

Dimensional Hausdorff Properties of Singular Continuous Spectra

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Abstract

We present an extension of the Gilbert-Pearson theory of subordinacy, which relates dimensional Hausdorff spectral properties of one-dimensional Schrödinger operators to the behavior of solutions of the corresponding Schrödinger equation. We use this theory to analyze the dimensional Hausdorff properties for several examples having singular-continuous spectrum, including sparse barrier potentials, the almost Mathieu operator and the Fibonacci Hamiltonian.

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Singular continuous spectra have been extensively studied recently within spectral theory. While their importance in Physics is still far from clear, they seem to occur for some one-body Hamiltonians associated with quasicrystals. Our interest here is in the classification and decomposition of such spectra with respect to dimensional Hausdorff measures. The measure-theoretical aspect of this point of view goes back to Rogers-Taylor [1], and it has been studied recently within spectral theory by Last [2] and by del-Rio et al. [3] who have shown that the singular-continuous spectrum which is produced by localized rank-one perturbations of Anderson-model Hamiltonians in the localized regime [4] must be purely zero-dimensional — in the sense that the associated spectral measures are supported on a set of zero Hausdorff dimension.

The main purpose of this paper is to report a general method for spectral analysis of one-dimensional Schrödinger operators from this point of view. It is a natural extension of the Gilbert-Pearson theory of subordinacy [5, 6], and it allows us to analyze the dimensional Hausdorff properties for a number of examples with singular-continuous spectrum. Below we describe the main ideas of our study and some of the main results. Mathematically complete proofs of these results will be given elsewhere [7].

Most of our discussion will be restricted to one-dimensional discrete (tight-binding) Schrödinger operators of the form

$$(H\psi)(n) = \psi(n+1) + \psi(n-1) + V(n)\psi(n) . \quad (1)$$

We shall consider two kinds of such operators: “line” operators acting on $\ell^2(\mathbb{Z})$ ($-\infty < n < \infty$), and “half-line” operators acting on $\ell^2(\mathbb{Z}^+)$ ($n > 0$), which are considered with a phase boundary condition of the form:

$$\psi(0) \cos \theta + \psi(1) \sin \theta = 0 , \quad (2)$$

where $-\pi/2 < \theta < \pi/2$.

Before formulating our main results, which would require a number of definitions, we would like to describe some of their applications. We stress at this point that the dimensional Hausdorff properties which we study are those which are associated with the spectral measures of the corresponding operators. The spectra themselves, as sets, are closed sets, and their dimensions may be larger than those which are associated with the spectral measures. A description of the precise spectral-theoretic scheme which underlies our study is given below.

We start with a somewhat artificial example of “half-line” operators with sparse barrier potentials. More specifically, we consider potentials which vanish for all n ’s outside a sparse

(fastly growing) sequence of points $\{L_n\}_{n=1}^\infty$ where $|V(L_n)| \rightarrow \infty$ as $n \rightarrow \infty$. Simon-Spencer [8] have shown that the Schrödinger operators corresponding to such potentials have no absolutely-continuous spectrum, and Gordon [9] has shown that if the $|V(L_n)|$'s grow sufficiently fast (compared to the growth of the L_n 's) then for (Lebesgue) a.e. boundary phase θ the corresponding operators have pure-point spectrum with exponentially decaying eigenfunctions. It is easy to see [10], however, that if the L_n 's grow sufficiently fast (compared to the growth of the $|V(L_n)|$'s) then, for every boundary phase θ , the spectrum in $(-2, 2)$ is purely singular-continuous, and Simon [11] has recently shown that if the growth is even faster then the spectrum in $(-2, 2)$ is purely one-dimensional, in the sense that the spectral measure does not give weight to sets of Hausdorff dimension less than 1. By applying the subordinacy theory described below we have shown:

Theorem 2. *Let $\alpha \in (0, 1)$. Let $L_n = 2^{(n^n)}$ and define a potential $V(k)$ for $k > 0$ by $V(L_n) = L_n^{(1-\alpha)/(2\alpha)}$; $V(k) = 0$ if $k \notin \{L_n\}_{n=1}^\infty$. Then:*

- (i) *For every boundary phase θ , the spectrum of the corresponding “half-line” discrete Schrödinger operator consists of the interval $[-2, 2]$ (which is the essential spectrum) along with some discrete point spectrum outside this interval.*
- (ii) *For every θ , the Hausdorff dimensionality of the spectrum in $(-2, 2)$ is bounded between dimensions α and $\beta \equiv 2\alpha/(1 + \alpha)$, in the sense that the restriction of the spectral measure to $(-2, 2)$ is supported on a set of Hausdorff dimension β and does not give weight to sets of Hausdorff dimension less than α .*
- (iii) *For Lebesgue a.e. θ , the spectrum in $[-2, 2]$ is of exact dimension α , namely, the restriction of the spectral measure to $[-2, 2]$ is supported on a set of Hausdorff dimension α and does not give weight to sets of Hausdorff dimension less than α .*

Remark. The result only requires the L_n 's to be sufficiently sparse (namely, to grow sufficiently fast). $L_n = 2^{(n^n)}$ is a particular choice for which the sufficient sparseness is easy to show. The result holds equally well for any sequence of L_n 's which is sparser (e.g., $L_n = 2^{(n^n)!}$).

Next we consider two examples of “line” operators with quasiperiodic potentials. The first is the almost Mathieu (also called Harper) operator $H_{\beta, \lambda, \theta}$, which is the operator of the form (1) on $\ell^2(\mathbb{Z})$ with potential $V(n) = V_{\beta, \lambda, \theta}(n) = \lambda \cos(2\pi\beta n + \theta)$, where λ, θ are any real numbers, and β is an irrational. Aubry and Andre [12] have conjectured that $H_{\beta, \lambda, \theta}$ has

purely absolutely-continuous spectrum whenever $|\lambda| < 2$, and purely point spectrum (with exponentially localized eigenfunctions) whenever $|\lambda| > 2$. While the $|\lambda| < 2$ part of this conjecture may be correct (so far, the existence of absolutely-continuous spectrum [13] and absence of point spectrum [14] have been established rigorously), the $|\lambda| > 2$ case turned out to be more delicate: Absolutely-continuous spectrum is absent [15], but both pure-point and singular-continuous spectra occur, depending on arithmetical properties of both β and θ [16]. It turns out, though, that if we concentrate on the dimensional Hausdorff properties of the spectral measures, rather than distinguishing between pure-point and singular-continuous spectra, the situation becomes simpler:

Theorem 3. *For $|\lambda| > 2$, every irrational β , and every θ , $H_{\beta,\lambda,\theta}$ has purely zero-dimensional spectrum, in the sense that its spectral measures are all supported on a set of zero Hausdorff dimension.*

Remarks.

- (i) The spectrum of $H_{\beta,\lambda,\theta}$, as a set, is known in this case ($|\lambda| > 2$) to have positive Lebesgue measure [17].
- (ii) The special case of Theorem 2 where β is a Liouville number has already been established by Last [2].
- (iii) The result extends to potentials of the form $V(n) = f(2\pi\beta n + \theta)$, where $f(x) \equiv \sum_{k=1}^N \lambda_k \cos(kx)$, in which case we prove that the spectrum is purely zero-dimensional whenever $|\lambda_N| > 2$. This is precisely the regime for which Herman's theorem [18] gives positivity of the Lyapunov exponent.

Our second “line” example is the Fibonacci Hamiltonian H_λ , which is the operator of the form (1) on $\ell^2(\mathbb{Z})$ with potential $V(n) = \lambda([(n+1)\omega] - [n\omega])$, where $\omega = (\sqrt{5} - 1)/2$ is the golden mean, and $[x] \equiv \max\{m \in \mathbb{Z} \mid m \leq x\}$. H_λ is the most studied of all one-dimensional quasicrystal models. It is known [19] that, for every $\lambda \neq 0$, it has purely singular-continuous spectrum, and, moreover, its spectrum (as a set) is a Cantor set of zero Lebesgue measure. By applying the subordinacy theory described below, we have shown:

Theorem 4. *For every λ there exists an $\alpha > 0$ such that H_λ has purely α -continuous spectrum, namely, its spectral measures do not give weight to sets of Hausdorff dimension less than α .*

Remark. There exists strong numerical evidence [20] that the spectrum of H_λ (as a set) has Hausdorff dimension strictly less than 1 (for every $\lambda \neq 0$), and this would imply that its spectrum must also be β -singular (see below) for some $\beta < 1$. So far, we have not succeeded in proving this conjecture, and we consider it as an interesting open problem.

Let us now describe the spectral-theoretic scheme in the context of which the above results should be understood.

Consider a separable Hilbert space $\mathcal{H}$, and a self adjoint operator H . Recall [21] that for each $\psi \in \mathcal{H}$, the spectral measure μ_ψ (also known to physicists as the local spectral density) is the unique Borel measure obeying $\langle \psi, f(H)\psi \rangle = \int f(x) d\mu_\psi(x)$ for any measurable function f . By Lebesgue's decomposition theorem, every Borel measure μ decomposes uniquely as: $\mu = \mu_{ac} + \mu_{sc} + \mu_{pp}$. The absolutely-continuous part, μ_{ac} , gives zero weight to sets of zero Lebesgue measure. The pure-point part, μ_{pp} , is a countable sum of atomic (Dirac) measures. The singular-continuous part, μ_{sc} , gives zero weight to countable sets and is supported on some set of zero Lebesgue measure (we say that a measure μ is supported on a set S if $\mu(\mathbb{R} \setminus S) = 0$). $\mu_s \equiv \mu_{sc} + \mu_{pp}$ is called the singular part of μ . $\mu_c \equiv \mu_{ac} + \mu_{sc}$ is called the continuous part of μ . Letting $\mathcal{H}_{ac} \equiv \{\psi \mid \mu_\psi \text{ is purely absolutely-continuous}\}$, $\mathcal{H}_{sc} \equiv \{\psi \mid \mu_\psi \text{ is purely singular-continuous}\}$, and $\mathcal{H}_{pp} \equiv \{\psi \mid \mu_\psi \text{ is purely pure-point}\}$, one obtains a decomposition: $\mathcal{H} = \mathcal{H}_{ac} \oplus \mathcal{H}_{sc} \oplus \mathcal{H}_{pp}$. $\mathcal{H}_{ac}$, $\mathcal{H}_{sc}$, and $\mathcal{H}_{pp}$ are closed (in norm), mutually orthogonal subspaces, which are invariant under H . The absolutely-continuous spectrum (σ_{ac}), singular-continuous spectrum (σ_{sc}), and pure-point spectrum (σ_{pp}) are defined as the spectra of the restrictions of H to the corresponding subspaces, and $\text{Spec}(H) \equiv \sigma = \sigma_{ac} \cup \sigma_{sc} \cup \sigma_{pp}$.

The above standard scheme of spectral theory can be extended, to further decompose the singular-continuous subspace, by using Hausdorff measures. Recall that for any subset S of $\mathbb{R}$ and $\alpha \in [0, 1]$, the α -dimensional Hausdorff measure, h^α , is given by

$$h^\alpha(S) \equiv \lim_{\delta \rightarrow 0} \inf_{\delta\text{-covers}} \sum_{\nu=1}^{\infty} |b_\nu|^\alpha, \quad (3)$$

where a δ -cover is a cover of S by a countable collection of intervals, $S \subset \bigcup_{\nu=1}^{\infty} b_\nu$, such that for each ν the length of b_ν is at most δ . h^1 coincides with Lebesgue measure, and h^0 is the counting measure (assigning to each set the number of points in it). If $\alpha < \beta$, then for any S , $h^\alpha(S) \geq h^\beta(S)$. Given any $\emptyset \neq S \subseteq \mathbb{R}$, there exists a unique $\alpha(S) \in [0, 1]$ such that $h^\alpha(S) = 0$ for any $\alpha > \alpha(S)$, and $h^\alpha(S) = \infty$ for any $\alpha < \alpha(S)$. $h^{\alpha(S)}(S)$ may be zero, finite, or infinite. This unique $\alpha(S)$ is called the Hausdorff dimension of S . A rich theory of decomposing measures with respect to Hausdorff measures has been developed by

Rogers and Taylor [1]. Below we shall only employ a small part of it. A much more detailed description has been given by Last [2].

Given $\alpha \in [0, 1]$, a measure μ is called α -continuous (αc) if $\mu(S) = 0$ for every set S with $h^\alpha(S) = 0$. It is called α -singular (αs) if it is supported on some set S with $h^\alpha(S) = 0$. We say that μ is one-dimensional (od) if it is α -continuous for every $\alpha < 1$. We say that it is zero-dimensional (zd) if it is α -singular for every $\alpha > 0$. μ is said to have exact dimension α (for $\alpha \in (0, 1)$) if, for every $\epsilon > 0$, it is both $(\alpha - \epsilon)$ -continuous and $(\alpha + \epsilon)$ -singular.

Absolutely-continuous measures are the same as 1-continuous measures, and, in particular, they are one-dimensional. Pure-point measures are zero-dimensional. A measure which is both α -singular and β -continuous for some $0 < \beta < \alpha < 1$ must be singular-continuous. But, there are also singular-continuous measures which are one-dimensional and which are therefore close to absolutely-continuous measures. Similarly, there are singular-continuous measures which are zero-dimensional and which are therefore close to pure-point measures.

Given a (positive, finite) measure μ and $\alpha \in [0, 1]$, we define

$$D_\mu^\alpha(x) \equiv \limsup_{\epsilon \rightarrow 0} \frac{\mu((x - \epsilon, x + \epsilon))}{(2\epsilon)^\alpha} \quad (4)$$

and $T_\infty \equiv \{x \mid D_\mu^\alpha(x) = \infty\}$. T_∞ must have $h^\alpha(T_\infty) = 0$ and thus, the restriction $\mu(T_\infty \cap \cdot) \equiv \mu_{\alpha s}$ is α -singular. Moreover, $\mu((\mathbb{R} \setminus T_\infty) \cap \cdot) \equiv \mu_{\alpha c}$ is α -continuous. Thus, each measure decomposes uniquely into an α -continuous part and an α -singular part: $\mu = \mu_{\alpha c} + \mu_{\alpha s}$. Moreover, an α -singular measure must have $D_\mu^\alpha(x) = \infty$ a.e. (with respect to it) and an α -continuous measure must have $D_\mu^\alpha(x) < \infty$ a.e.. It is important to note that $D_\mu^\alpha(x)$ is defined with a limsup. The corresponding limit need not exist.

We let $\mathcal{H}_{\alpha c} \equiv \{\psi \mid \mu_\psi \text{ is } \alpha\text{-continuous}\}$ and $\mathcal{H}_{\alpha s} \equiv \{\psi \mid \mu_\psi \text{ is } \alpha\text{-singular}\}$. $\mathcal{H}_{\alpha c}$ and $\mathcal{H}_{\alpha s}$ are mutually orthogonal closed subspaces which are invariant under H , and $\mathcal{H}$ decomposes as $\mathcal{H} = \mathcal{H}_{\alpha c} \oplus \mathcal{H}_{\alpha s}$. The α -continuous spectrum ($\sigma_{\alpha c}$) and α -singular spectrum ($\sigma_{\alpha s}$) are defined as the spectra of the restrictions of H to the corresponding subspaces, and $\sigma = \sigma_{\alpha c} \cup \sigma_{\alpha s}$. Note, in particular, that when we classify spectra as being α -singular, zero-dimensional, of exact dimension α etc., we always relate to the corresponding properties of the spectral measures.

The above scheme for spectral classification can be related to the dynamics of the underlying quantum systems. A detailed account of such relations has been given by Last [2].

It should be pointed out that certain fractal and multifractal studies of some operators with singular-continuous spectrum (including some of the examples we discussed above) have been carried out by several authors [20, 22]. While such studies are related to the above

decomposition theory, the relations are generally far from trivial, and we believe that they are only partial. One should exercise extreme care when attempting to interpret the results of such studies within the framework of the scheme discussed above.

From here on we shall restrict our discussion to one-dimensional tight-binding Schrödinger operators of the form (1). While we discuss discrete operators, the subordinacy results we describe are equally valid for continuous Schrödinger operators of the form $-\frac{d^2}{dx^2} + V$.

Consider first “half-line” operators, defined with a phase boundary condition of the form (2). For such operators, it is well known that the spectral measures for lattice-point vectors δ_n , where $\delta_n(m) = \delta_{nm}$, are all mutually equivalent (namely, they have the same sets of zero measure). Thus, the spectral problem reduces to analyzing a single spectral measure, which we choose to be $\mu = \mu_{\delta_1}$. The Gilbert-Pearson theory of subordinacy [5] relates the pointwise behavior of the spectral measure μ at some energy E to the behavior of solutions of the corresponding Schrödinger equation

$$\psi(n+1) + \psi(n-1) + V(n)\psi(n) = E\psi(n). \quad (5)$$

Given a solution of (5), we let $\|\psi\|_L$ denote the norm of the solution ψ over length L . While the solution is only defined for integer points, it is useful to consider the length L as a continuous variable (allowed to take any positive real value), and so we define:

$$\|\psi\|_L = \left[\sum_{n=1}^{[L]} |\psi(n)|^2 + (L - [L])|\psi([L] + 1)|^2 \right]^{1/2}, \quad (6)$$

where $[L]$ denotes the integer part of L . A (non-trivial) solution ψ of (5) is called a subordinate solution if for any other solution φ of (5), which is not a constant multiple of ψ , $\lim_{L \rightarrow \infty} \frac{\|\psi\|_L}{\|\varphi\|_L} = 0$. Note that a subordinate solution must be unique (up to constant multiples). The Gilbert-Pearson theory relates the decomposition of the spectral measure μ to subordinacy of solutions as follows: The absolutely-continuous part of μ is supported on the set of energies for which (5) has no subordinate solutions. (In fact, this set of energies is, up to a set of both Lebesgue and spectral measure zero, the set where μ has a finite non-vanishing derivative.) The singular part of μ is supported on the set of energies for which the solutions which obey the appropriate boundary condition (2) are subordinate. (In fact, this is precisely the set of energies for which μ has an infinite derivative: $\lim_{\epsilon \rightarrow 0} \mu((E - \epsilon, E + \epsilon))/(2\epsilon) = \infty$.)

Let us now denote by ψ_1 the solution of (5) which obeys the boundary condition (2) and has normalization $|\psi_1(0)|^2 + |\psi_1(1)|^2 = 1$. Let us denote by ψ_2 the solution of (5) which obeys the orthogonal boundary condition to (2), namely, $\psi_2(0)\sin\theta - \psi_2(1)\cos\theta = 0$, and has normalization $|\psi_2(0)|^2 + |\psi_2(1)|^2 = 1$. Our main result is the following:

Theorem 1. For any $\alpha \in (0, 1]$ and every $E \in \mathbb{R}$, $D_\mu^\alpha(E) = \infty$ if and only if

$$\liminf_{L \rightarrow \infty} \frac{\|\psi_1\|_L}{\|\psi_2\|_L^\beta} = 0 ,$$

where $\beta = \alpha/(2 - \alpha)$.

Remark. Theorem 1 is proven with the same ideas used by Gilbert-Pearson, but it requires some optimization of their analysis. As a by-product, we also get a simplified proof of their original results, as well as a strengthened version of some explicit bounds of the form obtained by Simon [23]. A key observation is to assign to each $\epsilon > 0$ a length $L(\epsilon)$ via the equality $\|\psi_1\|_{L(\epsilon)}\|\psi_2\|_{L(\epsilon)} = 1/(2\epsilon)$, for which we prove the explicit inequality

$$\frac{5 - \sqrt{24}}{|m(E + i\epsilon)|} < \frac{\|\psi_1\|_{L(\epsilon)}}{\|\psi_2\|_{L(\epsilon)}} < \frac{5 + \sqrt{24}}{|m(E + i\epsilon)|} ,$$

where $m(z)$ is the Weyl-Titchmarsh function obeying $m(z) = \langle \delta_1, (H - z)^{-1} \delta_1 \rangle = \int \frac{d\mu(x)}{x - z}$ (so $m(z)$ is the Borel transform of the spectral measure μ).

For spectral analysis, Theorem 1 can be combined with two further basic facts. The first is the existence of generalized eigenfunctions [24], from which one can show that for a.e. E with respect to the spectral measure μ , the solution ψ_1 must obey $\limsup_{L \rightarrow \infty} \frac{\|\psi_1\|_L}{L^{1/2} \ln L} < \infty$ and $\liminf_{L \rightarrow \infty} \frac{\|\psi_1\|_L}{L^{1/2}} < \infty$. The second is the constancy of the Wronskian $\psi_1(n+1)\psi_2(n) - \psi_2(n+1)\psi_1(n)$, which implies $\|\psi_1\|_L\|\psi_2\|_L \geq (L-1)/2$.

Recall that the upper Lyapunov exponent $\bar{\gamma}(E)$ is defined by:
 $\bar{\gamma}(E) \equiv \limsup_{L \rightarrow \infty} (1/L) \ln \|\Phi_L(E)\|$, where the $\Phi_L(E)$'s are the 2×2 transfer matrices defined by $\Phi_L(E) \equiv T_L(E)T_{L-1}(E) \cdots T_1(E)$, and

$$T_n(E) \equiv \begin{pmatrix} E - V(n) & -1 \\ 1 & 0 \end{pmatrix} .$$

A rather soft application of Theorem 1 is given by the following:

Corollary 1.1. Suppose that $\bar{\gamma}(E) > 0$ for every E (outside, possibly, a set of E 's of zero Hausdorff dimension), then for every boundary phase θ the corresponding spectral measure μ is zero-dimensional.

Corollary 1.1 is essentially a special case of:

Corollary 1.2. *Suppose that for some $\beta > 1$ and every E in some Borel set A (5) has a solution φ obeying $\limsup_{L \rightarrow \infty} \frac{\|\varphi\|_L^2}{L^\beta} > 0$. Then for every $\epsilon > 0$, the restriction $\mu(A \cap \cdot)$ is $(\alpha + \epsilon)$ -singular, where $\alpha = 2/(1 + \beta)$.*

Another general application of Theorem 1 relates averaged decay of generalized eigenfunctions (if it happens to occur) to dimensional Hausdorff properties:

Corollary 1.3. *Suppose that $\liminf_{L \rightarrow \infty} \frac{\|\psi_1\|_L^2}{L^\alpha} = 0$ for every E in some Borel set A . Then the restriction $\mu(A \cap \cdot)$ is α -singular.*

Remark. Note, in particular, that if the generalized eigenfunctions are averagely decaying, then μ must be singular.

It is well known that energies for which (5) has only bounded solutions must be associated with the absolutely-continuous part of the spectral measure μ (this is an immediate consequence of the Gilbert-Pearson theory, although it can also be shown by different means [23]). A certain extension of this principle is provided by the following:

Corollary 1.4. *Suppose that for some $1 \leq \beta < 2$ and every E in some Borel set A , every solution φ of (5) obeys $\limsup_{L \rightarrow \infty} \frac{\|\varphi\|_L^2}{L^\beta} < \infty$. Then the restriction $\mu(A \cap \cdot)$ is α -continuous, where $\alpha = 2 - \beta$.*

Remark. While Theorem 1 is also valid for continuous Schrödinger operators on $L^2(\mathbb{R})$, Corollaries 1.3 and 1.4 are not. Their proofs use the fact that the Wronskian in the discrete case involves only solutions, as opposed to the continuous case where the Wronskian also involves derivatives. Continuous Schrödinger operators may have absolutely-continuous spectrum along with decaying eigenfunctions (e.g., the potential $V(x) = -x$, for which the eigenfunctions are decaying Airy functions).

While Corollaries 1.1–1.4 can be applied to a variety of concrete examples, one can often obtain more detailed information by making a careful study of the ratio $\|\psi_1\|_L/\|\psi_2\|_L^\beta$, and applying Theorem 1 directly. Theorem 2 is obtained by such an analysis.

We now discuss briefly “line” operators. The spectral measures of a “line” operator can be constructed from those of corresponding two “half-line” operators (a left and a right), and while the relations are not completely trivial, they do allow an extension of the subordinacy theory to this case. Gilbert [6] has shown that the absolutely-continuous part of the spectral measures of a “line” operator is supported on the set of energies for which at least one of

the “half-line” problems has no subordinate solution. The singular part is supported on the set of energies for which (5) has a solution which is subordinate both to the right and to the left. The probing of dimensional Hausdorff properties is somewhat more delicate in this case since it involves a $\liminf$ rather than a limit. For example, in order for a “line” operator to have purely zero-dimensional spectrum, it is not sufficient that the right and left upper Lyapunov exponents are both positive. It is also needed that the length-scales, on the two sides, for which the norms of the transfer matrices are large will be correlated. Nevertheless, in concrete settings, such as ones discussed in Theorems 3 and 4, the required control can be obtained.

In conclusion we would like to remark the following: The classification of spectra with respect to dimensional Hausdorff measures extends the usual spectral classification in a natural way, and provides a useful way of distinguishing between different kinds of singular-continuous spectra. The subordinacy theory extends to this point of view in a natural way, and allows to answer the relevant spectral questions whenever the nature of the solutions of the corresponding Schrödinger equation is sufficiently well understood. We note, in particular, that singular-continuous spectrum which occurs in “close neighborhood” to Anderson localization (as in the case of the strongly coupled almost Mathieu operator or the rank-one perturbed Anderson model) tends to be purely zero-dimensional; while the singular-continuous spectrum of the Fibonacci Hamiltonian, which has been identified as having “critical states” in physics literature [20], is α -continuous for some positive α .

As we were completing this paper we learned of a preprint by Remlin [25] which obtains a restricted version of Theorem 1.

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