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with Degenerate Kinetic Energy**

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ASYMPTOTIC BEHAVIOR OF EIGENVALUES OF SCHRÖDINGER TYPE OPERATORS WITH DEGENERATE KINETIC ENERGY

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ABSTRACT. We study the eigenvalues of Schrödinger type operators $T + \lambda V$ and their asymptotic behavior in the small coupling limit $\lambda \rightarrow 0$, in the case where the symbol of the kinetic energy, $T(p)$, strongly degenerates on a non-trivial manifold of codimension one.

1. INTRODUCTION

In several recent papers attention has been drawn to Schrödinger type operators on $L^2(\mathbb{R}^n)$ of the form

$$H_\lambda = T(i\nabla) + \lambda V(x), \quad (1.1)$$

where the non-negative symbol $T(p)$ degenerates on a manifold S of codimension one, $V(x)$ is a real-valued potential, and $\lambda > 0$ denoting the coupling parameter. The degeneracy of T causes a high instability of the lower edge of the spectrum of H_λ and gives rise to spectral properties which are comparable to the case of Schrödinger operators in one dimension. Operators of the type (1.1) have appeared in the study of the roton spectrum of liquid helium II [13], matrix Hamiltonians in spintronics [4, 5, 6], as well in the elasticity theory [7, 8].

Typically, we think of $T(p)$ as originating from a smooth symbol, $P(p)$, which vanishes on S and has no critical points in the neighborhood of S , with

$$T(p) = |P(p)|^r \quad (1.2)$$

for some parameter $1 \leq r < \infty$. As pointed out by Laptev, Safronov and Weidl in [15], due to the singularity of the resolvent of T on S the spectrum of $T + \lambda V$ is mainly determined by the behavior of the potential V close to S . More precisely, an important role is played by an operator acting on functions on S , i.e., $\mathcal{V}_S : L^2(S) \rightarrow L^2(S)$, given by

$$(\mathcal{V}_S u)(p) = \frac{1}{\sqrt{|\nabla P(p)|}(2\pi)^{n/2}} \int_S \hat{V}(p-q) \frac{u(q)}{\sqrt{|\nabla P(q)|}} dq, \quad (1.3)$$

with dq being the Lebesgue measure on S and $\hat{V}(p) = (2\pi)^{-n/2} \int_{\mathbb{R}^n} e^{-ix \cdot p} V(x) dx$ denoting the Fourier transform of $V(x)$. In particular, it was shown in [15] that $T + \lambda V$ has infinitely many negative eigenvalues if V is negative.

Operators of the type (1.3) appeared already earlier in [3] in the study of scattering phases. They play a crucial role in the study of the non-linear Bardeen-Cooper-Schrieffer (BCS) gap equation of superfluidity [2, 16]. In fact, it was shown

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in [9, 11, 12] that the lowest eigenvalue of \mathcal{V}_S is related to the critical temperature for the existence of solutions of the BCS gap equation. In this case, $T(p)$ is roughly of the form $|p^2 - \mu|$ for $\mu > 0$, $p \in \mathbb{R}^3$, and hence S is the two dimensional sphere of radius $\sqrt{\mu}$.

The goal of the present paper is to generalize the results and techniques of [9, 11] to a large class of manifolds S and kinetic symbols $T(p)$. We shall show that corresponding to any negative eigenvalue, a_S^i , of the compact operator \mathcal{V}_S there exists a negative eigenvalue, $-e_i(\lambda)$, of $T + \lambda V$. Moreover, in Theorem 1 we study the asymptotic behavior of $e_i(\lambda)$ as $\lambda \rightarrow 0$ and show that

$$\lim_{\lambda \rightarrow 0} \lambda f(e_i(\lambda)) = -1/a_S^i, \quad (1.4)$$

where the function f depends on the value of r in (1.2) as

$$f(e) = \begin{cases} \frac{2\pi}{r \sin(\pi/r)} \frac{1}{e^{(r-1)/r}} & \text{if } r > 1 \\ 2 \ln[1 + 1/e] & \text{if } r = 1. \end{cases} \quad (1.5)$$

We shall also relate the eigenvector ψ_λ^i of H_λ corresponding to the eigenvalue $-e_i(\lambda)$ to the eigenvector u_i of \mathcal{V}_S with eigenvalue a_S^i . We shall find that after appropriate normalization ψ_λ^i converges to

$$\int_S e^{ix \cdot p} \frac{u_i(p)}{\sqrt{(2\pi)^n |\nabla P(p)|}} dp \quad (1.6)$$

in the limit $\lambda \rightarrow 0$ in a suitable sense.

If $1 \leq r < 2$ our methods enable us to find the next to leading order term of $\lambda f(e_i(\lambda))$ as $\lambda \rightarrow 0$. This is the content of Theorem 2.

2. MAIN RESULTS

We consider operators on $L^2(\mathbb{R}^n)$, $n \geq 2$, of the form

$$H_\lambda = T(i\nabla) + \lambda V(x). \quad (2.1)$$

The symbol of the kinetic operator, $T(p)$, attains its minimum on a manifold of codimension one. For convenience let us assume that the minimum value is zero, and let

$$S = \{p \in \mathbb{R}^n \mid T(p) = 0\}. \quad (2.2)$$

It is not being assumed that S is connected, but it should consist of only finitely many connected components. We shall further assume that there exists a $\sigma > 0$ and a compact neighborhood $\Omega \subset \mathbb{R}^n$ of S containing S , with the property that the distance of any point in S to the complement of Ω is at least σ . Moreover, we assume that

- (i) $T(p) = |P(p)|^r$ for some locally bounded, measurable function P , with $1 \leq r < \infty$, and $P \in C^2(\Omega)$,
- (ii) $|\nabla P|$ does not vanish in Ω ,
- (iii) for some constants $C_1 > 0$, $C_2 > 0$ and $s > 0$, $T \geq C_1 |p|^s + C_2$ for $p \notin \Omega$.

These assumptions appear naturally in all recent applications mentioned in the introduction. They could be relaxed in various ways, but we shall not try to do so in order to avoid unilluminating complications in the proofs.

Since S in (2.2) is the zero set of the function $P \in C^2(\Omega)$, and $\nabla P \neq 0$ in Ω by assumption, we conclude that S is a nice submanifold of codimension one. In

particular, if $V \in L^1(\mathbb{R}^n)$ then $\hat{V}(p)$ is a bounded, continuous function and hence (1.3) defines a compact (in fact, trace-class) operator V_S on $L^2(S)$.

In the following, it will be useful to introduce the operator $F_S : L^1(\mathbb{R}^n) \rightarrow L^2(S)$, which is obtained by restricting the Fourier transform to S and multiplying by $|\nabla P|^{-1/2}$, i.e.,

$$(F_S \varphi)(p) = \frac{1}{\sqrt{(2\pi)^n} \sqrt{|\nabla P(p)|}} \int_{\mathbb{R}^n} e^{-ix \cdot p} \varphi(x) dx \Big|_{p \in S}. \quad (2.3)$$

Its adjoint, $F_S^* : L^2(S) \rightarrow L^\infty(\mathbb{R}^n)$, is given by

$$(F_S^* u)(x) = \frac{1}{\sqrt{(2\pi)^n}} \int_S \frac{e^{ix \cdot p}}{\sqrt{|\nabla P(p)|}} u(p) dp. \quad (2.4)$$

Then \mathcal{V}_S is (1.3) equals $F_S V F_S^*$. Note that $V^{1/2} F_S^*$ is a bounded operator if $V \in L^1(\mathbb{R}^n)$.

For $i = 1, 2, \dots$, let $a_S^i < 0$ be the negative eigenvalues, counting multiplicity, of \mathcal{V}_S , and let u_i be its eigenvectors, i.e.,

$$\mathcal{V}_S u_i = a_S^i u_i, \quad u_i \in L^2(S). \quad (2.5)$$

The following theorem shows that it is possible to associate to any such a_S^i a negative eigenvalue $-e^i(\lambda)$ for H_λ . Moreover, we will recover the asymptotic behavior of $e_i(\lambda)$ in the limit $\lambda \rightarrow 0$. A similar statement can be made about the corresponding eigenvectors. The theorem is a generalization of [9, Theorem 1].

Theorem 1. *Let $T(p)$ satisfy the assumptions above, and let $V \in L^1(\mathbb{R}^n) \cap L^{n/s}(\mathbb{R}^n)$ if $n > s$, $V \in L^1(\mathbb{R}^n) \cap L^{1+\varepsilon}(\mathbb{R}^n)$ for some $\varepsilon > 0$ if $n = s$, and $V \in L^1(\mathbb{R}^n)$ if $n < s$. Additionally we assume that $\iint |V(x)| |x-y|^\kappa |V(y)| dx dy < \infty$, with $\kappa = 2$ if T is not a radial function, and $\kappa = 1$ if T is radial and $n = 2$. Then*

- (i) *for every negative eigenvalue $a_S^i < 0$ of \mathcal{V}_S , counting multiplicity, and every $\lambda > 0$, there is a negative eigenvalue $-e_i(\lambda) < 0$ of $H_\lambda = T + \lambda V$ such that*

$$\lim_{\lambda \rightarrow 0} \lambda f(e_i(\lambda)) = -1/a_S^i. \quad (2.6)$$

The function f is defined in (1.5).

- (ii) *for every eigenvector $\psi_\lambda^i \in L^2(\mathbb{R}^n)$ of H_λ , corresponding to the eigenvalue $-e_i(\lambda)$, there is an eigenvector $u_i \in L^2(S)$ of \mathcal{V}_S corresponding to a_S^i such that after appropriate normalization*

$$V^{1/2} \psi_\lambda^i \rightarrow V^{1/2} F_S^* u_i \quad \text{as } \lambda \rightarrow 0, \text{ strongly in } L^2(\mathbb{R}^n). \quad (2.7)$$

- (iii) *if $r < 2$ all other possible eigenvalues $-e_j(\lambda)$ of H_λ satisfy $f(e_j(\lambda)) \geq c\lambda^{-2}$ for some constant $c > 0$.*
- (iv) *if $r < 2$ and $\mathcal{V}_S \geq 0$, and there exists an δ such that also $F_S(V - \delta|V|)F_S^* \geq 0$ then $H_\lambda \geq 0$ for λ small enough.*

Equation (2.6) implies, in particular, that

$$e_i(\lambda) = \begin{cases} \left(\frac{2\pi}{r \sin(\pi/r)} \lambda |a_S^i| \right)^{r/(r-1)} (1 + o(1)) & \text{if } r > 1 \\ \exp \left(-\frac{1}{2\lambda |a_S^i|} (1 + o(1)) \right) & \text{if } r = 1 \end{cases} \quad (2.8)$$

as $\lambda \rightarrow 0$. On the other hand (iii) guarantees that all possible eigenvalues of H_λ not corresponding to a negative eigenvalue of \mathcal{V}_S satisfy

$$e_j(\lambda) \leq \begin{cases} \text{const } \lambda^{2r/(r-1)} & \text{if } 1 < r < 2 \\ \exp(-\text{const } \lambda^{-2}) & \text{if } r = 1. \end{cases} \quad (2.9)$$

The following immediate corollary of Theorem 1 generalizes results in [15, 18].

Corollary 1. *Let the assumptions be as in Theorem 1.*

- (i) *Then, for all $\lambda > 0$, the operator H_λ has at least as many negative eigenvalues as \mathcal{V}_S does.*
- (ii) *If $V(x) \leq 0$ and does not vanish a.e., then \mathcal{V}_S (and consequently H_λ) has infinitely many negative eigenvalues.*

Proof. The negative eigenvalues, $-e_i(\lambda)/\lambda$, of the operator $H_\lambda/\lambda = T/\lambda + V$ are monotonically decreasing in λ since $T \geq 0$. Consequently if $-e_i(\bar{\lambda}) < 0$ then $-e_i(\lambda)$ is necessarily negative for all $\lambda \geq \bar{\lambda}$. Thus (i) follows immediately from Theorem 1 (i).

If $V \leq 0$, then $\mathcal{V}_S \leq 0$ and all eigenvalues of \mathcal{V}_S are necessarily non-positive. We shall argue that 0 cannot be an eigenvalue of \mathcal{V}_S since for any non-zero function $\varphi \in L^2(S)$, $F_S^* \varphi$ can vanish at most on a subset of \mathbb{R}^n of codimension one. This follows from the fact that $(F_S^* \varphi)(x_1, \dots, x_n)$ is analytic in each component x_i , and therefore can only have isolated zeros in each component. Consequently $(\varphi, \mathcal{V}_S \varphi) = \int_{\mathbb{R}^n} |(F_S^* u)(x)|^2 V(x) dx < 0$ for any u . This implies (ii). \square

Remark 1. In the BCS gap equation of superfluidity at zero temperature [10, 9, 11, 12] the kinetic energy operator $T(p) = |p^2 - \mu|$ appears, with $\mu > 0$ being the chemical potential. In this case $r = 1$ and hence $f(e) = 2 \ln(1/e)$. Therefore, the eigenvalues of $T + \lambda V$ are exponentially small and satisfy $e_i(\lambda) \sim e^{-\frac{1}{2\lambda|a_S^i|}}$.

Remark 2. In the study of the roton spectrum in liquid helium [14] a kinetic energy of the type $T(p) = \frac{(|p| - p_0)^2}{2\mu} + \Delta$ arises, with $p_0, \mu, \Delta > 0$. In this case Theorem 1 implies that the eigenvalues depend quadratically on λ for small λ , i.e., $e_i(\lambda) - \Delta \sim (\lambda|a_S^i|)^2$, similar to the case of Schrödinger operators in one dimension [19].

Remark 3. The convergence property (2.7) can be particularly useful in the case where the manifold S is a sphere and the potential V is radial, since the eigenfunctions of \mathcal{V}_S are known explicitly. In the case $n = 3$, for instance, they are the spherical harmonics. If additionally $\hat{V} \leq 0$ then the constant function on S is the ground state of \mathcal{V}_S . This property was important in [11] where a precise characterization of the asymptotic behavior of the solution of the BCS gap equation of superfluidity was given.

Remark 4. In the case of trapped modes for an elastic plate in [7] a small coupling asymptotics was derived in the case where S is a circle in \mathbb{R}^2 .

In the following let $r < 2$. In this case, we shall now state a more precise characterization of the asymptotic behavior of the eigenvalues of H_λ as $\lambda \rightarrow 0$. More precisely, we will recover the next order in λ .

It will be shown in Lemma 2 that the quadratic form

$$(u, \mathcal{W}_S u) = \lim_{e \rightarrow 0} \left(u, F_S V \left(\frac{1}{T+e} - f(e) F_S^* F_S \right) V F_S^* u \right) \quad (2.10)$$

defines a bounded operator on $L^2(S)$. For $\lambda > 0$ let further

$$\mathcal{B}_S = \mathcal{V}_S - \lambda \mathcal{W}_S \quad (2.11)$$

and let $b_S^i(\lambda) < 0$ denote the negative eigenvalues of \mathcal{B}_S . The following theorem is a generalization of [11, Theorem 1].

Theorem 2. *Let T and V be as in Theorem 1 and assume that $r < 2$. Then*

- (i) *If $\lim_{\lambda \rightarrow 0} b_S^i(\lambda) < 0$ then $H_\lambda = T + \lambda V$ has, for small λ , a corresponding negative eigenvalue $-e_i(\lambda) < 0$, with*

$$\lim_{\lambda \rightarrow 0} \left[f(e_i(\lambda)) + \frac{1}{\lambda b_S^i(\lambda)} \right] = 0. \quad (2.12)$$

- (ii) *If the kernel of \mathcal{V}_S is not empty then there exists at least one corresponding negative eigenvalue of H_λ .*

Remark 5. If $a_S^i < 0$ is a non-degenerate eigenvalue of \mathcal{V}_S and u_i is the corresponding eigenvector, then first order perturbation theory implies that the corresponding eigenvalue of \mathcal{B}_S satisfies

$$b_S^i(\lambda) = a_S^i - \lambda(u_i, \mathcal{W}_S u_i) + o(\lambda). \quad (2.13)$$

Hence (2.12) can be rewritten in the form

$$\lim_{\lambda \rightarrow 0} \left[f(e_i(\lambda)) + \frac{1}{\lambda a_S^i} + \frac{(u_i, \mathcal{W}_S u_i)}{(a_S^i)^2} \right] = 0.$$

A similar expression holds in case $a_i < 0$ is k -fold degenerate, with $(u_i, \mathcal{W}_S u_i)$ replaced by the eigenvalues of the $k \times k$ matrix $(u_i^{(j)}, \mathcal{W}_S u_i^{(l)})$, where $u_i^{(j)}$ denotes the eigenvectors of \mathcal{V}_S corresponding to the eigenvalue a_S^i .

3. PROOFS

According to the Birman-Schwinger principle, the operator H_λ has a negative eigenvalue $-e < 0$ if and only if the compact operator

$$\lambda V^{1/2} \frac{1}{T + e} |V|^{1/2} \quad (3.1)$$

has an eigenvalue -1 . Here, we use the usual convention $V^{1/2} = \text{sgn}(V)|V|^{1/2}$. Note that $V^{1/2}(T + e)^{-1}|V|^{1/2}$ is actually a Hilbert-Schmidt operator for $e > 0$. This follows from the Hardy-Littlewood-Sobolev inequality [17, Theorem 4.3] and our assumptions on T and V .

More precisely, if

$$H_\lambda \psi_\lambda = -e \psi_\lambda \quad (3.2)$$

for $\psi_\lambda \in L^2(\mathbb{R}^n)$ and $e > 0$, then

$$\lambda V^{1/2} \frac{1}{T + e} |V|^{1/2} \phi_\lambda = -\phi_\lambda, \quad (3.3)$$

where $\phi_\lambda = V^{1/2} \psi_\lambda$. It is, in fact, not difficult to see $\phi_\lambda \in L^2$, since $|V|$ is infinitesimally form-bounded with respect to T under our assumptions on T and V . On the other hand (3.3) implies (3.2) by choosing

$$\psi_\lambda = \frac{1}{T + e} |V|^{1/2} \phi_\lambda \quad (3.4)$$

which is in $L^2(\mathbb{R}^3)$ since $T \geq 0$, $e > 0$ and the operator $|V|^{1/2}(T+e)^{-1}|V|^{1/2}$ is bounded.

Our results will rely on the fact that the singular part of the (3.1) as $e \rightarrow 0$ is governed by the operator $V^{1/2}F_S^*F_S|V|^{1/2}$, which is isospectral to $\mathcal{V}_S = F_SVF_S^*$.

In the following, let M_e denote the bounded operator

$$M_e = V^{1/2} \left(\frac{1}{T+e} - f(e)F_S^*F_S \right) |V|^{1/2}. \quad (3.5)$$

Proposition 1. *Assume that $1 + \lambda M_e$ is invertible. Then H_λ has an eigenvalue $-e < 0$ if and only if the selfadjoint operator*

$$F_S|V|^{1/2} \frac{\lambda f(e)}{1 + \lambda M_e} V^{1/2} F_S^* : L^2(S) \rightarrow L^2(S) \quad (3.6)$$

has an eigenvalue -1 . Furthermore, if $u \in L^2(S)$ is an eigenvector of (3.6) with eigenvalue -1 , then

$$\psi_\lambda = \frac{1}{T+e} |V|^{1/2} \frac{1}{1 + \lambda M_e} V^{1/2} F_S^* u \quad (3.7)$$

is an eigenvector of H_λ in $L^2(\mathbb{R}^n)$ with eigenvalue $-e < 0$.

Proof. According to the Birman-Schwinger principle discussed above, H_λ having an eigenvalue $-e < 0$ is equivalent to the fact that $\lambda V^{1/2} \frac{1}{T+e} |V|^{1/2} + 1$ has a zero eigenvalue. Using the definition of M_e in (3.5) this implies that

$$\begin{aligned} \lambda V^{1/2} \frac{1}{T+e} |V|^{1/2} + 1 &= \lambda f(e) V^{1/2} F_S^* F_S |V|^{1/2} + \lambda M_e + 1 \\ &= (1 + \lambda M_e) \left(\frac{\lambda f(e)}{1 + \lambda M_e} V^{1/2} F_S^* F_S |V|^{1/2} + 1 \right) \end{aligned} \quad (3.8)$$

has an eigenvalue 0. Under the assumption that $1 + \lambda M_e$ is invertible we conclude that

$$\frac{\lambda f(e)}{1 + \lambda M_e} V^{1/2} F_S^* F_S |V|^{1/2} \quad (3.9)$$

must have -1 as an eigenvalue. The fact that (3.9) is isospectral to (3.6), together with the observation that all the arguments work in either direction, implies the first part of the theorem. The second part of the theorem is an easy consequence of (3.4). \square

In order to apply Proposition 1 we need a bound on the operator M_e in (3.5). The bound we derive will be expressed in terms of the function

$$g(e) = \begin{cases} 1 & \text{if } 1 \leq r < 2 \\ 1 + \ln[1 + 1/e] & \text{if } r = 2 \\ 1 + e^{2-r} & \text{if } r > 2. \end{cases} \quad (3.10)$$

The following lemma is the basis for our analysis.

Lemma 1. *Let*

$$\mathcal{A}(V) = \begin{cases} \|V\|_{n/s} & \text{if } n > s \\ \|V\|_{1+\varepsilon} & \text{if } n = s \\ \|V\|_1 & \text{if } n < s. \end{cases} \quad (3.11)$$

Then

$$\|M_e\| \leq \text{const} \left(g(e) \left[\|V\|_1 + \left(\iint dx dy |V(x)| |x-y|^\kappa |V(y)| \right)^{1/2} \right] + \mathcal{A}(V) \right) \quad (3.12)$$

with $\kappa = 2$. If $T(p)$ is radial, then (3.12) holds with $\kappa = 0$ for $n \geq 3$ and $\kappa = 1$ for $n = 2$.

Let us postpone the proof of this lemma until the end of the section. The lemma says, in particular, that when $r < 2$ the family of operators M_e is uniformly bounded. The limit of M_e as $e \rightarrow 0$ actually exist in the operator norm topology. This is the content of the next lemma, whose proof will also be given at the end this section.

Lemma 2. *Assume that $r < 2$. Then the limit*

$$M_0 = \lim_{e \rightarrow 0} M_e \quad (3.13)$$

exists in the operator norm topology.

An explicit expression of M_0 will be given in the proof of Lemma 2. We note that the operator \mathcal{W}_S in (2.10) equals $\mathcal{W}_S = F_S |V|^{1/2} M_0 V^{1/2} F_S^*$.

We have now all tools in hand to prove our main theorems.

Proof of Theorem 1. By assumption, the operator \mathcal{V}_S has negative eigenvalues a_S^i with corresponding eigenfunctions $u_i \in L^2(S)$. We shall show that for every $a_S^i < 0$ and λ small enough there exists a function $e_i(\lambda) > 0$, with $\lim_{\lambda \rightarrow 0} \lambda f(e_i(\lambda)) = -1/a_S^i$, such that the selfadjoint operator (3.6) has an eigenvalue -1 for $e = e_i(\lambda)$. Because of Proposition 1 this implies (i).

For this purpose consider the selfadjoint operator

$$G(\lambda, e) = |V|^{1/2} \left(\frac{1}{1 + \lambda M_e} - 1 \right) |V|^{1/2}. \quad (3.14)$$

In terms of $G(\lambda, e)$, the operator (3.6) can be expressed as

$$\lambda f(e) (\mathcal{V}_S + F_S G(\lambda, e) F_S^*). \quad (3.15)$$

Let us first consider first the case $r < 2$, where $g(e) = 1$. According to Lemma 1, M_e is uniformly bounded and hence $1 + \lambda M_e$ is invertible for small λ . Therefore,

$$\|F_S G(\lambda, e) F_S^*\| \leq \text{const} \|V\|_1 \frac{\lambda \|M_e\|}{1 - \lambda \|M_e\|}, \quad (3.16)$$

where we used that $\|F_S |V| F_S^*\| \leq \text{const} \|V\|_1$.

Simple first order perturbation theory implies that for small λ , the operator (3.15) has negative eigenvalues $\lambda f(e)(a_S^i + O(\lambda))$. Moreover, the $O(\lambda)$ term depends continuously on e . Thus, for every $a_S^i < 0$ and $\lambda > 0$ small enough, there exists an $e_i(\lambda)$ such that $\lambda f(e_i(\lambda))(a_S^i + O(\lambda)) = -1$. This implies the statement.

A similar argument can be applied in the case $r \geq 2$. Although M_e is not uniformly bounded in this case, we see that for values of λ and e such that $\lambda f(e)$ is bounded, $\lambda g(e)$ goes to zero as λ and e go to zero. Because of Lemma 1 this implies that $\lambda \|M_e\| \rightarrow 0$ as $\lambda \rightarrow 0$ for such e . Hence we can again find a function $e_i(\lambda)$, with $\lim_{\lambda \rightarrow 0} \lambda f(e_i(\lambda)) = -1/a_S^i$, such that (3.15) has an eigenvalue -1 and for $e = e_i(\lambda)$. This concludes the proof of (i) in the general case.

In order to prove (ii) we shall again apply simple perturbation theory, which implies that for $e = e_i(\lambda)$ the eigenvector $u_\lambda^i \in L^2(S)$ of (3.15) corresponding to the eigenvalue -1 satisfies

$$u_\lambda^i = u_i + \eta_\lambda, \quad \lim_{\lambda \rightarrow 0} \|\eta_\lambda\|_2 = 0, \quad (3.17)$$

with u_i in the eigenspace of \mathcal{V}_S corresponding to the eigenvalue a_S^i . Applying the second part of Proposition 1, the eigenvector of H_λ corresponding to the eigenvalue $-e_i(\lambda)$ equals

$$\psi_\lambda^i = \frac{1}{T + e_i(\lambda)} |V|^{1/2} \frac{1}{1 + \lambda M_{e_i(\lambda)}} V^{1/2} F_S^* (u_i + \eta_\lambda). \quad (3.18)$$

Using the eigenvalue equation for ψ_λ , $(T + e_i(\lambda))\psi_\lambda^i = -\lambda V \psi_\lambda^i$, this can be rewritten as

$$-\lambda V^{1/2} \psi_\lambda^i = \frac{1}{1 + \lambda M_{e_i(\lambda)}} V^{1/2} F_S^* (u_i + \eta_\lambda). \quad (3.19)$$

Now $\lambda \|M_{e_i(\lambda)}\| \rightarrow 0$ as $\lambda \rightarrow 0$, and $F_S |V| F_S^*$ is bounded. After appropriate normalization, $V^{1/2} \psi_\lambda^i$ therefore converges to $V^{1/2} F_S^* u_i$ strongly in $L^2(\mathbb{R}^n)$, as claimed.

A simple perturbation argument leads to (iii). In fact, any negative eigenvalue of (3.15) which does not correspond to a negative eigenvalue of \mathcal{V}_S for $\lambda = 0$ can be at most as negative as $-\lambda f(e) \|F_S G(\lambda, e) F_S^*\| \geq -\text{const } \lambda^2 f(e)$ for some constant depending only on V . This can be easily seen using (3.16) and Lemma 1. Hence $\lambda^2 f(e) \geq \text{const}$ for such eigenvalues.

To see (iv) we use the operator inequality $G(\lambda, e) \geq -\text{const } \lambda |V|$ for small λ , which follows easily from (3.14) and Lemma 1. The operator in (3.15) is therefore bounded from below by

$$\lambda f(e) (\mathcal{V}_S - \text{const } \lambda F_S |V| F_S^*) = \lambda f(e) F_S (V - \text{const } \lambda |V|) F_S^*, \quad (3.20)$$

which is non-negative for λ small enough according to our assumption. \square

Proof of Theorem 2. Since $r < 2$ by assumption, Lemma 2 implies that M_e converges to M_0 in operator norm. Since $V^{1/2} F_S^*$ is a bounded operator, we conclude that also $F_S |V|^{1/2} M_e V^{1/2} F_S^*$ converges in operator norm to $F_S |V|^{1/2} M_0 V^{1/2} F_S^*$, which we shall denote by \mathcal{W}_S as in (2.10).

With $\mathcal{B}_S = \mathcal{V}_S - \lambda \mathcal{W}_S$ as in (2.11), the operator (3.6) can thus be rewritten as

$$\lambda f(e) (\mathcal{B}_S + \lambda F_S |V|^{1/2} W(\lambda, e) V^{1/2} F_S^*), \quad (3.21)$$

where

$$W(\lambda, e) = \frac{\lambda M_e^2}{1 + \lambda M_e} - M_e + M_0 \quad (3.22)$$

has the property that $\|W(\lambda, e)\| \rightarrow 0$ as $\lambda \rightarrow 0$ and $e \rightarrow 0$. If $b_S^i(\lambda)$ is a negative eigenvalue of \mathcal{B}_S , with $\lim_{\lambda \rightarrow 0} b_S^i(\lambda) < 0$, then a similar perturbation argument as in the proof of Theorem 1 implies that $\mathcal{B}_S + \lambda F_S W(\lambda, e) F_S^*$ has an eigenvalue with the asymptotic behavior $b_S^i(\lambda) + o(\lambda) + \lambda o(1)$, the last term going to zero as $e \rightarrow 0$. Given such a $b_S^i(\lambda)$, we can thus find an $e_i(\lambda)$, going to zero as $\lambda \rightarrow 0$, such that (3.21) has an eigenvalue 1 for $e = e_i(\lambda)$. In the limit $\lambda \rightarrow 0$, we conclude that

$$\lambda f(e_i(\lambda)) = -1/(b_S^i(\lambda) + o(\lambda)). \quad (3.23)$$

Using again Proposition 1 we obtain (i).

If \mathcal{V}_S has 0 as an eigenvalue, with corresponding eigenvector u_0 , then by the definition (2.11) of \mathcal{B}_S and the fact that $\mathcal{V}_S u_0 = F_S V F_S^* u_0 = 0$ we obtain that

$$(u_0, \mathcal{B}_S u_0) = -\lambda(u_0, \mathcal{W}_S u_0) = -\lambda \lim_{e \rightarrow 0} (u_0, F_S V \frac{1}{T+e} V F_S^* u_0). \quad (3.24)$$

The latter quantity is strictly negative, as can be seen by an analyticity argument similar to the proof of Corollary 1. In particular, if the kernel of \mathcal{V}_S is not empty then there is at least one corresponding negative eigenvalue of \mathcal{B}_S for small enough λ and e . Together with Proposition 1 this implies the existence of a corresponding negative eigenvalue of H_λ . \square

We are left with proving Lemmas 1 and 2.

Proof of Lemma 1. We note that $\tilde{M}_e = \text{sgn}(V)M_e$ is selfadjoint, and $\|M_e\| = \|\tilde{M}_e\|$. For $\psi \in L^2(\mathbb{R}^n)$, let $\varphi = |V|^{1/2}\psi$. By the definition of M_e in (3.5), we have

$$(\psi, \tilde{M}_e \psi) = \int_{\mathbb{R}^n} \frac{|\hat{\varphi}(p)|^2}{T(p)+e} dp - f(e) \int_S \frac{|\hat{\varphi}(p)|^2}{|\nabla P(p)|} dp. \quad (3.25)$$

By our assumptions on T , there exists a $\tau > 0$ such that

$$\Omega_\tau = \{p \in \mathbb{R}^3 \mid |P(p)| < \tau\} \quad (3.26)$$

is a subset of Ω . Recall that P is assumed to be twice differentiable on Ω , and hence also on Ω_τ . If S is not connected, we choose τ small enough such that Ω_τ has the same number of connected components as S . On Ω_τ , we will use the co-area formula to split the volume integral in the first term on the right side of (3.25) into integrals over the level sets

$$S_t = \{p \in \Omega_\tau \mid |P(p)| = T^{1/r}(p) = t\} \quad (3.27)$$

for $0 \leq t \leq \tau$. Note that $S_0 = S$. In fact, using the co-area formula we have

$$\int_{\Omega_\tau} \frac{|\hat{\varphi}(p)|^2}{T(p)+e} dp = \int_0^\tau dt \frac{1}{t^r + e} \int_{S_t} \frac{|\hat{\varphi}(p)|^2}{|\nabla P(p)|} dp, \quad (3.28)$$

where dp in the latter integral denotes the Lebesgue measure on S_t .

Recall that $T(p) = |P(p)|^r = 0$ on S , and $|\nabla P| \neq 0$ on Ω_τ . Hence every connected component of S_t consists of two disjoint surfaces, one lying outside S and one lying inside S . In order to bound (3.28) we make use of the following lemma.

Lemma 3. *Let $h : \Omega_\tau \rightarrow \mathbb{R}$, with $h \in C^1(\Omega_\tau)$, and let $0 < t < \tau$. Then*

$$\left| \int_{S_t} h(p) dp - 2 \int_S h(p) dp \right| \leq \int_0^t d\sigma \int_{S_\sigma} \frac{1}{|\nabla P(p)|} \left| \nabla \cdot \left(h(p) \frac{\nabla P(p)}{|\nabla P(p)|} \right) \right| dp. \quad (3.29)$$

Proof. Without loss of generality we can assume that S is connected. We shall write $S_t = S_t^o \cup S_t^i$, with $S_t^{i,o}$ lying inside and outside S , respectively. Let $\Omega_t^{o,i} = \bigcup_{0 \leq \sigma \leq t} S_\sigma^{o,i}$ denote the union of the sets $S_\sigma^{o,i}$ for $0 \leq \sigma \leq t$. By definition $\frac{\nabla P}{|\nabla P|}$ is a unit vector field which is orthogonal to the hypersurfaces S_σ^o and S_σ^i and points

either inward or outward, depending on P . Depending on the direction, we have

$$\int_{S_t^o} h(p) dp - \int_S h(p) dp = \pm \int_{\partial\Omega_t^o} h(p) \frac{\nabla P(p)}{|\nabla P(p)|} \cdot d\mathbf{S} \quad (3.30)$$

$$\int_{S_t^i} h(p) dp - \int_S h(p) dp = \mp \int_{\partial\Omega_t^i} h(p) \frac{\nabla P(p)}{|\nabla P(p)|} \cdot d\mathbf{S}. \quad (3.31)$$

Using Gauss' theorem we infer, for $q = o, i$,

$$\begin{aligned} \int_{\partial\Omega_t^q} h(p) \frac{\nabla P(p)}{|\nabla P(p)|} \cdot d\mathbf{S} &= \pm \int_{\Omega_t^q} \nabla \cdot \left(h(p) \frac{\nabla P(p)}{|\nabla P(p)|} \right) dp \\ &= \pm \int_0^t d\tau \int_{S_\tau^q} \frac{1}{|\nabla P(p)|} \nabla \cdot \left(h(p) \frac{\nabla P(p)}{|\nabla P(p)|} \right) dp, \end{aligned} \quad (3.32)$$

where the last equation follows again from the co-area formula. The rest is obvious. \square

We shall now apply Lemma 3 to the function $h(p) = |\hat{\varphi}(p)|^2 |\nabla P(p)|^{-1}$. Note that

$$\begin{aligned} |\hat{\varphi}(p)|^2 &= (2\pi)^{-n} \iint |V(x)|^{1/2} |V(y)|^{1/2} \psi(x)^* \psi(y) e^{ip \cdot (x-y)} dx dy \\ &\leq (2\pi)^{-n} \|V\|_1 \|\psi\|_2^2 \end{aligned} \quad (3.33)$$

uniformly in p by Schwarz's inequality. Similarly

$$|\nabla |\hat{\varphi}(p)|^2| \leq (2\pi)^{-n} \|\psi\|_2^2 \left(\iint |V(x)| |V(y)| |x-y|^2 dx dy \right)^{1/2}. \quad (3.34)$$

By assumption, there are constants $c, C > 0$ such that $|\nabla P| \geq c$ and $|\partial_i \partial_j P| \leq C$ for $1 \leq i, j \leq n$ on Ω_τ . Moreover, the measure of the sets S_t is uniformly bounded for $0 \leq t \leq \tau$. We conclude that

$$\begin{aligned} \left| \int_{S_t} \frac{|\hat{\varphi}(p)|^2}{|\nabla P(p)|} dp - 2 \int_S \frac{|\hat{\varphi}(p)|^2}{|\nabla P(p)|} dp \right| \\ \leq \text{const } t \|\psi\|_2^2 \left[\|V\|_1 + \left(\iint |V(x)| |V(y)| |x-y|^2 dx dy \right)^{1/2} \right]. \end{aligned} \quad (3.35)$$

By combining (3.35) with (3.28) and (3.25), we obtain the bound

$$\begin{aligned} \left| (\psi, \tilde{M}_e \psi) \right| &\leq \text{const} \int_0^\tau \frac{t dt}{t^r + e} \|\psi\|_2^2 \left[\|V\|_1 + \left(\iint |V(x)| |V(y)| |x-y|^2 dx dy \right)^{1/2} \right] \\ &\quad + \left| \int_0^\tau \frac{2 dt}{t^r + e} - f(e) \right| \int_S \frac{|\hat{\varphi}(p)|^2}{|\nabla P(p)|} dp + \int_{\Omega_\tau^c} \frac{|\hat{\varphi}(p)|^2}{T(p) + e} dp. \end{aligned} \quad (3.36)$$

It is easy to see that $\int_0^\tau t(t^2 + e)^{-1} dt \leq \text{const } g(e)$ for any fixed τ . Similarly, $|f(e) - 2 \int_0^\tau (t^r + e)^{-1} dt| \leq \text{const } g(e)$. The integral in the second term in (3.36) is bounded by $\|\psi\|_2^2 \|V\|_1$ using (3.33). Moreover, since $T(p) \geq \text{const}(1 + |p|^s)$ on Ω_τ^c , the last term in (3.36) can be bounded with the aid of the Hardy-Littlewood-Sobolev inequality [17, Theorem. 4.3] and Hölder's inequality as

$$\int_{\Omega_\tau^c} \frac{|\hat{\varphi}(p)|^2}{T(p) + e} dp \leq \text{const} \|\psi\|_2^2 \mathcal{A}(V) \quad (3.37)$$

with $\mathcal{A}(V)$ defined in (3.11). This proves (3.12) with $\kappa = 2$.

In the case when T is radial, the surfaces S_t are $n - 1$ dimensional spheres. In this case, we can obtain a better bound on $\|M_e\|$ in the following way. It is not necessary to obtain a pointwise bound on $\nabla|\hat{\varphi}(p)|^2$ but only on its spherical average. Using the fact that for $n \geq 2$

$$\begin{aligned} \frac{1}{|\mathbb{S}^{n-1}|} \int_{\mathbb{S}^{n-1}} e^{ik \cdot \omega} d\omega &= \frac{|\mathbb{S}^{n-2}|}{|\mathbb{S}^{n-1}|} \int_0^\pi e^{i|k| \cos \theta} (\sin \theta)^{n-2} d\theta \\ &= \pi \frac{\Gamma((n-1)/2)^2}{\Gamma(n/2)} \left(\frac{2}{|k|} \right)^{(n-2)/2} J_{(n-2)/2}(|k|), \end{aligned} \quad (3.38)$$

where $J_{(n-2)/2}$ is a Bessel function, as well as the bounds $J_{(n-2)/2}(|k|) \leq 1$, $J_{(n-2)/2}(|k|) \leq (|k|/2)^{(n-2)/2} \Gamma((n-1)/2)$ and the asymptotics $J_{(n-2)/2}(|k|) \sim |k|^{-1/2}$ for $|k| \rightarrow \infty$ [1], it is easy to see that

$$\left| \int_{\mathbb{S}^{n-1}} \nabla|\hat{\varphi}(p|\omega)|^2 d\omega \right| \leq \text{const} \|\psi\|_2^2 \left(\iint |V(x)||V(y)||x-y|^\kappa dx dy \right)^{1/2} \quad (3.39)$$

with $\kappa = 0$ for $n \geq 3$ and $\kappa = 1$ for $n = 2$. Using this bound instead of (3.34) and proceeding as above, we arrive at (3.12) with κ as stated. \square

Proof of Lemma 2. Let \tilde{M}_0 be defined via the quadratic form

$$\begin{aligned} (\psi, \tilde{M}_0 \psi) &= \int_0^\tau \frac{1}{t^r} \left(\int_{S_t} \frac{|\hat{\varphi}(p)|^2}{|\nabla P(p)|} dp - 2 \int_S \frac{|\hat{\varphi}(p)|^2}{|\nabla P(p)|} dp \right) + \int_{\Omega_\varepsilon} \frac{|\hat{\varphi}(p)|^2}{T(p)} dp \\ &\quad + C_\tau \int_S \frac{|\hat{\varphi}(p)|^2}{|\nabla P(p)|} dp, \end{aligned} \quad (3.40)$$

where $\varphi = |V|^{1/2} \psi$ and

$$C_\tau = \lim_{e \rightarrow 0} C_\tau(e), \quad C_\tau(e) = \int_0^\tau \frac{2}{t^r + e} dt - f(e), \quad (3.41)$$

which is finite for $1 \leq r < 2$. The notation is the same as in the proof of Lemma 1. With $\tilde{M}_e = \text{sgn}(V)M_e$ as before, we have

$$\begin{aligned} (\psi, (\tilde{M}_e - \tilde{M}_0) \psi) &= \int_0^\tau dt \left(\frac{1}{t^r + e} - \frac{1}{t^r} \right) \left(\int_{S_t} \frac{|\hat{\varphi}(p)|^2}{|\nabla P(p)|} dp - 2 \int_S \frac{|\hat{\varphi}(p)|^2}{|\nabla P(p)|} dp \right) \\ &\quad + \int_{\Omega_\varepsilon} |\hat{\varphi}(p)|^2 \left(\frac{1}{T(p) + e} - \frac{1}{T(p)} \right) dp \\ &\quad + (C_\tau(e) - C_\tau) \int_S \frac{|\hat{\varphi}(p)|^2}{|\nabla P(p)|} dp. \end{aligned} \quad (3.42)$$

From this representation and the various bounds derived in the proof of Lemma 1, it is easy to see that the right side goes to zero as $e \rightarrow 0$, and the convergence is uniform in ψ for fixed $\|\psi\|_2$. This implies that $\lim_{e \rightarrow 0} \|\tilde{M}_e - \tilde{M}_0\| = 0$, and hence also $\lim_{e \rightarrow 0} \|M_e - M_0\| = 0$ with $M_0 = \text{sgn}(V)\tilde{M}_0$. \square

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