ESI **The Erwin Schrödinger International Institute for Mathematical Physics**
Boltzmanngasse 9
A-1090 Wien, Austria

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M.J. Craddock
J.A. Hogan

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THE FRACTIONAL CLIFFORD-FOURIER KERNEL

M.J. CRADDOCK AND J.A. HOGAN

Abstract. The Clifford-Fourier transform was introduced by Brackx, De Schepper and Sommen and its kernel was computed in dimension $d = 2$ by the same authors. Here we compute the kernel of a fractional version of the transform when $d = 2$ and 4. In doing so we solve appropriate wave-type problems on spheres in two and four dimensions. We also give formulae for the solutions of these problems in all even dimensions and hence a means of computing the kernels of the fractional Clifford-Fourier kernels in even dimensions.

1. Introduction

Fourier analysis has become an indispensible tool in the sciences and many engineering disciplines. Time series such as those which arise from speech or music have been effectively treated by algorithms derived from Fourier analysis – the (fast) Fourier transform and wavelet transform being among the most celebrated. Complex analysis has also played an important role, contributing tools such as the analytic signal, the Paley-Wiener theorem and Hardy's theorem. Tensor products of the one-dimensional Fourier transform are often applied to higher-dimensional signals, such as grayscale images.

Fractional versions of the Fourier transform, in one- and higher dimensions, are growing in importance in signal processing (where they provide signal representations between the classical time and frequency representations) and physics (where they describe the evolution of images through lens systems and also have applications in quantum mechanics) [13]. Fractional kernels are in general more complicated than the Fourier kernel (which is an end-point case of the fractional kernels), and are computed through the Mehler formula which gives a closed form for a certain infinite weighted sum of products of Hermite functions [7].

Colour images, however, pose a different set of problems. Such images are composed of (at least) three channels rather than the one channel of grayscale images. Natural images contain very significant cross-channel correlations [12]. Any algorithm using the channel-by-channel paradigm is likely to be sub-optimal (especially for purposes of compression, but also for interpolation) for they do not see cross-channel correlations and are therefore unable to reduce them.

Electrical engineers have responded to the challenges posed by multi-channel signals by developing techniques through which such a signal can

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be treated as an algebraic whole rather than an ensemble of disparate, unrelated single-channel signals [6], [15], [16]. A colour image is now viewed as a signal taking values in the quaternions, an associative, non-commutative algebra. The algebra structure gives a meaning to the pointwise product of such signals. Fourier-type transforms of quaternionic signals were considered in [15] and [16]. Their advantage over the classical Fourier transform is that the kernels through which they act are quaternion-valued, and the transforms therefore "mix the channels" rather than acting on each channel separately. They fail, however, to have the kinds of covariance and convolution properties that are required for effective applications and do not extend, in any obvious way, to higher dimensions.

Brackx, De Schepper and Sommen [1] introduced a "Clifford"-Fourier transform for functions defined on $\mathbb{R}^d$ and taking values in the associated Clifford algebra $\mathbb{R}_d$. Like the classical Fourier transform, the Clifford-Fourier transform is defined as the exponential of a Hermite operator and has Hermite-type eigenfunctions. Through this operator-theoretic definition, many attractive properties may be determined. These include simple inversion and Plancherel theorems, as well as generalisations of classical results which describe the action on L^p-spaces and the Schwarz space of rapidly decreasing functions, and a Clifford generalisation of the classical result which states that the classical Fourier kernel is an eigenfunction of the partial differentiation operators $\partial/\partial x_j$ [1]. Generalisations of classical results which characterise translation-invariant bounded linear operators on L^2 and translation-invariant closed subspaces of L^2 are also known [11].

Unfortunately, computing the integral kernel through which the Clifford-Fourier transform acts is quite difficult since it is the exponential of a sum of non-commuting operators. In [2], the explicit f orm of the kernel is given in dimension $d = 2$ and in [4] a construction of the kernel in all even dimensions was given. The odd-dimensional kernels are still unknown.

In this paper we find explicit formulae for solutions of wave-type initial value problems on spheres in $\mathbb{R}^d$ and, as a special case, give a completely different construction of the even-dimensional Clifford-Fourier kernels. Furthermore, explicit formulae for the kernels of the fractionalisations of these operators are computed. It is also shown that once the 4-dimensional kernel is known, all even-dimensional kernels ($d \geq 6$) may be determined through a "method of ascent". In the preprint [3], the fractional Clifford-Fourier kernels are computed in even dimensions. The methods used there are very different from those employed in this paper. In particular, DeBie and De Schepper arrive at the fractional kernels without reference to the initial-value problems which are the focus of this paper, the solutions of which give the fractional kernels for particular initial values.

This paper is organised as follows. In section 2 we introduce those aspects of classical Fourier theory (Hermite functions and fractional Fourier transforms) that we intend to generalise as well as a brief introduction to Clifford algebra, the operator-theoretic construction of the Clifford-Fourier transform of Brackx, De Schepper and Sommen, and some results of classical harmonic analysis surrounding spherical harmonics. In section 3 we collect some results on the action of the angular Dirac operator which forms part of the Clifford-Hermite operator. Section 4 outlines the initial value problems satisfied by the fractional Clifford-Fourier kernels and describes how separation of variables leads to spherical harmonic expansions for these kernels, leading ultimately to expansions of the kernels in Legendre/Chebyshev polynomials. In section 5 we give explicit solutions of the initial value problems of section 4 (hence explicit expressions for the fractional Clifford-Fourier kernel) in dimensions $d = 2$ and $d = 4$. Finally in section 7 we describe the "method of ascent" through which solutions of the initial value problems of section 4 may be determined in dimensions $d \geq 5$ from the solution in dimension $d-2$. Hence, the kernel in even dimension $d \geq 6$ may be calculated (iteratively) from the kernel in dimension $d = 4$. We also show why the 2-dimensional kernel does not determine the 4-dimensional kernel.

2. Preliminaries

2.1. **Fractional Fourier transforms and the Hermite functions.** The classical Fourier transform (FT) is the integral operator $\mathcal{F} = \mathcal{F}_d : L^1(\mathbb{R}^d) \to L^\infty(\mathbb{R}^d)$ with kernel $K_d(x,y) = (2\pi)^{-d/2}e^{-i\langle x,y\rangle}$. Here $\langle x,y\rangle$ is the dot product of the vectors x and y in $\mathbb{R}^d$. The many applications of the FT and its discrete variants are well-documented (see for example [14]). Among the many reasons for the wide applicability of the FT, we include its attractive covariance properties, its energy conservation property, and the simple closed form of its kernel. By *covariance* we mean its simple interaction with the basic operators of harmonic analysis – translations, dilations and rotations (although its covariance under other actions such as "shears" is also becoming important in applications). By *energy conservation* we mean that the Fourier transform satisfies the Parseval identity, i.e., the Fourier transform extends to a unitary mapping of $L^2(\mathbb{R})$: $\int_{\mathbb{R}^d} \mathcal{F}f(y)\overline{\mathcal{F}g(y)}\,dy = \int_{\mathbb{R}^d} f(x)\overline{g(x)}\,dx$ for all $f,g \in L^2(\mathbb{R}^d)$.

The Hermite functions $\{h_n\}_{n=0}^\infty$ are defined on the real line by

$$h_n(x) = \frac{1}{\sqrt{2^n n!\sqrt{\pi}}}(-1)^n e^{x^2/2}\frac{d^n}{dx^n}e^{-x^2},$$

form an orthonormal basis for $L^2(\mathbb{R})$, and are eigenfunctions of the FT, i.e., $\mathcal{F}h_n = (-i)^n h_n$. Consequently, given $f \in L^2(\mathbb{R})$, its FT admits the expansion $\mathcal{F}f = \sum_{n=0}^\infty \langle f, h_n\rangle(-i)^n h_n$ which suggests a *fractionalisation* $\mathcal{F}_t$ of the FT, namely

$$\mathcal{F}_t f = \sum_{n=0}^\infty \langle f, h_n\rangle e^{-int} h_n \quad (t \in \mathbb{R}). \tag{1}$$

The operator defined by equation (1) is known as the *fractional Fourier transform* (frFT). It satisfies $\mathcal{F}_0 = \mathrm{id}$, $\mathcal{F}_{\pi/2} = \mathcal{F}$ and $\mathcal{F}_s \circ \mathcal{F}_t = \mathcal{F}_{s+t}$. Hence the mapping $t \in \mathbb{R} \to \mathcal{F}_t$ is a periodic embedding of the line into a one-parameter Lie group of unitary mappings of $L^2(\mathbb{R})$.

The Hermite operator $\mathcal{H}$ is defined by $\mathcal{H} = \frac{1}{2}\left(-\frac{d^2}{dx^2} + x^2 - 1\right)$ and has the Hermite functions as eigenfunctions, i.e., $\mathcal{H}h_n = nh_n$. From this we see that the FT may be written as the operator exponential

$$\mathcal{F} = \exp(-i(\pi/2)\mathcal{H}), \tag{2}$$

and the frFT may similarly be written as $\mathcal{F}_t = \exp(-it\mathcal{H})$ $(t \in \mathbb{R})$. From (1) we have an expression for the kernel $K_t(x, y)$ through which the frFT acts:

$$K_t(x, y) = \sum_{n=0}^{\infty} e^{-int} h_n(x) h_n(y).$$

This sum may be computed in closed form through the Mehler formula:

$$K_t(x, y) = \sqrt{\frac{-ie^{it} \csc t}{2\pi}} \exp(i(-(\csc t)xy + (\cot t)(x^2 + y^2)/2)) \quad (x, y \in \mathbb{R}),$$

an elegant proof of which may be found in [7]. Notice that when $t = \pi/2$ we recover the Fourier transform kernel $(2\pi)^{-1/2} e^{-i\langle x,y\rangle}$.

In higher dimensions, the Hermite functions, Hermite operator and frFT kernel have definitions based on tensor products of the one-dimensional definitions, so that in dimension d the Hermite operator takes the form $\mathcal{H}_d = \dfrac{1}{2}(-\Delta_d + |x|^2 - d)$ (where $\Delta_d = \sum_{j=1}^{d} \dfrac{\partial^2}{\partial x_j^2}$ is the Laplacian on $\mathbb{R}^d$, $x \in \mathbb{R}^d$) and for $x, y \in \mathbb{R}^d$, the frFT kernel becomes

$$K_{d,t}(x, y) = \left(\frac{-ie^{it} \csc t}{2\pi} \right)^{d/2} \exp(i(-(\csc t)\langle x, y\rangle + (\cot t)(|x|^2 + |y|^2)/2)).$$

2.2. **Clifford Algebra.** In this section we give a quick review of the basic concepts of Clifford algebra. The interested reader is referred to [5] for more details.

Let $\{e_1, e_2, \ldots, e_d\}$ be an orthonormal basis for d-dimensional euclidean space $\mathbb{R}^d$. The associative *Clifford algebra* $\mathbb{R}_d$ is the 2^d-dimensional algebra spanned by the collection

$$\bigcup_{j=1}^{d} \{e_A : A = \{i_1, i_2, \ldots, i_j\} \text{ with } 1 \le i_1 < i_2 < \cdots < i_j \le d\}$$

with algebraic properties $e_\emptyset = 1$ (the identity), $e_j^2 = -1$, and if $j < k$ then $e_{\{j,k\}} = e_j e_k = -e_k e_j$. Here $\emptyset$ is the null set and we often abuse notation and write $e_\emptyset = e_0 = 1$. Notice that for convenience we write $e_{\{j\}} = e_j$. In particular we have $\mathbb{R}_d = \{\sum_A x_A e_A;\ x_A \in \mathbb{R}\}$. Similarly, we have the complexified Clifford algebra $\mathbb{C}_d = \{\sum_A z_A e_A;\ z_A \in \mathbb{C}\}$. The canonical mapping of $\mathbb{R}^d$ into $\mathbb{R}_d$ maps the vector $(x_1, x_2, \ldots, x_d) \in \mathbb{R}^d$ to $\sum_{j=1}^{d} x_j e_j \in \mathbb{R}_d$. For this reason, elements of $\mathbb{R}_d$ of the form $\sum_{j=1}^{d} x_j e_j$ are also known as *vectors*. Notice that $\mathbb{R}_d$ decomposes as $\mathbb{R}_d = \Lambda_0 \oplus \Lambda_1 \oplus \ldots \Lambda_d$, where $\Lambda_j = \{\sum_{|A|=j} x_A e_A\}$. In particular, Λ_0 is the collection of scalars while Λ_1 is the collection of vectors. Given $x \in \mathbb{R}_d$ of the form $x = \sum_A x_A e_A$ and $0 \le p \le d$ we write $[x]_p$ to mean the "Λ_p-part" of x, i.e, $[x]_p = \sum_{|A|=p} x_A e_A$.

If $x, y \in \mathbb{R}_d$ are vectors, then $x^2 = -|x|^2$ (a scalar) and their Clifford product xy may be expressed as $xy = -\langle x, y\rangle + x \wedge y \in \Lambda_0 \oplus \Lambda_2$. Here $\langle x, y\rangle$ is the usual dot product of x and y while $x \wedge y$ is their *wedge product*. The linear involution $\overline{u}$ of $u \in \mathbb{R}_d$ is determined by the rules $\overline{x} = -x$ if $x \in \Lambda_1$ while $\overline{uv} = \overline{v}\,\overline{u}$ for all $u, v \in \mathbb{R}_d$.

As examples, note that $\mathbb{R}_1$ is identified algebraically with the field of complex numbers $\mathbb{C}$ while $\mathbb{R}_2$, which has basis $\{1, e_1, e_2, e_1e_2\}$ and whose typical element has the form $q = a + be_1 + ce_2 + de_1e_2$ (with $a, b, c, d \in \mathbb{R}$) is identifiable with the associative algebra of *quaternions* $\mathbb{H}$.

2.3. **The Clifford-Fourier transform.** In [1], Brackx, De Schepper and Sommen introduced the Clifford-Fourier transform in a manner analogous to that of equation (2). We now briefly outline the construction. The *angular momentum operators* $\mathcal{L}_{ij}$ ($1 \le i, j \le d$) are the differential operators defined by $\mathcal{L}_{ij} = x_i \dfrac{\partial}{\partial x_j} - x_j \dfrac{\partial}{\partial x_i}$ and the *angular Dirac operator* Γ by

$$\Gamma = \sum_{1 \le i < j \le d} e_i e_j \mathcal{L}_{ij} \tag{3}$$

(with e_i, e_j Clifford units in $\mathbb{R}_d$). The two Clifford-Hermite operators $\mathcal{H}_d^{\pm}$ are defined by

$$\mathcal{H}_d^{\pm} = \mathcal{H}_d \mp (\Gamma + d/2)$$

(where $\mathcal{H}_d$ is the classical d-dimensional Hermite operator) and corresponding *Clifford-Fourier transforms* $\mathcal{F}_d^{\pm}$ are defined by the operator exponentials $\mathcal{F}_d^{\pm} = \exp(-i(\pi/2)\mathcal{H}_d^{\pm})$. Since $\mathcal{H}_d$ commutes with Γ, we have $\mathcal{F}_d^{+}\mathcal{F}_d^{-} = \exp(-i\pi\mathcal{H}_d) = \tau$ where τ is the inversion $\tau f(x) = f(-x)$. Note also that

$$\begin{aligned}\mathcal{F}_d^{\pm} &= \exp(-i(\pi/2)\mathcal{H}_d^{\pm})\\ &= \exp(-i(\pi/2)(\mathcal{H}_d \mp (\Gamma + d/2)))\\ &= \exp(\pm i(\pi/2)(\Gamma + d/2))\exp(-i(\pi/2)\mathcal{H}_d) = \exp(\pm i(\pi/2)(\Gamma + d/2))\mathcal{F}_d\end{aligned}$$

where $\mathcal{F}_d$ is the classical d-dimensional FT which acts by integration against the scalar-valued kernel $K_d(x, y) = (2\pi)^{-d/2}e^{-i\langle x,y\rangle}$. Consequently, given $f \in L^2(\mathbb{R}^d, \mathbb{R}_d)$,

$$\mathcal{F}_d^{\pm} f(x) = \int_{\mathbb{R}^d} \exp(\pm i(\pi/2)(\Gamma_x + d/2))K_d(x, y)f(y)\, dy$$

from which we see that $\mathcal{F}_d^{\pm}$ acts by integration against the Clifford-valued kernel

$$C_d^{\pm}(x, y) = \exp(\pm i(\pi/2)(\Gamma_x + d/2))K_d(x, y). \tag{4}$$

In writing Γ_x, we mean to emphasise that Γ is acting on the x-variable. We define two *fractional Clifford Fourier transforms* (frCFT) by the operator exponential $\mathcal{F}_{d,t}^{\pm} = \exp(-it\mathcal{H}_d^{\pm})$. Then $\mathcal{F}_{d,t}^{+}\mathcal{F}_{d,t}^{-} = \mathcal{F}_{d,2t}$ and the frCFTs act by integration against the kernels

$$C_{d,t}^{\pm}(x, y) = \exp(\pm it(\Gamma_x + d/2))K_{d,t}(x, y) \tag{5}$$

with $K_{d,t}$ the classical d-dimensional frFT kernel.

2.4. **Spherical harmonics and Legendre polynomials.** A function $f \in C^2(\mathbb{R}^d)$ is said to be *harmonic* if $\Delta_d f \equiv 0$. Let Q_ℓ^d be the space of all harmonic polynomials of degree ℓ in d variables. A *spherical harmonic* of dimension d is the restriction to S^{d-1} of an harmonic polynomial in d variables. Let H_ℓ^d be the space of spherical harmonics of homogeneous degree ℓ in d variables. Groemer [10] shows that the dimension of H_ℓ^d is

$$\dim(H_\ell^d) = N(d, \ell) = \frac{2\ell + d - 2}{\ell + d - 2}\binom{\ell + d - 2}{d - 2}.$$

Given a function f defined on $\mathbb{R}^d$ and $r > 0$, we denote by $R_r f$ the restriction of f to the sphere rS^{d-1} with centre 0 and radius r. The homogeneous extension operator $X_r : C^k(rS^{d-1}) \to C^k(\mathbb{R}^d)$ is given by $X_r g(x) = g\left(\frac{rx}{|x|}\right)$ $(x \in \mathbb{R}^d)$. The *tangential Laplacian* Δ_T is now defined on $C^2(\mathbb{R}^d)$ by

$$\Delta_T F(x) = R_r \Delta X_r F(x) = \sum_{j=1}^{d} \frac{\partial^2}{\partial x_j^2} \left(F\left(\frac{rx}{|x|}\right)\right)\Big|_{r=|x|}.$$

The tangential Laplacian may also be considered as operating on functions defined on the unit sphere S^{d-1}, in which case it is known as the *Laplace-Beltrami operator*. Spherical harmonics are eigenfunctions of the Laplace-Beltrami operator. In fact, if $F \in H_\ell^d$, then $\Delta_T F = -\ell(\ell+d-2)F$. If $\ell \neq \ell'$ then the spaces H_ℓ^d and $H_{\ell'}^d$ are orthogonal with respect to the inner product $\langle F, G\rangle = \int_{S^{d-1}} F(\omega)\overline{G(\omega)}\, d\sigma(\omega)$ where $d\sigma$ is the surface area measure on S^{d-1}. Further, we have $L^2(S^{d-1}) = \oplus_{\ell=0}^\infty H_\ell^d$ and each $F \in L^2(S^{d-1})$ has a unique orthogonal expansion of the form $F = \sum_{\ell=0}^\infty F_\ell$ with each $F_\ell \in H_\ell^d$. By choosing an orthonormal basis $\{Y_{\ell,j}\}_{j=1}^{N(d,\ell)}$ for each H_ℓ^d, we have an expansion of $F \in L^2(S^{d-1})$ in spherical harmonics, namely $F = \sum_{\ell=0}^\infty \sum_{j=1}^{N(d,\ell)} \langle F, Y_{\ell,j}\rangle Y_{\ell,j}$.

The *Legendre polynomial* of dimension d and degree $\ell \geq 0$ is denoted P_ℓ^d and defined by $P_\ell^d(t) = \frac{(-1)^\ell}{2^\ell(\vartheta+1)(\vartheta+2)\dots(\vartheta+\ell)}(1-t^2)^{-\vartheta}\frac{d^\ell}{dt^\ell}(1-t^2)^{\ell+\vartheta}$ where here and subsequently, $\vartheta = (d-3)/2$. Notice that when $d = 3$ we have $\vartheta = 0$ and $P_\ell^3(t) = \frac{(-1)^\ell}{2^\ell \ell!}\frac{d^\ell}{dt^\ell}(1-t^2)^\ell = P_\ell(t)$, the standard Legendre polynomials of degree ℓ. The collection $\{P_\ell^d\}_{\ell=0}^\infty$ is orthogonal on the interval $[-1, 1]$ with respect to the weight function $(1-t^2)^\vartheta$, i.e.,

$$\int_{-1}^{1} P_\ell^d(t)P_{\ell'}^d(t)(1-t^2)^\vartheta\, dt = \delta_{\ell,\ell'}\frac{\sigma_d}{\sigma_{d-1}N(d,\ell)}$$

with $\sigma_d = 2\pi^{d/2}/\Gamma(d/2)$ the surface area of the $(d-1)$-sphere in $\mathbb{R}^d$.

Let $\mathcal{L}_d$ denote the second order differential operator defined by

$$\mathcal{L}_d = (1-t^2)\frac{d^2}{dt^2} - (d-1)t\frac{d}{dt}.$$

$\mathcal{L}_d$ is self-adjoint on $[-1, 1]$ with respect to the weight function $(1-t^2)^\vartheta$. Furthermore, each Legendre polynomial P_ℓ^d is an eigenfunction of $\mathcal{L}_d$ with eigenvalue $-\ell(\ell+d-2)$.

The connection between spherical harmonics and Legendre polynomials is made by the following result [10].

Theorem 1. *Let $d \geq 2$, $\ell \geq 0$ and $\{Y_{\ell,j}\}_{j=1}^{N(d,\ell)}$ be an orthonormal basis for H_ℓ^d. Then for all $u, v \in S^{d-1}$,*

$$\sum_{j=1}^{N(d,\ell)} \overline{Y_{\ell,j}(u)}Y_{\ell,j}(v) = \frac{N(d,\ell)}{\sigma_d}P_\ell^d(\langle u, v\rangle).$$

Inner products of zonal functions on the sphere (those of the form $f(v) = F(\langle u, v\rangle)$ for some fixed $u \in S^{d-1}$ and given $F : [-1, 1] \to \mathbb{C}$) with spherical harmonics have the following simple form [10].

Theorem 2 (Funk-Hecke theorem)**.** *If G is a bounded, integrable function on $[-1, 1]$ and $Y \in H^d_\ell$, then for any fixed $u \in S^{d-1}$, the function $g_u(v) = G(\langle u, v\rangle)$ $(v \in S^{d-1})$ is an integrable function on S^{d-1} and*

$$\int_{S^{d-1}} g_u(v)Y(v)\,d\sigma(v) = \sigma_{d-1}\alpha_{d,\ell}(G)Y(u)$$

where $\alpha_{d,\ell}(G) = \int_{-1}^{1} G(t)P^d_\ell(t)(1-t^2)^\vartheta\,dt$.

3. The angular Dirac operator

In this section we collect a series of results which describe the action of the angular Dirac operator Γ on the classes of functions of interest. Since the CFT and frCFT kernels are formed from exponentials of operators involving Γ, we are interested in the behaviour of powers of Γ. Given the obvious observation that Γ maps scalar-valued functions f into Λ_2-valued functions Γf, we might expect that $\Gamma^2 f$ would take values in $\Lambda_0 \oplus \Lambda_2 \oplus \Lambda_4$. This turns out to be false as the following result shows.

Proposition 3. *Let Γ_d denote the angular Dirac operator acting in $\mathbb{R}^d$ as defined in (3). Then, as an operator on $C^2(\mathbb{R}^d, \mathbb{R}_d)$, we have*

$$\Gamma_d^2 = -|x|^2\Delta_T + (2-d)\Gamma_d. \tag{6}$$

Remark 4. *As an immediate consequence we see that the frCFT kernel, being of the form $C_t(x, y) = e^{-it\Gamma_x} f(x, y)$ for a scalar-valued function f, takes values in $\Lambda_0 \oplus \Lambda_2$.*

The proof of Proposition 3 may be obtained by scaling equation (0.16) p. 140 of [5] so that it applies to spheres of arbitrary radius. We supply here a different proof which avoids any mention of the Dirac operator and uses the beautiful commutation properties of the angular momentum operators.

Proof. First note that $F \in C^2(\mathbb{R}^d, \mathbb{R}_d)$ may be decomposed as $F = \sum_A F_A e_A$ with each $F_A \in C^2(\mathbb{R}^d, \mathbb{R})$. Hence it is enough to prove the operator equation (6) on scalar-valued functions F. Given such an F, $\Gamma_d^2 F = [\Gamma_d^2 F]_0 + [\Gamma_d^2 F]_2 + [\Gamma_d^2 F]_4$, i.e., $\Gamma_d^2 F$ is the sum of a scalar-valued function, a 2-form and a 4-form. A straightforward computation gives

$$\Delta_T F = \Delta F - \frac{(d-1)}{|x|^2} EF - \frac{1}{|x|^2}\sum_{j,k=1}^{d} x_j x_k \frac{\partial^2 F}{\partial x_j x_k} \tag{7}$$

where E is the *Euler operator* $EF(x) = \sum_{j=1}^d x_j \dfrac{\partial F}{\partial x_j}$. Now

$$\begin{aligned}[\Gamma_d^2 F]_0 &= \sum_{1\le i<j\le d} e_i e_j e_i e_j \mathcal{L}_{ij}^2 F \\ &= -\sum_{1\le i<j\le d}\left[x_i^2\frac{\partial^2 F}{\partial x_j^2} - x_i\frac{\partial F}{\partial x_i} - 2x_ix_j\frac{\partial^2 F}{\partial x_i\partial x_j} - x_j\frac{\partial F}{\partial x_j} + x_j^2\frac{\partial^2 F}{\partial x_i^2}\right] \\ &= -\frac{1}{2}\left(\sum_{i=1}^d\sum_{j=1}^d - \sum_{1\le i\le d, i=j}\right)\left[x_i^2\frac{\partial^2 F}{\partial x_j^2} - x_i\frac{\partial F}{\partial x_i}\right. \\ &\qquad\qquad \left. - 2x_ix_j\frac{\partial^2 F}{\partial x_i\partial x_j} - x_j\frac{\partial F}{\partial x_j} + x_j^2\frac{\partial^2 F}{\partial x_i^2}\right] = S_1 + S_2.\end{aligned}$$

The sum S_1 simplifies to $S_1 = -|x|^2\Delta F + dEF + \sum_{j=1}^d\sum_{k=1}^d x_jx_k\dfrac{\partial^2 F}{\partial x_j\partial x_k}$ and S_2 becomes $S_2 = -EF$. From (7) we conclude that

$$[\Gamma_d^2 F]_0 = -|x|^2\Delta_d F + (d-1)EF + \sum_{i=1}^d\sum_{j=1}^d x_ix_j\frac{\partial^2 F}{\partial x_i\partial x_j} = -|x|^2\Delta_T F. \tag{8}$$

Next we show that

$$[\Gamma_d^2 F]_2 = (2-d)\Gamma_d F \tag{9}$$

for scalar-valued functions F. The proof is by induction on the dimension d. First, $\Gamma_2^2 = (e_1e_2\mathcal{L}_{12})^2 = -\mathcal{L}_{12}^2$ so that $[\Gamma_2^2 F]_2 = 0$ and (9) is verified for $d=2$. Suppose now that equation (9) holds for some $d\ge 2$. Then since $\Gamma_{d+1} = \Gamma_d + \sum_{i=1}^d e_ie_{d+1}\mathcal{L}_{i,d+1}$, we have

$$\begin{aligned}\Gamma_{d+1}^2 = \left(\Gamma_d + \sum_{i=1}^d e_ie_{d+1}\mathcal{L}_{i,d+1}\right)^2 &= \Gamma_d^2 + \sum_{i=1}^d \Gamma_d e_ie_{d+1}\mathcal{L}_{i,d+1} \\ &+ \sum_{i=1}^d e_ie_{d+1}\mathcal{L}_{i,d+1}\Gamma_d + \left(\sum_{i=1}^d e_ie_{d+1}\mathcal{L}_{i,d+1}\right)^2.\end{aligned}\tag{10}$$

We deal with the terms in (10) separately. The first term on the right-hand side of (10) is dealt with by the inductive hypothesis. Secondly, note that $\Gamma_d e_ie_{d+1}\mathcal{L}_{i,d+1} = \sum_{1\le j<k\le d} e_je_ke_ie_{d+1}\mathcal{L}_{jk}\mathcal{L}_{i,d+1}$ so that upon extracting the 2-form part of this operator we obtain

$$\begin{aligned}[\Gamma_d e_ie_{d+1}\mathcal{L}_{i,d+1}F]_2 &= \sum_{k>i} e_ie_ke_ie_{d+1}\mathcal{L}_{ik}\mathcal{L}_{i,d+1}F + \sum_{j<i} e_je_ie_ie_{d+1}\mathcal{L}_{ji}\mathcal{L}_{i,d+1}F \\ &= \sum_{k>i} e_ke_{d+1}\mathcal{L}_{ik}\mathcal{L}_{i,d+1}F - \sum_{j<i} e_je_{d+1}\mathcal{L}_{ji}\mathcal{L}_{i,d+1}F \\ &= \sum_{k\ne i} e_ke_{d+1}\mathcal{L}_{ik}\mathcal{L}_{i,d+1}F.\end{aligned}\tag{11}$$

Similarly, we have

$$[e_ie_{d+1}\mathcal{L}_{i,d+1}\Gamma_d F]_2 = \sum_{k\ne i} e_{d+1}e_k\mathcal{L}_{i,d+1}\mathcal{L}_{ik}F. \tag{12}$$

Combining (11) and (12) gives

$$
\begin{aligned}
&[\Gamma_d e_i e_{d+1}\mathcal{L}_{i,d+1}F + e_i e_{d+1}\mathcal{L}_{i,d+1}\Gamma_d F]_2 \\
&\qquad = \sum_{k\neq i} e_k e_{d+1}\mathcal{L}_{ik}\mathcal{L}_{i,d+1}F + \sum_{k\neq i} e_{d+1}e_k\mathcal{L}_{i,d+1}\mathcal{L}_{ik}F \\
&\qquad = \sum_{k\neq i} e_k e_{d+1}[\mathcal{L}_{ik},\mathcal{L}_{i,d+1}]F = \sum_{k\neq i} e_k e_{d+1}\mathcal{L}_{d+1,k}F \qquad (13)
\end{aligned}
$$

where we have used the fact that if $1 \leq i < j < k \leq d+1$, then the commutator $[\mathcal{L}_{ij},\mathcal{L}_{jk}] = \mathcal{L}_{ij}\mathcal{L}_{jk} - \mathcal{L}_{jk}\mathcal{L}_{ij} = \mathcal{L}_{ik}$. Turning now to the fourth term on the right hand side of (10)

$$
\left(\sum_{i=1}^{d} e_i e_{d+1}\mathcal{L}_{i,d+1}\right)^2 F = \sum_{i=1}^{d}\sum_{j=1}^{d} e_i e_j \mathcal{L}_{i,d+1}\mathcal{L}_{j,d+1}F,
$$

so that, upon extracting the 2-form part of this quantity, we have

$$
\begin{aligned}
\left[\left(\sum_{i=1}^{d} e_i e_{d+1}\mathcal{L}_{i,d+1}\right)^2 F\right]_2 &= \sum_{1\leq i\leq d, 1\leq j\leq d, i\neq j} e_i e_j\mathcal{L}_{i,d+1}\mathcal{L}_{j,d+1}F \\
&= \sum_{1\leq i<j\leq d} (e_i e_j \mathcal{L}_{i,d+1}\mathcal{L}_{j,d+1} + e_j e_i \mathcal{L}_{j,d+1}\mathcal{L}_{i,d+1})F \\
&= \sum_{1\leq i<j\leq d} e_i e_j [\mathcal{L}_{i,d+1},\mathcal{L}_{j,d+1}]F = -\sum_{1\leq i<j\leq d} e_i e_j \mathcal{L}_{ij}F = -\Gamma_d F. \qquad (14)
\end{aligned}
$$

Combining equations (10), (13) and (14) gives

$$
\begin{aligned}
[\Gamma_{d+1}^2 F]_2 &= (2-d)\Gamma_d F - \sum_{i=1}^{d}\sum_{1\leq k\leq d, k\neq i} e_k e_{d+1}\mathcal{L}_{k,d+1}F - \Gamma_d F \\
&= (1-d)\Gamma_d F - \sum_{i=1}^{d}\sum_{1\leq k\leq d, k\neq i} e_k e_{d+1}\mathcal{L}_{k,d+1}F. \qquad (15)
\end{aligned}
$$

On the other hand, the second term on the right hand side of (15) may be written as

$$
\begin{aligned}
&\sum_{i=1}^{d}\sum_{1\leq k\leq d, k\neq i} e_k e_{d+1}\mathcal{L}_{k,d+1} \\
&= (d-1)\sum_{i=1}^{d} e_i e_{d+1}\mathcal{L}_{i,d+1} = (d-1)(\Gamma_{d+1}-\Gamma_d). \qquad (16)
\end{aligned}
$$

Combining equations (15) and (16) gives

$$
\begin{aligned}
[\Gamma_{d+1}^2 F]_2 &= (1-d)(\Gamma_d + \sum_{i=1}^{d} e_i e_{d+1}\mathcal{L}_{i,d+1})F \\
&= (1-d)(\Gamma_d + \Gamma_{d+1} - \Gamma_d)F = (1-d)\Gamma_{d+1}F.
\end{aligned}
$$

Finally we must show that $[\Gamma_d^2 F]_4 = 0$. In dimensions $d = 2$ or 3, this is clear since in these cases the Clifford algebra $\mathbb{R}_d$ contains no 4-forms. For

$d \geq 4$, $[\Gamma_d^2 F]_4$ may be written as

$$[\Gamma_d^2 F]_4 = \sum_{1 \leq i < j < k < \ell \leq d} e_i e_j e_k e_\ell \mathcal{L}_{ijk\ell} F \tag{17}$$

with $\mathcal{L}_{ijk\ell}$ a differential operator composed from the six angular momentum operators $\mathcal{L}_{ij}$, $\mathcal{L}_{ik}$, $\mathcal{L}_{i\ell}$, $\mathcal{L}_{jk}$, $\mathcal{L}_{j\ell}$ and $\mathcal{L}_{k\ell}$. By a direct computation we find that

$$\mathcal{L}_{ij}\mathcal{L}_{k\ell} - \mathcal{L}_{ik}\mathcal{L}_{j\ell} + \mathcal{L}_{i\ell}\mathcal{L}_{jk} = 0.$$

Also, if i, j, k, ℓ are distinct, the commutator $[\mathcal{L}_{ij}, \mathcal{L}_{k\ell}] = 0$. Furthermore, in the expansion of $[\Gamma_d^2]_4$ in (17), we have

$$\begin{aligned} e_i e_j e_k e_\ell \mathcal{L}_{ijk\ell} &= e_i e_j \mathcal{L}_{ij} e_k e_\ell \mathcal{L}_{k\ell} + e_k e_\ell \mathcal{L}_{k\ell} e_i e_j \mathcal{L}_{ij} + e_i e_k \mathcal{L}_{ik} e_j e_\ell \mathcal{L}_{j\ell} \\ &\quad + e_j e_\ell \mathcal{L}_{j\ell} e_i e_k \mathcal{L}_{ik} + e_i e_\ell \mathcal{L}_{i\ell} e_j e_k \mathcal{L}_{jk} + e_j e_k \mathcal{L}_{jk} e_i e_\ell \mathcal{L}_{i\ell} \\ &= 2 e_i e_j e_k e_\ell (\mathcal{L}_{ij}\mathcal{L}_{k\ell} - \mathcal{L}_{ik}\mathcal{L}_{j\ell} + \mathcal{L}_{i\ell}\mathcal{L}_{jk}) = 0. \end{aligned}$$

The proof is complete. □

In order to compute $\exp(it\Gamma_d)$ we need to compute powers of Γ_d. Iterating Proposition 3 allows us to compute the scalar and 2-form parts of $\exp(it\Gamma_d)f$ (f real-valued) in terms of the action of powers of $(-|x|^2\Delta_T)$ on f.

Proposition 5. *Let Γ_d and Δ_T be as above. Then for all integers $n \geq 1$,*

$$\Gamma_d^n = p_n(-|x|^2\Delta_T) + q_n(-|x|^2\Delta_T)\Gamma_d \tag{18}$$

where p_n, q_n are polynomials satisfying the recurrence relation

$$\begin{pmatrix} p_{n+1}(t) \\ q_{n+1}(t) \end{pmatrix} = \begin{pmatrix} 0 & t \\ 1 & 2-d \end{pmatrix} \begin{pmatrix} p_n(t) \\ q_n(t) \end{pmatrix}; \quad \begin{pmatrix} p_0(t) \\ q_0(t) \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}. \tag{19}$$

Proof. The proof is by induction on n. Since $\Gamma_d^2 = -|x|^2\Delta_T + (2-d)\Gamma_d$, we are assured of the existence of polynomials p_n, q_n for which (18) is satisfied. With p_n, q_n as in the statement of the proposition, the cases $n = 0$ and $n = 1$ are immediate. Suppose then that (18) is valid for some value of $n \geq 1$. Then since Γ commutes with $|x|^2\Delta_T$, an application of Proposition 3 gives

$$\begin{aligned} \Gamma_d^{n+1} &= p_n(-|x|^2\Delta_T)\Gamma_d + q_n(-|x|^2\Delta_T)\Gamma_d^2 \\ &= p_n(-|x|^2\Delta_T)\Gamma_d + q_n(-|x|^2\Delta_T)(-|x|^2\Delta_T + (2-d)\Gamma_d) \\ &= (p_n(-|x|^2\Delta_T) + (2-d)q_n(-|x|^2\Delta_T))\Gamma_d - |x|^2\Delta_T q_n(-|x|^2\Delta_T) \\ &= p_{n+1}(-|x|^2\Delta_T) + q_{n+1}(-|x|^2\Delta_T)\Gamma_d, \end{aligned}$$

so that the matrix recurrence relation (19) is satisfied. The proof is complete. □

We will be required to compute the action of the operator exponential $\exp(it\Gamma_d)$ on real-valued functions of the form $f(x) = F(|x|^2, \langle x, y \rangle)$ for some fixed $y \in \mathbb{R}^d$ and $F \in C^2(\mathbb{R}^2)$. To do this we need to understand the action of Γ_d and Δ_T on such functions.

Proposition 6. *Let* $F = F(t_1, t_2) \in C^2(\mathbb{R}^2)$*,* $y \in \mathbb{R}^d$ *be fixed and* $f(x) = F(|x|^2, \langle x, y\rangle)$*. Then*

$$\begin{aligned}\Delta_T f(x) &= |x|^{-2}(|x|^2|y|^2 - \langle x, y\rangle^2)\frac{\partial^2 F}{\partial t_2^2}(|x|^2, \langle x, y\rangle) \\ &\qquad + (1-d)\langle x, y\rangle \frac{\partial F}{\partial t_2}(|x|^2, \langle x, y\rangle), \text{ and} \\ \Gamma_d f(x) &= (x \wedge y)\frac{\partial F}{\partial t_2}(|x|^2, \langle x, y\rangle).\end{aligned}$$

The proof of Proposition 6 is through direct computation. As a consequence, we have that if $F \in C^\infty(\mathbb{R})$ and $f(x) = F(\langle x, y\rangle)$ for some fixed $y \in \mathbb{R}^d$, then $\exp(it\Gamma_d)f(x) = u + v$ with u taking values in Λ_0 and v taking values in Λ_2. Furthermore, from Proposition 6 we see that $u = u(|x|^2, \langle x, y\rangle)$ and $v = (x \wedge y)w(|x|^2, \langle x, y\rangle)$ with w taking values in Λ_0.

From now on we simplify notation by writing Γ for Γ_d when the dimension d is clear, but when appropriate we write Γ_x to emphasise that Γ is acting on the x-variable. Computation of the frCFT kernels requires knowledge of the action of Γ on several other types of functions. We now outline the required results but suppress the proofs which are obtained through direct computation.

Proposition 7. *Let* $G \in C^1([0, \infty), \mathbb{C})$*. Then if* $x, y \in \mathbb{R}^d$*,*

$$\Gamma_x(G(|x \wedge y|)) = -\frac{(x \wedge y)}{|x \wedge y|}\langle x, y\rangle G'(|x \wedge y|).$$

Proposition 8. *Let* $f \in C^1([0, \infty), \mathbb{C})$ *and* $g \in C^1(\mathbb{R}^d, \mathbb{C}_d)$*. Then*

$$\Gamma_x(f(|x|)g(x)) = f(|x|)\Gamma_x g(x).$$

Proposition 9. *Let* $f \in C^1(\mathbb{R}, \mathbb{C})$*,* $g \in C^1(\mathbb{R}^d, \mathbb{C}_d)$ *and* $y \in \mathbb{R}^d$ *be fixed. Then*

$$\Gamma_x(f(\langle x, y\rangle)g(x)) = f'(\langle x, y\rangle)(x \wedge y)g(x) + f(\langle x, y\rangle)\Gamma_x g(x).$$

4. Initial value problems

We are now in a position to write down wave equation-type initial value problems for the scalar part and 2-form part of the frCFT kernel.

Proposition 10. *Let* $f(x) = F(\langle x, y\rangle)$ *for some fixed* $y \in \mathbb{R}^d$ *and* $u = u(x, t)$*,* $w = w(x, t)$ *be real-valued functions such that*

$$\exp(it\Gamma)f(x) = u(x, t) + \Gamma w(x, t).$$

Then u *satisfies the initial value problem*

$$\begin{aligned}&\frac{\partial^2 u}{\partial t^2} + i(d-2)\frac{\partial u}{\partial t} = |x|^2\Delta_T u \quad (x \in \mathbb{R}^d,\ t > 0) \\ &u(x, 0) = f(x) \quad (x \in \mathbb{R}^d) \\ &\left.\frac{\partial u}{\partial t}\right|_{t=0} = 0 \quad (x \in \mathbb{R}^d),\end{aligned} \tag{20}$$

and w satisfies the initial value problem

$$\begin{aligned} &\frac{\partial^2 w}{\partial t^2} + i(d-2)\frac{\partial w}{\partial t} = |x|^2 \Delta_T w \quad (x \in \mathbb{R}^d,\ t > 0) \\ &w(x,0) = 0 \quad (x \in \mathbb{R}^d) \\ &\left.\frac{\partial w}{\partial t}\right|_{t=0} = i f(x) \quad (x \in \mathbb{R}^d). \end{aligned} \tag{21}$$

Proof. Since $u = [\exp(it\Gamma)f]_0$ we have

$$\frac{\partial u}{\partial t} = i[\Gamma \exp(it\Gamma)f]_0 \tag{22}$$

and, by Proposition 3 and equation (22),

$$\begin{aligned} \frac{\partial^2 u}{\partial t^2} &= [-\Gamma^2 \exp(it\Gamma)f]_0 \\ &= [|x|^2 \Delta_T \exp(it\Gamma)f + (d-2)\Gamma \exp(it\Gamma)f]_0 \\ &= |x|^2 \Delta_T u + (d-2)[\Gamma \exp(it\Gamma)f]_0 = |x|^2 \Delta_T u - i(d-2)\frac{\partial u}{\partial t}. \end{aligned}$$

Putting $t = 0$ in the definition of u gives $u(x,0) = f(x)$. Putting $t = 0$ in (22) and recalling that $\Gamma f \in \Lambda_2$ gives $\left.\frac{\partial u}{\partial t}\right|_{t=0} = 0$.

With $v = [\exp(it\Gamma)f]_2$ we have, from Proposition 5

$$\begin{aligned} v = \sum_{j=0}^{\infty} \frac{(it)^j}{j!}[\Gamma^j f]_2 &= \sum_{j=0}^{\infty} \frac{(it)^j}{j!}[p_j(-|x|^2\Delta_T)f + q_j(-|x|^2\Delta_T)\Gamma f]_2 \\ &= \sum_{j=0}^{\infty} \frac{(it)^j}{j!}[q_j(-|x|^2\Delta_T)\Gamma f]_2 = \Gamma w \end{aligned}$$

with $w = \sum_{j=0}^{\infty} \frac{(it)^j}{j!} q_j(-|x|^2\Delta_T)f$. Now, another application of Proposition 5 gives

$$\begin{aligned} \frac{\partial w}{\partial t} &= i\sum_{j=0}^{\infty} \frac{(it)^j}{j!} q_{j+1}(-|x|^2\Delta_T)f \\ &= i\sum_{j=0}^{\infty} \frac{(it)^j}{j!}[p_j(-|x|^2\Delta_T)f + (2-d)q_j(-|x|^2\Delta_T)f] \\ &= i\sum_{j=0}^{\infty} \frac{(it)^j}{j!} p_j(-|x|^2\Delta_T)f + i(2-d)w. \end{aligned} \tag{23}$$

Yet another application of Proposition 5 to (23) gives

$$\begin{aligned}\frac{\partial^2 w}{\partial t^2} &= i(2-d)\frac{\partial w}{\partial t} - \sum_{j=0}^{\infty} \frac{(it)^j}{j!} p_{j+1}(-|x|^2\Delta_T) f \\ &= i(2-d)\frac{\partial w}{\partial t} - \sum_{j=0}^{\infty} \frac{(it)^j}{j!} [-|x|^2\Delta_T q_j(-|x|^2\Delta_T) f] \\ &= i(2-d)\frac{\partial w}{\partial t} + |x|^2\Delta_T w.\end{aligned}$$

Putting $t = 0$ in the definition of w gives $w(x, 0) = 0$ since $q_0(t) \equiv 0$. Also, since $q_1(t) \equiv 1$, we have $\left.\frac{\partial w}{\partial t}\right|_{t=0} = if$. □

The initial value problems of Proposition 10 may be viewed as wave-type problems on spheres centred at the origin in $\mathbb{R}^d$. Solutions to problems of this type may be found through a variety of methods. Gonzalez and Zhang [8] show that solutions can be computed by taking mean values of the initial data over the intersection of the sphere with hyperplanes. The integral formulae they develop seem impossible to compute in closed form. It is also worth noting that the formulae are different in even and odd dimension d. Since we are only interested in the situation in which the data takes the form $f(x) = F(\langle x, y\rangle)$, it is possible to use the classical method of separation of variables to write down infinite sum expansions for the solution. In even dimension we are able to compute these sums in closed form.

Proposition 11. *Let $f(x) = F(\langle x, y\rangle) = F(z\cos\theta)$ with $z = |x||y|$, $F_z(s) = F(zs)$ ($|s| \le 1$), $\cos\theta = \langle x, y\rangle/z$, and $u = u_d(z, \theta, t)$, $w = w_d(z, \theta, t)$ be the unique solutions of the initial problems of Proposition 10. Then, with $\alpha_{d,\ell}$ as in Theorem 2, u_d and w_d admit the expansions*

$$u_d = c_d \sum_{\ell=0}^{\infty} \left(\frac{(\ell+d-2)e^{i\ell t} + \ell e^{i(2-d-\ell)t}}{2\ell+d-2} \right) \alpha_{d,\ell}(F_z) N(d,\ell) P_\ell^d(\cos\theta) \tag{24}$$

$$w_d = c_d \sum_{\ell=0}^{\infty} \left(\frac{e^{i\ell t} - e^{i(2-d-\ell)t}}{2\ell+d-2} \right) \alpha_{d,\ell}(F_z) N(d,\ell) P_\ell^d(\cos\theta) \tag{25}$$

where $c_d = \sigma_{d-1}/\sigma_d = \Gamma(d/2)/(\sqrt{\pi}\Gamma((d-1)/2))$.

Proof. We seek solutions of the initial value problems of Proposition 10 via the method of separation of variables. Note first that the collection

$$Y_{\ell,m}^{(r)}(x) = r^{-(\ell+(d-1)/2)} Y_{\ell,m}(x) \quad (0 \le \ell < \infty,\ 1 \le m \le N(d,\ell),\ r = |x|)$$

is an orthonormal basis for $L^2(rS^{d-1})$, the space of square-integrable functions on the sphere of radius r, centred at the origin, and equipped with the surface measure. Here the collection $Y_{\ell,m}$ ($1 \le m \le N(d,\ell)$) is an orthonormal basis for the space H_ℓ^d of spherical harmonics in d-variables of degree ℓ. Because of the orthogonality of the spaces $\{H_\ell^d\}_{\ell=0}^\infty$, we have $\int_{rS^{d-1}} Y_{\ell,m}^{(r)}(x)\overline{Y_{\ell',m'}^{(r)}(x)}\, d\sigma(x) = \delta_{\ell,\ell'}\delta_{mm'}$. We look then for solutions of (20) of the form

$$u(x,t)=\sum_{\ell=0}^{\infty}\sum_{m=0}^{N(d,\ell)} a_{\ell,m}(t)Y_{\ell,m}^{(r)}(x). \tag{26}$$

Rewriting (7) as $\Delta_T f(x)=\Delta f(x)+(2-d)|x|^{-2}Ef(x)-|x|^{-2}E^2f(x)$ and observing that if f is homogeneous of degree ℓ, then $Ef(x)=\ell f(x)$, we find

$$|x|^2\Delta_T Y_{\ell,m}^{(r)}(x)=\ell(2-d-\ell)Y_{\ell,m}^{(r)}(x).$$

With u as in (26) satisfying the differential equation of the initial value problem (20), we have

$$\begin{aligned}\frac{\partial^2 u}{\partial t^2}+i(d-2)\frac{\partial u}{\partial t} &= \sum_{\ell,m}(a''_{\ell,m}(t)+i(d-2)a'_{\ell,m}(t))Y_{\ell,m}^{(r)}(x)\\ &= |x|^2\Delta_T u\\ &= \sum_{\ell,m} a_{\ell,m}(t)|x|^2\Delta_T Y_{\ell,m}^{(r)}(x)\\ &= \sum_{\ell,m} a_{\ell,m}(t)\ell(2-d-\ell)Y_{\ell,m}^{(r)}(x)\end{aligned}$$

from which we conclude that the functions $a_{\ell,m}(t)$ satisfy the differential equation $a''_{\ell,m}(t)+i(d-2)a'_{\ell,m}(t)-\ell(2-d-\ell)a_{\ell,m}(t)=0$, which has solutions $a_{\ell,m}(t)=c_{\ell,m}e^{i\ell t}+b_{\ell,m}e^{i(2-d-\ell)t}$ with $c_{\ell,m}$ and $b_{\ell,m}$ constants. The solution u of the initial value problem (20) then takes the form

$$u=\sum_{\ell,m}[c_{\ell,m}e^{i\ell t}+b_{\ell,m}e^{i(2-d-\ell)t}]Y_{\ell,m}^{(r)}(x). \tag{27}$$

Differentiating both sides of (27) with respect to t gives

$$\frac{\partial u}{\partial t}=\sum_{\ell,m} i[\ell c_{\ell,m}e^{i\ell t}+(2-d-\ell)b_{\ell,m}e^{i(2-d-\ell)t}]Y_{\ell,m}^{(r)}(x).$$

Setting $t=0$ and applying the second of the initial conditions of (20) gives $b_{\ell,m}=\dfrac{-\ell c_{\ell,m}}{2-d-\ell}$. Substituting this into (27) gives

$$u=\sum_{\ell=0}^{\infty}\sum_{m=0}^{N(d,\ell)} c_{\ell,m}\left(e^{i\ell t}-\frac{\ell}{2-d-\ell}e^{i(2-d-\ell)t}\right)Y_{\ell,m}^{(r)}(x).$$

Putting $t=0$ into this last equation, evaluating both sides at $t=0$ and applying the first initial condition of (20) gives

$$\sum_{\ell=0}^{\infty}\sum_{m=0}^{N(d,\ell)} c_{\ell,m}\left(\frac{2-d-2\ell}{2-d-\ell}\right)Y_{\ell,m}^{(r)}(x)=f(x).$$

The orthonormality of the spherical harmonics $Y_{\ell,m}^{(r)}$ now gives the coefficients $c_{\ell.m}$ as $c_{\ell,m}=((\ell+d-2)/(2\ell+d-2))\int_{rS^{d-1}} f(x)\overline{Y_{\ell,m}^{(r)}(x)}\,d\sigma(x)$ so that u

takes the form

$$u = \sum_{\ell,m} \left(\frac{(\ell+d-2)e^{i\ell t} + \ell e^{i(2-d-\ell)t}}{2\ell+d-2} \right) \times \left(\int_{rS^{d-1}} f(\xi) \overline{Y^{(r)}_{\ell,m}(\xi)}\, d\sigma(\xi) \right) Y^{(r)}_{\ell,m}(x). \tag{28}$$

Now we specialise to the case where $f(\xi) = F(\langle \xi, y \rangle)$ for some fixed $y \in \mathbb{R}^d$. An application of the Hecke-Funk theorem (Theorem 2) gives

$$\int_{rS^{d-1}} f(\xi) \overline{Y^{(r)}_{\ell,m}(\xi)}\, d\sigma(\xi) = r^{(d-1)/2} \sigma_{d-1} \alpha_{d,\ell}(F_z) \overline{Y_{\ell,m}(y/|y|)} \tag{29}$$

where $z = |x||y|$. Substituting this into (28) and applying Theorem 1 now gives equation (24).

The solution w of (21) has an expansion of the form (27), but the condition $w(x,0) = 0$ gives $c_{\ell,m} = -b_{\ell,m}$ so that

$$w = \sum_{\ell,m} c_{\ell,m} [e^{i\ell t} - e^{i(2-d-\ell)t}] Y^{(r)}_{\ell,m}(x). \tag{30}$$

Then $\left.\dfrac{\partial w}{\partial t}\right|_{t=0} = i \sum_{\ell,m} c_{\ell,m} (2\ell + d - 2) Y^{(r)}_{\ell,m}(x) = i f(x)$ from which we find that $c_{\ell,m} = (2\ell + d - 2)^{-1} \int_{rS^{d-1}} f(\xi) \overline{Y^{(r)}_{\ell,m}(\xi)}\, d\sigma(\xi)$. Substituting this into (30) and applying (29) and Theorem 1 gives equation (25). □

5. Closed form solutions of the initial value problems

Evaluating the sums (24) and (25) in closed form for general data $f(x) = F(\langle x, y \rangle)$ is the subject of the next few results. When the dimension d is even, this problem is tractable, but when d is odd, no such closed form is known. The problem seems to be the nature of the Legendre polynomials in even and odd dimension.

Theorem 12. *Let u, w be as in (24) and (25) respectively, $\nu = (d-2)/2$, $\alpha_{d,\ell}$ be as in Theorem 2, c_d be as in Proposition 11, and $\beta_{d,\ell}(G) = \int_{-1}^{1} \mathcal{L}_d G(s)(1-s^2)^{(d-3)/2} P^d_\ell(s)\, ds$. Then we have*

$$\frac{\partial}{\partial t}\left(e^{i\nu t} \frac{\partial u}{\partial t} \right) = c_d \sum_{\ell=0}^{\infty} \cos((\ell+\nu)t) \beta_{d,\ell}(F_z) N(d,\ell) P^d_\ell(\cos\theta) \tag{31}$$

$$\frac{\partial}{\partial t}(e^{i\nu t} w) = i c_d \sum_{\ell=0}^{\infty} \cos((\ell+\nu)t) \alpha_{d,\ell}(F_z) N(d,\ell) P^d_\ell(\cos\theta). \tag{32}$$

Proof. Differentiating (24) with respect to t and multiplying the result by $e^{i\nu t}$ gives

$$e^{i\nu t} \frac{\partial u}{\partial t} = -c_d \sum_{\ell=0}^{\infty} \frac{\ell(\ell+2\nu)}{\ell+\nu} \sin((\ell+\nu)t) \alpha_{d,\ell}(F_z) N(d,\ell) P^d_\ell(\cos\theta).$$

Differentiating again with respect to t and using the eigenvalue property $\mathcal{L}_d P_\ell^d = -\ell(\ell+d-2)P_\ell^d$ and the self-adjointness of $\mathcal{L}_d$ gives

$$\begin{aligned}\frac{\partial}{\partial t}\left(e^{i\nu t}\frac{\partial u}{\partial t}\right) &= c_d\sum_{\ell=0}^{\infty}\cos((\ell+\nu)t)\left(\int_{-1}^{1}F(zs)(1-s^2)^{(d-3)/2}\mathcal{L}_d P_\ell^d(s)\,ds\right)\\ &\qquad\times N(d,\ell)P_\ell^d(\cos\theta)\\ &= c_d\sum_{\ell=0}^{\infty}\cos((\ell+\nu)t)\left(\int_{-1}^{1}\mathcal{L}_d F(zs)(1-s^2)^{(d-3)/2}P_\ell^d(s)\,ds\right)\\ &\qquad\times N(d,\ell)P_\ell^d(\cos\theta)\end{aligned}$$

which gives (31). Equation (32) is derived from (25) in a similar manner. □

For the moment we restrict attention to the cases $d = 2$ and $d = 4$. When $d = 2$, the initial value problem (20) collapses to the wave problem on the circle:

$$\begin{aligned}&\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial \theta^2} \quad (0 \le \theta < 2\pi)\\ &u(\theta,0) = f(\theta) \quad (0 \le \theta < 2\pi)\\ &\left.\frac{\partial u}{\partial t}\right|_{t=0} = 0 \quad (0 \le \theta < 2\pi)\end{aligned} \tag{33}$$

which has d'Alembert solution $u = \frac{1}{2}[f(\theta+t)+f(\theta-t)]$. Similarly, the initial value problem (21) collapses to

$$\begin{aligned}&\frac{\partial^2 w}{\partial t^2} = \frac{\partial^2 w}{\partial \theta^2} \quad (0 \le \theta < 2\pi)\\ &w(\theta,0) = 0 \quad (0 \le \theta < 2\pi)\\ &\left.\frac{\partial w}{\partial t}\right|_{t=0} = if(\theta) \quad (0 \le \theta < 2\pi)\end{aligned} \tag{34}$$

which has d'Alembert solution $w = \frac{i}{2}\int_{\theta-t}^{\theta+t} f(\varphi)\,d\varphi$.

When $d = 4$, summing the series solution (24) and (25) is significantly more complicated. In this case, (24) becomes

$$\begin{aligned}u = c\sum_{\ell=0}^{\infty}&\left(\int_0^{\pi}F(z\cos\varphi)P_\ell^4(\cos\varphi)\sin^2\varphi\,d\varphi\right)\\ &\times(\ell+1)[(\ell+2)e^{i\ell t}+\ell e^{-i(\ell+2)t}]P_\ell^4(\cos\theta).\end{aligned} \tag{35}$$

The Legendre polynomials P_ℓ^4 satisfy $P_\ell^4(\cos\theta) = \dfrac{\alpha\sin((\ell+1)\theta)}{(\ell+1)\sin\theta}$ with α a constant – the value of which is, for the moment, unimportant. Equation (35) then becomes

$$\begin{aligned}u = c\alpha^2\sum_{\ell=0}^{\infty}&\left(\int_0^{\pi}F(z\cos\varphi)\sin\varphi\sin((\ell+1)\varphi)\,d\varphi\right)\\ &\times[(\ell+2)e^{i\ell t}+\ell e^{-i(\ell+2)t}]\frac{\sin((\ell+1)\theta)}{(\ell+1)\sin\theta}.\end{aligned} \tag{36}$$

Putting $t = 0$ in (36) gives

$$\begin{aligned} u(\theta,0) &= \frac{2c\alpha^2}{\sin\theta}\sum_{\ell=0}^{\infty}\left(\int_0^\pi F(z\cos\varphi)\sin\varphi\sin((\ell+1)\varphi)\,d\varphi\right)\sin((\ell+1)\theta) \\ &= \pi c\alpha^2 F(z\cos\theta) \end{aligned}$$

where we have used the fact that $\{\sqrt{2/\pi}\sin((\ell+1)\theta)\}_{\ell=0}^{\infty}$ is an orthonormal basis for $L^2[0,\pi]$. However $u(\theta,0) = F(z\cos\theta)$, so we see that $c = 1/(\pi\alpha^2)$. After differentiating with respect to t and multiplying by e^{it}, we have

$$\begin{aligned} e^{it}\frac{\partial u}{\partial t} = -\frac{2}{\pi}\sum_{\ell=0}^{\infty}&\left(\int_0^\pi F(z\cos\varphi)\sin\varphi\sin((\ell+1)\varphi)\,d\varphi\right) \\ &\times \ell(\ell+2)\frac{\sin((\ell+1)t)\sin((\ell+1)\theta)}{(\ell+1)\sin\theta} \end{aligned}$$

and, after another differentiation with respect to t,

$$\begin{aligned} \frac{\partial}{\partial t}\left(e^{it}\frac{\partial u}{\partial t}\right) &= -\frac{2}{\pi}\sum_{\ell=0}^{\infty}\left(\int_0^\pi F(z\cos\varphi)\sin\varphi\sin((\ell+1)\varphi)\,d\varphi\right)\ell(\ell+2) \\ &\qquad \times \frac{\cos((\ell+1)t)\sin((\ell+1)\theta)}{\sin\theta} \\ &= -\frac{1}{\pi}\sum_{\ell=0}^{\infty}\left(\int_0^\pi F(z\cos\varphi)\sin\varphi\sin((\ell+1)\varphi)\,d\varphi\right)\ell(\ell+2) \\ &\qquad \times \frac{[\sin((\ell+1)(\theta+t)) + \sin((\ell+1)(\theta-t))]}{\sin\theta}. \end{aligned} \tag{37}$$

The eigenfunction property of $\{P_\ell^4\}_{\ell=0}^{\infty}$ may be written

$$-\ell(\ell+2)(1-s^2)^{1/2}P_\ell^4(s) = \frac{d}{ds}\left[(1-s^2)^{3/2}\frac{d}{ds}P_\ell^4(s)\right]$$

so that with $G(s) = \dfrac{d}{ds}\left[(1-s^2)^{3/2}\dfrac{d}{ds}(F(zs))\right]$, the integral on the right hand side of (37) becomes

$$\begin{aligned} -\ell(\ell+2)&\int_0^\pi F(z\cos\varphi)\sin\varphi\sin((\ell+1)\varphi)\,d\varphi \\ &= -\frac{\ell(\ell+2)(\ell+1)}{\alpha}\int_0^\pi F(z\cos\varphi)\sin^2\varphi P_\ell^4(\cos\varphi)\,d\varphi \\ &= -\frac{\ell(\ell+2)(\ell+1)}{\alpha}\int_{-1}^1 F(zs)(1-s^2)^{1/2}P_\ell^4(s)\,ds \\ &= \frac{(\ell+1)}{\alpha}\int_{-1}^1 F(zs)\frac{d}{ds}\left[(1-s^2)^{3/2}\frac{d}{ds}P_\ell^4(s)\right]ds \\ &= \frac{(\ell+1)}{\alpha}\int_{-1}^1 G(s)P_\ell^4(s)\,ds \\ &= \int_0^\pi G(\cos\varphi)\sin((\ell+1)\varphi)\,d\varphi. \end{aligned}$$

Substituting this into (37) and using the orthogonality and completeness of the functions $\{\sqrt{2/\pi}\sin((\ell+1)\varphi)\}_{\ell=1}^{\infty}$ on $[0,\pi]$ gives

$$\frac{\partial}{\partial t}\left(e^{it}\frac{\partial u}{\partial t}\right) = \frac{1}{\pi\sin\theta}\sum_{\ell=0}^{\infty}\left(\int_0^{\pi} G(\cos\varphi)\sin((\ell+1)\varphi)\,d\varphi\right) \times [\sin((\ell+1)(\theta+t))+\sin((\ell+1)(\theta-t))]$$

$$= \frac{[g(\theta+t)+g(\theta-t)]}{2\sin\theta} \tag{38}$$

where $g(\theta)=G(\cos\theta)$. Integrating both sides of (38) with respect to t gives

$$e^{it}\frac{\partial u}{\partial t} = \frac{1}{2\sin\theta}\int_{\theta-t}^{\theta+t} g(u)\,du + H(\theta)$$

with H an arbitrary function of the single variable θ, but the initial condition $\left.\frac{\partial u}{\partial t}\right|_{t=0} = 0$ gives $H\equiv 0$. Hence

$$u = \frac{1}{2\sin\theta}\int_0^t e^{-is}\int_{\theta-s}^{\theta+s} g(u)\,du\,ds + J(\theta) \tag{39}$$

with J an arbitrary function of the single variable θ. Applying the initial condition $u(z,\theta,0) = F(z\cos\theta)$ gives $J(\theta) = F(z\cos\theta)$. Integrating the outer integral on the right hand side of (39) by parts gives

$$\int_0^t e^{-is}\int_{\theta-s}^{\theta+s} g(u)\,du\,ds = ie^{-it}\int_{\theta-t}^{\theta+t} g(u)\,du - i\int_0^t e^{-is}[g(\theta+s)+g(\theta-s)]\,ds. \tag{40}$$

But $G(s) = (1-s^2)^{3/2}\frac{d^2}{ds^2}(F(zs)) - 3s(1-s^2)^{1/2}\frac{d}{ds}(F(zs))$, so

$$g(u) = G(\cos u) = -\frac{z}{\sin u}\frac{d}{du}[\sin^3 uF'(z\cos u)],$$

and therefore, with an integration by parts we have

$$\begin{aligned}\int_0^t e^{-is}g(\theta+s)\,ds &= -ze^{i\theta}\int_{\theta}^{\theta+t}\frac{e^{-iu}}{\sin u}\frac{d}{du}\left[(\sin^3 u)F'(z\cos u)\right]du \\ &= -ze^{i\theta}\left[(\cot u - i)\sin^3 uF'(z\cos u)\Big|_{u=\theta}^{u=\theta+t} - \int_{\theta}^{\theta+t}\frac{d}{du}(\cot u - i)\sin^3 uF'(z\cos u)\,du\right] \\ &= -ze^{-it}\sin^2(\theta+t)F'(z\cos(\theta+t)) + z\sin^2\theta F'(z\cos\theta) \\ &\quad + e^{i\theta}(F(z\cos(\theta+t)) - F(z\cos\theta)).\end{aligned} \tag{41}$$

A similar computation gives

$$\begin{aligned}\int_0^t e^{-is}g(\theta-s)\,ds &= -z\sin^2\theta F'(z\cos\theta) + ze^{-it}\sin^2(\theta-t)F'(z\cos(\theta-t)) \\ &\quad + e^{-i\theta}(F(z\cos\theta) - F(z\cos(\theta-t))).\end{aligned} \tag{42}$$

Also, yet another integration by parts yields

$$\begin{aligned}\int_{\theta-t}^{\theta+t} g(u)\,du &= \int_{\theta-t}^{\theta+t} G(\cos u)\,du \\ &= \int_{\cos(\theta+t)}^{\cos(\theta-t)} \frac{G(s)}{\sqrt{1-s^2}}\,ds \\ &= \int_{\cos(\theta+t)}^{\cos(\theta-t)} \frac{1}{\sqrt{1-s^2}}\frac{d}{ds}[(1-s^2)^{3/2}\frac{d}{ds}(F(zs))]\,ds \\ &= z\sin^2(\theta-t)F'(z\cos(\theta-t)) - z\sin^2(\theta+t)F'(z\cos(\theta+t)) \\ &- \cos(\theta-t)F(z\cos(\theta-t)) + \cos(\theta+t)F(z\cos(\theta+t)) \\ &+ \int_{\cos(\theta+t)}^{\cos(\theta-t)} F(zs)\,ds. \end{aligned} \tag{43}$$

Combining equations (40)-(43) now gives

$$\begin{aligned}\int_0^t e^{-is}\int_{\theta-s}^{\theta+s} g(u)\,du\,ds &= e^{-it}\sin(\theta+t)F(z\cos(\theta+t)) \\ &+ e^{-it}\sin(\theta-t)F(z\cos(\theta-t)) \\ &- 2\sin\theta F(z\cos\theta) + ie^{-it}\int_{\cos(\theta+t)}^{\cos(\theta-t)} F(zs)\,ds.\end{aligned}$$

Applying this to (39) now gives

$$\begin{aligned}u = \frac{e^{-it}}{2\sin\theta}\Bigg[&\sin(\theta+t)F(z\cos(\theta+t)) + \sin(\theta-t)F(z\cos(\theta-t)) \\ &+ i\int_{\cos(\theta+t)}^{\cos(\theta-t)} F(zs)\,ds\Bigg].\end{aligned}$$

This completes the computation of $u = u_4$. The Λ_2-part of $e^{it\Gamma}f$ is Γw with w given by (25). When $d = 4$, this sum becomes

$$\begin{aligned}w = \frac{e^{-it}c\alpha^2}{\sin\theta}\sum_\ell \frac{\sin(\ell+1)t}{\ell+1}\left(\int_0^\pi F(z\cos\varphi)\sin((\ell+1)\varphi)\sin\varphi\,d\varphi\right) \\ \times \sin((\ell+1)\theta)\end{aligned}$$

so that

$$\begin{aligned}\frac{\partial}{\partial t}(e^{it}w) = \frac{c\alpha^2}{\sin\theta}\sum_\ell \left(\int_0^\pi F(z\cos\varphi)\sin((\ell+1)\varphi)\sin\varphi\,d\varphi\right) \\ \times \cos((\ell+1)t)\sin((\ell+1)\theta)\end{aligned}$$

and by putting $t = 0$ in this equation and using the orthogonality and completeness of $\{\sqrt{2/\pi}\sin(\ell+1)\varphi\}_{\ell=0}^\infty$ in $L^2[0,\pi]$, we find that the constant

c is in fact $c = -2i/(\pi\alpha^2)$. Hence

$$\begin{aligned}\frac{\partial}{\partial t}(e^{it}w) &= \frac{2i}{\pi}\sum_{\ell}\left(\int_0^{\pi} F(z\cos\varphi)\sin((\ell+1)\varphi)\sin\varphi\, d\varphi\right) \\ &\qquad\qquad\times \frac{\cos((\ell+1)t)\sin((\ell+1)\theta)}{\sin\theta} \\ &= \frac{i}{2}\left[\frac{\sin(\theta+t)F(z\cos(\theta+t)) + \sin(\theta-t)F(z\cos(\theta-t))}{\sin\theta}\right].\end{aligned}$$

Integrating both sides of this equation with respect to t gives

$$e^{it}w = \frac{i}{2\sin\theta}\int_0^t [\sin(\theta+s)F(z\cos(\theta+s)) + \sin(\theta-s)F(z\cos(\theta-s)]\, ds + K(\theta)$$

but putting $t = 0$ and applying the initial condition $w(x,0) = 0$ gives $K \equiv 0$ so that

$$w = \frac{ie^{-it}}{2\sin\theta}\int_{\cos(\theta+t)}^{\cos(\theta-t)} F(zs)\, ds.$$

We've shown the following.

Theorem 13. *Let $u = u(z,\theta,t)$, $w = w(z,\theta,t)$ be the unique solutions of the initial value problems (20) and (21) of Proposition 10 where $d = 4$, $f(x) = F(\langle x,y\rangle)$ for some fixed $y \in \mathbb{R}^4$ and $z = |x||y|$. Then*

$$\begin{aligned}u = \frac{e^{-it}}{2\sin\theta}\Bigg[&\sin(\theta+t)F(z\cos(\theta+t)) + \sin(\theta-t)F(z\cos(\theta-t)) \\ &\qquad + i\int_{\cos(\theta+t)}^{\cos(\theta-t)} F(zs)\, ds\Bigg]\end{aligned}$$

and

$$w = \frac{ie^{-it}}{2\sin\theta}\int_{\cos(\theta+t)}^{\cos(\theta-t)} F(zs)\, ds.$$

6. Closed forms for the fractional Clifford-Fourier transform kernels

Recall from section 2 that in d dimensions, the fractional Clifford-Fourier transform kernels $C^{\pm}_{d,t}(x,y)$ may be written as

$$C^{\pm}_{d,t}(x,y) = e^{\pm itd/2}\exp(\pm it\Gamma_x)K_{d,t}(x,y)$$

where $K_{d,t}(x,y)$ is the classical (d-dimensional) fractional Fourier transform kernel as defined in section 2. When $d = 2$, $z = |x||y|$, $\cos\theta = \langle x,y\rangle/z$ and $f(\theta) = K_{2,t}(x,y)$ (y, $|x|$ fixed), the d'Alembert solution of (33) gives the scalar part of the frCFT kernel $C^+_{2,t}(x,y)$ to be

$$\begin{aligned} u_2(\theta,t) &= -\frac{ie^{2it}}{4\pi\sin t}e^{i\cot(t)(|x|^2+|y|^2)/2}[e^{-iz\csc(t)\cos(\theta+t)}+e^{-iz\csc(t)\cos(\theta-t)}] \\ &= -\frac{ie^{2it}}{4\pi\sin t}e^{i\cot(t)(|x|^2+|y|^2)/2}e^{-iz\csc(t)\cos(\theta)\cos(t)} \\ &\qquad\qquad \times [e^{iz\csc(t)\sin(\theta)\sin(t)}+e^{-iz\csc(t)\sin(\theta)\sin(t)}] \\ &= -\frac{ie^{2it}}{4\pi\sin t}e^{i\cot(t)(|x|^2+|y|^2-2\langle x,y\rangle)/2}[e^{iz\sin(\theta)}+e^{-iz\sin(\theta)}] \\ &= -\frac{ie^{2it}}{2\pi\sin t}e^{i\cot(t)|x-y|^2/2}\cos(z\sin(\theta)) \\ &= -\frac{ie^{2it}}{\sin t}e^{i\cot t|x-y|^2/2}\cos(|x\wedge y|). \end{aligned}$$

Similarly, with w as in (21) and $d=2$ we have $w(\theta,t)=\frac{i}{2}\int_{\theta-t}^{\theta+t}K_{2,t}(\varphi,t)\,d\varphi$ so that the Λ_2- part of the frCFT kernel takes the form

$$\begin{aligned} v=\Gamma w = e_1e_2\frac{\partial w}{\partial\theta} &= \frac{e_1e_2}{4\pi\sin t}e^{2it}e^{i\cot(t)(|x|^2+|y|^2)/2} \\ &\qquad\qquad \times [e^{-iz\csc(t)\cos(\theta+t)}-e^{-iz\csc(t)\cos(\theta-t)}] \\ &= \frac{ie_1e_2}{2\pi\sin t}e^{2it}e^{i\cot t|x-y|^2/2}\sin(|x\wedge y|). \end{aligned}$$

Consequently the 2-dimensional frCFT kernel is

$$\begin{aligned} C_{2,t}(x,y) &= -\frac{ie^{2it}}{2\pi\sin t}e^{i\cot(t)|x-y|^2/2}[\cos(|x\wedge y|)-e_1e_2\sin(|x\wedge y|)] \\ &= -\frac{ie^{2it}}{2\pi\sin t}e^{i\cot(t)|x-y|^2/2}e^{-(x\wedge y)} \end{aligned}$$

and, putting $t=\pi/2$ we obtain the Clifford-Fourier kernel in 2 dimensions: $C_2(x,y)=C_{2,\pi/2}(x,y)=i\dfrac{e^{-(x\wedge y)}}{2\pi}$.

When $d=4$, the scalar part $u_4(\theta,t)$ of the frCFT kernel takes the form $u_4=u_4^{(1)}+u_4^{(2)}$ with

$$\begin{aligned} u_4^{(1)} &= -\frac{e^{3it}}{8\pi^2\sin\theta\sin^2 t}e^{i\cot(t)(|x|^2+|y|^2)/2} \\ &\qquad \times [\sin(\theta+t)e^{-iz\csc(t)\cos(\theta+t)}+\sin(\theta-t)e^{-iz\csc(t)\cos(\theta-t)}] \\ &= -\frac{e^{3it}}{4\pi^2\sin\theta\sin^2 t}e^{i\cot(t)(|x|^2+|y|^2)/2}e^{-iz\cot(t)\cos(\theta)} \\ &\qquad \times [\sin(\theta)\cos(t)\cos(z\sin\theta)+i\cos(\theta)\sin(t)\sin(z\sin\theta)] \\ &= -\frac{e^{3it}}{4\pi^2\sin\theta\sin^2 t}e^{i\cot t|x-y|^2/2} \\ &\qquad \times [\sin(\theta)\cos(t)\cos(|x\wedge y|)+i\cos(\theta)\sin(t)\sin(|x\wedge y|)] \\ &= -\frac{e^{3it}}{4\pi^2\sin^2 t}e^{i\cot(t)|x-y|^2/2}\left[\cos(t)\cos(|x\wedge y|)+i\langle x,y\rangle\sin(t)\frac{\sin(|x\wedge y|)}{|x\wedge y|}\right] \end{aligned}$$

and

$$\begin{aligned}
u_4^{(2)} &= -\frac{ie^{it}}{8\pi^2\sin\theta}\int_{\cos(\theta+t)}^{\cos(\theta-t)}\frac{e^{2it}}{\sin^2 t}e^{i\cot(t)(|x|^2+|y|^2)/2}e^{-izs\csc(t)}\,ds\\
&= -\frac{ie^{3it}}{4\pi^2\sin t}e^{i\cot(t)|x-y|^2/2}\frac{\sin(z\sin\theta)}{z\sin\theta}\\
&= -\frac{ie^{3it}}{4\pi^2\sin t}e^{i\cot(t)|x-y|^2/2}\frac{\sin(|x\wedge y|)}{|x\wedge y|}
\end{aligned}$$

so that the scalar part of the four-dimensional frCFT kernel becomes

$$u_4 = -\frac{e^{3it}}{4\pi^2\sin t}e^{i\cot t|x-y|^2/2}\left[\cot(t)\cos(|x\wedge y|)+i\left(\frac{\langle x,y\rangle}{|x\wedge y|}+1\right)\frac{\sin(|x\wedge y|)}{|x\wedge y|}\right].$$

The 2-form part v_4 of the frCFT kernel is given by $v_4 = \Gamma w_4$ with

$$\begin{aligned}
w_4 &= -\frac{ie^{it}}{8\pi^2\sin\theta\sin^2 t}e^{2it}e^{i\cot(t)(|x|^2+|y|^2)/2}\int_{\cos(\theta+t)}^{\cos(\theta-t)}e^{-izs\csc(t)}\,ds\\
&= \frac{e^{3it}}{8\pi^2 z\sin\theta\sin t}e^{i\cot(t)(|x|^2+|y|^2)/2}[e^{-iz\csc(t)\cos(\theta-t)} - e^{-iz\csc(t)\cos(\theta+t)}]\\
&= \frac{-ie^{3it}}{4\pi^2 z\sin\theta\sin t}e^{i\cot(t)|x-y|^2/2}\sin(z\sin\theta)\\
&= \frac{-ie^{it}}{4\pi^2\sin t}e^{i\cot(t)|x-y|^2/2}\frac{\sin(|x\wedge y|)}{|x\wedge y|}.
\end{aligned}$$

Hence

$$v_4 = \Gamma_x w_4 = -\frac{ie^{3it}}{4\pi^2\sin t}e^{i\cot(t)(|x|^2+|y|^2)/2}\Gamma_x\left[e^{-i\cot(t)\langle x,y\rangle}\frac{\sin(|x\wedge y|)}{|x\wedge y|}\right]$$

where we have applied Proposition 8. Propositions 9 and 7 now give

$$\begin{aligned}
v_4 &= -\frac{ie^{3it}}{4\pi^2\sin t}e^{i\cot(t)(|x|^2+|y|^2)/2}\Gamma_x\left(e^{-(i/2)\cot(t)\langle x,y\rangle}\frac{\sin(|x\wedge y|)}{|x\wedge y|}\right)\\
&= -\frac{ie^{3it}}{4\pi^2\sin t}e^{i\cot(t)(|x|^2+|y|^2)/2}\frac{x\wedge y}{|x\wedge y|}e^{-i\cot(t)\langle x,y\rangle}\\
&\times\left[-i\cot(t)\sin(|x\wedge y|)-\langle x,y\rangle\left(\frac{|x\wedge y|\cos(|x\wedge y|)-\sin(|x\wedge y|)}{|x\wedge y|^2}\right)\right]\\
&= \frac{ie^{3it}}{4\pi^2\sin t}e^{i\cot(t)|x-y|^2/2}\frac{x\wedge y}{|x\wedge y|}\\
&\times\left[i\cot(t)\sin(|x\wedge y|)+\langle x,y\rangle\left(\frac{|x\wedge y|\cos(|x\wedge y|)-\sin(|x\wedge y|)}{|x\wedge y|^2}\right)\right].
\end{aligned}$$

The four-dimensional frCFT kernel is then

$$\begin{aligned}
&C_{4,t}(x,y) = u_4(x,y,t)+v_4(x,y,t)\\
&= \frac{e^{3it}}{4\pi^2\sin t}e^{i\cot(t)|x-y|^2/2}\\
&\times\left[e^{x\wedge y}\left(-\cot t+i\frac{\langle x,y\rangle x\wedge y}{|x\wedge y|^2}\right)-i\frac{x\wedge y}{|x\wedge y|^2}(x\wedge y+\langle x,y\rangle)\frac{\sin(|x\wedge y|)}{|x\wedge y|}\right]
\end{aligned}$$

where we have used the fact that $(x \wedge y)^2 = -|x \wedge y|^2$ and hence $e^{x\wedge y} = \cos(|x\wedge y|) + \frac{x \wedge y}{|x \wedge y|}\sin(|x\wedge y|)$. Putting $t = \pi/2$ gives the four-dimensional CFT kernel

$$C_4(x,y) = \frac{1}{4\pi^2}\left[e^{x\wedge y}\frac{\langle x,y\rangle(x\wedge y)}{|x\wedge y|^2} + \frac{(x\wedge y)}{|x\wedge y|^2}(\langle x,y\rangle + (x\wedge y))\frac{\sin(|x\wedge y|)}{|x\wedge y|}\right].$$

7. Method of Ascent

In section 5 it was shown how solutions of the "d-dimensional" initial value problems (20) and (21) – which take place on spheres in $\mathbb{R}^d$ – may be computed via expansions in Legendre polynomials P_ℓ^d, and how closed forms for these expansions may be determined from addition formulae satisfied by these polynomials, at least in dimensions 2 and 4. In this section we give details of a "method of ascent" for computing solutions of the $(d+2)$-dimensional versions of the initial value problems (20) and (21) in terms of solutions of the d-dimensional versions of those same initial value problems, with different initial data ($d \geq 3$). Since we have closed forms the solutions of the 4-dimensional initial-value problem (Theorem 13), we have as an immediate consequence a means of computing closed forms for the solutions of the even-dimensional problems. As a special case we can compute (iteratively) all even-dimensional frCFT kernels.

We start with the expansion (24) of the solution $u_d(x, y.t) = u_d(z, \theta, t)$ of the initial value problem (20). The solutions of the initial-value problems (20) and (21) may of course be viewed as time-evolutions of the initial data F. We therefore, when convenient, write $u_d(x, y, t) = u_d(F)(x, y, t)$ and $w_d(x, y, t) = w_d(F)(x, y, t)$.

It is more convenient now to work with a different normalisation of the Legendre polynomials P_ℓ^d. We use instead the Gegenbauer polynomials C_ℓ^ν which are related to the Legendre polynomials by

$$C_\ell^\nu = \binom{\ell+d-3}{d-3}P_\ell^d \qquad (\nu = (d-2)/2), \tag{44}$$

and satisfy the differential recurrence relation [10]

$$\frac{d}{d\omega}C_\ell^\nu(\omega) = 2\nu C_{\ell-1}^{\nu+1}(\omega) \quad (\ell \geq 1) \quad \text{with} \quad \frac{d}{d\omega}C_0^\nu(\omega) = 0. \tag{45}$$

We'll also use the Fourier transform relation

$$\int_{-1}^{1}(1-x^2)^{\mu-1/2}e^{iax}C_n^\mu(x)\,dx = \frac{\pi 2^{1-\mu}i^n\Gamma(2\mu+n)}{n!\Gamma(\mu)}a^{-\mu}J_{\mu+n}(a) \tag{46}$$

which is valid provided $\Re\mu > -1/2$ [9]. Here $J_{\mu+n}$ is a Bessel function of the first kind and we understand that the function $K_{n,\mu}(a) = a^{-\mu}J_{n+\mu}(a)$ is an odd function if n is odd and is an even function if n is even. The data $F_z(s)$ given by

$$F_z(s) = \begin{cases} F(zs) & \text{if } |s| \leq 1 \\ 0 & \text{else} \end{cases}$$

has Fourier expansion

$$F_z(s) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\widehat{F_z}(\xi)e^{i\xi s}\,d\xi \tag{47}$$

so that an application of (44) and (46) with $\mu = \nu = (d-2)/2$ gives

$$\alpha_{d,\ell}(F_z) = \frac{i^\ell (d-3)!}{2^\nu \Gamma(\nu)} \int_{-\infty}^{\infty} \widehat{F_z}(\xi) \frac{J_{\ell+\nu}(\xi)}{\xi^\nu}\, d\xi, \tag{48}$$

which will be valid provided $d \geq 3$. We write to F_z' to mean $(F_z)'$. Note that $F_z' = z(F')_z$.

Theorem 14. *Let $u_d = u_d(z,\theta,t)$ and $w_d(z,\theta,t)$ be the unique solutions of the initial value problems (20) and (21) of Proposition 10 where $f(x) = F(\langle x,y\rangle)$ for some fixed $y \in \mathbb{R}^d$ and $z = |x||y|$. Then $u_d = u_d^e + u_d^o$ where $e^{i\nu t}u_d^e$ is the odd part (with respect to t) of $e^{i\nu t}u_d$ and $e^{i\nu t}u_d^o$ is the odd part (with respect to t) and if $d \geq 3$,*

$$u_{d+2}^e(F') = \frac{e^{-it}}{z}\frac{\partial u_d^e(F)}{\partial \omega}; \quad u_{d+2}^o(F') = \frac{d}{d-2}\frac{e^{-it}}{z}\frac{\partial u_d^o(F)}{\partial \omega} \tag{49}$$

$$w_{d+2}(F') = \frac{e^{-it}}{z}\frac{\partial w_d(F)}{\partial \omega}. \tag{50}$$

Proof. From (24) and (48) we have

$$u_d(z,\theta,t) = c_d \frac{2^{\nu-2}\Gamma(\nu+1/2)}{\nu\sqrt{\pi}} \sum_{\ell=0}^{\infty} ((\ell+d-2)e^{i\ell t} + \ell e^{-i(\ell+d-2)t}) \times i^\ell \left(\int_{-\infty}^{\infty} \widehat{F_z}(\xi)\xi^{-\nu} J_{\ell+\nu}(\xi)\, d\xi \right) C_\ell^\nu(\omega)$$

where we've used the fact that $(d-3)! = \Gamma(2\nu)$ and the Gamma function doubling formula [9] $\Gamma(2x) = 2^{2x-1}\Gamma(x)\Gamma(x+1/2)/\sqrt{\pi}$. This expansion for $u_d(z,\theta,t)$ may be rearranged to obtain $u_d = u_d^e + u_d^o$ where

$$u_d^e = \frac{\gamma_d}{\nu} e^{-i\nu t} \sum_{\ell=0}^{\infty} i^\ell (\ell+\nu)\cos((\ell+\nu)t) \left(\int_{-\infty}^{\infty} \widehat{F_z}(\xi) \frac{J_{\ell+\nu}(\xi)}{\xi^\nu}\, d\xi \right) C_\ell^\nu(\omega) \tag{51}$$

and

$$u_d^o = i\gamma_d e^{-i\nu t} \sum_{\ell=0}^{\infty} i^\ell \sin((\ell+\nu)t) \left(\int_{-\infty}^{\infty} \widehat{F_z}(\xi) \frac{J_{\ell+\nu}(\xi)}{\xi^\nu}\, d\xi \right) C_\ell^\nu(\omega) \tag{52}$$

with $\gamma_d = c_d 2^{\nu-1}\Gamma(\nu+1/2)/\sqrt{\pi}$. Note that $e^{i\nu t}u_d^e$ is the odd part (with respect to t) of $e^{i\nu t}u_d$ and $e^{i\nu t}u_d^o$ is the odd part (with respect to t) of $e^{i\nu t}u_d$. We now apply (45) to obtain

$$\begin{aligned}\frac{\partial u_d^e(F)}{\partial \omega} &= 2\gamma_d e^{-i\nu t} \sum_{\ell=1}^{\infty} i^\ell (\ell+\nu)\cos((\ell+\nu)t) \\ &\quad \times \left(\int_{-\infty}^{\infty} \widehat{F_z}(\xi)\xi^{-\nu} J_{\ell+\nu}(\xi)\, d\xi \right) C_{\ell-1}^{\nu+1}(\omega) \\ &= 2\gamma_d e^{-i\nu t} \sum_{\ell=0}^{\infty} i^{\ell+1}(\ell+\nu+1)\cos((\ell+\nu+1)t) \\ &\quad \times \left(\int_{-\infty}^{\infty} \widehat{F_z}(\xi)\xi^{-\nu} J_{\ell+\nu+1}(\xi)\, d\xi \right) C_\ell^{\nu+1}(\omega).\end{aligned}$$

Integration by parts gives $\widehat{(F'_z)}(\xi) = F(z)e^{-i\xi} - F(-z)e^{i\xi} + i\xi\widehat{F_z}(\xi)$ and applying (46) with $a = -1$ yields

$$\int_{-\infty}^{\infty} e^{-i\xi}\xi^{-d/2}J_{\ell+d/2}(\xi)\,d\xi = (\xi^{-d/2}J_{\ell+d/2})\hat{\ }(1)$$
$$= c_{\ell,d}(\chi_{[-1,1]}(x)(1-x^2)^{d/2-1/2}C_\ell^{d/2}(x))\big|_{x=1} = 0.$$

Similarly, $\int_{-\infty}^{\infty} e^{i\xi}\xi^{-d/2}J_{\ell+d/2}(\xi)\,d\xi = 0$. Hence,

$$\frac{\partial u_d^e(F)}{\partial\omega} = 2\gamma_d e^{-i\nu t}z\sum_{\ell=0}^{\infty} i^\ell(\ell+\nu+1)\cos((\ell+\nu+1)t)$$
$$\times\left(\int_{-\infty}^{\infty} -i\widehat{(F')_z}(\xi)\frac{J_{\ell+\nu+1}(\xi)}{\xi^{\nu+1}}\,d\xi\right)C_\ell^{\nu+1}(\omega)$$
$$= z\frac{d}{d-1}\frac{c_d}{c_{d+2}}e^{it}u_{d+2}^e(F') \tag{53}$$

since $F'_z = z(F')_z$. However, $\sigma_d = 2\pi^{d/2}/\Gamma(d/2)$, so $\dfrac{c_d}{c_{d+2}} = \dfrac{d-1}{d}$ and therefore, from (53) we conclude that u_d^e satisfies the first equation of (49).

The odd part, $u_d^o(F)$ of $u_d(F)$ takes the form (52) so that an application of (45) gives

$$\frac{\partial u_d^o(F)}{\partial\omega} = 2i\nu\gamma_d e^{-i\nu t}\sum_{\ell=1}^{\infty} i^\ell\sin((\ell+\nu)t)\left(\int_{-\infty}^{\infty}\widehat{F_z}(\xi)\frac{J_{\ell+\nu}(\xi)}{\xi^\nu}\,d\xi\right)C_{\ell-1}^{\nu+1}(\omega)$$
$$= 2i\nu z\gamma_d e^{-i\nu t}\sum_{\ell=0}^{\infty} i^\ell\sin((\ell+\nu+1)t)\left(\int_{-\infty}^{\infty}\widehat{(F')_z}(\xi)\frac{J_{\ell+\nu+1}(\xi)}{\xi^{\nu+1}}\,d\xi\right)C_\ell^{\nu+1}(\omega)$$
$$= e^{it}z\frac{c_d}{c_{d+2}}\frac{\nu}{\nu+1/2}u_{d+2}^o(F') = e^{it}z\left(\frac{d-2}{d}\right)u_{d+2}^o(F')$$

from which we conclude that u_d^o satisfies the second equation of (49).

From (25) we have

$$w_d(F) = \frac{i\gamma_d}{\nu}e^{-i\nu t}\sum_{\ell=0}^{\infty} i^\ell\sin((\ell+\nu)t)\left(\int_{-\infty}^{\infty}\widehat{F_z}(\xi)\frac{J_{\ell+\nu}(\xi)}{\xi^\nu}\,d\xi\right)C_\ell^\nu(\omega).$$

Differentiating this equation with respect to ω and applying (45) yields

$$\frac{\partial w_d(F)}{\partial\omega} = 2i\gamma_d z e^{-i\nu t}\sum_{\ell=0}^{\infty} i^\ell\sin((\ell+\nu+1)t)$$
$$\times\left(\int_{-\infty}^{\infty}\widehat{(F')_z}(\xi)\frac{J_{\ell+\nu+1}(\xi)}{\xi^{\nu+1}}\,d\xi\right)C_\ell^{\nu+1}(\omega)$$
$$= ze^{it}\frac{c_d}{c_{d+2}}\frac{\nu+1}{\nu+1/2}w_{d+2}(F') = ze^{it}w_{d+2}(F').$$

This gives (50). □

These results are valid only when $d \geq 3$ – note for example that the second of the equations in (49) is undefined when $d = 2$. Similarly, the right hand side of (24) is undefined when $d = 2$. The two-dimensional case is slightly different from those of higher dimensions and, as we will show, the solution of the initial-value problem (20) in dimension $d = 4$ cannot be fully determined from the solution in dimension $d = 2$. The means that to compute the higher even-dimensional solutions, it is not enough to start with the 2-dimensional solution – one must start with the 4-dimensional solution, which is given by Theorem 13. To explore this, we need first to write down the two-dimensional version of (24).

In dimension $d = 2$, an application of separation of variables to the initial value problem (33) with $f(\theta) = F(z\cos\theta)$ gives

$$u_2(z,\theta,t) = \frac{1}{\pi}\int_0^{2\pi} f(\varphi)\,d\varphi + \frac{2}{\pi}\sum_{\ell=1}^{\infty}\left(\int_0^{2\pi} f(\varphi)\cos(\ell\varphi)\,d\varphi\right)\cos(\ell\theta)\cos(\ell t)$$

$$= \frac{1}{\pi}\int_{-1}^{1}\frac{F(zs)}{\sqrt{1-s^2}}\,ds + \frac{2}{\pi}\sum_{\ell=1}^{\infty}\left(\int_{-1}^{1} F(zs)\frac{T_\ell(s)}{\sqrt{1-s^2}}\,ds\right)\cos(\ell\theta)\cos(\ell t) \tag{54}$$

where $\{T_\ell\}_{\ell=0}^{\infty}$ are the Chebyshev polynomials $T_\ell(s) = \cos(\ell\cos^{-1} s)$ ($|s| \leq 1$). As in the higher dimensional case we apply the Fourier expansion (47) to the integrals on the right hand side of (54) and apply Fubini to the resulting double integrals as well as the identity $\int_{-1}^{1}\frac{T_\ell(s)e^{is\xi}}{\sqrt{1-s^2}}\,ds = \pi i^\ell J_\ell(\xi)$ ($\ell \geq 0$) [9] to obtain (with $\omega = \cos\theta$)

$$u_2(z,\omega,t) = \frac{1}{2}\int_{-\infty}^{\infty}\widehat{F_z}(\xi)J_0(\xi)\,d\xi$$

$$+\sum_{\ell=1}^{\infty} i^\ell\left(\int_{-\infty}^{\infty}\widehat{F_z}(\xi)J_\ell(\xi)\,d\xi\right)T_\ell(\omega)\cos(\ell t). \tag{55}$$

Observe from (55) that $u_2^o(F) \equiv 0$. Consequently, the odd part u_4^o of the four-dimensional solution cannot be obtained from u_2^o.

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School of Mathematical Sciences, University of Technology Sydney, PO Box 123, Broadway NSW 2007, Australia

E-mail address: Mark.Craddock@uts.edu.au

School of Mathematical and Physical Sciences, University of Newcastle, Callaghan NSW 2308, Australia

E-mail address: jeff.hogan@newcastle.edu.au