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Special Invariant Operators I.

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SPECIAL INVARIANT OPERATORS I.

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Abstract. The aim of the first part of a series of papers is to give a description of invariant differential operators on manifolds with an almost Hermitian symmetric structure of the type G/B which are defined on bundles associated to the reducible but undecomposable representation of the parabolic subgroup B of the Lie group G. One example of an operator of this type is the Penrose's local twistor transport. In this part general theory is presented, and conformally invariant operators are studied in more details.

1. Introduction.

There is a series of papers by R.Baston, M.Eastwood, T.Bailey and others describing invariant operators on manifolds with almost Hermitian symmetric structure (AHS - structure) (see [Baston I,II,1991]) of the type G/B acting on bundles associated to a reducible, but indecomposable representation of B.

The class of these operators include local twistor transport, Thomas operator and other interesting invariant operators. Let us call these operators special operators. The aim of the paper is to give more explicit, purely geometrical description of these special operators using the theory of G-structures of higher order, and to extend some of results presented there.

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2. Definition and basic properties of AHS-structures.

Let $\mathfrak{g}$ be a simple (complex, real) $\|1\|$- graded Lie algebra with decomposition

$$\mathfrak{g} = \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1.$$

It means

$$[\mathfrak{g}_i, \mathfrak{g}_j] = \mathfrak{g}_{i+j}$$

with $\mathfrak{g}_2 = \mathfrak{g}_{-2} = 0$.

It follows from the above properties that $\mathfrak{g}_0$ is reductive and $\mathfrak{g}_{-1}, \mathfrak{g}_1$ are abelian. The complete classification of complex, simple, $\|1\|$-graded Lie algebras is given in ([Ba1]), to consider the real ones, the real forms of corresponding complex algebras

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must be considered. We shall restrict below to study of some real cases, the full discussion of all possible cases will be published later.

Suppose in the following, that not only Lie algebras but Lie groups are given i.e let G be a (real or complex) Lie group with Lie algebra $\mathfrak{g}$, let B be a parabolic subgroup of G with Lie algebra $\mathfrak{g}_0 \oplus \mathfrak{g}_1$, and let B_0 be a subgroup of B with Lie algebra $\mathfrak{g}_0$. Then G/B is a Hermitian symmetric space. The group B can be considered in a natural way as a subgroup of the second jet group $G^2(n)$, with respect to the action of B on $T_o(G/B) \equiv \mathfrak{g}_{-1}$.

2.1 Definition. *An AHS-structure on a manifold M is a G-structure of the second order $\mathcal{B}'$ which is a reduction of the first prolongation of a G-structureof the first order $\mathcal{B}_0$ (reduction of the frame bundle) with the group B_0 on M (see*[Sternberg]*).*

2.2 Definition. *An AHS-structure $\mathcal{B}_0$ on M is called 1-flat if $\mathcal{B}_0$ admits a torsionless connection (it means that the first structure function of the G-structure vanishes).*

2.3 Remarks.

A. All conformal and projective structures are 1-flat. For other AHS-structures the condition of 1-flatness is a restrictive condition on structure.

B. If AHS-structure is 1-flat, we can consider also an AHS-structure as a holonomic G-structure $\mathcal{B}$ of second order (reduction of frames of second order) with the structural group B on M. The structure $\mathcal{B}$ of the second order defines G-structure of first order $\mathcal{B}_0$ with the structural group B_0 and projection

$$\pi_0 : \mathcal{B} \longrightarrow \mathcal{B}_0.$$

If AHS-structure is not 1-flat, then also a principal fibre bundle $\mathcal{B}$ with the same properties as above can be constructed. The construction will be described in Sect.6.

The definitions used in this text are generalization of the definition of Baston in ([Baston]), where only complex holomorphic G-structures are considered.

2.5 Remark. *We shall also use some covering spaces of the coresponding frame bundles (frames of first and also second order). It will be done especially for spin-manifolds. It needs only simple modification of the presented theory which is quite natural.*

3. Flat models.

3.1. Any AHS-structure has a homogeneous model (called flat) which is 1-flat. The homogeneous model is given by

$$\pi : G \to G/B.$$

Here G is a principal fibre bundle on M = G/B with structural group B and the bundle $\mathcal{B}_0$ is sitting inside it.

Let us denote by $\tilde{G} \to M$ the principal fibre bundle with the structural group G associated to $\pi : G \to M$ by

$$\tilde{\pi} : \tilde{G} = G \times_\iota G \to M$$

where the action ι of B on G is the left multiplication.

Let $\omega : TG \to \mathfrak{g}$ be the Maurer-Cartan form on G, it is also (normal) Cartan connection on $\pi : G \to M$.

Then there is a uniquely defined an (ordinary) connection $\tilde{\omega}$ on the principal bundle $\tilde{\pi} : \tilde{G} \to M$ constructed from ω, we shall call it the AHS-connection (as in general situation).

The construction of $\tilde{\omega}$ can be described in the following way.

The principal fibre bundle G is embedded canonically into $\tilde{G}$ by:

$$i(g) = [(g, e)].$$

Let us identify G with its canonical image $i(G) \subset \tilde{G}$.

For any $g \in G \subset \tilde{G}$ we have a decomposition

$$T_g(\tilde{G}) = T_g(G) \oplus \mathfrak{g}_{-1}$$

with projection on the second factor given by

$$\sigma_{-1} : T_g(\tilde{G}) \longrightarrow \mathfrak{g}_{-1}$$

Let us define $\tilde{\omega}$ by

$$\tilde{\omega} = \omega + \sigma_{-1}$$

on G and let us extend it equivariantely to $\tilde{G}$.

3.2 Theorem. *$\tilde{\omega}$ is a connection on $\tilde{G}$ called the AHS-connection .*

Proof. The fact that $\tilde{\omega}$ is a connection follows from a simple direct computation.

Let $\rho : G \to Aut(\mathbb{E})$ be an irreducible representation of the group G, let

$$\tilde{E} := \tilde{G} \times_\rho \mathbb{E}$$

be the corresponding associated bundle. Then the AHS-connection induces the associated covariant derivative

$$\tilde{\nabla} : \Gamma(\tilde{E}) \to \Gamma(\tilde{E} \otimes T^*(M))$$

on $\tilde{E}$.

In the definition of the AHS-connection only the structure of flat AHS-manifold is used and it is (the simplest case of) canonical invariant connection on AHS-manifold.

3.3 Lemma. *Let $\tilde{E} := \tilde{G} \times_\rho \mathbb{E}$ and $E := G \times_{\rho'} \mathbb{E}$, then there is the canonical isomorphism of the fibre bundles $\phi : \tilde{E} \to E$.*

Proof. The isomorphism ϕ is defined by the correspondence

$$v \in \mathbb{E} \to \{((g, 1), v)\} \in \tilde{E} \leftrightarrow \{(g, v)\} \in E$$

which is B-equivariant (resp. B_0- equivariant).

The bundles E and $\tilde{E}$ will be identified by ϕ, then on this bundle also the invariant covariant differential

$$\tilde{\nabla} : \Gamma(E) \to \Gamma(E \otimes T^*(M))$$

can be defined in the standard way.

Moreover in the flat case also the following construction is available and will be used in some of the following papers.

Let $s \in \Gamma(E)$ be a section of E, and let $\Phi(s) : \tilde{G} \to \mathbb{E}$ be the corresponding equivariant function.

For any $X \in \mathfrak{g}_{-1}$, let ξ_X be the left invariant vector field on G induced by X. It gives a uniquely defined horizontal vector field $h(X)$ on $\tilde{G}$ with respect to $\tilde{\omega}$. On the subbundle $G \subset \tilde{G}$, it has the form

$$h(X) = \xi_X - X^*,$$

where X^* is the canonical vertical field on $\tilde{G}$ given by X.

Let ρ' be the restriction of the representation ρ to B (resp. to B_0) (it is not irreducible in general).

We have now simple and natural formula for $\tilde{\nabla}$ coming from the normal Cartan connection ω on G.

If the section $s \in \Gamma(E)$ is represented by an equivariant function $\Phi(s) : G \to \mathbb{E}$, then

$$\Phi(\tilde{\nabla}_{\tilde{\pi}_*(h(X))}s) = h(X)\Phi(s) = \xi_X\Phi(s) - \rho'(X)(s),\ X \in \mathfrak{g}_{-1}.$$

4. Classical AHS-structures.

Let $\mathbb{K}$ be or $\mathbb{R}$ or $\mathbb{C}$. Then we have the following examples of AHS-structures coming from classical simple Lie algebras. The list is complete for complex case, but not all real cases are listed. The other real structures will be discussed in details elsewhere. There are two cases for exceptional (nonclassical) simple algebras which will not be presented here.

4.1 Conformal structures.

Let $\mathfrak{g} = so(m+2,\mathbb{C})$, $\mathfrak{g}_0 = co(m,\mathbb{C})$ and $\mathfrak{g}_1 = \mathfrak{g}_{-1} = \mathbb{C}^m$. A standard real form is $\mathfrak{g} = so(m+1,1), \mathfrak{g}_0 = co(m,\mathbb{R})$ and $\mathfrak{g}_1 = \mathfrak{g}_{-1}\mathbb{R}^m$.

Then $G = SO(m+2,\mathbb{C})$ or $Spin(m+2,\mathbb{C})$ and the flat model G/B is the quadric (complexified,compactified Minkowski space) in the complex case; G = SO(m+1,1) or Spin(m+1,1) and the flat model is an m-dimensional conformal sphere in the real case .

4.2 Almost Grassmanian structures.

Let $\mathfrak{g} = sl(p+q,\mathbb{C})$, $\mathfrak{g}_0 = s(sl(p,\mathbb{K}) \oplus sl(q,\mathbb{K}))$, or unitary algebras . $\mathfrak{g}_1 = \mathfrak{g}_{-1} = \mathbb{K}^{pq}$. The flat model is the complex resp. the real Grassmanian $Gr_p(\mathbb{K}^{p+q})$.

There are two special cases:

A. Projective structures.

For p = 1 (or q = 1) the flat model is the projective space $\mathbb{K}P^p$.

B. Almost quaternionic structures.

For p = 2 (or q = 2) we have structures which (under additional conditions) include quaternionic, quaternionic Kaehler and Hyperkaehler structures .

4.3 Almost Lagrangian structures.
Let $\mathfrak{g} = sp(n,\mathbb{K})$ or $u(n)$, $\mathfrak{g}_0 = gl(n,\mathbb{K})$ and $\mathfrak{g}_{-1} = \mathfrak{g}_1 = K^N$ with $N = \frac{1}{2}n(n+1)$.

The flat models are the Lagrangian Grassmanians $LGr(\mathbb{K}^{2n})$, resp. $Sp(n,\mathbb{R})/U(n)$.

4.4 Almost spinorial structures..
Let $\mathfrak{g} = so(2n,\mathbb{K})$ and $\mathfrak{g}_0 = gl(n,\mathbb{K})$, resp. $u(n)$.

The flat models are the manifold of reduced pure spinors $\mathcal{S}_n = SO(2n,\mathbb{C})/Gl(n,\mathbb{C})$ resp. $SO^*(2n,\mathbb{R})/U(n)$.

5. Spencer cohomology.

5.1. The Spencer cohomology is a cohomology theory of the Lie algebra $\mathfrak{g}_{-1}$ with coefficients in a graded $\mathfrak{g}$-module $\mathbb{F}$. Denote by $\circ$ the action of $\mathfrak{g}$ on $\mathbb{F}$. So we have

$$\mathbb{F} = \oplus_i \mathbb{F}_i, \ \mathfrak{g}_i \circ \mathbb{F}_j \subset \mathbb{F}_{i+j}.$$

The bigraded chain complex $C^{p,q}(\mathbb{F})$ is defined by

$$C^{p,q}(\mathbb{F}) = \mathbb{F}_q \otimes \Lambda^p \mathfrak{g}^*_{-1}$$

with the differential

$$\partial : C^{p,q}(\mathbb{F}) \longrightarrow C^{p+1,q-1}(\mathbb{F})$$

given by

$$(\partial c)(X_0, ..., X_q) = \sum_0^q X_i \circ c(X_0, ...\hat{X}_i, ...X_q).$$

The cohomology of the complex $(C^{p,q}(\mathbb{F}), \partial)$ will be denoted by $H^{p,q}(\mathbb{F})$ and the direct sum

$$H^p(\mathfrak{g}_{-1}, \mathbb{F}) = \oplus_q H^{p,q}(\mathbb{F})$$

is the Lie algebra cohomology of $\mathfrak{g}_{-1}$ with coefficients in $\mathbb{F}$.

Because ∂ intertwines the action of $\mathfrak{g}_0$, these cohomology groups are $\mathfrak{g}_0$-modules.

If $<,>$ is the Killing form on $\mathfrak{g}$, and $\{\xi_i\}$ basis of $\mathfrak{g}_{-1}$ and $\{\eta^j\}$ (dual) basis of $\mathfrak{g}_1$ with

$$< \xi_i, \eta^j >= \delta_i^j,$$

we can define the adjoint operator

$$\partial^* : C^{p,q}(\mathbb{F}) \longrightarrow C^{p-1,q+1}(\mathbb{F})$$

by setting

$$(\partial^* c)(X_1, \dots, X_{q-1}) = \sum_i \eta^i \circ c(\xi_i, X_1, \dots, X_{q-1}).$$

We get a Hodge theory on cohomology ([K]), namely we define

$$\Delta = \partial\partial^* + \partial^*\partial$$

and we say that $c \in C^{p,q}(\mathbb{F})$ is harmonic if $\Delta c = 0$.

There is a decomposition

$$C^{p,q}(\mathbb{F}) = \operatorname{im}\partial \oplus \ker\Delta \oplus \operatorname{im}\partial^*$$

and a unique harmonic representative in each cohomology class. Because ∂ and ∂^* intertwine the action of $\mathfrak{g}_0$ we have that Δ acts by scalars on each irreducible $\mathfrak{g}_0$-submodule of $C^{p,q}(\mathbb{F})$.

Moreover Δ is invertible on $C^{p,q}(\mathbb{F})$ if and only if $H^{p,q}(\mathbb{F}) = 0$.

All representations of $\mathfrak{g}_0$ constructed above $\mathbb{F}, \mathbb{F}_i, C^{p,q}(\mathbb{F}), H^{p,q}(\mathbb{F})$ give representations of B_0 and define associated bundles $F, F_i, C^{p,q}(F), H^{p,q}(F)$ to $\mathcal{B}_0$ on M.

Moreover we have well-defined operators amond these bundles induced from the operators $\partial, \partial^*, \Delta$. We denote these operators with same letters.

5.2 Remark. *The Spencer cohomology groups $H^{p,q}(\mathbb{F})$ may be computed using Borel-Bott-Weil theorem (see e.g* [Kostant]*). In almost all our case the condition for invertibility of some operators (i.e vanishing of certain group $H^{p,q}(\mathbb{F})$) will be satisfied. It will be discussed for any structure separately.*

6. The AHS-connection.

6.1 The canonical Cartan connection. There is certain amount of facts known on Cartan connections on manifolds with AHS structure. These facts are well-known in 1-flat cases, while the other cases where first treated in [Baston, 1991]. His description was based on covariant derivatives defined on certain vector bundles on M, while it is more natural from our point of view to consider Cartan connections as forms on pricipal fibre bundles. A systematic and detailed description of this approach was recently given in [Čap, Slovak, Souček, I,II; 1994]. In this part, corresponding facts will be shortly reviewed, all details can be found in the quoted papers.

Let M be a manifold with an AHS-structure, i.e. suppose that we have a B_0-structure $\mathcal{B}_0$ on M (a G-structure of the first order). Then it is possible to construct a B-structure $(\mathcal{B}, \theta)$ on M which is, by definition, a principal fiber bundle $\mathcal{B} \to M$ with the group B equipped with the soldering form $\theta = \theta_{-1} + \theta_0 \in \Omega^1(\mathcal{B}, \mathfrak{g}_{-1} \oplus \mathfrak{g}_0)$ with the properties

(1) $\theta_{-1}(\xi) = 0$ if and only if ξ is a vertical vector.
(2) $\theta_0(\zeta_{Y+Z}) = Y$ for all $Y \in \mathfrak{g}_0$, $Z \in \mathfrak{g}_1$.
(3) $(R_b)^*\theta = \operatorname{Ad}(b^{-1})\theta$ for all $b \in B$ where Ad means the action on the vector space $\mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \simeq \mathfrak{g}/\mathfrak{g}_1$ induced by the adjoint action.

By construction, $\mathcal{B}$ is a principal fiber bundle over $\mathcal{B}_0$ with the group B_1.

We shall consider now Cartan connections on the bundle $\mathcal{B}$. Let us recall that a $\mathfrak{g}$-valued one form ω on $\mathcal{B}$ is called a Cartan connection if it has the following properties:

(1) $\omega(\zeta_X) = X$ for all $X \in \mathfrak{b}$

(2) $(R_b)^*\omega = \mathrm{Ad}(b^{-1}) \circ \omega$ for all $b \in B$
(3) $\omega|_{T_u\mathcal{B}} \colon T_u\mathcal{B} \to \mathfrak{g}$ is a bijection for all $u \in \mathcal{B}$

Any Cartan connection splits into three components $\omega = \omega_{-1} + \omega_0 + \omega_1$.

There is a special class of Cartan connections called admissible Cartan connections. They are characterized by the property $\omega_{-1} + \omega_0 = \theta$. For all structures considered (with exception of the simplest case $\mathfrak{g}_0 = \mathfrak{sl}(2)$), there is a canonical Cartan connection. Its existence and uniqueness follows, in principle, from the fact that the second prolongation of the structrure in question is trivial. The canonical Cartan connection will be used below for the construction of invariant operators between bundles corresponding to certain reducible, but indecomposable, representations of the group B. Even if these operators will be described in a simple way using covariant derivatives associated to standard connections on a big principal fibre bundle over M, it is important to express them with help of more standard objects, namely to describe them using a distinguished class of affine connection on the first order principal bundle $\mathcal{B}_0$ (in the case of the conformal structure, these are just Levi-Civita connections given by a choice of a metric inside the conformal class). To explain that below, we need to recall now a few more facts concerning the relations between connections chosen in this special class of affine connections and the canonical Cartan connection.

There is a well defined class of connections on $\mathcal{B}_0$ related to the given second order structure $(\mathcal{B}, \theta)$. It consists of connections on $\mathcal{B}_0$ with a harmonic torsion. They are parametrized by one-forms on M. Let us call these connections harmonic connections.

In the context of Spencer cohomology, the class can be described in the following way. The torsion τ of any connection γ on $\mathcal{B}_0$ is a B_0-invariant map

$$\tau_\theta : \mathcal{B}_0 \to \mathfrak{g}_{-1} \otimes \Lambda^2 \mathfrak{g}^*_{-1} = C^{2,-1}.$$

There is the $\mathfrak{g}_0$-invariant decomposition

$$C^{2,-1} = \mathrm{im}\, \partial \oplus \ker \Delta.$$

Note that the boundary operator ∂ is trivial on $C^{2,-1}$, hence $\ker \Delta$ coincides with $\ker \partial^*$. The connection γ is called harmonic, if its torsion belongs to $\ker \Delta$. The part of the connection lying in $\mathrm{Ker}\Delta$ is invariant of the structure (it does not depend on a choice of the connection) and is called Weyl torsion tensor of the AHS-structure. It can be shown that the space of all harmonic connections is nonempty and that it is in bijective correspondence with the space of all (global) B_0-equivariant sections of the principal bundle $\mathcal{B} \to \mathcal{B}_0$. Moreover, fixing one harmonic connections, the space of all of them can be identified with the space of one-forms on M.

To describe the correspondence between equivariant sections and harmonic connections more precisely, suppose that σ be a B_0-equivariant section of $\mathcal{B}_0$. Then $\gamma = \sigma^*\theta_0$ is a harmonic principal connection on $\mathcal{B}_0$. Moreover, the section σ defines uniquely a Cartan connection ω_γ characterized by the properties $\omega_\gamma = \theta \oplus \omega_1$ and $\omega_1(T_\sigma(T\mathcal{B}_0)) = 0$. The connection ω_γ and the affine connection $\theta_{-1} \oplus \gamma$ are then σ-related.

Suppose now that we have two admissible Cartan connections ω and ω'. Their difference vanishes on vertical vectors and has values in $\mathfrak{g}_1$, hence there is a map Γ from $\mathcal{B}$ to $\mathfrak{g}^*_{-1} \otimes \mathfrak{g}_1$ such that $\omega' = \omega + \Gamma \circ \theta_1$. The equivariant map Γ represents a tensor field on M called the deformation tensor.

6.2 The AHS connection.

Given the canonical Cartan connection ω on $\mathcal{B}$, we are going to construct its canonical unique extension to the (standard) connection $\widetilde{\omega}$ called AHS-connection on associated extended principal fibre bundle

$$\widetilde{\mathcal{B}} = \mathcal{B} \times_\iota G$$

given by the left multiplication of B on G.

Moreover the principal fibre bundle $\mathcal{B}$ can be embedded canonically into $\widetilde{\mathcal{B}}$ by

$$i(r) = ((r, e)).$$

The existence and uniqueness of the connection $\widetilde{\omega}$ follows from the theorem:

6.3 Theorem. *Let ω be a (non necessary canonical) Cartan connection on $\mathcal{B}$, let $\widetilde{\mathcal{B}}$ be the associated extended principal fibre bundle with group G and let i be the embedding of $\mathcal{B}$ into $\widetilde{\mathcal{B}}$ as above. Then there exist the unique connection $\widetilde{\omega}$ on $\widetilde{\mathcal{B}}$ such that $i^*(\widetilde{\omega}) = \omega$.*

Proof. Let $\tilde{r}$ be a point in $\widetilde{\mathcal{B}}$, then there exist $r \in B$ and $g \in G$ such that $\widetilde{r} = R_g r$. Let $\tilde{X}$ be an element of the tangent space $T_{\tilde{r}}\widetilde{\mathcal{B}}$, then its translation $R_{g^{-1}*}(\tilde{X})$ is an element of $T_r\widetilde{\mathcal{B}}$ and can be uniquely expressed as

$$R_{g^{-1}*}(\tilde{X}) = X_r + A_r^*$$

with $X_r \in T_r(B)$ and where A_r^* is the value at r of the canonical vertical field corresponding to some $A \in g_{-1}$.

The value of the connection form $\widetilde{\omega}$ on $\tilde{X}$ is defined by

$$\widetilde{\omega}(\tilde{X}) = Ad(g^{-1})(\omega(X_r) + A).$$

The proof of the fact that $\widetilde{\omega}$ is a connection form is technical and will be omitted. The uniqueness of $\widetilde{\omega}$ is clear from the definition.

We can use also the following alternative description of the AHS connection which is more useful for the further use.

For any $r \in \mathcal{B}$ we have a decomposition

$$T_r(\tilde{\mathcal{B}}) = T_r(\mathcal{B}) \oplus \mathfrak{g}_{-1}$$

with projection on the second factor

$$\sigma_{-1} : T_r(\tilde{\mathcal{B}}) \longrightarrow \mathfrak{g}_{-1}$$

For $X \in T_r(\tilde{\mathcal{B}}), X = X^T + \sigma_{-1}(X)$, with $X^T \in T_r(\mathcal{B})$, we define

$$\tilde{\omega}(X) = \omega(X^T) + \sigma_{-1}(X)$$

on $\mathcal{B}$ and we extend $\tilde{\omega}$ equivariantly to $\tilde{\mathcal{B}}$.

Then $\tilde{\omega}$ is then called the AHS-connection on $\tilde{\mathcal{B}}$.

The following description of the horizontal lift $\tilde{X}_1$ of a vector field X will be used below:

Take $x \in M, X \in T_x(M), r \in \mathcal{B}_0$ with $\pi(r) = x$, we have the horizontal lift $\tilde{X}$ of X to $\pi^{-1}(x) \subset \mathcal{B}_0$ with respect to the connection ω, thus $\omega_{-1}(\tilde{X}) = \Phi(X)$ as mapping

$$\pi^{-1}(x) \to \mathfrak{g}_{-1}.$$

Define

$$\tilde{X}_1(r) = \tilde{X}(r) - \sigma_{-1}(\tilde{X}(r))^*$$

then we have $\widetilde{\omega}(\tilde{X}_1) = 0$ and the field $\tilde{X}_1$ is the horizontal lift of X restricted on $\mathcal{B}_0$.

Let $(\rho, \mathbb{E})$ be a representation of G, and

$$E := \widetilde{\mathcal{B}} \times_\rho \mathbb{E}$$

the associated vector bundle.

6.4 Remark. *In general the tangent (and also cotangent) bundle is not associated to the principal fibre bundle $\widetilde{\mathcal{B}}$. It is associated always to the principal fibre bundle $\mathcal{B}_0$ with respect to the standard representation σ of B_0 on $\mathbb{C}^n$, it can be associated to the principal fibre bundle $\mathcal{B}$ with respect to the extension of the representation σ from B_0 to B trivially on well defined part B_1 of B ($B = B_0 \hat{\oplus} B_1$, the semidirect product), but there is (in general) no extension of this action to G.*

The connection $\widetilde{\omega}$ induces a covariant derivative $\widetilde{\nabla}$ on the space of sections $\Gamma(E)$ and differential operator

$$\widetilde{\nabla} : \Gamma(E) \to \Gamma(E \otimes T^*(M))$$

is an invariant operator on M of special type.

It is defined as follows: Let us recall that $\mathcal{B}$ is canonically and B-equivariantly embedded into $\widetilde{\mathcal{B}}$. In any point $\tilde{r} \in \widetilde{\mathcal{B}}$ we have horizontal space $\bar{H}_r \subset T_{\tilde{r}}(\widetilde{\mathcal{B}})$ equivariant with respect of the action of the group G. We can lift any vector $X \in T_{\pi(r)}(M)$ horizontally to the field $\tilde{X}$ on $\tilde{\pi}^{-1}(\tilde{\pi}(r))$.

Let $s \in \Gamma(E)$ be a section of the bundle E , and

$$\Phi(s) : \widetilde{\mathcal{B}} \to \mathbb{E}$$

the corresponding equivariant function.

Then $\tilde{X}\Phi(s)$ is a function on $\widetilde{\mathcal{B}}$ with values in $\mathbb{E}$ which is $\tilde{B}$-equivariant and define a section of E as associated bundle to $\widetilde{\mathcal{B}}$.

Fixing harmonic connection ω_0 on $\mathcal{B}_0$, we have an B_0 invariant embeddings of principal fiber bundles on M

$$\mathcal{B}_0 \subset \mathcal{B} \subset widetilde\mathcal{B}$$

and E is also associated bundle to $\mathcal{B}_0$.

We shall use the following description of the operator $\widetilde{\nabla}$ using the covariant derivative ∇ with respect to ω_0 :

Decompose $\mathbb{E}$ into irreducible components with respect to the representation of the subgroup B_0

$$\mathbb{E} = \oplus_i \mathbb{E}_i.$$

Let us remark that these components are mixed with respect to the actions of groups B and G.

Denote the associated vector bundles on M by

$$E_i := \mathcal{B}_0 \times_{\rho_i} \mathbb{E}_i.$$

Then we have

$$E = \oplus_i E_i.$$

The AHS-connection (covariant derivative) $\widetilde{\nabla}$ on E is expressed in terms of background connection and "additional" terms coming from the forms σ_{-1}, σ_1 as follows

Let $X \in T_x(M)$ be a tangent vector, denote $F_x = \pi^{-1}(x) \subset \mathcal{B}_0$ the fiber over x. Then we have a horizontal lift $\widetilde{X}$ of X as a vector field defined in any point of F_x and an equivariant mapping $\Phi(X) : F_x \to \mathfrak{g}_{-1}$ defined by X. Moreover the map

$$\Gamma : \mathcal{B} \to Hom(\mathfrak{g}_{-1}, \mathfrak{g}_1)$$

define the map

$$Q(X) = \Gamma \circ \Phi(X) : F_x \to \mathfrak{g}_1.$$

Let s be a section of E given in a neigborhood U of x, let

$$\Phi(s) : \pi^{-1}(U) \to \mathbb{E}$$

be the equivariant function defined by s.

Then we get a formula for covariant derivative with respect to the AHS-connection:

$$\Phi(\widetilde{\nabla}_X s)(r) = \widetilde{X}(r)\Phi(s) - \rho(\Phi(X)(r))\Phi(s)(r) + \rho(Q(X)(r))\Phi(s)(r).$$

Denote $Q_{-1}(X), Q_1(X)$ the corresponding operators, so we have:

$$\widetilde{\nabla}_X s = \nabla_X s - Q_{-1}(X)s + Q_1(X)s.$$

Any representation space of G is a graded $\mathfrak{g}$-module, we have for $\mathbb{E}$ also decomposition with respect to the grading:

$$\mathbb{E} = \mathbb{E}^0 \oplus \mathbb{E}^1 \oplus ... \oplus \mathbb{E}^k$$

All spaces $\mathbb{E}^i$ are $\mathfrak{g}_0$ modules (not necessary ireducible), they consist of sums of some $\mathbb{E}_i$.

If $E^i = \mathcal{B}_0 \times_{\rho^i} \mathbb{E}^i$ are corresponding bundles and if we decompose the space of sections

$$\Gamma(E) = \Gamma(E^0) \oplus ... \oplus \Gamma(E^k)$$

then for each $i \in \{0, ..., k\}$ and $X \in T_x(M)$ we have

$$\nabla_X : \Gamma(E^i) \to \Gamma(E^i)$$

and

$$Q_{-1}(X) : \Gamma(E^i) \to \Gamma(E^{i-1}),\ Q_1(X) : \Gamma(E^i) \to \Gamma(E^{i+1}).$$

7. Conformal structures.

The complex conformal geometry.

Take $\mathfrak{g} = so(m+2, \mathbb{C})$,$\mathfrak{g}_0 = co(m, \mathbb{C}) = so(m, \mathbb{C}) \oplus \mathbb{C}$ and $\mathfrak{g}_1 = \mathfrak{g}_{-1} = \mathbb{C}^m$.

The corresponding groups are: $G = SO(m+2, \mathbb{C})$ (or $Spin(m+2, \mathbb{C})$), $B_0 = CO(m, \mathbb{C})$ (or its corresponding universal covering), and the flat model G/B is an m-dimensional quadric (the complexified,compactified Minkowski space) in the complex projective space.

The real conformal geometry.

Take $\mathfrak{g} = so(m+1, 1)$, $\mathfrak{g}_0 = co(m, \mathbb{R}) = so(m, \mathbb{R}) \oplus \mathbb{R}$ and $\mathfrak{g}_1 = \mathfrak{g}_{-1} = \mathbb{R}^m$.

The corresponding groups are $G = SO(m+1, 1, \mathbb{R})$ (or $Spin(m+1, 1, \mathbb{R})$), $B_0 = CO(m, \mathbb{R})$ (or its universal covering) and the flat model G/B is the m-dimensional conformal sphere.

The special conformally invariant operators.

Let M be a manifold endowed with conformal structure, let g be a fixed metric from the given conformal class and ω the Levi-Civita connection of g. Let Ric be the Ricci curvature tensor and R the scalar curvature of the metric g.

In the conformal case, it is possible to give an explicit description of Γ in terms of the Riemann curvature tensor. Let us define a symmetric 2-form P on M by

$$P = \frac{1}{n-2}(Ric - \frac{1}{2(n-1)} R.g)$$

Then P defines an B_0-equivariant map

$$\Phi(P) : \mathcal{B}_0 \to \odot^2 \mathfrak{g}_1.$$

Using the duality between $\mathfrak{g}_1$ and $\mathfrak{g}_{-1}$ and the symmetry of P, we obtain desired B_0-equivariant map

$$\Gamma : \mathcal{B}_0 \to \mathfrak{g}_1 \otimes \mathfrak{g}_{-1}^* = Hom(\mathfrak{g}_{-1}, \mathfrak{g}_1),$$

which is the deformation tensor for a couple given by the canonical Cartan connection ω and any admissible Cartan connection ω_γ induced by any harmonic affine connection.

In the next part special invariant operators which correspond to all fundamental representations of the group $G = Spin(m+2, \mathbb{C})$ (i.e to the representations of Lie algebra $\mathfrak{g} = so(m+2, \mathbb{C})$) will be described.

7.4 Example 1 : Spinor representation..

The following description of spinors and spinor representation coming out from Clifford algebra calculus (see [DSS]) will be used here.

A. Even dimensional case (m = 2n).

Let

$$\{e_0, e_1, \dots, e_n; e_{n+1}, \dots, e_{2n}, e_{2n+1}\}$$

be the basis of $\mathbb{R}^{2n+2}$ with

$$\mathcal{Q}(e_0, e_0) = \mathcal{Q}(e_{2n+1}, e_{2n+1}) = 0, \mathcal{Q}(e_0, e_{2n+1}) = 1, \mathcal{Q}(e_i, e_j) = -2\delta_{ij}$$

for i,j = ,1,...,2n.

There is the standard embedding $\mathbb{R}^{2n+2} \subset \mathcal{C}^R_{2n+2}$ and the relations in the real Clifford algebra $\mathcal{C}^R_{2n+2}$:

$$e_0.e_{2n+1} + e_{2n+1}.e_0 = -2; e_0^2 = e_{2n+1}^2 = 0, e_j.e_k + e_k.e_j = -2\delta_{ij}$$

Moreover after complexification we get

$$\mathbb{R}^{2n+2} \subset \mathbb{C}^{2n+2} \subset \mathcal{C}^C_{2n+2}.$$

Let us construct the canonical embeddings

$$so(2n+1,1) \longrightarrow so(2n+2,\mathbb{C}) \longrightarrow \mathcal{C}^C_{2n+2}.$$

as follows:

Consider first the natural isomorphism:

$$so(2n+2,\mathbb{C}) \equiv \Lambda^2(\mathbb{C}^{2n+2})$$

given by

$$u \wedge v \mapsto \phi(u \wedge v)(x) = 2(\mathcal{Q}(v,x)u - \mathcal{Q}(u,x)v); u, v, x \in \mathbb{C}^{2n+2}$$

and thus define an embedding

$$\Lambda^2(\mathbb{C}^{2n+2}) \to \mathcal{C}^{\mathbb{C}}_{2n+2}$$

given by

$$u \wedge v \mapsto u.v - \mathcal{Q}(u,v)1; u, v \in \mathbb{C}^{2n+2}$$

After an easy computation the desired embedding of Lie algebra

$$\iota : so(2n+1,1) \to \mathcal{C}^C_{2n+2}$$

has the following form

$$\iota(V_\alpha) = \frac{1}{2} e_0.e_\alpha; \text{ for } V_\alpha \in \mathfrak{g}_{-1};$$

$$\iota(\eta_\alpha) = \frac{1}{2} e_{2n+1}.e_\alpha; \text{for } \eta_\alpha \in \mathfrak{g}_1;$$

and

$$\iota(A_0) = \frac{1}{2}(e_{2n+1}.e_0 + 1);\ i(A_{jk}) = \frac{1}{2} e_j.e_k; 1 \le j < k \le 2n$$

for the elements from $\mathfrak{g}_0$.

Let us introduce the isotropical basis of $\mathbb{C}^{2n+2}$,

$$\{f_0, f_1, \ldots, f_n; \hat{f}_0, \hat{f}_1, \ldots, \hat{f}_n\}$$

with

$$\mathcal{Q}(f_i, \hat{f}_i) = 1, \mathcal{Q}(f_i, f_i) = \mathcal{Q}(\hat{f}_i, \hat{f}_i) = 0; \text{ for } i = 0, 1, ..., n$$

The transformation relations are given by

$$f_0 = e_0, \hat{f}_0 = e_{2n+1}, f_j = \frac{1}{2i}(e_j + ie_{n+j}), \hat{f}_j = \frac{1}{2i}(e_j - ie_{n+j})$$

and for the inverse transformation by

$$e_0 = f_0, e_{2n+1} = \hat{f}_0, e_j = i(f_j + \hat{f}_j), e_{n+j} = i(f_j - \hat{f}_j).$$

Putting together we get the following realization of $\mathfrak{g}$ in Clifford algebra $\mathcal{C}^C_{2n+2}$:

$$\mathfrak{g}_{-1} = \{E_j = i\hat{f}_0.(f_j + \hat{f}_j),\ E_{n+j} = i\hat{f}_0.(f_j - \hat{f}_j); j = 1, \ldots, n\}$$

$$\mathfrak{g}_1 = \{E^j = if_0.(f_j + \hat{f}_j),\ E^{n+j} = if_0.(f_j - \hat{f}_j); j = 1, \ldots, n\}$$

$$\mathfrak{g}_0 = \{E_{j,k} = (f_j + \hat{f}_j).(f_k + \hat{f}_k)\} \cup \{E_{j,n+k} = (f_j + \hat{f}_j).(f_k - \hat{f}_k)\}$$
$$\cup\{E_{n+j,n+k} = (f_j - \hat{f}_j).(f_k - \hat{f}_k)\} \cup \{\hat{f}_0.f_0 + 1\}.$$

Let us denote

$$N^p = \{(j_1, ..., j_k); j_1 < j_2 ... < j_k; j_i \in \{1, ..., n\}; k \le n, k \equiv p (\mathrm{mod}\, 2)\};\ p = 0, 1$$

and

$$\bar{N}^p = \{(j_1, ..., j_k); j_1 < j_2 ... < j_k; j_i \in \{0, 1, ..., n\}; k \le n, k \equiv p (\mathrm{mod}\, 2)\};\ p = 0, 1.$$

One of possible choice of the spinor space is

$$\mathbb{S} = \{(\sum_{J \in \bar{N}^0} a_J.\hat{f}_J).I\}$$

with grading

$$\mathbb{S}_0 = \{(\hat{f}_0. \sum_{J \in N^1} a_J.\hat{f}_J).I\};\ \mathbb{S}_1 = \{(\sum_{J \in N^0} a_J.\hat{f}_J).I\}$$

where $I = f_0.\hat{f}_0.f_1\hat{f}_1...f_n\hat{f}_n$ is an idempotent in the Clifford algebra $\mathcal{C}^C_{2n+2}$.

The action of an element $A \in \mathfrak{g}$ on an element $s \in \mathbb{S}$ is given by the left multiplication $A.s$ in the Clifford algebra $\mathcal{C}^C_{2n+2}$.

Let us describe the action of $\mathfrak{g}_{-1}$ and $\mathfrak{g}_1$ on $\mathbb{S}$ in more details:

For the elements

$$E_j = i\hat{f}_0.(f_j + \hat{f}_j), E_{n+j} = i\hat{f}_0.(f_j - \hat{f}_j)$$

from $\mathfrak{g}_{-1}$ we have

$$E_j.s_0 = E_{n+j}.s_0 = 0.$$

for any $s_0 \in \mathbb{S}_0$. Let

$$s_1 \in \mathbb{S}_1,\ \ s_1 = (\sum_{J \in N^0} a_J.\hat{f}_J).i.$$

It can be written in the form

$$s_1 = (\sum_{J',j\in J'} a_{J'}.\hat{f}_{J'}).I + (\sum_{J'',j\in J''} a_{J''}.\hat{f}_{J''}).I,$$

then

$$E_j.s_1 = i\hat{f}_0.(\sum_{J',j\in J'} 2a_{J'}.\hat{f}_{J'-j}).I + (\sum_{J'',j\in J''} a_{J''}.\hat{f}_{j\,J''}).I$$

$$E_{n+j}.s_0 = i\hat{f}_0.(\sum_{J',j\in J'} 2a_{J'}.\hat{f}_{J'-j}).I - (\sum_{J'',j\in J''} a_{J''}.\hat{f}_{j\,J''}).I.$$

Let us provide a similar computation for $\mathfrak{g}_1$. For the elements

$$E^j = if_0.(f_j + \hat{f}_j), E^{n+j} = if_0.(f_j - \hat{f}_j)$$

from $\mathfrak{g}_1$, we have

$$E^j.s_1 = E^{n+j}.s_1 = 0.$$

for any $s_1 \in \mathbb{S}_1$. Let

$$s_0 \in \mathbb{S}_0, s_0 = (\hat{f}_0.\sum_{J\in N^1} a_J.\hat{f}_J).I.$$

It can be written in the form

$$s_0 = \hat{f}_0.(\sum_{J'j\in J'} a_{J'}.\hat{f}_{J'}).I + \hat{f}_0(\sum_{J'',j\in J''} a_{J''}.\hat{f}_{J''}).I,$$

then

$$E^j.s_0 = i(\sum_{J'j\in J'} 2a_{J'}.\hat{f}_{J'-j}).I + (\sum_{J'',j\in J''} a_{J''}.\hat{f}_{j\,J''}).I$$

$$E^{n+j}.s_0 = i(\sum_{J'j\in J'} 2a_{J'}.\hat{f}_{J'-j}).I - (\sum_{J'',j\in J''} a_{J''}.\hat{f}_{j\,J''}).I.$$

Let M be a conformal spin manifold, S the vector bundle associated to the representation $\mathbb{S}$ of G , then

$$S = S_0 \oplus S_1$$

is the total spinor bundle on M.

Let ∇^i be the spin covariant derivative on (sections of) S_i, obtained from Levi-Civita connection .

The special operator $\widetilde{\nabla}$ on spinor fields can be defined in the following way:

Let $r = \{r_1, ..., r_{2n}\}$ be a (spin) frame in $x \in M$, and

$$s = (s_0, s_1) \in \Gamma(S_0) \oplus \Gamma(S_1)$$

a section of S defined on a neighborhood of x.

Then for $\alpha, 1 \leq \alpha \leq 2n$ we define

$$\widetilde{\nabla}_{r_\alpha}(s_0, s_1) = (\nabla^0_{r_\alpha} s_0 - E_\alpha . s_1; \nabla^1_{r_\alpha} s_1 + \sum_{\beta=1}^{2n} P_{\alpha\beta} E^\beta . s_0),$$

where

$$\Gamma(r)(E_\alpha) = \sum_{\beta=1}^{2n} P_{\alpha\beta} E^\beta.$$

B. Odd case m = 2n-1.

Let

$$\{p_0, p_1, \dots, p_{2n-1}, p_{2n}\}$$

be a basis of $\mathbb{R}^{2n+1}$ with

$$\mathcal{Q}(p_0, p_0) = \mathcal{Q}(p_{2n}, p_{2n}) = 0, \mathcal{Q}(p_0, p_{2n}) = 1,$$

$$\mathcal{Q}(p_j, p_k) = \delta_{jk} \text{for} j, k = 1, ..., 2n-1.$$

We have the embedding $\mathbb{R}^{2n+1} \subset \mathcal{C}^R_{2n+1}$ and the relations in Clifford algebra:

$$p_0 . p_{2n} + p_{2n} . p_0 = -2; p_0^2 = p_{2n}^2 = 0, p_j . p_k + p_k . p_j = \delta_{ij},$$

there is a natural embedding

$$\mathcal{C}^R_{2n+1} \subset \mathcal{C}^R_{2n+2}$$

given by

$$p_j \equiv e_0 . e_{2n} \text{for } 0 \leq j \leq 2n-1, p_{2n} \equiv e_{2n+1} . e_{2n}$$

onto even part of $\mathcal{C}^R_{2n+2}$. As in the even case we have the natural embeddings

$$\mathbb{R}^{2n+1} \subset \mathbb{C}^{2n+1} \subset \mathcal{C}^C_{2n+1}.$$

Moreover we can also suppose

$$\mathcal{C}^C_{2n+1} \subset \mathcal{C}^C_{2n+2}.$$

Similarly as in the even dimensional case we can construct the map

$$so(2n+1, C) \longrightarrow \mathcal{C}^C_{2n+1}$$

and finally we get the embedding

$$so(2n, 1) \longrightarrow \mathcal{C}^C_{2n+2}$$

given on $\mathfrak{g}_0$ by:

$$P = (p_{2n}.p_0 + 1) \mapsto E = (\hat{f}_0.f_0 + 1); P_{j,k} = p_j.p_k \mapsto -E_{jk} = -(f_j + \hat{f}_j)(f_k + \hat{f}_k)$$

$$P_{j,n+k} = p_j.p_{n+k} \mapsto -E_{j,n+k} = -(f_j + \hat{f}_j)(f_k - \hat{f}_k)$$

$$P_{j+n,k+n} = p_j.p_k \mapsto -E_{n+j,n+k} = -(f_j - \hat{f}_j)(f_k - \hat{f}_k)$$

and on $\mathfrak{g}_{-1}$ by

$$P_j = p_{2n}.p_j \mapsto E_j = i\hat{f}_0(f_j + \hat{f}_j); P_{n+k} = p_{2n}.p_{n+k} \mapsto E_{n+k} = i\hat{f}_0(f_k - \hat{f}_k);$$

and finally on $\mathfrak{g}_1$ by

$$P^j = p_{2n}.p_j \mapsto E^j = if_0(f_j + \hat{f}_j); P^{n+k} = p_{2n}.p_{n+k} \mapsto E^{n+k} = if_0(f_k - \hat{f}_k);$$

with $1 \leq j, k \leq n, n+j, n+k \leq 2n-1$.

The spinor space is the same as in even-dimensional case (2n+2) and the action of $\mathfrak{g}$ is given by the restriction of the corresponding action for even-dimensional case.

Using the notation and results coming out from the even case we get a description of the special operator $\widetilde{\nabla}$ for the odd case as follows.

Let M be a conformal spin manifold of dimension 2n-1 , S the bundle associated to the representation $\mathbb{S}$ of G , then

$$S = S_0 \oplus S_1$$

is the whole spin bundle on M. Let ∇^i be spin covariant derivative on S_i constructed from the Levi-Civita connection on M.

The special operator $\widetilde{\nabla}$ on spinor fields is defined in the following way: Let $r = \{r_1, ..., r_{2n-1}\}$ be a (spin) frame in $x \in M$, and

$$s = (s_0, s_1) \in \Gamma(S_0) \oplus \Gamma(S_1)$$

a section of S defined on a neighborhood of x.

Then for α, $1 \leq \alpha \leq 2n-1$, we have

$$\widetilde{\nabla}_{r_\alpha}(s_0, s_1) = (\nabla^0_{r_\alpha} s_0 - E_\alpha.s_1; \nabla^1_{r_\alpha} s_1 + \sum_{\beta=1}^{2n-1} P_{\alpha\beta} E^\beta.s_0).$$

7.5 Example 2 : The fundamental vector representation.

Let us denote
$$\mathbb{C}^{n+2} = \mathbb{V} = \mathbb{V}_{-1} \oplus \mathbb{V}_0 \oplus \mathbb{V}_1$$
with
$$\mathbb{V}_{-1} = \mathbb{C}; \mathbb{V}_0 = \mathbb{C}^n; \mathbb{V}_1 = \mathbb{C}.$$
Then $\mathbb{V}$ is a representation space of $\mathfrak{g}$, through left matrix multiplication on column vector from $\mathbb{C}^{n+2}$.

Let $Z_\alpha = (0,..,0,1,0,..,0), \alpha = 0,1,...,n,n+1$, with 1 on α^{th}-place be the standard basis of $\mathbb{V}$.

Then the action of $\mathfrak{g}_{-1}$ and $\mathfrak{g}_1$ on $\mathbb{V}$ is given by:

$$E_k.Z_{n+1} = 0, E_k.Z_i = -Z_{n+1}, E_k.Z_0 = Z_k$$

$$E^k.Z_0 = 0, E^k.Z_i = -Z_0, E^k.Z_{n+1} = Z_k.$$

Let V be the associated bundle to $\mathbb{V}$, and

$$V = V_0 \oplus V_1 \oplus V_2$$

be the grading of V, dim V_0 = dimV_2 = 1, dim V_1 = n, and ∇^i the covariant derivative associated with Levi-Civita connection ω on V_i. Then for section

$$w = (w_0; \{w_i\}; w_{n+1}) \in \Gamma(V_0) \oplus \Gamma(V_1) \oplus \Gamma(V_2)$$

we get

$$\widetilde{\nabla}_{e_i} w = (\nabla^0_{e_i} w_0 - \delta_{ij} w_j; \{\nabla^1_{e_i} w_j + P_{ij} w_0 - \delta_{ij} w_{n+1}\}; \nabla^2_{e_i} w_{n+1} + P_{ij} w_j)$$

which is just the Penrose local twistor transport ([Ba1]).

7.6 Example: The fundamental representations $\Lambda^k\mathbb{V}$.

The space $\Lambda^k\mathbb{V}$ as a representation space of $\mathfrak{g}_0$ with respect to the action induced from the action on $\mathbb{V}$ from Ex.2. has the following grading:

$$\Lambda^k\mathbb{V} = \Lambda^k_0\mathbb{V} \oplus \Lambda^k_1\mathbb{V} \oplus \Lambda^k_2\mathbb{V}$$

with
$$\Lambda^k_0\mathbb{V} = E_0 \wedge \Lambda^{k-1}\mathbb{V}_0; \Lambda^k_1\mathbb{V} = E_0 \wedge E_{n+1} \wedge \Lambda^{k-2}\mathbb{V}_0 \oplus \Lambda^k\mathbb{V}_0;$$
$$\Lambda^k_2\mathbb{V} = E_0 \wedge \Lambda^{k-1}\mathbb{V}_0.$$
We need to have the action of the elements of basis of $\mathfrak{g}_{-1}$ and $\mathfrak{g}_1$ on these spaces.

Let $Z_\alpha = (0,..,0,1,0,..,0), \alpha = 0,1,...,n,n+1$, with 1 on α^{th}-place be the standard basis of $\mathbb{V}$. Then we have (see Ex.2.above)

$$E_k.Z_{n+1} = 0, E_k.Z_i = -Z_{n+1}, E_k.Z_0 = Z_k$$

$$E^k.Z_0 = 0, E^k.Z_i = -Z_0, E^k.Z_{n+1} = Z_k.$$

And moreover we have:

$$E_k.(Z_0 \wedge Z_{i_1} \wedge Z_{i_2} \wedge \ldots \wedge Z_{i_l}) = Z_k \wedge Z_{i_1} \wedge Z_{i_2} \wedge \ldots \wedge Z_{i_l} +$$

$$+ Z_0 \wedge Z_{n+1} \wedge \sum_{j=1}^{l} (-1)^{j+1} Z_{i_1} \wedge .. \wedge \hat{Z_{i_j}} \wedge \ldots \wedge Z_{i_l}$$

$$E_k.(Z_{i_1} \wedge Z_{i_2} \wedge \ldots \wedge Z_{i_l}) = -Z_{n+1} \wedge \sum_{j=1}^{l} (-1)^{j} Z_{i_1} \wedge .. \wedge \hat{Z_{i_j}} \wedge \ldots \wedge Z_{i_l}$$

$$E_k.(Z_0 \wedge Z_{n+1} \wedge Z_{i_1} \wedge Z_{i_2} \wedge \ldots \wedge Z_{i_l}) = -Z_{n+1} \wedge Z_k \wedge Z_{i_1} \wedge Z_{i_2} \wedge \ldots \wedge Z_{i_l}$$

$$E_k.(Z_{n+1} \wedge Z_{i_1} \wedge Z_{i_2} \wedge \ldots \wedge Z_{i_l}) = 0$$

and

$$E^k.(Z_0 \wedge Z_{i_1} \wedge Z_{i_2} \wedge \ldots \wedge Z_{i_l}) = 0$$

$$E^k.(Z_{i_1} \wedge Z_{i_2} \wedge \ldots \wedge Z_{i_l}) = Z_0 \wedge \sum_{j=1}^{l} (-1)^{j} Z_{i_1} \wedge .. \wedge \hat{Z_{i_j}} \wedge \ldots \wedge Z_{i_l}$$

$$E^k.(Z_0 \wedge Z_{n+1} \wedge Z_{i_1} \wedge Z_{i_2} \wedge \ldots \wedge Z_{i_l}) = Z_0 \wedge Z_k \wedge Z_{i_1} \wedge Z_{i_2} \wedge \ldots \wedge Z_{i_l}$$

$$E^k.(Z_{n+1} \wedge Z_{i_1} \wedge Z_{i_2} \wedge \ldots \wedge Z_{i_l}) = Z_k \wedge Z_{i_1} \wedge Z_{i_2} \wedge \ldots \wedge Z_{i_l} +$$

$$-Z_0 \wedge Z_{n+1} \wedge \sum_{j=1}^{l} (-1)^{j} Z_{i_1} \wedge .. \wedge \hat{Z_{i_j}} \wedge \ldots \wedge Z_{i_l}$$

Let M be a conformal manifold, $\Lambda^k V$ the bundle associated to the representation $\Lambda^k \mathbb{V}$ of G , then we have

$$\Lambda^k V = \Lambda^k_0 V \oplus \Lambda^k_1 V \oplus \Lambda^k_2 V$$

its grading, denote ∇^i covariant derivative on (sections of) $\Lambda^k_i V$ induced by Levi-Civita connection ω .

The special operator $\widetilde{\nabla}$ on sections of $\Lambda^k V$ is defined in the following way: Let $r = \{r_1, ..., r_{2n}\}$ be a frame in $x \in M$, and

$$s = (s_0, s_1, s_2) \in \Gamma(\Lambda^k_0 V) \oplus \Gamma(\Lambda^k_1 V) \oplus \Gamma(\Lambda^k_2 V)$$

a section of S defined on a neighborhood of x. Then for $j, 1 \leq j \leq n$ we have

$$\widetilde{\nabla}_{r_j}(s_0, s_1, s_2) = (\nabla^0_{r_j} s_0 - E_j.s_1; \nabla^1_{r_j} s_1 - E_j.s_2 + \sum_{k=1}^{2n} P_{jk} E^k.s_0, \nabla^2_{r_j} s_2 + \sum_{k=1}^{2n} P_{jk} E^k.s_1).$$

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