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Residues of Eisenstein Series and Generalized Shalika Models for SO_{4n}

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1 Introduction

Let π be an irreducible unitary cuspidal automorphic representation of $\mathrm{GL}_{2n}(\mathbb{A})$, where $\mathbb{A}$ is the ring of adeles associated to a number field k. The exterior square L-function $L(s,\pi,\Lambda^2)$ has been studied extensively by the Langlands-Shahidi method ([Sh88]) and by the Rankin-Selberg method (the work of Jacquet-Shalika in [JS90]). The known properties involving π and $L(s,\pi,\Lambda^2)$ are summarized in Theorem 2.2 in [Jng06]. The main result of [JS90] (see also Part (5) of Theorem 2.2 in [Jng06]) says that the existence of the pole at $s=1$ of the (partial) exterior square L-function $L(s,\pi,\Lambda^2)$ is characterized by the nonvanishing of the Shalika period attached to π.

In this paper, we show by a different method that the nonvanishing of the Shalika period attached to π implies the existence of the pole at $s=1$ of the exterior square L-function $L(s,\pi,\Lambda^2)$. This method was initiated by Jacquet and Rallis in [JR92] and developed in [Jng98]. The more detailed account of this method as one useful approach to determine the existence of poles of Eisenstein series or related automorphic L-functions can be found in [GJR04a], [Jng-ecun], and [Jng-aim]. It is an interesting and important problem to find suitable period conditions which yield the existence poles of Eisenstein series or related automorphic L-functions in general. The relations between periods of automorphic forms and the Langlands functoriality conjectures, and applications to number theory can be found in [J97], [Hd87], [HLR86], [Hr94], [GP92], [GP94], [JMR99], [JLR99], [GJR01], [GJR04b], [GJR05], [Od01], [JLZ06], for instance.

Following the idea of this approach, we find a generalized Shalika period for automorphic forms on $\mathrm{SO}_{4n}(\mathbb{A})$, which SO_{4n} is the k-split even special orthogonal group. The definition is given in §4 for the global generalized Shalika periods for automorphic forms on $\mathrm{SO}_{4n}(\mathbb{A})$, and in §2 for the local generalized Shalika functionals for irreducible admissible representations of p-adic group $\mathrm{SO}_{4n}(k_v)$, where k_v is the localization of the number field k at a finite local place v of k.

Some basic results have been established in §2 and §3 for the local generalized Shalika functionals, which classify the unitarily induced representation $I(s,\tau)$ of $\mathrm{SO}_{4n}(k_v)$ from the supercuspidal datum (M,τ) attached to the standard Siegel parabolic subgroup

$$P = MN = \mathrm{GL}_{2n}N$$

of SO_{4n}. More precisely, it is proved in Theorem 1, §3, that $I(s,\tau)$ admits a non-zero local generalized Shalika functional if and only if the irreducible supercuspidal representation τ of $\mathrm{GL}_{2n}(k_v)$ (GL_{2n} is the Levi part of P) admits a non-zero Shalika functional and s must be 1. Moreover, in this case, the non-zero generalized Shalika functional on $I(1,\tau)$ must factor through the unique Langlands quotient of $I(1,\tau)$.

The global version of Theorem 1 has been established in §4 for the generalized Shalika periods, which is Theorem 3. It shows that the cuspidal datum (M,π) has a non-zero Shalika period if and only if the residue at $s=1$ of the Eisenstein series on $\mathrm{SO}_{4n}(\mathbb{A})$ associated to (M,π) has a nonzero generalized Shalika period. As consequence, we obtain two applications. The first is to determine the existence of the pole at $s=1$ of the Eisenstein series in terms of the nonvanishing of the Shalika period for the cuspidal datum, from which the Eisenstein series is built. By Theorem 4.11 of [K05], the relevant local intertwining operator can be normalized in terms of the Shahidi local factor. Hence we obtain as the second application that the nonvanishing of the Shalika period for π implies the existence of the pole at $s = 1$ of the exterior square L-function $L(s, \pi, \Lambda^2)$.

It is worthwhile to mention that the Fourier coeeficients attached to the generalized Shalika model considered in this paper for SO_{4n} is analogue of the Fourier coefficients considered for symplectic groups Sp_{2n} by Piatetski-Shapiro and Rallis in [PSR88] in order to obtain a new way to study automorphic L-functions and by J.-S. Li [Li89] to study the distinguished cuspidal automorphic representations of $\mathrm{Sp}_{2n}(\mathbb{A})$, and that for the quasi-split unitary groups in the more recent work [Q]. We consider the generalized Bessel models for a certain family of residual representations of $\mathrm{SO}_{4n}(\mathbb{A})$. It is interesting to consider the generalized Shalika models for cuspidal auotmorphic representations of $\mathrm{SO}_{4n}(\mathbb{A})$ as well.

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2 Generalized Shalika Models

Let k be a nonarchimedean local field of characteristic zero, ψ be a nontrivial character of k. Denote by $G = \mathrm{SO}_{4n}$ the k-split even special orthogonal group of rank $2n$, with respect to the non-degenerate symmetric form given by

$$\begin{pmatrix} & 1_{2n} \\ 1_{2n} & \end{pmatrix}.$$

We consider the Siegel parabolic subgroup P of G given by

$$P = \mathrm{GL}_{2n} \cdot V. \tag{2.1}$$

Let $\mathcal{A}_{2n}$ be the set of skew-symmetric matrices of degree $2n$. We may write elements of V as

$$v = \begin{pmatrix} 1_{2n} & z \\ & 1_{2n} \end{pmatrix}, \text{ with } z \in \mathcal{A}_{2n}.$$

Let $b \in \mathcal{A}_{2n}$ be a nonsingular skew-symmetric matrix, define a character ψ^b of V by

$$\psi^b(v) = \psi(\tfrac{1}{2}\mathrm{tr}(bz)). \tag{2.2}$$

The stabilizer of the character ψ^b in GL_{2n} is Sp_{2n}^b, the symplectic group with respect to b,

$$\mathrm{Sp}_{2n}^b = \{g \in \mathrm{GL}_{2n} | \ {}^t gbg = b\}. \tag{2.3}$$

Form a group

$$H^b := \mathrm{Sp}_{2n}^b \cdot V. \tag{2.4}$$

It is clear ψ^b can be extended to be a character of H^b by

$$\psi^b(g,v) := \psi^b(v), \quad (g,v) \in H^b. \tag{2.5}$$

Let

$$\varepsilon_l = \begin{pmatrix} & & 1 \\ & \cdot^{\cdot^{\cdot}} & \\ 1 & & \end{pmatrix}_{l\times l}, \qquad J_{2l} = \begin{pmatrix} & \varepsilon_l \\ -\varepsilon_l & \end{pmatrix}. \tag{2.6}$$

Then $J_{2n} \in \mathcal{A}_{2n}$ is nonsingular skew-symmetric matrix. We will drop the superscript "b" of ψ^b, H^b and Sp_{2n}^b if $b = J_{2n}$.

One of the main problems in this section is to decompose the flag variety $P\backslash \mathrm{SO}_{4n}$ into H orbits. Firstly we have the following Bruhat decomposition.

Proposition 2.1. *The group* SO_{4n} *decompose into* $2n+1$ *cosets with respect to* (P,P)*, and their representative are given as follows:*

$$\omega_j = \begin{pmatrix} 1_{2n-j} & & & \\ & & & 1_j \\ & & 1_{2n-j} & \\ & 1_j & & \end{pmatrix}, \quad j = 0, \dots, 2n.$$

It reduces to calculate decomposition

$$P\backslash P\,\omega_j\,P/H$$

for $j = 0, \dots, 2n$. In the following, we will denote by $Q_{n_1,\dots,n_k}$ the standard parabolic subgroup of GL_{2n} with Levi part

$$\mathrm{GL}_{n_i} \times \mathrm{GL}_{n_2} \times \cdots \times \mathrm{GL}_{n_k}.$$

Let $V_{n_1,\dots,n_k}$ be the nilpotent radical of $Q_{n_1,\dots,n_k}$.

Consider the double coset decomposition

$$P\backslash P\,\omega_j\,P/H = \omega_j^{-1} P \omega_j \cap P\backslash\ P\ /H \tag{2.7}$$

One can check that

$$\omega_j^{-1} P \omega_j \cap P = Q_{2n-j,\,j} \cdot V_j \tag{2.8}$$

where V_j is the subgroup of V consisting of elements of the following type:

$$\begin{pmatrix} 1 & z \\ & 1 \end{pmatrix}, \qquad \text{with } z = \begin{pmatrix} x & y \\ -{}^t y & 0_j \end{pmatrix}.$$

Since P normalize V, we obtain

$$P \backslash P \omega_j P / H \cong Q_{2n-j,\,j} \backslash \mathrm{GL}_{2n} / \mathrm{Sp}_{2n}. \tag{2.9}$$

If $0 \le j \le n$, the set representatives of $Q_{2n-j,\,j} \backslash \mathrm{GL}_{2n}/\mathrm{Sp}_{2n}$ can be chosen as follows:

$$\gamma_i = \begin{pmatrix} 1_i & & & & \\ & 0 & 1_{2(n-j)+r/2} & 0 & \\ & 0 & 0 & 1_{r/2} & \\ & 1_{r/2} & 0 & 0 & \\ & & & & 1_{i+r/2} \end{pmatrix} \tag{2.10}$$

with $0 \le i \le j$ such that $r = j - i$ is even. We notice here γ_i is chosen such that

$$\eta_i := \gamma_i \, J_{2n} \, {}^t\gamma_i = \begin{pmatrix} & & & \varepsilon_i \\ & J_{2(n-j)+r} & 0 & \\ & 0 & J_r & \\ -\varepsilon_i & & & \end{pmatrix} \tag{2.11}$$

Similarly, if $n \le j \le 2n$, the set of representatives of $Q_{2n-j,\,j} \backslash \mathrm{GL}_{2n}/\mathrm{Sp}_{2n}$ can be chosen as

$$\gamma_i' = \begin{pmatrix} 1_{i+r/2} & & & & \\ & 0 & 0 & 1_{r/2} & \\ & 1_{j-i} & 0 & 0 & \\ & 0 & 1_{r/2} & 0 & \\ & & & & 1_i \end{pmatrix} \tag{2.12}$$

for $0 \le i \le 2n - j$ such that $r = j - i$ is even. γ_i' is chosen so that

$$\eta_i := \gamma_i' \, J_{2n} \, {}^t\gamma_i' = \begin{pmatrix} & & & \varepsilon_i \\ & J_{2(n-j)+r} & 0 & \\ & 0 & J_r & \\ -\varepsilon_i & & & \end{pmatrix} \tag{2.13}$$

In conclusion, our computation implies the following result

Proposition 2.2. *Notation as above.*

$$P\backslash\mathrm{SO}_{4n}/H=\bigcup_{j=0}^{n}\bigcup_{\substack{i=0\\ i\equiv j(\mathrm{mod}\,2)}}^{j}P\omega_j\gamma_iH\cup P\omega_{2n-j}\gamma_i'H. \tag{2.14}$$

The character ψ^{η_i} will be used in the computation of Fourier coefficients of Eisenstein series. Before we study properties of double cosets of $P\backslash\mathrm{SO}_{4n}/H$, we give a formula of it. Let $z\in\mathcal{A}_{2n}$. Write z as

$$z=\begin{pmatrix} S & T\\ -{}^tT & V\end{pmatrix} \tag{2.15}$$

with $S\in\mathrm{M}_{(2n-j)\times(2n-j)}$ and

$$S=\begin{pmatrix} s_1 & s_2\\ s_3 & s_4\end{pmatrix},\ T=\begin{pmatrix} t_1 & t_2\\ t_3 & t_4\end{pmatrix},\ V=\begin{pmatrix} v_1 & v_2\\ v_3 & v_4\end{pmatrix}. \tag{2.16}$$

It is clear that

$$z\eta_i=\begin{pmatrix} -t_2\varepsilon_i & * & * & *\\ * & s_4J_{2(n-j)+r} & * & *\\ * & * & v_1J_r & *\\ * & * & * & -t_2\varepsilon_i\end{pmatrix}. \tag{2.17}$$

Hence

$$\psi^{\eta_i}(z)=\psi(\tfrac{1}{2}\mathrm{tr}(z\eta_i))=\psi(-\mathrm{tr}(t_2\varepsilon_i)+\tfrac{1}{2}s_4J_{2(n-j)+r}+\tfrac{1}{2}v_1J_r). \tag{2.18}$$

Let $0\le j\le 2n$, $0\le i\le\min(j,2n-j)$ so that $r=j-i$ is even. It can be checked directly that $Q_{2n-j,\,j}\cap\mathrm{Sp}_{2n}^{\eta_i}$ has a Levi component consists of elements of the following form

$$\begin{pmatrix} A & & & \\ & D & & \\ & & E & \\ & & & F\end{pmatrix} \tag{2.19}$$

with $A\in\mathrm{GL}_{2n}$, $D\in\mathrm{Sp}_{2(n-j)+r}$, $E\in\mathrm{Sp}_r$, and $F=\varepsilon_i\ {}^tA^{-1}\varepsilon_i$. The unipotent radical of $Q\cap\mathrm{Sp}_{2n}^{\eta_i}$ is a subgroup of $V_{i,2(n-j)+r,r,i}$ consists of elements of the following type

$$\begin{pmatrix} 1 & X & S & T \\ & 1 & 0 & V \\ & & 1 & Y \\ & & & 1 \end{pmatrix} \tag{2.20}$$

satisfying $X = \varepsilon_i\, {}^tV J_{2(n-j)+r}$, $S = \varepsilon_i\, {}^tY J_r$ and

$${}^tT\varepsilon_i + {}^tV J_{2(n-j)+r} V + {}^tY J_r Y - \varepsilon_i T = 0. \tag{2.21}$$

Note that

$$P \cap \omega_j \gamma_i H \gamma_i^{-1} \omega_j^{-1} = \omega_j (P \cap H^{\eta_i}) \omega_j^{-1}. \tag{2.22}$$

Hence $P \cap \omega_j \gamma_i H \gamma_i^{-1} \omega_j^{-1}$ consists of elements of the following form

$$\begin{pmatrix} a & at & & \\ & q & & \\ & & {}^ta^{-1} & \\ & & -{}^tq^{-1}\,{}^tt & {}^tq^{-1} \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 & v & b \\ & 1 & -{}^tb & 0 \\ & & 1 & 0 \\ & & & 1 \end{pmatrix} \tag{2.23}$$

such that

$$\begin{pmatrix} a & ab \\ & {}^tq^{-1} \end{pmatrix} \in Q_{2n-j,\, j} \cap \mathrm{Sp}_{2n}^{\eta_i}. \tag{2.24}$$

since the effect of $\mathrm{Ad}(\omega_j^{-1})$ on $P \cap \omega_j \gamma_i H \gamma_i^{-1} \omega_j^{-1}$ is

$$\begin{aligned} &\begin{pmatrix} a & at & & \\ & q & & \\ & & {}^ta^{-1} & \\ & & -{}^tq^{-1}\,{}^tt & {}^tq^{-1} \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 & v & b \\ & 1 & -{}^tb & 0 \\ & & 1 & 0 \\ & & & 1 \end{pmatrix} \\ \xrightarrow{\mathrm{Ad}(\omega_j^{-1})} &\begin{pmatrix} a & ab & & \\ & {}^tq^{-1} & & \\ & & {}^ta^{-1} & \\ & & -q\,{}^tb & q \end{pmatrix} \begin{pmatrix} 1 & 0 & v - t\,{}^tb + b\,{}^tt & t \\ & 1 & -{}^tt & 0 \\ & & 1 & 0 \\ & & & 1 \end{pmatrix}. \end{aligned} \tag{2.25}$$

Let L^{η_i} be the subgroup of GL_{2n} consists of elements of the following type

$$(a, q; t) = \begin{pmatrix} a & at \\ & q \end{pmatrix}, \qquad \text{with } t \in \mathrm{M}_{(2n-j)\times j} \tag{2.26}$$

and

$$a = \begin{pmatrix} A & AX \\ & D \end{pmatrix}, \quad q = \begin{pmatrix} E & \\ \varepsilon_i A \varepsilon_i Y & \varepsilon_i A \varepsilon_i \end{pmatrix} \tag{2.27}$$

with $A \in \mathrm{GL}_i$, $D \in \mathrm{Sp}_{2(n-j)+r}$, $E \in \mathrm{Sp}_r$, $X \in \mathrm{M}_{i,\,2(n-j)+r}$, $Y \in \mathrm{M}_{i,\,r}$. It is clear from (2.23) and (2.24) that

$$L^{\eta_i} = Q_{2n-j,\,j} \cap \omega_j \gamma_i H \gamma_i^{-1} \omega_j. \tag{2.28}$$

Proposition 2.3. *Let $0 \le j \le 2n$. For $0 \le i \le \min(j, 2n-j)$ such that $r = j - i$ is even, the dimension of the algebraic variety $P\omega_j\gamma_i H$ is*

$$d(j) = n^2 - (j-n)^2. \tag{2.29}$$

In particular, open double cosets are $P\omega_n\gamma_i H$ with $0 \le i \le n$ so that $n - i$ is even.

Proof. The mapping $x \mapsto x\gamma_i^{-1}$ induced an homeomorphism

$$P\backslash P\omega_j\gamma_i H \to P\backslash P\omega_j H^{\eta_i}. \tag{2.30}$$

Since $H^{\eta_i} = \gamma_i H \gamma_i^{-1}$, $\mathrm{Ad}\gamma_i$ induce an homeomorphism

$$P\backslash P\omega_j H \to P\backslash P\omega_j H^{\eta_i}. \tag{2.31}$$

Hence the dimension of the algebraic variety $P\backslash P\omega_j\gamma_i H$ is the same as that of $P\backslash P\omega_j H$.

Assume $i = j$. Since

$$\dim P\backslash P\omega_j\gamma_j H = \dim H - \dim P \cap \omega_j H^{\eta_j} \omega_j^{-1}. \tag{2.32}$$

By the structure of $Q_{2n-j} \cap \omega_j \mathrm{Sp}_{2n}^{\eta_j} \omega_j^{-1}$ in (2.23), (2.24), we see that

$$\dim P \cap \omega_j H^{\eta_j} \omega_j^{-1} = (j-n)^2 + 3n^2. \tag{2.33}$$

Since $\dim H = 4n^2$, the result follows. □

2.4 Admissible double cosets

Let $0 \le j \le 2n$ be an even number. If $i = 0$, then L^{η_0} (cf. (2.26), (2.27)) is the subgroup of $Q_{2n-j,j}$ consists of elements of the following type:

$$\begin{pmatrix} A & X \\ & D \end{pmatrix}, \quad \text{with } A \in \mathrm{Sp}_{2n-j}, D \in \mathrm{Sp}_j, X \in M_{2n-j,j}. \tag{2.34}$$

We will write L for L^{η_0} if no confusion is caused. Let θ be the trivial character of L.

Let γ be a representative of a double coset in $U\backslash \mathrm{GL}_{2n}/L$, where U is the standard maximal unipotent subgroup of GL_{2n}. We say that γ is admissible if for all $u \in U \cap \gamma L \gamma^{-1}$ we have

$$\psi_U(u) = 1 \tag{2.35}$$

where ψ_U is a generic character of U.

Proposition 2.5. *Let $0 \le j \le 2n$ be an even number. Then every double coset in $U\backslash \mathrm{GL}_{2n}/L$ is not admissible.*

Proof. We prove the proposition for $0 \le j \le n$, the proof for $j > n$ is similar and we omit it.

Let ψ_U, ψ'_U be two generic characters of U. Then there is a diagonal matrix h in GL_{2n} such that

$$\psi_U(u) = \psi'_U(huh^{-1}), \quad u \in U.$$

Since h normalizes U, $U\gamma L$ is an admissible double coset for ψ'_U if and only if $Uh\gamma L$ is admissible double coset for ψ_U. Without loss of generality, we assume that the generic character ψ_U is given by

$$\psi_U(u) = \psi(\sum_{i=1}^{2n-1} u_{i,i+1}) \tag{2.36}$$

where ψ is a nontrivial character of k.

Let W be the Weyl group of GL_{2n}, define

$$W_j = W \cap Q_{2n-j,j}, \tag{2.37}$$

If we identify W with the symmetric group S_{2n}, then $W_j = S_{2n-j} \times S_j$. It is well known that

$$B\backslash \mathrm{GL}_{2n}/Q_{2n-j,j} = W/W_j. \tag{2.38}$$

and

$$Q_{2n-j,\,j}\backslash \mathrm{GL}_{2n}/Q_{2n-j,\,j} = W_j\backslash W/W_j. \tag{2.39}$$

Take a set of representatives of $Q_{2n-j,\,j}\backslash \mathrm{GL}_{2n}/Q_{2n-j,\,j}$ as follows:

$$w_m = \begin{pmatrix} 1_{2n-j-m} & 0 & 0 & 0 \\ 0 & 0 & 1_m & 0 \\ 0 & 1_m & 0 & 0 \\ 0 & 0 & 0 & 1_{j-m} \end{pmatrix}, \text{ for } 0 \le m \le j. \tag{2.40}$$

Then representatives of $B\backslash \mathrm{GL}_{2n}/Q_{2n-j,\,j}$ can be chosen to be of the following form

$$ww_m, \quad \text{ for some } w \in W_j. \tag{2.41}$$

Let γ be a representative of $U\backslash \mathrm{GL}_{2n}/L$. Then γ can be chosen as

$$hww_my \tag{2.42}$$

where h is a diagonal element, $y \in \mathrm{GL}_{2n-j} \times \mathrm{GL}_j$ be a representative of $Q_{2n-j,j}/L$. We need to show that ψ_U is nontrivial on $U \cap \gamma L \gamma^{-1}$.

If $m = 0$, then we take $\gamma = hy$ with a diagonal matrix h and $y \in \mathrm{GL}_{2n-j} \times \mathrm{GL}_j/\mathrm{Sp}_{2n-j} \times \mathrm{Sp}_j$. For $z \in k$, let

$$u(z) = (u_{p,q}) \in U \tag{2.43}$$

be such that $u_{p,p} = 1$ for $1 \le p \le 2n$ and if $p \neq q$, $u_{p,q} = 0$ unless $p = 2n - j, q = 2n - j + 1$, and $u_{2n-j,\,2n-j+1} = z$. Assume that $h = \mathrm{diag}(h_1, \ldots, h_{2n})$, then

$$h^{-1}u(z)h = u(\frac{h_{2n-j+1}}{h_{2n-j}}z) \in L, \tag{2.44}$$

and $y^{-1}h^{-1}u(z)hy \in L$. Hence $u(z) \in U \cap \gamma^{-1}L\gamma$. Since we can take z sufficiently large such that $\psi(z) \neq 1$, ψ_U is nontrivial on $U \cap \gamma L \gamma^{-1}$.

Let $m > 0$, then $\mathrm{Ad}(w_m^{-1})$ acts on $Q_{2n-j,\,j}$ by

$$\begin{pmatrix} A & B & S & T \\ C & D & U & V \\ & & E & F \\ & & G & H \end{pmatrix} \xrightarrow{\mathrm{Ad}(w_m^{-1})} \begin{pmatrix} A & S & B & T \\ 0 & D & 0 & F \\ C & U & C & V \\ 0 & G & 0 & H \end{pmatrix}. \tag{2.45}$$

Here matrices in $Q_{2n-j,j}$ is written in blocks of partition

$$(2n-j-m, m, m, j-m). \tag{2.46}$$

If $m \neq 2n-j$, from (2.45), we see that the B part and V part of $Q_{2n-j,\,j}$ is nonzero. Let $w = \alpha\beta$ for some $\alpha \in S_{2n-j}$, $\beta \in S_j$. If α satisfies

$$\begin{array}{rcl} \alpha(1), \dots, \alpha(m) & \geq & 2n-j-m, \\ \alpha(m+1), \dots, \alpha(2n-j) & \leq & 2n-j-m, \end{array} \tag{2.47}$$

and β satisfies similar condition:

$$\begin{array}{rcl} \beta(1), \dots, \beta(m) & \geq & j-m, \\ \beta(m+1), \dots, \beta(j) & \leq & j-m, \end{array} \tag{2.48}$$

then $\mathrm{Ad}(w)^{-1}$ acts on $Q_{2n-j,\,j}$ by

$$\begin{pmatrix} A' & B' & S' & T' \\ C' & D' & U' & V' \\ & & E' & F' \\ & & G' & H' \end{pmatrix} \xrightarrow{\mathrm{Ad}w^{-1}} \begin{pmatrix} D^* & C^* & V^* & U^* \\ B^* & A^* & T^* & S^* \\ & & H^* & G^* \\ & & F^* & E^* \end{pmatrix}. \tag{2.49}$$

Here the matrix on the left side is written in blocks of partition

$$(m, 2n-j-m, j-m, m), \tag{2.50}$$

while the matrix on the right is in blocks of partition

$$(2n-j-m, m, m, j-m). \tag{2.51}$$

The U' part of the matrix on the left is moved to U^* part of the matrix on the right by performing column and row permutations to ${}^tU'$. Note that ψ is nontrivial on the U' part on the left matrix of (2.49), which is move to U^* part of matrix on the right, which is invariant under $\mathrm{Ad}(w_m)^{-1}$ as indicated in (2.45). Hence ψ is nontrivial on $U \cap \gamma L \gamma^{-1}$.

If one of (2.47) and (2.48) is not satisfied, say, α does not satisfy (2.47), we claim that there exists $u \in U$ such that $\psi(u) \neq 1$ and

$$w^{-1}uw = \begin{pmatrix} 1 & b & 0 & 0 \\ & 1 & 0 & 0 \\ & & 1 & 0 \\ & & & 1 \end{pmatrix} \tag{2.52}$$

for some $b \in M_{2n-j-m,\,m}$. Here the matrix is written in blocks of partition $(2n-j-m, m, m, j-m)$. Obviously, if this claim is true, then ψ_U is nontrivial on $U \cap \gamma L \gamma^{-1}$.

In fact, if such u doesn't exist, then for $1 \leq \ell \leq 2n-j-1$, either

$$\alpha(\ell) \leq 2n-j-m,\ \alpha(\ell+1) \leq 2n-j-m, \tag{2.53}$$

or

$$\alpha(\ell) > 2n-j-m. \tag{2.54}$$

Then there is ℓ_0 such that

$$\begin{aligned} \alpha(1), \dots, \alpha(\ell_0) &> 2n-j-m \\ \alpha(\ell_0+1), \dots, \alpha(2n-j) &\leq 2n-j-m. \end{aligned} \tag{2.55}$$

Hence $\ell_0 = m$ and α satisfies (2.47). This contradicts to our assumption.

Now let $m = 2n-j$, then $j = m = n$. Let

$$y = \begin{pmatrix} a_1 & \\ & a_2 \end{pmatrix}, \qquad \text{with } a_1, a_2 \in \mathrm{GL}_n. \tag{2.56}$$

Denote temporarily the symplectic group for $a_i J_i {}^t a_i$ by $\mathrm{Sp}_j(a_i)$, then

$$\mathrm{Sp}_j(a_i) = a_i \mathrm{Sp}_j a_i^{-1}, \text{ for } i = 1, 2. \tag{2.57}$$

It is clear that

$$yLy^{-1} = \{ \begin{pmatrix} A & U \\ & D \end{pmatrix} \mid A \in \mathrm{Sp}_j(a_1), D \in \mathrm{Sp}_j(a_2), U \in \mathrm{M}_{n\times n} \}, \tag{2.58}$$

and $Utww_n yL$ is admissible if and only if $Utww_n(yLy^{-1})$ is admissible.

Recall the action of $\mathrm{Ad}(w_n)^{-1}$ on $Q_{n,n}$ is

$$\begin{pmatrix} A & X \\ & D \end{pmatrix} \xrightarrow{\mathrm{Ad}(w_n)^{-1}} \begin{pmatrix} D & \\ X & A \end{pmatrix}, \qquad \text{with } A, D \in \mathrm{GL}_n. \tag{2.59}$$

Decompose $w = w_1 w_2$, with $w_1, w_2 \in S_n$. Let u be an element of U, then

$$u = \begin{pmatrix} u_1 & u_2 \\ & u_4 \end{pmatrix} \tag{2.60}$$

with u_1, u_4 in the maximal unipotent subgroups U_0 of GL_n. Then

$$[(ww_n)^{-1}]u[ww_n] = \begin{pmatrix} w_2^{-1}u_4w_2 & \\ * & w_1^{-1}u_1w_1 \end{pmatrix} \tag{2.61}$$

It is well known that there is no admissible double coset of $U_0\backslash\mathrm{GL}_n/Sp_n$, equivalently, for every $a \in \mathrm{GL}_n$ there exists $u' \in U_0$ such that

$$\psi_{U_0}(u) \neq 1, \quad \text{ and } a^{-1}u'a \in \mathrm{Sp}_n. \tag{2.62}$$

Here ψ_{U_0} is the restriction of ψ_U to U_0, which is a generic character of U_0. Now let $a = w_1a_2^{-1}$, $u_1 \in U_0$ such that $\psi_{U_0}(u) \neq 1$ and

$$a_2w_1^{-1}u_1w_1a_2^{-1} \in \mathrm{Sp}_n, \tag{2.63}$$

which is equivalent to $w_1^{-1}u_1w_1 \in \mathrm{Sp}(a_2)$. Choose above u_1 sufficiently large, we see that $Utww_n(yLy^{-1})$ is not admissible. □

Corollary 2.6. *Let τ be an irreducible admissible generic representation of GL_{2n}. Then we have*

$$\dim_{\mathbb{C}} \mathrm{Hom}_L(\tau, \theta) = 0. \tag{2.64}$$

In particular, any irreducible generic cuspidal automorphic representation of $\mathrm{GL}_{2n}(\mathbb{A})$ has no nonzero (L, θ)-periods.

Proof. The proof is similar to the proof of Corollary 3.3 in [GJR04a], we will not give details here. □

3 Local Functionals

Let k be a p-adic field, ψ be a nontrivial character of k.

Definition 3.1. Let (σ, V) be an irreducible representation of SO_{4n}. We say that σ has a local generalized Shalika functional if $\mathrm{Hom}_{H^b}(\sigma, \psi^b)$ is nonzero for some nonsingular b in $\mathcal{A}_{2n}$, i.e. there is a nonzero linear functional l_{ψ^b} on V such that

$$l_{\psi^b}(\sigma(h)v) = \psi^b(h)l_{\psi^b}(v), \text{ for } h \in H^b,\, v \in V.$$

(see (2.5) for definition of H^b and ψ^b). It is clear if σ has a general Shalika model for some b, so it has for all nonsingular b in $\mathcal{A}_{2n}$.

Let S be the Shalika group of GL_{2n} consisting of element of the following type

$$\begin{pmatrix} g & gu \\ & g \end{pmatrix}, \qquad \text{with } g \in \mathrm{GL}_n, u \in \mathrm{M}_{n\times n}. \tag{3.1}$$

Let ψ_S be the character of S defined by

$$\psi_S(\begin{pmatrix} g & gu \\ & g \end{pmatrix}) = \psi(\mathrm{tr}(u)). \tag{3.2}$$

Let (τ, E) be an irreducible representation of GL_{2n}. As in [JR96], we say that τ has a Shalika functional if $\mathrm{Hom}_S(\tau, \psi_S)$ is nonzero, i.e. there is a nonzero linear functional l on E such that

$$l(\tau(x)v) = \psi_S(x)l(v), \text{ for all } x \in S, v \in E. \tag{3.3}$$

Recall that $P = \mathrm{GL}_{2n} \cdot V$ is the Siegel parabolic subgroup of SO_{4n}. Let $\alpha : \mathrm{GL}_{2n} \to \mathbb{C}^\times$ be the character of defined by

$$\alpha(g) = |\det g|^{\frac{1}{2}}, \quad g \in \mathrm{GL}_{2n}. \tag{3.4}$$

Let (τ, E) be an irreducible admissible unitary generic representation of GL_{2n}. For $s \in \mathbb{C}$, consider the normalized induced representation

$$I(s, \tau) = \mathrm{Ind}_P^{\mathrm{SO}_{4n}}(\tau \otimes \alpha^s). \tag{3.5}$$

Theorem 3.1. *Notations as above. The induced representation $I(s, \tau)$ admits a local generalized Shalika functional if and only if τ has a Shalika functional and $s = 1$. In this case, the nontrivial Shalika functional factors through the unique Langlands quotient of $I(s, \tau)$ at $s = 1$.*

Proof. Let $\mathcal{V}$ be the space of smooth E-valued functions $f : G \to E$ satisfying

$$f(pg) = \tau(p)\delta^{\frac{1}{2}}(p)f(g), \qquad \text{for } p \in P, g \in \mathrm{SO}_{4n}. \tag{3.6}$$

Here $\delta(p)$ is the modular function of P. Then $I(s, \tau)$ acts on $\mathcal{V}$ by

$$[I(s, \tau)(g)]f(x) = f(xg), \text{ for } x, g \in \mathrm{SO}_{4n} \tag{3.7}$$

For $0 \le j \le n$, let $\mathcal{V}_j$ be the invariant P subspace of $\mathcal{V}$, consisting of functions compactly supported modulo P, and

$$\mathrm{supp}\ f \subset \bigcup\nolimits_{k\le j} [P\omega_k P \cup P\omega_{2n-k}P]. \tag{3.8}$$

Then $\mathcal{V} = \mathcal{V}_n \supset \cdots \supset \mathcal{V}_0$ is a decreasing filtration of $\mathcal{V}$ as P-spaces. Let J_ξ be the subspace of $\mathcal{V}$ consisting of functions supported on double coset $P\xi H$ and compactly supported modulo P. Recall that double cosets $P\backslash \mathrm{SO}_{4n}/H$ are computed at section 2, more explicitely, they are

$$P\omega_j\gamma_i H, \quad P\omega_{2n-j}\gamma_i' H, \tag{3.9}$$

for $0 \le j \le n$ and $0 \le i \le n$ so that $j-i$ is even, where γ_i and γ_i' are matrices defined at (2.10) and (2.12).

Then the following sequence is exact

$$0 \to \mathcal{V}_{j-1} \to \mathcal{V}_j \to \bigoplus_i J_{\omega_j\gamma_i} \oplus J_{\omega_{2n-j}\gamma_i'} \to 0 \tag{3.10}$$

where i runs through all $0 \le i \le j$ so that $j-i$ is even.

Let $l : I(s,\tau) \to \mathbb{C}$ be a linear linear functional on $\mathcal{V}$ satisfying

$$l([I(s,\tau)(h)]f) = \psi(h)l(f), \tag{3.11}$$

for all $h \in H, f \in \mathcal{V}$. By the localization principle of Gelfand and Kazhdan, l is generated by distributions supported on J_ξ satisfying (3.11). So without loss of generality, we assume that l is supported on $P\omega_j\gamma_i H$ for some $0 \le j \le n$ and $0 \le i \le j$ so that $j-i$ is even.

As in the proof of Proposition 2.3, $P \cap \omega_j H^{\eta_i}\omega_j^{-1}$ consists of elements of the following form

$$x = \begin{pmatrix} g & \\ & {}^tg^{-1} \end{pmatrix} \begin{pmatrix} 1 & u \\ & 1 \end{pmatrix} \in P, \tag{3.12}$$

here

$$g = \begin{pmatrix} a & at \\ & q \end{pmatrix} \in Q_{2n-j,\,j},\ u = \begin{pmatrix} v & b \\ -{}^tb & 0 \end{pmatrix} \tag{3.13}$$

such that

$$\begin{pmatrix} a & ab \\ & {}^tq^{-1} \end{pmatrix} \in Q_{2n-j,\,j} \cap \mathrm{Sp}^{\eta_i}. \tag{3.14}$$

Set

$$H_0 = \gamma_i^{-1}\omega_j^{-1}[P \cap \omega_j H^{\eta_i}\omega_j^{-1}]\omega_j\gamma_i. \tag{3.15}$$

Then H_0 is a subgroup of H. Denote the restriction of the character ψ of H (cf. (2.5)) on H_0 by ψ again. Let $C_\psi^\infty(H, H_0)$ be the set of smooth function on H compactly supported modular H_0 and left ψ-invariant. Let H_c be the unipotent group of H consists of elements of the following type

$$\begin{pmatrix} g & gu \\ & {}^t g^{-1} \end{pmatrix}, \text{ with } g = \gamma_i^{-1} \left(\begin{smallmatrix} 1 & 0 \\ z & 1 \end{smallmatrix}\right) \gamma_i \in \mathrm{Sp}_{2n}, \quad u = \gamma_i^{-1} \left(\begin{smallmatrix} 0 & 0 \\ 0 & z' \end{smallmatrix}\right) \gamma_i \in \mathcal{A}_{2n}. \tag{3.16}$$

Here $z \in \mathrm{M}_{j\times(2n-j)}, z' \in \mathrm{M}_{j\times j}$. Then $H_0 \cap H_c = \{1\}$ and $H = H_0 \cdot H_c$. Let $C_c^\infty(H_c)$ be the space of smooth function on H_c with compact support. Then restriction induced an homeomorphism

$$C_\psi^\infty(H, H_0) \cong C_c^\infty(H_c) \tag{3.17}$$

Let dh be the Haar measure on H_c. Define a surjective mapping $\mathcal{P}_{\omega_j\gamma_i} : J_{\omega_j\gamma_i} \to E$ by

$$\mathcal{P}_{\omega_j\gamma_i}(f) = \int_{H_c} f(h)dh. \tag{3.18}$$

Since l is supported on $P\omega_j\gamma_i H$ and satisfies (3.11), there is a linear functional $l' : E \to \mathbb{C}$ such that the following diagram is commutative

$$\begin{array}{ccc} J_{\omega_j\gamma_i} & \xrightarrow{\mathcal{P}_{\omega_j\gamma_i}} & E \\ {\scriptstyle l}\downarrow & & \downarrow{\scriptstyle l'} \\ \mathbb{C} & \xrightarrow{=} & \mathbb{C} \end{array}$$

and satisfies

$$l'(\tau(x)\alpha^s(x)\delta_P^{\frac{1}{2}}(x)v) = \psi(\gamma_i^{-1}\omega_j^{-1}x\omega_j\gamma_i)|\det \mathrm{Ad}(\gamma_i^{-1}\omega_j^{-1}x\omega_j\gamma_i)|_{H_c}|l'(v), \tag{3.19}$$

for $x \in L^{\eta_i}, v \in E$. We note here that

$$\det(\gamma_i^{-1}\omega_j^{-1}x\omega_j\gamma_i)|_{H_c} = \det(a)^{2i}, \text{ for } x \in L^{\eta_i}. \tag{3.20}$$

If $i \neq 0$ and $2(n-j)+r \neq 0$, for $X \in M_{i\times 2(n-j)+r}$, $Y \in M_{r\times i}$, define

$$a = \begin{pmatrix} 1 & X \\ & 1 \end{pmatrix}, \quad q = \begin{pmatrix} 1 & \\ Y & 1 \end{pmatrix}. \tag{3.21}$$

We choose $0 \neq X$ such that there is $b \in M_{(2n-j)\times j}$ such that

$$\begin{pmatrix} a & ab \\ & {}^t q^{-1} \end{pmatrix} \in P \cap \omega_j H^{\eta_i} \omega_j^{-1}. \tag{3.22}$$

For $t = \left(\begin{smallmatrix} t_1 & t_2 \\ t_3 & t_4 \end{smallmatrix}\right) \in \mathrm{M}_{2n-j,\,j}$, define

$$x = \begin{pmatrix} 1 & X & t_1 & t_2 \\ & 1 & t_3 & t_4 \\ & & 1 & 0 \\ & & Y & 1 \end{pmatrix} \in L^{\eta_i}. \tag{3.23}$$

Then simple computation shows that

$$\psi(\gamma_i^{-1}\omega_j^{-1}x\omega_j\gamma_i) = \psi(-\mathrm{tr}(t_2\varepsilon_i)). \tag{3.24}$$

Let

$$t' = \begin{pmatrix} t'_1 & t'_2 \\ t'_3 & t'_4 \end{pmatrix} \in \mathrm{M}_{2n-j,\ j}$$

be such that $\psi(-\mathrm{tr}(Xt'_4\varepsilon_i)) \neq 1$. If we let

$$x' = \begin{pmatrix} 1 & 0 & t'_1 & t'_2 \\ & 1 & t'_3 & t'_4 \\ & & 1 & \\ & & 0 & 1 \end{pmatrix} \tag{3.25}$$

then

$$\begin{aligned} \psi(\gamma_i^{-1}\omega_j^{-1}xx'\omega_j\gamma_i) &= \psi(-\mathrm{tr}[(t_2+t'_2)\varepsilon_i])\psi(-\mathrm{tr}(Xt'_4\varepsilon_i)) \\ &\neq \psi(\gamma_i^{-1}\omega_j^{-1}x\omega_j\gamma_i)\cdot\psi(\gamma_i^{-1}\omega_j^{-1}x'\omega_j\gamma_i). \end{aligned} \tag{3.26}$$

This contradicts (3.19), which states that the function

$$\psi(\gamma_i^{-1}\omega_j^{-1}x\omega_j\gamma_i), \quad x \in L^{\eta_i}$$

is a character, hence in this case $l' = 0$.

If $i = 0$, then (3.19) becomes

$$l'(\tau(x)v) = l'(v), \text{ for } x \in L^{\eta_0}. \tag{3.27}$$

This is impossible by Proposition 2.5. So in this case, $l' = 0$.

If $2(n-j)+r=0$, then $r=0$ and $n=j=i$. Denote the group L^{η_n} defined at (2.28) simply by L. Then L is the subgroup of GL_{2n} consisting of elements of the following type

$$\begin{pmatrix} g & gu \\ & \varepsilon_n g \varepsilon_n \end{pmatrix}, \text{ with } g \in \mathrm{GL}_n, u \in \mathrm{M}_{n\times n}. \tag{3.28}$$

Hence $\alpha(x)=|\det(g)|$. By (2.17),

$$\psi(\gamma_n^{-1}\omega_n^{-1}x\omega_n\gamma_n)=\psi(-u\varepsilon_n), \tag{3.29}$$

The equation (3.19) of l' is

$$l'(\tau(x)\alpha^{s-1})=\psi(-u\varepsilon_n)l'(v), \quad v\in E. \tag{3.30}$$

Let

$$x_0=\begin{pmatrix} 1 & \\ & -\varepsilon_n \end{pmatrix}. \tag{3.31}$$

Then $L\cong \mathrm{Ad}(x_0)(S)$, where S is the Shalika subgroup of GL_{2n}, and

$$\psi(\gamma_n^{-1}\omega_n^{-1}x\omega_n\gamma_n)=\psi_S(\mathrm{Ad}(x_0^{-1})(x)). \tag{3.32}$$

Here ψ_S is the character of S defined at (3.2). Hence if l' is nonzero and supported on $P\omega_n\gamma_n H$, then $\pi\cdot\alpha^{s-1}$ has a Shalika functional. In [JR96], Jacquet and Rallis shows that if an irreducible representation of GL_{2n} has a Shalika functional, then it is self-dual. So $s=1$ and π is self-dual.

Conversely, if π is self-dual irreducible representation of GL_{2n} having a Shalika functional l', we can construct a generalized Shalika functional l of $\mathrm{Ind}_P^{\mathrm{SO}_{4n}}(\pi\alpha)$ by define it be zero on

$$\mathcal{V}_{n-1}\bigoplus\bigoplus_{\substack{0\le i<n\\ i\equiv n \bmod 2}} J_{\omega_n\gamma_i}. \tag{3.33}$$

and on $J_{\omega_n\gamma_n}$ by

$$l(f):=l'(\mathcal{P}_{\omega_n\gamma_n}(f)), \text{ for } f\in J_{\omega_n\gamma_n}. \tag{3.34}$$

From the argument above, we see that if l is a generalized Shalika functional of $\mathrm{Ind}_P^{\mathrm{SO}_{4n}}(\pi\alpha)$, then it factors through $\mathcal{V}_{n-1}$ which is the maximal P-invariant subspace of $\mathcal{V}$. As the Jacquet functor is exact, l factors through the langlands quotient of $\mathrm{Ind}_P^{\mathrm{SO}_{4n}}(\pi\alpha)$. □

4 Global Periods

Let k be a number field and $\mathbb{A}$ be the ring of adeles of k. We shall consider Eisenstein series on $\mathrm{SO}_{4n}(\mathbb{A})$ associated to the maximal parabolic subgroup $P_{2n} = M_{2n}U_{2n}$ and irreducible cuspidal automorphic representation π of the Levi part $M_{2n}(\mathbb{A}) \cong \mathrm{GL}_{2n}(\mathbb{A})$. Here SO_{4n} is the k-split even special orthogonal group. The location of the pole of the Eisenstein series is expected to be determined in terms of the location of poles of the exterior square L-function $L(s, \pi, \Lambda^2)$ by the Langlands theory of the constant terms of Eisenstein series. We shall use (generalized) Shalika model to realize the residue of the Eisenstein series, as a by-product, we determine the location of the poles of the Eisenstein series by means of the Shalika model of the cuspidal data.

Let $K = \prod_v K_v$ be the maximal compact subgroup of $\mathrm{SO}_{4n}(\mathbb{A})$ such that $\mathrm{SO}_{4n}(\mathbb{A})$ has the Iwasawa decomposition

$$\mathrm{SO}_{4n}(\mathbb{A}) = P_{2n}(\mathbb{A})K.$$

In particular, for each finite local place v, $K_v = \mathrm{SO}_{4n}(\mathcal{O}_v)$, where $\mathcal{O}_v$ is the ring of integers in the local field k_v. Then the Langlands decomposition of $\mathrm{SO}_{4n}(\mathbb{A})$ is

$$\mathrm{SO}_{4n}(\mathbb{A}) = U_{2n}(\mathbb{A})M^1_{2n}A^+_{2n}K.$$

Let A_{2n} be the (split) center of M_{2n}, the unique reduced root in $\Phi^+(P_{2n}, A_{2n})$ can be identified with simple root α_{2n}. As normalized in [Sh88], we denote

$$\tilde{\alpha}_{2n} :=< \rho_{P_{2n}}, \alpha_{2n} >^{-1} \rho_{P_{2n}}, \tag{4.1}$$

where $\rho_{P_{2n}}$ is half of the sum of all positive root in U_{2n} and $< \cdot, \cdot >$ is the usual Killing-Cartan form for the root system of SO_{4n}. We let

$$\mathfrak{a}_{M_{2n}} = Hom_{\mathbb{R}}(X(M_{2n}), \mathbb{R}), \quad \mathfrak{a}^*_{M_{2n}} = X(M_{2n}) \otimes \mathbb{R}, \tag{4.2}$$

where $X(M_{2n})$ denotes the group of all rational characters of M_{2n}. Since P_{2n} is maximal, $\mathfrak{a}^*_{2n}$ is of one dimension. We identify $\mathbb{C}$ with $\mathfrak{a}^*_{2n,\mathbb{C}}$ via $s \mapsto s\tilde{\alpha}_{2n}$.

For simplicity, we use the notation without the indication $2n$. Let $H_P : M \mapsto \mathfrak{a}_M$ be the map defined as follows, for any $\chi \in \mathfrak{a}^*_M$,

$$H_P(m)(\chi) = \prod_v |\chi(m_v)|_v \tag{4.3}$$

for $m \in M(\mathbb{A})$. It follows that H_P is trivial on M^1. This map H_P can be extended as a function over $\mathrm{SO}_{4n}(\mathbb{A})$ via the Iwasawa decomposition or the Langlands decomposition above. By direct computation, we know that

$$H_P(m)(s) = |\det_{2n} a|^{\frac{s}{2}}, \quad H_P(m)(\rho_P) = |\det_{2n} a|^{\frac{2n-1}{2}} \tag{4.4}$$

where $s \in \mathbb{C}$ and $m = m(a) = \mathrm{diag}(a, a') \in M$ with $a \in \mathrm{GL}_{2n}$.

Let π be an irreducible cuspidal automorphic representation of $\mathrm{GL}_{2n}(\mathbb{A})$ with trivial central character. Then (M, π) is a cuspidal datum attached the Levi subgroup M. Let $\phi(g)$ be a complex-valued smooth function on $\mathrm{SO}_{4n}(\mathbb{A})$ which is left $U(\mathbb{A})M(k)$-invariant and right K-finite. Writing by the Langlands decomposition

$$g = um^1 a\kappa \in U(\mathbb{A})M^1 A_M^+ K,$$

we assume that

$$\phi(g) = \phi(m^1\kappa). \tag{4.5}$$

If we fix a $\kappa \in K$, the map

$$m^1 \mapsto \phi(m^1\kappa)$$

defines a $K \cap M^1$-finite vector in the space of cuspidal representation π of $M(\mathbb{A})$. We set

$$F(g;\phi,s) := H_P(g)(s + \rho_P)\phi(g).$$

Attached to such a function $F(g;\phi,s)$, we define an Eisenstein series

$$E(g;\phi,s) := \sum_{\gamma \in P(k)\backslash \mathrm{SO}_{4n}(k)} F(\gamma g;\phi,s) \tag{4.6}$$

From the general theory of Eisenstein series [MW95], this Eisenstein series converges absolutely for the real part $Re(s) > \frac{2n-1}{2}$ and has a meromorphic continuation to the whole s-plane with finitely many possible simple poles for $Re(s) > 0$. By the Langlands theory of the constant terms of Eisenstein series, the existence of the poles on the positive half plan of this Eisenstein series should be detected in general by means of that of the constant terms of the Eisenstein series.

In the case under consideration, the only nonzero constant term of the Eisenstein series $E(g;\phi,s)$ is the one along the maximal parabolic subgroup P, i.e.

$$E_P(g;\phi,s) := \int_{U(k)\backslash U(\mathbb{A})} E(ug;\phi,s)du \tag{4.7}$$

where du is the Haar measure on $U(k)\backslash U(\mathbb{A})$, which is normalized so that the total volume equals one. By assuming the real part of s large, we have

$$E_P(g;\phi,s) = F(g;\phi,s) + F(g;M(s,w_{2n})(\phi),-s) \tag{4.8}$$

where $M(s,w_{2n})$ is the standard (global) intertwining operator attached to the maximal parabolic subgroup P and the Weyl element w_{2n}, which has the property that $w_{2n}Mw_{2n}^{-1} = M$ and $w_{2n}Uw_{2n}^{-1} = U^-$ (the opposite of U). This equation holds for all $s \in \mathbb{C}$ by meromorphic continuation. The intertwining operator $M(s,w_{2n})$ can be expressed as

$$M(s,w_{2n}) = \otimes_v A(s,\pi_v,w_{2n}) \tag{4.9}$$

by following the notations in [Sh90], where $A(s,\pi_v,w_{2n})$ is the local intertwining operator

$$\mathrm{Ind}_{P(k_v)}^{\mathrm{SO}_{4n}(k_v)}(\pi_v \otimes |\mathrm{det}_{2n}a|^{\frac{s}{2}}) \to \mathrm{Ind}_{P(k_v)}^{\mathrm{SO}_{4n}(k_v)}(w_{2n}(\pi_v) \otimes |\mathrm{det}_{2n}a|^{-\frac{s}{2}}).$$

Then, one can write

$$A(s,\pi_v,w_{2n}) = \frac{L(s,\pi_v,\Lambda^2)}{L(1+s,\pi_v,\Lambda^2)\epsilon(s,\pi_v,\Lambda^2,\psi_v)} \cdot N(s,\pi_v,w_{2n}).$$

Proposition 4.1 (Theorem 4.11 [K05]). *Assume that π_v is the local v-component of an irreducible cuspidal automorphic representation π of $M(\mathbb{A}) = \mathrm{GL}_{2n}(\mathbb{A})$. Then the normalized local intertwining operator $N(s,\pi_v,w_{2n})$ is holomorphic and nonzero for the real part $Re(s) \geq \frac{1}{2}$.*

By Proposition 4.1, we express the intertwining operator $M(s,w_{2n})$ as

$$M(s,w_{2n}) = \frac{L(s,\pi,\Lambda^2)}{L(1+s,\pi,\Lambda^2)\epsilon(s,\tau,\Lambda^2)} \cdot N(s,\pi,w_{2n}). \tag{4.10}$$

The constant term in (4.8) can be expressed as

$$\begin{aligned} E_P(g;\phi,s) &= F(g;\phi,s) \\ &\quad + \frac{L(s,\pi,\Lambda^2)}{L(1+s,\pi,\Lambda^2)\epsilon(s,\tau,\Lambda^2)} \cdot F(g;N(s,\pi,w_{2n})(\phi),-s). \end{aligned} \tag{4.11}$$

The analytic properties of the exterior square L-functions $L(s,\pi,\Lambda^2)$ are summerized in [Jng06], which in particular says that $L(s,\pi,\Lambda^2)$ has meromorphic continuation to the whole complex plane and has a possible simple pole at $s = 1$. Hence the Eisenstein series $E(g; \phi, s)$ is holomorphic for the real part of s greater than $\frac{1}{2}$, except possibly at $s = 1$ where it may have a simple pole. $E(g; \phi, s)$ has a simple pole at $s = 1$ if and only if the exterior square L-functions $L(s, \pi, \Lambda^2)$ has a simple pole at $s = 1$. This is the method that one determines the location of poles of Eisenstein series in terms of the location of the poles of the relevant L-functions.

4.1 Residues of Eisenstein Series

In this section, we want to show that the location of the possible simple pole of the Eisenstein series may be determined by means of the generalized Shalika model of the residue. Let s_0 be a real number greater than $\frac{1}{2}$ and define

$$E_{s_0}(g; \phi) := \mathrm{Res}_{s=s_0} E(g; \phi, s).$$

We want to calculate the following period integral

$$\mathcal{P}_{H,\psi}(E_{s_0}(\cdot; \phi)) = \int_{H(k)\backslash H(\mathbb{A})} E_{s_0}(r; \phi)\psi^{-1}(r)dr. \tag{4.12}$$

It is clear that one might write the period as

$$\begin{aligned} \mathcal{P}_{H,\psi}(E_{s_0}(\cdot, \phi)) &= \int_{\mathrm{Sp}_{2n}(k)\backslash \mathrm{Sp}_{2n}(\mathbb{A})} \int_{V(k)\backslash V(\mathbb{A})} E_{s_0}(vx; \phi)\psi_V(v)dvdx \qquad (4.13) \\ &= \int_{\mathrm{Sp}_{2n}(k)\backslash \mathrm{Sp}_{2n}(\mathbb{A})} \mathrm{Res}_{s=s_0}[\int_{V(k)\backslash V(\mathbb{A})} E(vx; \phi, s)\psi_V(v)dv]dx. \end{aligned}$$

The convergency of the integration along the variable x needs to be justified via the Arthur's truncation method. We are going to calculate the inner integration first, which is the Fourier coeffient of the Eisenstein series.

4.2 Fourier Coefficients of Eisenstein Series

We shall study in this section the Fourier coefficient of the Eisenstein series

$$\int_{V(k)\backslash V(\mathbb{A})} E(vh; \phi, s)\psi_V(v)dv, \tag{4.14}$$

which occurs as an inner integration of (4.13). In the following we assume the real part of s be large, so that the Eisenstein series converges absolutely and uniformly on every compact subset in $\mathrm{SO}_{4n}(k)\backslash\mathrm{SO}_{4n}(\mathbb{A})$. By Proposition 2.2, we have

$$\begin{aligned}(4.14) \quad &= \int_{V(k)\backslash V(\mathbb{A})} \sum_{\gamma\in P(k)\backslash \mathrm{SO}_{4n}(k)} F(\gamma vh;\phi,s)\psi_V(v)dv \\ &= \sum_{j=0}^{2n} \sum_{\gamma\in P^{\omega_j}\backslash P/V} \int_{V^{\omega_j,\gamma}\backslash V(\mathbb{A})} F(\omega_j\gamma vh;\phi,s)\psi_V(v)dv \qquad (4.15)\end{aligned}$$

where $P^{\omega_j}=\omega_j^{-1}P\omega_j\cap P$ and $V^{\omega_j,\gamma}=(\omega_j\gamma)^{-1}P\omega_j\gamma\cap V$. In the discussion in §2 and Proposition 2.2, we write $P^{\omega_j}\backslash P/H$ as a disjoint union

$$P^{\omega_j}\backslash P/H=\cup_i P^{\omega_j}\gamma_i\mathrm{Sp}_{2n}V, \tag{4.16}$$

where i runs through $0\le i\le \min(j,2n-j)$ so that $r=j-i$ be even. Here for $j>n$, we write γ_i for γ_i' there. For each given ω_j, we set

$$\mathrm{Sp}_{2n,\,\gamma_i}:=\gamma_i^{-1}P^{\omega_j}\gamma_i\cap\mathrm{Sp}_{2n}. \tag{4.17}$$

Then we obtain

$$(4.15)=\sum_{j=0}^{2n}\sum_{\gamma_i}\sum_{\epsilon_i\in\mathrm{Sp}_{2n,\gamma_i}\backslash\mathrm{Sp}_{2n}}\int_{V^{\gamma_i,\epsilon_i}\backslash V(\mathbb{A})} f(\omega_j\gamma_i\epsilon_i vh;\phi,s)\psi_V(v)dv. \tag{4.18}$$

where $V^{\gamma_i,\epsilon_i}=(\gamma_i\epsilon_i)^{-1}P^{\omega_j}(\gamma_i\epsilon_i)\cap V$. Since Sp_{2n} normalizes V and stabilizes the character ψ_V, we have $V^{\gamma_i,\epsilon_i}=V^{\omega_j,\gamma_i}$ and we can express (4.18) as

$$(4.18)=\sum_{j=0}^{2n}\sum_{\gamma_i}\sum_{\epsilon_i\in\mathrm{Sp}_{2n,\gamma_i}\backslash\mathrm{Sp}_{2n}}\int_{V^{\gamma_i}\backslash V(\mathbb{A})} f(\omega_j\gamma_i v\epsilon_i h;\phi,s)\psi_V(v)dv. \tag{4.19}$$

In the following we will discuss (4.19) for each ω_j case by case.

If $j=0$, then $\omega_0=e$, $\gamma_i=\epsilon_i=e$. Hence

$$\begin{aligned}(4.19)_{j=0} \quad &= \int_{V(k)\backslash V(\mathbb{A})} F(vh;\phi,s)\psi_V(v)dv \qquad (4.20)\\ &= F(h;\phi,s)\int_{V(k)\backslash V(\mathbb{A})}\psi_V(v)dv=0, \qquad (4.21)\end{aligned}$$

since ψ_V is nontrivial on $V(k)\backslash V(\mathbb{A})$.

Let $0 \le j \le n$. The double coset representatives γ_i of $P^{\omega_j}\backslash P/H$ are given in (2.10), for $0 \le i \le j$ such that $r = j - i$ is even. Let $V_0 \subset V$ be the subgroup consists of elements of the following form

$$u = \gamma_i^{-1} \begin{pmatrix} 1 & 0 & v & 0 \\ & 1 & 0 & 0 \\ & & 1 & 0 \\ & & & 1 \end{pmatrix} \gamma_i, \text{ with } v \in \mathrm{M}_{2n-j,\,j}. \tag{4.22}$$

It is clear that

$$\omega_j \gamma_i u \gamma_i^{-1} \omega_j = \begin{pmatrix} 1 & 0 & v & 0 \\ & 1 & 0 & 0 \\ & & 1 & 0 \\ & & & 1 \end{pmatrix} \in V(\mathbb{A}), \tag{4.23}$$

hence $V_0 \subset V^{\omega_j, \gamma_i}$. We notice that

$$\psi_V(u) = \psi^{\eta_i} \begin{pmatrix} 1 & 0 & v & 0 \\ & 1 & 0 & 0 \\ & & 1 & 0 \\ & & & 1 \end{pmatrix}. \tag{4.24}$$

By the formula of ψ^{η_i} in (2.17), if $2(n-j) + r \neq 0$, then ψ_V is nontrivial on $V_0(\mathbb{A})$, we then obtain

$$\int_{V_0(k)\backslash V_0(\mathbb{A})} F(\omega_j \gamma_i v \epsilon_i h; \phi, s) \psi_V(v) dv \tag{4.25}$$

$$= \quad F(\omega_j \gamma_i \epsilon_i h; \phi, s) \int_{V_0(k)\backslash V_0(\mathbb{A})} \psi_V(v) dv = 0 \tag{4.26}$$

This proves the integral in (4.19) vanishes when $0 \le j \le n$, $i \neq n$, i.e.

$$(4.19)_{j<n} \quad = \quad \sum_{\gamma_i} \sum_{\epsilon_i} \int_{V^{\omega_j, \gamma_i}\backslash V(\mathbb{A})} F(\omega_j \gamma_i v \epsilon_i h; \phi, s) \psi_V(v) dv = 0 \tag{4.27}$$

$$\begin{aligned} (4.19)_{j=n} \quad &= \quad \sum_{\gamma_i \neq \gamma_n} \sum_{\epsilon_i} \int_{V^{\omega_n, \gamma_i}\backslash V(\mathbb{A})} F(\omega_j \gamma_i v \epsilon_i h; \phi, s) \psi_V(v) dv \\ &+ \quad \sum_{\epsilon_n \in \mathrm{Sp}_{2n,\gamma_n}\backslash \mathrm{Sp}_{2n}} \int_{V^{\omega_n, \gamma_n}\backslash V(\mathbb{A})} F(\omega_n \gamma_n v \epsilon_n h; \phi, s) \psi_V(v) dv \\ &= \quad \sum_{\epsilon_n} \int_{V^{\omega_n, \gamma_n}\backslash V(\mathbb{A})} F(\omega_n \gamma_n v \epsilon_n h; \phi, s) \psi_V(v) dv \end{aligned} \tag{4.28}$$

Since $\gamma_n = 1_{2n}$, simple computation shows that V^{ω_n,γ_n} is the subgroup of V consists of elements of V of the following type:

$$\begin{pmatrix} 1 & 0 & v & b \\ & 1 & -b^t & 0 \\ & & 1 & 0 \\ & & & 1 \end{pmatrix} \tag{4.29}$$

and

$$\mathrm{Sp}_{2n,\gamma_n} = \{ \begin{pmatrix} g & gu \\ 0 & \varepsilon_n\, {}^t g^{-1} \varepsilon_n \end{pmatrix} \in \mathrm{Sp}_{2n} \mid g \in \mathrm{GL}_n, u \in M_{n\times n} \} \tag{4.30}$$

is the Siegel parabolic of Sp_{2n}, we change notation now and denote it by P_0. Hence

$$(4.19)_{j=n} = \sum_{\epsilon_n \in P_0(k)\backslash \mathrm{Sp}_{2n}(k)} \int_{V^{\omega_n,\gamma_n}\backslash V(\mathbb{A})} F(\omega_n v \epsilon_n h; \phi, s)\psi_V(v)dv \tag{4.31}$$

The proof of vanish of (4.19) for $j > n$ is similar to that of $j < n$, hence,

$$(4.19)_{j>n} = \sum_{\gamma_i}\sum_{\epsilon_i} \int_{V^{\omega_j,\gamma_i}\backslash V(\mathbb{A})} F(\omega_j \gamma_i v \epsilon_i h; \phi, s)\psi_V(v)dv = 0 \tag{4.32}$$

In other words, we have proved that

Proposition 4.3. *The ψ_V-Fourier coefficient of the Eisenstein series $E(vh;\phi,s)$ has the following expression:*

$$\int_{V(k)\backslash V(\mathbb{A})} E(vh;\phi,s)\psi_V(v)dv = \sum_{\epsilon_n \in P_0(k)\backslash \mathrm{Sp}_{2n}(k)} \int_{V^{\omega_n,\gamma_n}\backslash V(\mathbb{A})} F(\omega_n v \epsilon_n h; \phi, s)\psi_V(v)dv. \tag{4.33}$$

Since V is abelian,

$$V = V^{\omega_n,\gamma_n} \times V_{\gamma_n} \tag{4.34}$$

where V_{γ_n} is the subgroup of V consists of elements of the following form

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ & 1 & 0 & u \\ & & 1 & 0 \\ & & & 1 \end{pmatrix}. \tag{4.35}$$

Define

$$V_{n,n} = \omega_n V^{\omega_n,\gamma_n} \omega_n^{-1} \cap P, \tag{4.36}$$

which is the unipotent radical of the parabolic $P_{n,n}$ of GL_{2n}. The character $\psi_{n,n}$ of $V_{n,n}$ defined by

$$\psi_{n,n} \begin{pmatrix} 1 & x \\ & 1 \end{pmatrix} = \psi(-x\varepsilon_n) \tag{4.37}$$

is the character induced from ψ_V by means of conjugation by ω_n. Hence the integral in (4.33) can be expressed as

$$\begin{aligned} \mathcal{F}(g;\phi,s) : &= \int_{V^{\omega_n,\gamma_n}(k)\backslash V(\mathbb{A})} F(\omega_n v g;\phi,s)\psi_V(v)dv \\ &= \int_{V_{\gamma_n}(\mathbb{A})} \int_{V_{n,n}(k)\backslash V_{n,n}(\mathbb{A})} F(v\omega_n v_2 g;\phi,s)dvdv_2. \end{aligned} \tag{4.38}$$

It is clear the first integral over $V_{n,n}(k)\backslash V_{n,n}(\mathbb{A})$ is the $\psi_{n,n}$-Fourier coefficient of the irreducible cuspidal automorphic representation π and the second integration over $V_{\gamma_n}(\mathbb{A})$ is the intertwining operator associated to the Weyl element ω_n in this case.

For the given irreducible cuspidal representation π of $\mathrm{GL}_{2n}(\mathbb{A})$, the $\psi_{n,n}$-Fourier coefficient

$$\mathcal{W}^{\psi_{n,n}}(h,\varphi_\pi) := \int_{V_{n,n}(k)\backslash V_{n,n}(\mathbb{A})} \varphi_\pi(vh)\psi_{n,n}(v)dv \tag{4.39}$$

generates an automorphic representation of the centralizer GL_n of the character $\psi_{n,n}$, which is denoted by $\mathcal{W}(\pi,\psi_{n,n})$.

It is easy to see that the function $\mathcal{F}(g;\phi,s)$ is invariant under the left translation by the unipotent radical N_0 of P_0. If $m(h)$ denotes the element of the Levi subgroup of P_0, then the function

$$m(h) \mapsto \mathcal{F}(m(h)g;\phi,s) \tag{4.40}$$

belongs to the space of automorphic representation

$$|\det(h)|^{s+n}\mathcal{W}(\pi,\psi_{n,n}) \tag{4.41}$$

of GL_n. In fact, assume $h \in \mathrm{GL}_n$, then

$$m(h) = \begin{pmatrix} h &&& \\ & \varepsilon_n\, {}^t h^{-1} \varepsilon_n && \\ && {}^t h^{-1} & \\ &&& \varepsilon_n h \varepsilon_n \end{pmatrix} \in P_0.$$

Here elements in GL_{2n} are identified as those in M. Note that the Jacobian of $m(h)$ on V_{γ_n} is $|\det h|^{-(n-1)}$. Hence (4.38) becomes

$$\mathcal{F}(m(h)g;\phi,s) = |\det h|^{-(n-1)} \int\limits_{V_{\gamma_n}(\mathbb{A})} \int\limits_{V_{n,n}(k)\backslash V_{n,n}(\mathbb{A})} F(v\omega_n m(h)v_2 g;\phi,s)dvdv_2$$

By (2.25),

$$\omega_n m(h)\omega_n^{-1} = \begin{pmatrix} h & & & \\ & \varepsilon_n h \varepsilon_n & & \\ & & {}^t h^{-1} & \\ & & & \varepsilon_n\ {}^t h^{-1}\varepsilon_n \end{pmatrix},$$

which is a centralizer of $\psi_{n,n}$. Since its Jacobian on $V_{n,n}$ is 1, if we define $\phi'(x) = \phi(m(h)x)$, then by (4.4)

$$\begin{aligned} \mathcal{F}(m(h)g;\phi,s) &= |\det h|^{-(n-1)} H_P(m(h))(s+\rho_P)\mathcal{F}(g;\phi',s) \\ &= |\det h|^{-(n-1)} |\det h^2|^{\frac{s}{2}} |\det h^2|^{\frac{2n-1}{2}} \mathcal{F}(g;\phi',s) \\ &= |\det h|^{s+n}\mathcal{F}(g;\phi',s). \end{aligned}$$

It follows that the function in (4.33) is an Eisenstein series of $\mathrm{Sp}_{2n}(\mathbb{A})$ of form

$$\sum_{\epsilon\in P_0(k)\backslash \mathrm{Sp}_{2n}(k)} \mathcal{F}(\epsilon h;\phi,s). \tag{4.42}$$

where $\mathcal{F}(g;\phi,s)$ is defined as in (4.38).

4.4 Period Identity

We continue here the explicit calculation of the period $\mathcal{P}_{H,\phi}(E_{s_0}(\cdot;\phi)$ of the residue $E_{s_0}(\cdot;\phi)$ defined in (4.12). By (4.13), (4.33) and (4.42), we have

$$\mathcal{P}_{H,\phi}(E_{s_0}(\cdot;\phi) = \int_{\mathrm{Sp}_{2n}(k)\backslash \mathrm{Sp}_{2n}(\mathbb{A})} \mathrm{Res}_{s=s_0} \sum_{\epsilon\in P_0(k)\backslash \mathrm{Sp}_{2n}(k)} \mathcal{F}(\epsilon h;\phi,s)dh. \tag{4.43}$$

It is clear that integral (4.43) may not be convergent , so we use the Arthur truncation method to regularize the integral. By §I.2.13 of [MW95], we apply the truncation method to the automorphic function

$$I_{n,s}(h) := \sum_{\epsilon\in P_0(k)\backslash \mathrm{Sp}_{2n}(k)} \mathcal{F}(\epsilon h;\phi,s)$$

and its residue at $s = s_0$

$$I_{n,s_0}(h) := \mathrm{Res}_{s=s_0} \sum_{\epsilon \in P_0(k)\backslash \mathrm{Sp}_{2n}(k)} \mathcal{F}(\epsilon h; \phi, s)$$

Recall from §I.2.13 of [MW95], for any locally L^1-function ϕ on the quotient $\mathrm{Sp}_{2n}(k)\backslash \mathrm{Sp}_{2n}(\mathbb{A})$, the truncated function, which is also a locallly L^1-function, is given by

$$\Lambda^T \phi(g) := \sum_{B \subset P = MN \subset \mathrm{Sp}_{2n}} (-1)^{r(G)-r(M)} \sum_{\gamma \in P\backslash \mathrm{Sp}_{2n}} \phi_P(\gamma g) \hat{\tau}_P(\log_M(m_P(\gamma g) - T_M) \tag{4.44}$$

where $P = MN$ are standard parabolic subgroups of G, which may be equal to G, and ϕ_P is the constant term of ϕ along N, and the other notations are the same as in [MW95].

Applying (4.44) to the automorphic function $I_{n,s}(h)$, we obtain the rapidly decreasing function $\Lambda^T I_n(h, s)$ on the fundamental domain for $\mathrm{Sp}_{2n}(k)\backslash \mathrm{Sp}_{2n}(\mathbb{A})$. Hence we have

$$\mathrm{Res}_{s=s_0} \int_{\mathrm{Sp}_{2n}(k)\backslash \mathrm{Sp}_{2n}(\mathbb{A})} \Lambda^T I_{n,s}(h) dh = \int_{\mathrm{Sp}_{2n}(k)\backslash \mathrm{Sp}_{2n}(\mathbb{A})} \Lambda^T I_{n,s_0}(h) dh. \tag{4.45}$$

As in Page 353, [GJR04a], we write

$$\Phi_s(h) := \Lambda^T I_{n,s}(h) - I_{n,s}(h). \tag{4.46}$$

Then we have

$$\mathcal{P}_{\mathrm{Sp}_{2n}}(I_{n,s}) = \mathrm{Res}_{s=s_0} \int_{\mathrm{Sp}_{2n}(k)\backslash \mathrm{Sp}_{2n}(\mathbb{A})} \Lambda^T I_{n,s}(h) dh - \int_{\mathrm{Sp}_{2n}(k)\backslash \mathrm{Sp}_{2n}(\mathbb{A})} \Phi_{s_0}(h) dh. \tag{4.47}$$

As in (6,14) of [GJR04a], we write

$$\begin{aligned} \Lambda^T I_{n,s}(h) &= I_{n,s}(h) \\ &- \sum_{\gamma \in P_0\backslash \mathrm{Sp}_{2n}} \mathcal{F}(\gamma h; \phi, s) \hat{\tau}_{P_0}(\log_{M_{P_0}}(m_{P_0}(\gamma h)) - T_{M_{P_0}}) \\ &+ \sum_{\gamma \in P_0\backslash \mathrm{Sp}_{2n}} \mathcal{F}(\gamma h; \phi, s) \hat{\tau}_{P_0}(\log_{M_{P_0}}(m_{P_0}(\gamma h)) - T_{M_{P_0}}) \\ &+ \Phi_s(h), \end{aligned} \tag{4.48}$$

It follows that the period

$$\begin{aligned}
&\int_{\mathrm{Sp}_{2n}(k)\backslash\mathrm{Sp}_{2n}(\mathbb{A}))} \Lambda^T I_{n,s}(h)dh && (4.49)\\
= &\int_{\mathrm{Sp}_{2n}(k)\backslash\mathrm{Sp}_{2n}(\mathbb{A})} \sum_{\gamma\in P_0\backslash\mathrm{Sp}_{2n}} \mathcal{F}(\gamma h;\phi,s)(1-\hat{\tau}_{P_0}(\log_{M_{P_0}}(m_{P_0}(\gamma h))-T_{M_{P_0}}))dh\\
&+\int_{\mathrm{Sp}_{2n}(k)\backslash\mathrm{Sp}_{2n}(\mathbb{A})} \sum_{\gamma\in P_0\backslash\mathrm{Sp}_{2n}} \mathcal{F}(\gamma h;\phi,s)\hat{\tau}_{P_0}(\log_{M_{P_0}}(m_{P_0}(\gamma h))-T_{M_{P_0}})dh\\
&+\int_{\mathrm{Sp}_{2n}(k)\backslash\mathrm{Sp}_{2n}(\mathbb{A})} \Phi_s(h)dh.
\end{aligned}$$

Following the same argument as in [JR92], [Jng98], [GJR01], and [GJR04a], we expect that the residue at $s=s_0$ of the following as a (meromorphic function in s)

$$\int_{\mathrm{Sp}_{2n}(k)\backslash\mathrm{Sp}_{2n}(\mathbb{A})} \sum_{\gamma\in P_0\backslash\mathrm{Sp}_{2n}} \mathcal{F}(\gamma h;\phi,s)\hat{\tau}_{P_0}(\log_{M_{P_0}}(m_{P_0}(\gamma h))-T_{M_{P_0}})+\Phi_s(h)dh$$

is equal to

$$\int_{\mathrm{Sp}_{2n}(k)\backslash\mathrm{Sp}_{2n}(\mathbb{A})} \Phi_{s_0}(h))dh.$$

The justification of this expectation involves some detailed computations of relevant intertwining operators and will be omitted here.The main idea of such explicit calculation is the same as the one we are going to deal with, which is the second integral in (4.49). It follows that

$$\mathcal{P}_{\mathrm{Sp}_{2n}}(I_{n,s_0}) = \mathrm{Res}_{s=s_0}\int_{\mathrm{Sp}_{2n}(k)\backslash\mathrm{Sp}_{2n}(\mathbb{A})} \sum_{\gamma\in P_0\backslash\mathrm{Sp}_{2n}} \mathcal{F}(\gamma h;\phi,s)(1-\tau_c(H(\gamma h)))dh, \tag{4.50}$$

where $\tau_c(H(g)) = \hat{\tau}_{P_0}(\log_{M_{P_0}}(m_{P_0}(\gamma h))-c_{M_{P_0}})$ as defined in §4.1, [Jng98]. Since the function $\mathcal{F}(\gamma h;\phi,s)(1-\tau_c(H(\gamma h)))$ is truncated, the summation over $\gamma\in P_0\backslash\mathrm{Sp}_{2n}$ is finite, which is absolutely convergent in particular. Hence the integral in (4.50) is equal to

$$\int_{P_0(k)\backslash\mathrm{Sp}_{2n}(\mathbb{A})} \mathcal{F}(h;\phi,s)(1-\tau_c(H(h)))dh \tag{4.51}$$

By the Iwasawa decomposition $\mathrm{Sp}_{2n}(\mathbb{A}) = P_0(\mathbb{A})K_{\mathrm{Sp}_{2n}}$, where $K_{\mathrm{Sp}_{2n}}$ is the standard maximal compact subgroup of $\mathrm{Sp}_{2n}(\mathbb{A})$, the Haar measure on $\mathrm{Sp}_n(\mathbb{A})$ at $g = nm\kappa$ can be chosen as

$$|\det m|^{-(n+1)} dndmd\kappa.$$

By the Langlands decomposition

$$\mathrm{GL}_n(\mathbb{A}) = \mathrm{GL}_n(\mathbb{A})^1 \cdot A^+,$$

we deduce that the integral in (4.51) is equal to

$$\int_{K_{\mathrm{Sp}_{2n}} \times \mathrm{GL}_n(k)\backslash \mathrm{GL}_n(\mathbb{A})} \mathcal{F}(m\kappa;\phi,s)(1-\tau_c(H(m)))|\det m|^{-(n+1)} dmd\kappa. \quad (4.52)$$

It is easy to calculate the following

$$\begin{aligned} |\det m|^{-(n+1)} &= |a|_{\mathbb{R}}^{-nd(n+1)} \\ \mathcal{F}(m\kappa;\phi,s) &= |a|_{\mathbb{R}}^{nd(s+n)} \mathcal{F}(m^1\kappa;\phi,s) \\ (1-\tau_c(H(mw_2v_1))) &= (1-\tau_c(H(t_a))) \end{aligned}$$

where d is the number of the real archimedean places of the number field k. The second equation comes from (4.41) by the assumption that ϕ is A^+-invariant. We deduce that the integral in (4.52) is equal to

$$\lambda(s) \cdot \int_{K_{\mathrm{Sp}_{2n}} \times \mathcal{Z}_n(\mathbb{A})\mathrm{GL}_n(k)\backslash \mathrm{GL}_n(\mathbb{A})} \mathcal{F}(m\kappa;\phi,s) dmd\kappa. \quad (4.53)$$

where $\mathcal{Z}_n$ denotes the center of GL_n and the function $\lambda(s)$ is given by

$$\lambda(s) := \mathrm{vol}(\mathbb{A}^1/k) \cdot \int_{\mathbb{R}^+} |a|^{nd(s-1)}(1-\tau_c(H(t_a)))da^\times.$$

It is easy to check that

$$\int_{\mathbb{R}^+} |a|^{nd(s-1)}(1-\tau_c(H(t_a)))da^\times = \frac{c^{nd(s-1)}}{s-1}.$$

Since the integral

$$\int_{K_{\mathrm{Sp}_{2n}} \times \mathcal{Z}_n(\mathbb{A})\mathrm{GL}_n(k)\backslash \mathrm{GL}_n(\mathbb{A})} \mathcal{F}(m\kappa;\phi,s) dmd\kappa$$

represents a holomorphic in s, the possible of (4.53) can only come from the function $\lambda(s)$, which has the only simple pole at $s = 1$. Hence from (4.50) and (4.53) we obtain that s_0 must be 1 and

$$\mathcal{P}_{\mathrm{Sp}_{2n}}(I_{n,1}) = \frac{\mathrm{vol}(\mathbb{A}^1/k)}{nd} \int_{K_{\mathrm{Sp}_{2n}} \times \mathcal{Z}_n(\mathbb{A})\mathrm{GL}_n(k)\backslash \mathrm{GL}_n(\mathbb{A})} \mathcal{F}(m\kappa; \phi, s) dm d\kappa. \quad (4.54)$$

By (4.38), it is easy to see that the integration in variable m in (4.54) yields the following integral

$$\int_{V_{\gamma_n}(\mathbb{A})} \int_{\mathcal{Z}_n(\mathbb{A})\mathrm{GL}_n(k)\backslash \mathrm{GL}_n(\mathbb{A})} \int_{V_{n,n}(k)\backslash V_{n,n}(\mathbb{A})} F(vmw_n v_{\gamma_n}\kappa; \phi, s) \psi_{V_{n,n}}(v) dv dm dv_{\gamma_n}. \quad (4.55)$$

It follows that the integration over $S_n = V_{n,n} \ltimes \mathrm{GL}_n$ is the Shalika period for the cuspidal datum (GL_{2n}, π). We set

$$\mathcal{P}_{S_n,\psi}(F(\cdot;\phi))(g) := \int_{S_n(k)\backslash S_n(\mathbb{A})} F(vmg; \phi, s) \psi_{n,n}(v) dv dm. \quad (4.56)$$

Hence we obtain the main identity relating the 'inner period', the Shalika period for the cuspidal datum on GL_{2n} to the 'outer' period, the generalized Shalika period for the residue of Eisenstein series on SO_{4n} associated to the given cuspidal datum.

Theorem 4.2. *The period $\mathcal{P}_{H,\psi}(E_{s_0}(\cdot, \phi))$ is zero unless $s_0 = 1$. If $s_0 = 1$, we have*

$$\mathcal{P}_{H,\psi}(E_1(\cdot, \phi)) = \frac{\mathrm{vol}(\mathbb{A}^1/k)}{nd} \int_{K_{\mathrm{Sp}_{2n}} \times V_{\gamma_n}(\mathbb{A})} \mathcal{P}_{S_n,\psi}(F(\cdot;\phi))(w_n v_{\gamma_n}\kappa) dv_{\gamma_n} d\kappa.$$

It follows from Theorem 4 above and the argument in Page 179-180, [JR92], in Theorems 5.2, 5.5, [Jng98], and in Theorem 3.2, [GJR01] that the non-vanishing of the Shalika period $\mathcal{P}_{S_n,\psi}(F(\cdot;\phi))$ is equivalent to the nonvanishing of the generalized Shalika period $\mathcal{P}_{H,\psi}(E_1(\cdot, \phi))$. By means of Proposition 2 and the remarks afterwards, we obtain a sufficient condition (in terms of period) for the existence of the pole at $s = 1$ of the (complete) exterior square L-function $L(s, \pi, \Lambda^2)$, which is a theorem of Jacquet and Shalika in [JS90] proved by a different method, i.e. the Rankin-Selberg integral representation method. We record it here for completeness.

Corollary 4.3. *Let π be an irreducible cuspidal automorphic representation of $\mathrm{GL}_{2n}(\mathbb{A})$. If the Shalika period*

$$\mathcal{P}_{S_n,\psi}(\varphi_\pi) = \int_{S_n(k)\backslash S_n(\mathbb{A})} \varphi_\pi(vh)\psi_{n,n}(v)dvdh$$

does not vanish for some $\varphi_\pi \in \pi$, then

1. *the Eisenstein series $E(g;\phi_\pi,s)$ has a pole at $s=1$, and*
2. *the exterior square L-function $L(s,\pi,\Lambda^2)$ has a pole at $s=1$.*

References

[GJR01] Ginzburg, D.; Jiang, D.; Rallis, S., *Nonvanishing of the central critical value of the third symmetric power L-functions.* Forum Math. 13 (2001), no. 1, 109–132.

[GJR04a] Ginzberg, D.; Jiang, D.; Rallis, S. *Periods on residul representations of* SO_{2l}. Manuscipta Math., 113 (2004), 319-358.

[GJR04b] Ginzburg, D.; Jiang, D.; Rallis, S., *Non-vanishing of the central value of the Rankin-Selberg L-functions. I.*. Journal of AMS. 17 (2004), 679-722.

[GJR05] Ginzburg, D.; Jiang, D.; Rallis, S., *Non-vanishing of the central value of the Rankin-Selberg L-functions. II.*. Automorphic Representations, L-Functions and Applications: Progress and Prospects. Ohio State University Mathematical Research Institute Publications 11. 157–191, 2005.

[GP92] Gross, B.; Prasad, D. *On the decomposition of a representation of* SO_n *when restricted to* SO_{n-1}. Can. J. Math. Vol. 44 (1992), 974–1002.

[GP94] Gross, B.; Prasad, D. *On irreducible representations of* $\mathrm{SO}_{2n+1} \times \mathrm{SO}_{2m}$. Canad. J. Math. 46 (1994), no. 5, 930–950.

[Hd87] Harder, G. *Eisenstein cohomology of arithmetic groups. The case* GL_2. Invent. Math. 89 (1987), no. 1, 37–118.

[HLR86] Harder, G.; Langlands, R. P.; Rapoport, M. *Algebraische Zyklen auf Hilbert-Blumenthal-Flachen.* (German) J. Reine Angew. Math. 366 (1986), 53–120.

[Hr94] Harris, M. *Hodge-de Rham structures and periods of automorphic forms.* Motives (Seattle, WA, 1991), 573–624, PSPM, 55, Part 2, AMS, Providence, RI, 1994.

[J97] Jacquet, H. *Automorphic spectrum of symmetric spaces.* PSPM AMS, 61(1997), 443–455.

[JLR99] Jacquet, H.; Lapid, E.; Rogawski, J. *Periods of automorphic forms.* J. Amer. Math. Soc. 12 (1999), no. 1, 173–240.

[JR92] Jacquet, H.; Rallis, S. *Symplectic periods.* J. Reine Angew. Math. 423 (1992), 175–197.

[JR96] Jacquet, H.; Rallis, S. *Uniqueness of linear periods.* Compositio Math. 102 (1996), no. 1, 65–123.

[JS90] Jacquet, H.; Shalika, J. *Exterior square L-functions.* Automorphic forms, Shimura varieties, and L-functions, Vol. II (Ann Arbor, MI, 1988), 143–226, Perspect. Math., 11, Academic Press, Boston, MA, 1990.

[Jng98] Jiang, D. *G_2-periods and residual representations.* J. Reine Angew. Math. 497 (1998), 17–46.

[Jng06] Jiang, D. *On the fundamental automorphic L-functions of* $\mathrm{SO}(2n+1)$. Int. Math. Res. Not. 2006, 64069, 1–26.

[Jng-ecun] Jiang, D. *Periods of automorphic forms.* Accepted by Proceedings of the Conference on Complex geometry, Algebra, and Combinatoric, ECNU.

[Jng-aim] Jiang, D. *Residues of Eisenstein series and related problems* Accepted by the proceedings of AIM workshop on Eisenstein Series.

[JLZ06] Jiang, D.; Li, J.-S.; Zhang, S.-W. *Periods and distribution of cycles on Hilbert modular varieties.* Pure Appl. Math. Q. 2 (2006), no. 1, 219–277.

[JMR99] Jiang, D.; Mao, Z; Rallis, S. *A relative Kuznietsov trace formula on* G_2. Manuscripta Math. 99 (1999), no. 3, 411–423.

[K05] Kim, H. *On local L-functions and normalized intertwining operators*. Canad. J. Math. 57 (2005), no. 3, 535–597.

[Li89] Li, Jian-Shu *Distinguished cusp forms are theta series*. Duke Math. J. 59 (1989), no. 1, 175–189.

[MW95] Moeglin, C.; Waldspurger, J.-L. *Spectral decomposition and Eisenstein series* Cambridge Tracts in Mathematics, 113. Cambridge University Press, Cambridge, 1995.

[Od01] Oda, T. *Modular cycles on arithmetic quotients of classical domains*. Proc. of Japanese-German Seminar. Hakuba, 2001, 142–152.

[PSR88] Piatetski-Shapiro, I.; Rallis, S. *A new way to get Euler products.* J. Reine Angew. Math. 392 (1988), 110–124.

[Q] Qin, Y.-J. *An integral representation of automorphic L-function for quasi-split unitary groups.* Ph.D Thesis, Hong Kong University of Science and Technology, 2004.

[Sh88] Shahidi, F. *On the Ramanujan conjecture and the finiteness of poles of certain L-functions*, Ann. of Math. (2)(1988)(127), 547-584.

[Sh90] Shahidi, F. *A proof of Langlands' conjecture on Plancherel measures; Complementary series for p-adic groups.* Annals of Math. 132 (1990), 273–330.