathcal{A}}(\Omega_{\text{Der}}^1(\mathcal{A}), \mathcal{A})$ one can characterize the pair $(\Omega_{\text{Der}}^1(\mathcal{A}), d)$ consisting of the diagonal bimodule $\Omega_{\text{Der}}^1(\mathcal{A})$ and the derivation d of \mathcal{A} into $\Omega_{\text{Der}}^1(\mathcal{A})$ by the following universal property [19]: for any derivation δ of \mathcal{A} into a diagonal bimodule M over \mathcal{A} , there is a unique bimodule homomorphism $i_\delta : \Omega_{\text{Der}}^1(\mathcal{A}) \rightarrow M$ such that $\delta = i_\delta \circ d$. This means that if δ is a derivation of \mathcal{A} into a diagonal bimodule M , the bimodule homomorphism $j_\delta : \Omega_u^1(\mathcal{A}) \rightarrow M$ factorizes through the canonical bimodule homomorphism of $\Omega_u^1(\mathcal{A})$ onto $\Omega_{\text{Der}}^1(\mathcal{A})$. Recall that the underlying bimodule of the module of sections of a vector bundle over a manifold V is diagonal and that $\Omega_{\text{Der}}^1(\mathcal{C}^\infty(V))$ is the space of 1-forms on $V!$, so the above result generalizes a well known result of differential geometry.

Let X be a derivation of \mathcal{A} , then one defines an antiderivation i_X of degree -1 of $\underline{\Omega}_{\text{Der}}(\mathcal{A})$ by setting $(i_X\omega)(X_1, \dots, X_{n-1}) = \omega(X, X_1, \dots, X_{n-1})$ for $\omega \in \underline{\Omega}_{\text{Der}}^n(\mathcal{A})$ and $X_i \in \text{Der}(\mathcal{A})$. The mapping $X \mapsto i_X$ is an operation, in the sense of H. Cartan [3], of the Lie algebra $\text{Der}(\mathcal{A})$ in the graded differential algebra $\underline{\Omega}_{\text{Der}}(\mathcal{A})$, i.e. one has $i_X i_Y + i_Y i_X = 0$ and, if one sets $L_X = di_X + i_X d$, $L_X i_Y - i_Y L_X = i_{[X, Y]}$ and $L_X L_Y - L_Y L_X = L_{[X, Y]}$. Furthermore

$L_X \upharpoonright \mathcal{A} = X$ so $X \mapsto L_X$ is a Lie algebra homomorphism of $\text{Der}(\mathcal{A})$ into the derivations of degree zero of $\underline{\Omega}_{\text{Der}}(\mathcal{A})$ which extends the action of $\text{Der}(\mathcal{A})$ on \mathcal{A} . The differential subalgebra $\Omega_{\text{Der}}(\mathcal{A})$ is stable by the $i_X, X \in \text{Der}(\mathcal{A})$, so one has by restriction an operation of $\text{Der}(\mathcal{A})$ in $\Omega_{\text{Der}}(\mathcal{A})$. The operation $X \mapsto i_X$ is of course the generalization of the interior product (or contraction) of forms by vector fields while L_X generalizes the Lie derivative on forms.

5 Noncommutative symplectic structures

It is well known that the structural similarity between classical mechanics and quantum mechanics is the most apparent if one uses the hamiltonian approach for the former and that this is important for the problems of classical and semiclassical limits. In this context the appropriate generalization of the Poisson structures is also well known. A *Poisson bracket* on \mathcal{A} is a Lie algebra structure $(x, y) \mapsto \{x, y\}$ on \mathcal{A} satisfying $\{x, yz\} = \{x, y\}z + y\{x, z\}$ for any elements x, y and z of \mathcal{A} . Such a Poisson bracket is *real* if furthermore one has $\{x, y\}^* = \{x^*, y^*\}$ for $x, y \in \mathcal{A}$. For any \mathcal{A} , there is the standard real Poisson bracket $\{x, y\} = i[x, y]$ ($= i(xy - yx)$). Although this bracket is trivial for a commutative algebra it is, up to a real factor, the most common Poisson bracket occuring in quantum mechanics. In classical hamiltonian mechanics, the Poisson bracket is associated with the symplectic structure of the phase space. It is the aim of this section to describe the generalization of symplectic structures for \mathcal{A} and to show its relevance for quantum mechanics [14], [15], [24].

The first thing to do is to generalize the notion of a nondegenerate two-form. An element ω of $\underline{\Omega}_{\text{Der}}^2(\mathcal{A})$ will be said to be *nondegenerate* if, for

any $x \in \mathcal{A}$, there is a derivation $\text{Ham}(x) \in \text{Der}(\mathcal{A})$ such that one has $\omega(X, \text{Ham}(x)) = X(x)$ for any $X \in \text{Der}(\mathcal{A})$. Notice that if ω is nondegenerate then $X \mapsto i_X \omega$ is an injective linear mapping of $\text{Der}(\mathcal{A})$ into $\underline{\Omega}_{\text{Der}}^1(\mathcal{A})$ but that the converse is not true; the condition for ω to be nondegenerate is stronger than the injectivity of $X \mapsto i_X \omega$. If V is a manifold, an element $\omega \in \underline{\Omega}_{\text{Der}}^2(\mathcal{C}^\infty(V))$ is an ordinary 2-form on V and it is nondegenerate in the above sense if and only if the 2-form ω is nondegenerate in the classical sense (i.e. everywhere nondegenerate).

Let $\omega \in \underline{\Omega}_{\text{Der}}^2(\mathcal{A})$ be nondegenerate, then for a given $x \in \mathcal{A}$ the derivation $\text{Ham}(x)$ is unique and $x \mapsto \text{Ham}(x)$ is a linear mapping of \mathcal{A} into $\text{Der}(\mathcal{A})$. Define then an antisymmetric bilinear bracket on \mathcal{A} by $\{x, y\} = \omega(\text{Ham}(x), \text{Ham}(y))$. One has $\{x, yz\} = \{x, y\}z + y\{x, y\}$ for $x, y, z \in \mathcal{A}$, however the bracket $(x, y) \mapsto \{x, y\}$ is a Lie bracket, (i.e. satisfies the Jacobi identity), if and only if $d\omega = 0$. A closed nondegenerate element ω of $\underline{\Omega}_{\text{Der}}^2(\mathcal{A})$ will be called a *symplectic structure for \mathcal{A}* . Let ω be a symplectic structure for \mathcal{A} , then the corresponding bracket $(x, y) \mapsto \{x, y\} = \omega(\text{Ham}(x), \text{Ham}(y))$ is a Poisson bracket on \mathcal{A} and one has $[\text{Ham}(x), \text{Ham}(y)] = \text{Ham}(\{x, y\})$, i.e. Ham is a Lie-algebra homomorphism of $(\mathcal{A}, \{, \})$ into $\text{Der}(\mathcal{A})$. If furthermore ω is real, i.e. $\omega = \omega^*$, then this Poisson bracket is real and $\text{Ham}(x^*) = (\text{Ham}(x))^*$ for any $x \in \mathcal{A}$. We shall refer to the above bracket as the Poisson bracket associated to the symplectic structure ω .

If V is a manifold, a symplectic structure for $\mathcal{C}^\infty(V)$ is just a symplectic form on V . Since there are manifolds which do not admit symplectic form, one cannot expect that an arbitrary \mathcal{A} admits a symplectic structure.

Assume that \mathcal{A} has a trivial center $Z(\mathcal{A}) = \mathbb{C}\mathbb{1}$ and that all its derivations are inner (i.e. of the form $ad(x), x \in \mathcal{A}$). Then one defines an element ω of $\underline{\Omega}_{\text{Der}}^2(\mathcal{A})$ by setting $\omega(ad(ix), ad(iy)) = i[x, y]$. It is easily seen that ω is a real symplectic structure for which one has $\text{Ham}(x) = ad(ix)$ and $\{x, y\} = i[x, y]$. Although a little tautological, this construction is relevant for quantum mechanics.

Let \mathcal{A} be, as above, a complex unital $*$ -algebra with a trivial center and only inner derivations and assume that there exists a linear form τ on \mathcal{A} which is central, i.e. $\tau(xy) = \tau(yx)$, and normalized by $\tau(\mathbb{1}) = 1$. Then one defines an element $\theta \in \underline{\Omega}_{\text{Der}}^1(\mathcal{A})$ by $\theta(ad(ix)) = x - \tau(x)\mathbb{1}$. One has $(d\theta)(ad(ix), ad(iy)) = i[x, y]$, i.e. $\omega = d\theta$, so in this case the symplectic form ω is exact. As examples of such algebras one can take $\mathcal{A} = M_n(\mathbb{C})$, (a factor of type I_n), with $\tau = \frac{1}{n}$ trace, or $\mathcal{A} = \mathcal{R}$, a von Neumann algebra which is a factor of type II_1 with τ equal to the normalized trace. The algebra $M_n(\mathbb{C})$ is the algebra of observables of a quantum spin $s = \frac{n-1}{2}$ while \mathcal{R} is the algebra used to describe the observables of an infinite assembly of quantum spin; two typical types of quantum systems with no classical counterpart.

Let us now consider the C.C.R. algebra (canonical commutative relations) \mathcal{A}_{CCR} [14]. This is the complex unital $*$ -algebra generated by two hermitian elements q and p satisfying the relation $[q, p] = i\hbar\mathbb{1}$. This algebra is the algebra of observables of the quantum counterpart of a classical system with one degree of freedom. We keep here the positive constant \hbar (the Planck constant) in the formula for comparison with classical mechanics, although the algebra for $\hbar \neq 0$ is isomorphic to the one with $\hbar = 1$. We restrict here attention to one degree of freedom to simplify the notations but the discussion

extends easily to a finite number of degrees of freedom. This algebra has again only inner derivations and a trivial center so $\omega(ad(\frac{i}{\hbar}x), ad(\frac{i}{\hbar}y)) = \frac{i}{\hbar}[x, y]$ defines a symplectic structure for which $\text{Ham}(x) = ad(\frac{i}{\hbar}x)$ and $\{x, y\} = \frac{i}{\hbar}[x, y]$ which is the standard quantum Poisson bracket. In this case one can express ω in terms of the generators q and p and their differentials :

$$\omega = \sum_{n \geq 0} \left(\frac{1}{i\hbar}\right)^n \frac{1}{(n+1)!} [\dots [dp, \underbrace{p, \dots, p}_n] \dots [dq, \underbrace{q, \dots, q}_n]]$$

Notice that this formula is meaningful because if one inserts two derivations $ad(ix), ad(iy)$ in it, only a finite number of terms contribute in the sum. For $\hbar = 0$, q and p commute and the algebra reduces to the algebra of complex polynomial functions on the phase space \mathbb{R}^x . Furthermore the limit of $\{x, y\} = \frac{i}{\hbar}[x, y]$ at $\hbar = 0$ reduces to the usual classical Poisson bracket as well known and, by using the above formula, one sees that the formal limit of ω at $\hbar = 0$ is $dpdq$.

6 Derivations and Connections

In this section \mathcal{C} is a complex unital commutative $*$ -algebra and \mathcal{A} is a complex unital $*$ -algebra which is to be considered as the noncommutative generalization of \mathcal{C} . Our aim is to discuss the theory of connections on the various objects which generalize the \mathcal{C} -modules when \mathcal{C} is replaced by \mathcal{A} in the framework of the differential calculus based on derivations as generalization of vector fields [18] (cf. Section 0.4). In most parts of the following the involution is not involved and therefore, in the definitions and results where the reality conditions do not enter, one may assume that \mathcal{C} and its noncommutative counterpart \mathcal{A} are simply algebras (instead of $*$ -algebras).

As generalizations of the category of \mathcal{C} -modules when \mathcal{C} is replaced by \mathcal{A} we consider the four following categories (cf. Section 0.2), the category $\mathfrak{C}_{(0,0)}$ of $Z(\mathcal{A})$ -modules, the category $\mathfrak{C}_{(1,0)}$ of left \mathcal{A} -modules, the category $\mathfrak{C}_{(0,1)}$ of right \mathcal{A} -modules and the category $\mathfrak{C}_{(1,1)}$ of central bimodules over \mathcal{A} i.e. of left $\mathcal{A} \otimes_{Z(\mathcal{A})} \mathcal{A}^{op}$ -modules. In each of these categories, one has a direct sum and if M is an object of any of these categories, it has a canonical underlying structure of $Z(\mathcal{A})$ -module. The labelling of these categories by elements $\alpha = (i, j)$ of $\mathbb{Z}_{\neq} \times \mathbb{Z}_{\neq}$ will be very convenient to deal with the duality and tensor products. In $\mathbb{Z}_{\neq} \times \mathbb{Z}_{\neq}$ one defines an involutive mapping $\alpha \mapsto \alpha'$ by $(i, j)' = (1 - i, 1 - j)$, i.e. $\alpha' = \alpha + (1, 1)$. Correspondingly one has a duality $M \mapsto M'$ of \mathfrak{C}_{α} into $\mathfrak{C}_{\alpha'}$, where $M' = M^*$ if M is a left or right \mathcal{A} -module and $M' = M^{*\mathcal{A}}$ if M is a $Z(\mathcal{A})$ -module or a central bimodule over \mathcal{A} . Another bit of notation will be convenient; we set $A_0 = Z(\mathcal{A})$ and $A_1 = \mathcal{A}$. Using this notation, an object of $\mathfrak{C}_{(i,j)}$ is a (A_i, A_j) -bimodule (of a specific kind) and we can define tensor products $\mathfrak{C}_{(i,j)} \times \mathfrak{C}_{(j,k)} \rightarrow \mathfrak{C}_{(i,j)} \tilde{\otimes} \mathfrak{C}_{(j,k)} \subset \mathfrak{C}_{(i,k)}$ by $M \tilde{\otimes} N = M \otimes_{A_j} N$ if M is an object of $\mathfrak{C}_{(i,j)}$ and N an object of $\mathfrak{C}_{(j,k)}$ (one verifies that $M \tilde{\otimes} N$ is then an object of $\mathfrak{C}_{(i,k)}$).

Let M be an object of $\mathfrak{C}_{(i,j)}$. A *connection on M* is a linear mapping ∇ , $X \mapsto \nabla_X$, of $\text{Der}(\mathcal{A})$ into the linear endomorphism of M such that one has for any $m \in M$ and any $X \in \text{Der}(\mathcal{A})$

$$\begin{cases} \nabla_{zX}(m) = z\nabla_X(m), & \forall z \in Z(\mathcal{A}) \\ \nabla_X(a_i m a_j) = X(a_i) m a_j + a_i \nabla_X(m) a_j + a_i m X(a_j), & \forall a_i \in A_i, \forall a_j \in A_j \end{cases}$$

remembering that M is canonically a $Z(\mathcal{A})$ -module and that since $Z(\mathcal{A}) = A_0$ is stable by $\text{Der}(\mathcal{A})$, $\text{Der}(\mathcal{A})$ acts by derivations on $Z(\mathcal{A}) = A_0$ and on $\mathcal{A} = A_1$. It should be stressed that elements of $A_0 = Z(\mathcal{A})$ can be moved to the other side. Given ∇ as above, *the curvature R of ∇* is the bilinear

antisymmetric mapping $(X, Y) \mapsto R_{X,Y}$ of $\text{Der}(\mathcal{A}) \times \text{Der}(\mathcal{A})$ into the linear endomorphisms of M defined by $R_{X,Y}(m) = \nabla_X(\nabla_Y(m)) - \nabla_Y(\nabla_X(m)) - \nabla_{[X,Y]}(m)$, $\forall X, Y \in \text{Der}(\mathcal{A})$, $\forall m \in M$. One has $R_{zX,Y}(m) = zR_{X,Y}(m)$ and $R_{X,Y}(a_i m a_j) = a_i R_{X,Y}(m) a_j$, $\forall m \in M$, $\forall X, Y \in \text{Der}(\mathcal{A})$, $\forall z \in Z(\mathcal{A})$, $\forall a_i \in A_i$, $\forall a_j \in A_j$. More precisely, R is an antisymmetry $Z(\mathcal{A})$ -bilinear mapping of $\text{Der}(\mathcal{A}) \times \text{Der}(\mathcal{A})$ into the $Z(\mathcal{A})$ -module $\text{Hom}_{\mathfrak{C}_{(i,j)}}(M, M)$, ($\text{Hom}_{\mathfrak{C}_{(i,j)}}$ being the morphisms in $\mathfrak{C}_{(i,j)}$).

There is an obvious connection $\nabla_1 \oplus \nabla_2$ on the direct sum $M_1 \oplus M_2$ of two objects M_1 and M_2 of $\mathfrak{C}_{(i,j)}$ equipped with connections ∇_1 and ∇_2 .

Let M be an object of $\mathfrak{C}_{(i,j)}$ then its dual M' is an element of $\mathfrak{C}_{(i,j)'}$ and we denote by $(m, m') \mapsto \langle m, m' \rangle \in \mathcal{A}$ the bilinear duality bracket obtained by evaluation, $\langle, \rangle: M \times M' \rightarrow \mathcal{A}$. Then, for any connection ∇ on M , there is unique *dual connection* ∇' on M' such that $X(\langle m, m' \rangle) = \langle \nabla_X(m), m' \rangle + \langle m, \nabla'_X(m') \rangle$, $\forall m \in M, \forall m' \in M'$ and $\forall X \in \text{Der}(\mathcal{A})$. Indeed the above equality defines ∇' uniquely and one checks that it is a connection. In general, the mapping $\nabla \mapsto \nabla'$ is not injective nor surjective. However if the canonical mapping of M into its bidual M'' , (which is a morphism of $\mathfrak{C}_{(i,j)}$), is injective, then ∇'' is an extension of ∇ and therefore $\nabla \mapsto \nabla'$ is injective and of course bijective whenever $M = M''$. An object M of $\mathfrak{C}_{(i,j)}$ will be called *diagonal* if the canonical morphism of M in M'' is injective. This generalizes the notion introduced in Section 0.2 (for $\mathfrak{C}_{(1,1)}$) and the terminology is suggested by the following. The algebra \mathcal{A} itself can be considered as an object of $\mathfrak{C}_{(i,j)}$ when it is equipped with the canonical corresponding underlying structure and the same is true for \mathcal{A}^I where I is an arbitrary set, since $\mathfrak{C}_{(i,j)}$ has arbitrary products and, more generally, arbitrary projective limits. Then M is diagonal

if and only if there is an injective $\mathfrak{C}_{(i,j)}$ -morphism of M into \mathcal{A}^I , for some set I . Finally let us notice that any projective limit of diagonal objects is diagonal and that dual objects are diagonal, i.e. if M is an object of $\mathfrak{C}_{(i,j)}$, then its dual M' is a diagonal object of $\mathfrak{C}_{(i,j)'$.

Let M_1 be an object of $\mathfrak{C}_{(i,j)}$ and M_2 be an object of $\mathfrak{C}_{(j,k)}$ and let ∇^1 be a connection on M_1 and ∇^2 be a connection on M_2 . Then, for any $X \in \text{Der}(\mathcal{A})$, $D_X = \nabla_X^1 \otimes id_{M_2} + id_{M_1} \otimes \nabla_X^2$ is such that that it maps into itself the subspace of $M_1 \otimes M_2$ generated by the elements $m_1 a_j \otimes m_2 - m_1 \otimes a_j m_2$, with $m_1 \in M_1$, $m_2 \in M_2$ and $a_j \in A_j$. It follows that the D_X pass to the quotient and define linear endomorphisms ∇_X of $M_1 \tilde{\otimes} M_2$ and one verifies that ∇ so defined is a connection on the object $M_1 \tilde{\otimes} M_2$ of $\mathfrak{C}_{(i,k)}$. This connection will be referred to as *the tensor product of ∇_1 and ∇_2* .

Thus we have defined connections on $Z(\mathcal{A})$ -modules, on left and right \mathcal{A} -modules and on central bimodules over \mathcal{A} and we have also defined dual and tensor product of such connections. Let us now come to the problems of reality for such connections. As pointed out in Section 0.2, the notion of reality makes sense only for $*$ -modules over $Z(\mathcal{A})$ or for $*$ -bimodules over \mathcal{A} . So let M be either a $*$ -module over $Z(\mathcal{A})$ or a $*$ -bimodule over \mathcal{A} which is central. If ∇ is a connection on M one can define another one ∇^* , *its conjugate*, by setting $\nabla_X^*(m) = (\nabla_{X^*}(m^*))^*$ and ∇ will be said to be a *real connection* if $\nabla = \nabla^*$. Let M' be the dual of M , i.e. $M' = M^{*\mathcal{A}}$ in this case, then there is a unique involution $m' \mapsto m'^*$ on M' such that $\langle m, m' \rangle^* = \langle m^*, m'^* \rangle$ and, equipped with this involution, M' is a (central) $*$ -bimodule over \mathcal{A} if M is a $*$ -module over $Z(\mathcal{A})$ or a $*$ -module over $Z(\mathcal{A})$ if M is a central $*$ -bimodule over \mathcal{A} . Furthermore, one has $(\nabla^*)' = (\nabla')^*$, so the dual connection of a real connection is real.

7 Linear connections

In classical differential geometry, a connection on the tangent bundle, or equivalently on the cotangent bundle, of a manifold is usually called a linear connection. Although this terminology is a little misleading, we shall nevertheless use it for the corresponding noncommutative generalizations. Within the framework of Section 0.4 and Section 0.6, one sees that there are three natural definitions of such generalizations. First a connection on $\Omega_{\text{Der}}^1(\mathcal{A})$, second a connection on $\text{Der}(\mathcal{A})$ and third a connection on $\underline{\Omega}_{\text{Der}}^1(\mathcal{A})$. However, as explained in Section 0.4, $\Omega_{\text{Der}}^1(\mathcal{A})$ is a diagonal bimodule with $\text{Der}(\mathcal{A})$ as \mathcal{A} -dual, i.e. $\text{Der}(\mathcal{A}) = (\Omega_{\text{Der}}^1(\mathcal{A}))'$ with the notation of Section 0.6, and $\underline{\Omega}_{\text{Der}}^1(\mathcal{A})$ is the \mathcal{A} -dual of $\text{Der}(\mathcal{A})$, i.e. $\underline{\Omega}_{\text{Der}}^1(\mathcal{A}) = (\text{Der}(\mathcal{A}))' = (\Omega_{\text{Der}}^1(\mathcal{A}))''$. Therefore, it follows from the discussion of the previous section that, by duality, there is an injective mapping of the (affine) space of connections on $\Omega_{\text{Der}}^1(\mathcal{A})$ into the space of connections on $\text{Der}(\mathcal{A})$ and that there is also an injective mapping of the space of connections on $\text{Der}(\mathcal{A})$ into the space of connections on $\underline{\Omega}_{\text{Der}}^1(\mathcal{A})$. Thus all these connections may be imbedded into the connections on $\underline{\Omega}_{\text{Der}}^1(\mathcal{A})$. A real connection on $\underline{\Omega}_{\text{Der}}^1(\mathcal{A})$ will be called a *linear connection on \mathcal{A}* . The connections on $\text{Der}(\mathcal{A})$ form a subclass of connections on $\underline{\Omega}_{\text{Der}}^1(\mathcal{A})$ and an even smaller subclass consists of connections on $\Omega_{\text{Der}}^1(\mathcal{A})$. Given a connection ∇ on $\underline{\Omega}_{\text{Der}}^1(\mathcal{A})$, one defines a bimodule homomorphism $T : \underline{\Omega}_{\text{Der}}^1(\mathcal{A}) \rightarrow \underline{\Omega}_{\text{Der}}^2(\mathcal{A})$, its *torsion*, by setting $(T\omega)(X, Y) = (d\omega)(X, Y) - \nabla_X(\omega)(Y) + \nabla_Y(\omega)(X)$ for $X, Y \in \text{Der}(\mathcal{A})$ and $\omega \in \underline{\Omega}_{\text{Der}}^1(\mathcal{A})$. If ∇ comes from a connection on $\Omega_{\text{Der}}^1(\mathcal{A})$, (by biduality), T restricted to $\Omega_{\text{Der}}^1(\mathcal{A})$, is a bimodule homomorphism of $\Omega_{\text{Der}}^1(\mathcal{A})$ into $\Omega_{\text{Der}}^2(\mathcal{A})$. If ∇ is the dual of a connection, again denoted by ∇ , on $\text{Der}(\mathcal{A})$, its torsion can be identified with the $Z(\mathcal{A})$ -bilinear antisymmetric mapping T of

$\text{Der}(\mathcal{A}) \times \text{Der}(\mathcal{A})$ into $\text{Der}(\mathcal{A})$ defined by $T(X, Y) = \nabla_X(Y) - \nabla_Y(X) - [X, Y]$, $\forall X, Y \in \text{Der}(\mathcal{A})$. For a more complete discussion as well as for the notion of Levi-Civita connection of a generalization of pseudo-riemannian metric, we refer to [18].

8 Conclusion: General differential calculi

The above notions of connections are natural ones when one uses the differential calculus based on derivations as generalization of vector fields. However, for some purposes, (see e.g. in [8]), it is useful to use other differential calculi and therefore, it is natural to ask for a definition of connections adapted to such calculi. Let Ω be a differential calculus over \mathcal{A} . There is then a well known useful definition of an Ω -connection on a left (or right) \mathcal{A} -module [6]. The problem arises when one tries to define an Ω -connection on a bimodule over \mathcal{A} such as Ω^1 . This problem is unavoidable if one wishes to generalize linear connections since the natural structure of Ω^1 is that of a bimodule. Some authors, e.g. [5], define a connection on Ω^1 to be a left module Ω -connection on Ω^1 . Besides the fact that it is unnatural to privilege part of a bimodule structure, this definition has two drawbacks if one thinks of it as a generalization of linear connections. First, one cannot introduce then the notion of reality which generalizes the classical notion of reality of a linear connection in differential geometry because the involution of Ω^1 is linked to its bimodule structure, (so the conjugate of a left \mathcal{A} -module connection on Ω^1 is rather a right \mathcal{A} -module connection). Second, one cannot, in general, define the tensor product over \mathcal{A} of such a connection with a connection say on a left \mathcal{A} -module although it is very desirable to have such a tensor product, e.g. for the description of the generalization of the classical coupling

of gravitation with a field coupled to a Yang-Mills field. A definition of linear connections for general differential calculi which takes into account the complete bimodule structure of Ω^1 has been proposed by J. Mourad [26] and further generalized to other bimodules [17]. This definition involves a generalization of the permutations in tensor products and with it the question of reality can be addressed. Furthermore, in this framework, tensor products of connections are defined straightforwardly and it has been recently shown [2], (see in appendix A of [2]), that conversely, in order that tensor products of connections exist in a very general sense, one has to use this definition of connections for bimodules.

References

- [1] N. Bourbaki, *Algèbre I*, Chapitre III. Paris, Hermann 1970.
- [2] K. Bresser, F. Müller-Hoissen, A. Dimakis, A. Sitarz, *Noncommutative geometry of finite groups*, Preprint GOET-TP 95/95.
- [3] H. Cartan, *Notion d'algèbre différentielle; application aux groupes de Lie et aux variétés où opère un groupe de Lie* and *La transgression dans un groupe de Lie et dans un espace fibré principal*, Colloque de topologie (Bruxelles 1950), Paris, Masson 1951.
- [4] H. Cartan, S. Eilenberg, *Homological algebras*, Princeton University Press 1973.
- [5] A.H. Chamseddine, J. Fröhlich, O. Grandjean, *The gravitational sector in the Connes-Lott formulation of the standard model*, J. Math. Phys. **36** (1995) 6255.

- [6] A. Connes, *Non-commutative differential geometry*, Publ. IHES **62** (1986), 257.
- [7] A. Connes, *C* algèbres et géométrie différentielle*, C.R. Acad. Sci. Paris, **290**, Série A (1980), 599-604.
- [8] A. Connes, *Non-commutative geometry*, Academic Press, 1994.
- [9] A. Connes, *Geometry from the spectral point of view*, Lett. Math. Phys. **34** (1995) 203-238.
- [10] A. Connes, *Noncommutative geometry and reality*, Preprint IHES/M/95/52 (1995).
- [11] R. Coquereaux, *Higgs fields and superconnections*, in Differential Geometric Methods in Theoretical Physics, Rapallo 1990 (C. Bartocci, U. Bruzzo, R. Cianci, eds), Lecture Notes in Physics **375**, Springer Verlag 1991.
R. Coquereaux, R. Häußling, F. Scheck, *Algebraic connections on parallel universes*, Int. J. Mod. Phys. **A10** (1995) 89-98.
- [12] P.A.M. Dirac, *On quantum algebras*, Proc. Camb. Phil. Soc. **23** (1926) 412.
- [13] M. Dubois-Violette, *Dérivations et calcul différentiel non commutatif*, C.R. Acad. Sci. Paris, **307**, Série I (1988), 403-408.
- [14] M. Dubois-Violette, *Non-commutative differential geometry, quantum mechanics and gauge Theory*, in Differential Geometric Methods in Theoretical Physics, Rapallo 1990 (C. Bartocci, U. Bruzzo, R. Cianci, eds), Lecture Notes in Physics **375**, Springer-Verlag 1991.

- [15] M. Dubois-Violette, R. Kerner, J. Madore, *Non-commutative differential geometry of matrix algebras*, J. Math. Phys. **31** (1990) 316.
- [16] M. Dubois-Violette, R. Kerner, J. Madore, *Gauge bosons in a non-commutative geometry*, Phys. Lett. **B217** (1989) 485.
M. Dubois-Violette, R. Kerner, J. Madore, *Classical bosons in a non-commutative geometry*, Class. Quantum Grav. **6** (1989) 1709.
M. Dubois-Violette, R. Kerner, J. Madore, *Non-commutative differential geometry and new models of gauge theory*, J. Math. Phys. **31** (1990) 323.
- [17] M. Dubois-Violette, T. Masson, *On the first order operators in bimodules*, Preprint LPTHE-Orsay 95/56, q-alg 9507028.
- [18] M. Dubois-Violette, P.W. Michor, *Connections on central bimodules in noncommutative geometry*, Preprint L.P.T.H.E.-Orsay 94/100 and ESI-preprint 210, to appear in Journal of Geometry and Physics.
- [19] M. Dubois-Violette, P.W. Michor, *Dérivations et calcul différentiel non-commutatif.II*, C.R. Acad. Sci. Paris, **319**, Série I (1994), 927-931.
- [20] M. Dubois-Violette, P.W. Michor, *More on the Frölicher-Nijenhuis bracket in noncommutative differential geometry*, To appear.
- [21] M. Karoubi, *Homologie cyclique des groupes et algèbres*, C.R. Acad. Sci. Paris **297**, Série I, (1983) 381-384.
- [22] D. Kastler, *Lectures on Alain Connes' non commutative geometry and applications to fundamental interactions*, in Infinite dimensional geometry, noncommutative geometry, operator algebras, fundamental inter-

- actions; Saint-François, Guadeloupe 1993 (R. Coquereaux, M. Dubois-Violette, P. Flad, eds), World Scientific 1995.
- [23] J.L. Koszul, *Fibre bundles and differential geometry*, Tata Institute of Fundamental Research, Bombay, 1960.
- [24] J. Madore, *Noncommutative differential geometry and its physical applications*, Cambridge University Press 1995.
- [25] G. Malsiniotis, *Le langage des espaces et des groupes quantiques*, Comm. Math. Phys. **151** (1993) 275.
- [26] J. Mourad, *Linear connections in noncommutative geometry*, Class. Quantum Grav. **12** (1995) 965-974.
- [27] S.L. Woronowicz, *Differential calculus on compact matrix pseudogroups (quantum groups)*, Commun. Math. Phys. **122** (1989) 125-170.