

Four Dimensional Integrable Theories

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Four dimensional integrable theories ¹

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Abstract. There exist many four dimensional integrable theories. They include self-dual gauge and gravity theories, all their extended supersymmetric generalisations, as well the full (non-self-dual) N=3 super Yang-Mills equations. We review the harmonic space formulation of the twistor transform for these theories which yields a method of producing explicit connections and metrics. This formulation uses the concept of harmonic space analyticity which is closely related to that of quaternionic analyticity.

1. Introduction

Many Lorentz invariant four dimensional exactly solvable nonlinear theories are known. The most remarkable of these are those admitting the Penrose-Ward twistor transform [1], which may be thought of as an analogue of the transformation to action-angle variables for hamiltonian dynamical systems, in the sense that it involves a transformation to variables in which the dynamics is trivial, reducing the problem to that of inverting the transformation. Further, the solution methods for many lower-dimensional completely integrable systems, like the inverse scattering transform for the KdV equation, may be thought of as reductions of the twistor transform [2], so the prospect has arisen, of a unification of the various existing methods of solving two dimensional systems as different manifestations of the twistor transform for self-dual Yang-Mills (SDYM).

The twistor transform, which takes its most dramatic form in its application to the solution of the self-dual Yang-Mills and Einstein equations has been found to have a remarkably clear realisation in the language of ‘harmonic spaces’ ([3]-[7]). In fact harmonic or twistor spaces admit supersymmetrisation, yielding a remarkably simple supersymmetrisation of the SDYM and Einstein equations, which is much more straightforward, and moreover independent of the N-extension (where N is the number of independent supersymmetries), than the supersymmetrisations of the corresponding full non-self-dual theories, for which the supersymmetrisation for each extension N has to be considered anew. All N-extended supersymmetric theories may

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therefore be treated on an equal footing [8, 9]. Moreover, for the self-dual super Yang-Mills theories, there exists a remarkable ‘matreoshka’-like nested structure [8] in which the $N = 0$ solution data may be dressed-up to higher N solution data in a basically algebraic fashion using solutions of first-order equations.

The list of four-dimensional theories (which may equally well be considered to be in complexified space or in real spaces of Euclidean (4,0) or Kleinian (2,2) signature) amenable to the twistor transform is therefore quite large and includes

- Self-dual Yang-Mills (SDYM) equations, for any semisimple gauge group.
- All N -extended ($N = 1, \dots, 4$) supersymmetrisations of the latter.
- Self-dual Einstein equations, with or without cosmological constant.
- All N -extended Poincaré and conformal self-dual supergravities.
- The full (i.e. non-self-dual) $N=3$ super-Yang-Mills theory (even in the Minkowskian (3,1) signature).

In this talk, we shall describe the harmonic space versions of the twistor transform for all the above theories. The crucial feature allowing the applicability of the twistor transform to field theories is the possibility of presenting the equations of motion in the form of algebraic constraints amongst the components of some curvature tensor, the paradigmatic example being the Yang-Mills self-duality equations. In particular, the constraints take the form

$$[\mathcal{D}_{\alpha i}, \mathcal{D}_{\beta j}] = \epsilon_{\alpha\beta} F_{ij},$$

where α, β are spinor indices of some group having skew-symmetric invariant $\epsilon_{\alpha\beta}$, i, j are some other indices or labels, and F_{ij} are the non-zero curvatures representing the obstruction to Frobenius’ integrability. Twistor or harmonic space is an auxiliary space in which the curvature is zero in some ‘analytic’ subspaces, allowing the use of ‘Frobenius variables’ to reduce the system. In the harmonic space setting a transformation to such variables converts the system to a set of Cauchy-Riemann-like (CR) equations, thereby reducing the problem to that of reconstructing the original variables from the ‘analytic’ data (satisfying these CR equations). The crucial idea of harmonic space analyticity is closely related to the concepts of quaternionic and Fueter analyticity, to which Feza Gürsey, whom we all loved so dearly, devoted so much attention. It is therefore especially appropriate to present these ideas at this meeting dedicated to his memory. In fact it was precisely in Feza’s last paper (with V. Ogievetsky and M. Evans)[4] that the intimate relation between quaternionic and harmonic space analyticities was clarified. That paper was completed shortly after Feza’s untimely death and we feel it appropriate to quote the dedication to Feza contained in its manuscript, which Physical Review refused to include in the published version.

“Feza Gürsey, a fine human being and outstanding physicist, passed away on April 13, 1992. He is a coauthor of the present paper, which is one of a series of his works devoted to quaternionic aspects of four-dimensional field theories, a field in which he was a pioneer. Feza enthusiastically participated in the writing of this paper, even as he fought the disease to which he finally succumbed. Sadly, he did not live long enough to approve the paper’s final version, and so bears no responsibility for whatever shortcomings it may possess. It was a great joy and privilege to work with Feza, and to benefit from his fertile mind and keen intelligence. The experience of working with him and the wonderful personality of Feza Gürsey will abide forever in the memories of the two other authors. ”

2. From 2D complex to 4D quaternionic analyticity

In two-dimensional Euclidean space the two *real* coordinates may be quite naturally combined into a *single complex number* $x^\mu = \{x^1, x^2\} \rightarrow z = x^1 + ix^2$ and the most general conformal coordinate transformation in two dimensions is the *analytic* transformation

$$z' = f(z), \quad \bar{z}' = \bar{f}(\bar{z}). \quad (1)$$

In virtue of the Cauchy-Riemann condition, $\frac{\partial}{\partial \bar{z}} f(z) = 0$, its d'Alembertian vanishes, $\frac{\partial}{\partial z} \frac{\partial}{\partial \bar{z}} f(z) = 0$.

Similarly naturally, four dimensional coordinates may be combined into a quaternion. In spinor notation we have

$$x^\mu \rightarrow q = x^{\alpha\dot{\alpha}} = \begin{pmatrix} x^0 - ix^3 & -ix^1 - x^2 \\ -ix^1 + x^2 & x^0 + ix^3 \end{pmatrix} = x^0 + e_a x^a \quad (2)$$

where the Pauli matrices represent the algebra of the quaternionic units, $e_a = -i\sigma_a$

$$e_a e_b = -\delta_{ab} + \epsilon_{abc} e_c. \quad (3)$$

Analytic transformations (1) are fundamental to 2D-conformal field theories. Feza Gürsey often wondered whether there exist 4D theories in which some form of quaternionic analyticity plays a correspondingly crucial rôle [10], [11]. However, the notion of quaternionic analyticity is rather delicate and there are several possible forms, some of which being too restrictive to be applicable to field theories. For instance, the straightforward generalisation of the Cauchy-Riemann condition

$$\frac{\partial}{\partial \bar{q}} f = \frac{1}{2} \left(\frac{\partial}{\partial x^0} + \frac{1}{3} e_a \frac{\partial}{\partial x^a} \right) f = 0 \quad (4)$$

where $\frac{\partial}{\partial \bar{q}}$ satisfies $\frac{\partial}{\partial \bar{q}} q = 0$ and $\frac{\partial}{\partial \bar{q}} \bar{q} = 1$ is well known (see e.g. [12]) to allow only a linear solution $f = a + qb$, with constant quaternions a and b , because of the noncommutativity of quaternions.

Fueter quaternion analyticity [13, 14], however, is less restrictive. This defines an analytic function of a quaternion q , as a Weierstrass-like series

$$f(q) = \sum a_n q^n, \quad (5)$$

where the coefficients a_n are real or complex numbers (or quaternions, but multiplying q^n on only one side, e.g. left as in (5)). Such a function obeys a Cauchy-Riemann-like condition, of the *third order* in derivatives and is therefore in general not a harmonic function ($\square f(q) \neq 0$), although it is bi-harmonic ($\square^2 f(q) = 0$). Moreover, it is not invariant under $SO(4)$ rotations [14].

In self-dual and $N = 2$ supersymmetric theories, however, manifolds of quaternionic character namely quaternionic-Kähler and hyper-Kähler manifolds naturally arise [15]. In these theories hyper-Kähler and quaternionic structures are related to yet another notion of analyticity, namely harmonic-space analyticity, which we shall explain.

3. Harmonic space

Harmonic space [3] is essentially an enlargement of four dimensional space-time, which may be thought of in terms of the coset space $\frac{\text{Poincaré group}}{\text{Lorentz group}}$, to coset space $\frac{\text{Poincaré group}}{SU(2) \times U(1)} = \frac{\text{Poincaré group}}{\text{Lorentz group}} \times \frac{SU(2)}{U(1)}$ (for the case of signature (4,0)). This space has additional coordinates

parametrising the two-sphere $S^2 = \frac{SU(2)}{U(1)}$. Of course, one could choose polar (θ, ϕ) or stereographic (z, \bar{z}) coordinates to describe this sphere. However, it is in practice very useful to use a more abstract parametrisation using two fundamental representations of the $SU(2)$ algebra, $u_{\dot{\alpha}}^{\pm}$ (where $\dot{\alpha}$ is an $SU(2)$ spinor index and \pm denote $U(1)$ charges), which are just spin $\frac{1}{2}$ spherical harmonics of S^2 , defined up to the $U(1)$ equivalence $u_{\dot{\alpha}}^{\pm} \sim e^{\pm\gamma} u_{\dot{\alpha}}^{\pm}$; $\gamma \in \mathbb{C}$ and satisfying the equations $\epsilon_{\dot{\alpha}\dot{\beta}} u^{+\dot{\alpha}} u^{-\dot{\beta}} = 1$. The further hermiticity condition $u_{\dot{\alpha}}^{-} = \overline{u^{+\dot{\alpha}}}$ yields two independent real variables. In the complexified setting, however, $u_{\dot{\alpha}}^{+}$ and $u_{\dot{\alpha}}^{-}$ are independent and an appropriate equivalence relation holds [4].

4. Self-dual Yang-Mills

The usual self-duality condition for the Yang-Mills field strength

$$F_{\mu\nu} = \frac{1}{2} \epsilon_{\mu\nu\rho\sigma} F_{\rho\sigma} , \quad (6)$$

basically says that the (0,1) part of the gauge field vanishes. This is better expressed in terms of 2-spinor notation in the form: $f_{\dot{\alpha}\dot{\beta}} = 0$ which is equivalent to the statement that the field strength curvature only contains the (1,0) Lorentz representation, i.e.

$$[\mathcal{D}_{\alpha\dot{\alpha}} , \mathcal{D}_{\beta\dot{\beta}}] = \epsilon_{\dot{\alpha}\dot{\beta}} f_{\alpha\beta} . \quad (7)$$

Now multiplying (7) by the two commuting spinors $u^{+\dot{\alpha}}, u^{+\dot{\beta}}$, one can compactly represent it as the vanishing of a curvature

$$[\nabla_{\alpha}^{+}, \nabla_{\beta}^{+}] = 0 , \quad (8)$$

where $\nabla_{\alpha}^{+} \equiv u^{+\dot{\alpha}} \nabla_{\alpha\dot{\alpha}}$, with linear system

$$\nabla_{\alpha}^{+} \varphi = 0 . \quad (9)$$

This is precisely the Belavin-Zakharov-Ward linear system for SDYM. Now the $u^{+\dot{\alpha}}$ are actually harmonics [3] on S^2 and it is better to consider these equations in an auxiliary space with coordinates $\{x^{\pm\alpha} \equiv x^{\alpha\dot{\alpha}} u_{\dot{\alpha}}^{\pm} , u_{\dot{\alpha}}^{\pm} ; u^{+\dot{\alpha}} u_{\dot{\alpha}}^{-} = 1\}$, where the harmonics are defined up to a $U(1)$ phase, and gauge covariant derivatives in this harmonic space are

$$\nabla_{\alpha}^{+} = \partial_{\alpha}^{+} + A_{\alpha}^{+} = \frac{\partial}{\partial x^{-\alpha}} + A_{\alpha}^{+} . \quad (10)$$

In this space (8) is actually not equivalent to the self-duality conditions. We also need

$$[D^{++}, \nabla_{\alpha}^{+}] = 0 , \quad (11)$$

where D^{++} is a harmonic space derivative which acts on negatively-charged harmonics to yield their positively-charged counterparts, i.e. $D^{++} u_{\dot{\alpha}}^{-} = u_{\dot{\alpha}}^{+}$, whereas $D^{++} u_{\dot{\alpha}}^{+} = 0$. This operator, in a fixed parametrisation, has also been considered by Newman (e.g. [16]). In ordinary x-space, when the harmonics are treated as parameters, the condition (11) is actually incorporated in the definition of ∇_{α}^{+} as a *linear* combination of the covariant derivatives. The system (8,11) is now *equivalent* to SDYM and has been considered by many authors, e.g. [4, 5]; the equivalence holding in spaces of signature (4,0) or (2,2), or in complexified space. In this regard, we should note that for real spaces, our understanding is completely clear for the Euclidean signature. For the (2,2) signature, the situation is richer and more intricate due to the noncompact nature of the rotation group and our present considerations concern only those signature (2,2) configurations which may be obtained by Wick rotation of (4,0) configurations.

Now, in (11) the covariant derivative (10) has pure-gauge form

$$\nabla_{\alpha}^{+} = \partial_{\alpha}^{+} + \varphi \partial_{\alpha}^{+} \varphi^{-1}. \quad (12)$$

and D^{++} is ‘short’ i.e. has no connection. This choice of frame (the ‘central’ frame) is actually inherited from the four-dimensional x-space and is not the most natural one for harmonic space. We may however change coordinates to a basis in which ∇_{α}^{+} is ‘short’ and D^{++} is ‘long’ (i.e. acquires a Lie-algebra-valued connection) instead. Namely,

$$\begin{aligned} \nabla_{\alpha}^{+} &= \partial_{\alpha}^{+} \\ \mathcal{D}^{++} &= D^{++} + V^{++}, \end{aligned} \quad (13)$$

a change of frame tantamount to a gauge transformation by the ‘bridge’ φ in (9). In this basis (the ‘analytic’ frame) the SDYM system (8,11) remarkably takes the form of a Cauchy-Riemann (CR) condition

$$\frac{\partial}{\partial x^{-\alpha}} V^{++} = 0 \quad (14)$$

expressing independence of half the x-coordinates. In virtue of passing to this basis the nonlinear SDYM equations (6) are in a sense trivialised: Any ‘analytic’ (i.e. satisfying (14)) function $V^{++} = V^{++}(x^{+\alpha}, u^{\pm})$ corresponds to some self-dual gauge potential. From any such V^{++} , by solving the *linear* equation

$$D^{++} \varphi = \varphi V^{++} \quad (15)$$

for the bridge φ , a self-dual vector potential may be recovered from the harmonic expansion:

$$\varphi \partial_{\alpha}^{+} \varphi^{-1} = u^{+\dot{\alpha}} A_{\alpha\dot{\alpha}}; \quad (16)$$

the linearity in the harmonics $u^{+\dot{\alpha}}$ being guaranteed by (11).

Solving (15) for an *arbitrary* analytic gauge algebra valued function V^{++} yields the *general* local self-dual solution. This correspondence between self-dual gauge potentials and holomorphic prepotentials V^{++} is a convenient tool for the explicit construction of local solutions of the self-duality equations.

Furthermore, in the analytic subspace of harmonic space (with coordinates $\{x^{+\alpha}, u_{\alpha}^{\pm}\}$), there exists an especially simple presentation of the infinite-dimensional symmetry group acting on solutions of the self-duality equations. It is the (apparently trivial) transformation $V^{++} \rightarrow V^{++\prime} = g^{++}$, where g^{++} depends in an arbitrary way on V^{++} and its derivatives as well as on the analytic coordinates themselves, modulo gauge transformations $V^{++} \rightarrow e^{-\lambda}(V^{++} + D^{++})e^{\lambda}$, where λ is also an arbitrary analytic function.

5. Supersymmetric self-dual Yang-Mills theories

Yang-Mills theories can be supersymmetrised to couple successively lower spin fields to the vector field. Since extended super Yang-Mills theories are massless theories, the components are classified by helicity and we have the following representation content in theories up to N=3:

$$\begin{array}{l} \text{helicity :} \\ N = 0 \\ N = 1 \\ N = 2 \\ N = 3 \end{array} \quad \begin{array}{cccccccc} 1 & \frac{1}{2} & 0 & -\frac{1}{2} & \frac{1}{2} & 0 & -\frac{1}{2} & -1 \\ f_{\alpha\beta} & & & & & & & f_{\dot{\alpha}\dot{\beta}} \\ f_{\alpha\beta} & \lambda_{\alpha} & & & & & \lambda_{\dot{\alpha}} & f_{\dot{\alpha}\dot{\beta}} \\ f_{\alpha\beta} & \lambda_{\alpha}^i & \overline{W} & & & W & \lambda_{\dot{\alpha}i} & f_{\dot{\alpha}\dot{\beta}} \\ f_{\alpha\beta} & \lambda_{\alpha}^i & W_i & \chi_{\dot{\alpha}} & \chi_{\alpha} & W^i & \lambda_{\dot{\alpha}i} & f_{\dot{\alpha}\dot{\beta}} \end{array} \quad (17)$$

In real Minkowski space fields in the left and right triangles are related by CPT conjugation but in complexified space or in a space with signature (4,0) or (2,2), we may set fields in one of the triangles to zero without affecting fields in the other triangle. If we set all the fields in the right (left) triangle to zero, the equations of motion reduce to the super (anti-) self-duality equations. For instance, the self-duality equations for the N=3 theory take the form

$$\begin{aligned}
\epsilon^{\beta\gamma}\mathcal{D}_{\gamma\dot{\beta}}f_{\alpha\beta} &= 0 \\
\epsilon^{\gamma\beta}\mathcal{D}_{\gamma\dot{\beta}}\lambda_{\beta}^i &= 0 \\
\epsilon^{\dot{\gamma}\dot{\alpha}}\mathcal{D}_{\alpha\dot{\gamma}}\chi_{\dot{\alpha}} &= -[\lambda_{\alpha}^k, W_k] \\
\mathcal{D}_{\alpha\dot{\beta}}\mathcal{D}^{\alpha\dot{\beta}}W_i &= \frac{1}{2}\epsilon_{ijk}\{\lambda^{\alpha j}, \lambda_{\alpha}^k\}.
\end{aligned} \tag{18}$$

We see that the spin 1 source current actually factorises into parts from the two triangles, so it manifestly vanishes for super self-dual solutions. The first equation in (18) is just the Bianchi identity for self-dual field-strengths. So apart from the self-duality condition (6), we have one equation for zero-modes of the covariant Dirac operator in the background of a self-dual vector potential (having (6) as integrability condition) and two further non-linear equations. However, any given self-dual vector potential actually *determines* the general (local) solution of the rest of the equations. This is the most striking consequence of the matreoshka phenomenon: the N=0 core determining the properties of the higher-N theories. Another consequence is that many conserved currents identically vanish in the super self-dual sector. For instance, since self-duality always implies the *source-free* second order Yang-Mills equations, the spin 1 source current vanishes for the entire matreoshka. Further, the usual Yang-Mills stress tensor clearly vanishes for self-dual fields:

$$T_{\alpha\dot{\alpha},\beta\dot{\beta}} \equiv f_{\alpha\dot{\beta}}f_{\alpha\beta} = 0 ;$$

and as a consequence of this, once one has put on further layers of the matreoshka, the super-currents generating supersymmetry transformations, which contain the stress tensor as well as its superpartners also identically vanish for super self-dual fields.

In N-independent form, (18) can be conveniently written as the following super curvature constraints in chiral superspace:

$$\begin{aligned}
\{\bar{\mathcal{D}}_{\dot{\alpha}}^i, \bar{\mathcal{D}}_{\dot{\beta}}^j\} &= \epsilon_{\dot{\alpha}\dot{\beta}}W^{ij} \\
\{\mathcal{D}_{\alpha i}, \mathcal{D}_{\beta j}\} &= 0 = [\mathcal{D}_{\alpha i}, \nabla_{\alpha\beta}] \\
\{\mathcal{D}_{\alpha j}, \bar{\mathcal{D}}_{\dot{\beta}}^i\} &= 2\delta_j^i\nabla_{\alpha\dot{\beta}}.
\end{aligned} \tag{19}$$

Having expressed the super self-duality equations in this form, the supersymmetrisation of the harmonic-twistor construction is straightforward. In harmonic superspaces with coordinates

$$\{x^{\pm\alpha} \equiv u_{\dot{\beta}}^{\pm}x^{\alpha\dot{\beta}}, \bar{\vartheta}_i^{\pm} \equiv u_{\dot{\alpha}}^{\pm}\bar{\vartheta}_i^{\dot{\alpha}}, \vartheta^{\alpha i}, u_{\alpha}^{\pm}\},$$

these take the form

$$\begin{aligned}
\{\mathcal{D}_{\alpha i}, \mathcal{D}_{\beta j}\} &= 0 = \{\bar{\mathcal{D}}^{+i}, \bar{\mathcal{D}}^{+j}\} \\
[\nabla_{\alpha}^+, \nabla_{\beta}^+] &= 0 = [\bar{\mathcal{D}}^{+i}, \nabla_{\alpha}^+] \\
\{\mathcal{D}_{\alpha j}, \bar{\mathcal{D}}^{+i}\} &= 2\delta_j^i\nabla_{\alpha}^+ \\
[\mathcal{D}_{\alpha i}, \nabla_{\beta}^+] &= 0,
\end{aligned} \tag{20}$$

where the gauge covariant derivatives are given by

$$\mathcal{D}_{\alpha i} = D_{\alpha i} + A_{\alpha i}, \quad \bar{\mathcal{D}}^{+i} = u^{+\dot{\alpha}}\bar{\mathcal{D}}_{\dot{\alpha}}^i = \bar{D}^{+i} + \bar{A}^{+i}, \quad \nabla_{\alpha}^+ = u^{+\dot{\alpha}}\nabla_{\alpha\dot{\alpha}} = \partial_{\alpha}^+ + A_{\alpha}^+, \tag{21}$$

and satisfy the equations

$$[D^{++}, \mathcal{D}_{\alpha i}] = [D^{++}, \bar{\mathcal{D}}^{+i}] = [D^{++}, \nabla_{\alpha}^+] = 0 . \quad (22)$$

The equations (20,22) are *equivalent* to (19) and (20) are consistency conditions for the following system of linear equations

$$\begin{aligned} \mathcal{D}_{\alpha i} \varphi &= 0 \\ \bar{\mathcal{D}}^{+i} \varphi &= 0 \\ \nabla_{\alpha}^+ \varphi &= 0, \end{aligned} \quad (23)$$

This system is extremely redundant, φ allowing the following transformation under the gauge group

$$\varphi \rightarrow e^{-\tau(x^{\alpha\dot{\alpha}}, \bar{\vartheta}_i^{\dot{\alpha}} \vartheta^{\alpha i})} \varphi e^{\lambda(x^{+\alpha}, \bar{\vartheta}_i^+, u_{\dot{\alpha}}^{\pm})}, \quad (24)$$

where τ and λ are arbitrary functions of the variables shown, without affecting the constraints (20). These constraints therefore allow an economic choice of chiral-analytic basis in which the bridge ϕ and the prepotential V^{++} depend only on *positively* $U(1)$ -charged, *barred* Grassmann variables, viz. $\bar{\vartheta}_i^+$, being independent of $\vartheta^{i\alpha}$ and $\bar{\vartheta}_i^-$. In this basis, φ too is independent of $\vartheta^{i\alpha}$ and $\bar{\vartheta}_i^-$; its non-analyticity manifesting itself in its dependence on $x^{-\alpha}$. Moreover, consistently with the commutation relations (20), the covariant spinor derivatives take the form $\mathcal{D}_{\alpha i} = \frac{\partial}{\partial \vartheta^{\alpha i}}$, $\bar{\mathcal{D}}^i = 2\vartheta^{\alpha i} \nabla_{\alpha}^+$. The super self-duality conditions (20,22) are therefore equivalent to the *same* system of equations as the N=0 SDYM equations, viz. (9,11), except that now φ and A_{α}^+ are chiral superfields depending on $\{x^{\pm\alpha}, \bar{\vartheta}_i^+, u_{\dot{\alpha}}^{\pm}\}$ [8]. As for the N=0 case, we may express this system in the form of analyticity conditions for the harmonic space connection superfield V^{++} :

$$\frac{\partial}{\partial x^{-\alpha}} V^{++}(x^{+\alpha}, \bar{\vartheta}^{+i}, u_{\dot{\alpha}}^{\pm}) = 0. \quad (25)$$

and the bridge φ to the central basis may be found by solving (15). Fields solving (18) may then be obtained by inserting solutions φ of (15) into the expression

$$\varphi \partial_{\alpha}^+ \varphi^{-1} = u^{+\dot{\alpha}} A_{\alpha\dot{\alpha}}(x^{\alpha\dot{\alpha}}, \bar{\vartheta}_i^{\dot{\alpha}}), \quad (26)$$

(the left side being guaranteed to be linear in u^+), and expanding the superfield vector potential on the right to obtain the component multiplet satisfying (18) thus:

$$A_{\alpha\dot{\beta}}(x, \bar{\vartheta}) = A_{\alpha\dot{\beta}}(x) + \bar{\vartheta}_{\dot{\beta}i} \lambda_{\alpha}^i(x) + \epsilon^{ijk} \bar{\vartheta}_{\dot{\alpha}j} \bar{\vartheta}_i^{\dot{\alpha}} \nabla_{\alpha\dot{\beta}} W_k(x) + \epsilon^{ijk} \bar{\vartheta}_{\dot{\alpha}i} \bar{\vartheta}_j^{\dot{\alpha}} \bar{\vartheta}_k^{\dot{\gamma}} \nabla_{\alpha\dot{\gamma}} \chi_{\dot{\beta}}(x). \quad (27)$$

It is remarkable that super self-duality implies the absence of higher-order terms in $\bar{\vartheta}$. In fact any N=0 solution completely and recursively determines its higher-N extensions [8].

The most general infinite-dimensional group of transformations of super-self-dual solutions acquires a transparent form in the analytic harmonic superspace with coordinates $\{x^{+\alpha}, \bar{\vartheta}^{+i}, u_{\dot{\alpha}}^{\pm}\}$. As for the $N = 0$ case, it is given by the transformation

$$V^{++} \rightarrow V^{++'} = g^{++}(V^{++}, x^{+\alpha}, \bar{\vartheta}^{+i}, u_{\dot{\alpha}}^{\pm}), \quad (28)$$

where g^{++} is an arbitrary doubly $U(1)$ -charged analytic algebra-valued functional, modulo gauge transformations $V^{++} \rightarrow e^{-\lambda}(V^{++} + D^{++})e^{\lambda}$, where λ is also an arbitrary analytic function. This group has an interesting subgroup of transformations

$$V^{++} \rightarrow V^{++'} = V^{++}(x^{+'}, \bar{\vartheta}^{+'}, u'), \quad (29)$$

induced by diffeomorphisms of the analytic harmonic superspace

$$x^{+\alpha'} = x^{+\alpha'}(x^+, \bar{\vartheta}^+, u), \quad \bar{\vartheta}^{+i'} = \bar{\vartheta}^{+i'}(x^+, \bar{\vartheta}^+, u), \quad u' = u'(x^+, \bar{\vartheta}^+, u). \quad (30)$$

6. N=3 (non-self-dual) super Yang-Mills theory

As we have seen, the spin 1 source currents of all super self-dual theories vanish because they factorise into parts from the two triangles in (17). It turns out that we can restore these source currents and solve the full (i.e. non-self-dual) super Yang-Mills equations by intermingling self-dual and anti-self-dual holomorphic data [17]; and this works *exactly* for the N=3 case. Again the crucial feature is the presentability of the thrice-extended super Yang-Mills equations in the form of the super-curvature constraints [18]

$$\begin{aligned}\{\mathcal{D}_{i\alpha}, \mathcal{D}_{j\beta}\} &= \epsilon_{\alpha\beta} W_{ij} \\ \{\bar{\mathcal{D}}_{\dot{\alpha}}^i, \bar{\mathcal{D}}_{\dot{\beta}}^j\} &= \epsilon_{\dot{\alpha}\dot{\beta}} W^{ij} \\ \{\mathcal{D}_{i\alpha}, \bar{\mathcal{D}}_{\dot{\beta}}^j\} &= 2\delta_i^j \nabla_{\alpha\dot{\beta}},\end{aligned}\tag{31}$$

where $\mathcal{D}_A \equiv \partial_A + A_A = (\nabla_{\alpha\dot{\beta}}, \mathcal{D}_{i\alpha}, \bar{\mathcal{D}}_{\dot{\beta}}^j), i, j = 1, 2, 3$, are gauge-covariant super-derivatives. These constraints are purely kinematical for N=1,2 but are equivalent to the dynamical equations for the component fields for N=3 [19]. Now in order to present these as zero-curvature conditions in some harmonic space, the appearance of invariants of both simple parts of the Lorentz group $(\epsilon_{\alpha\beta}, \epsilon_{\dot{\alpha}\dot{\beta}})$ requires the ‘harmonisation’ of the entire Lorentz group. This allows the consideration of all possible signatures, with the corresponding harmonic spaces being given by:

$$\begin{aligned}\textit{Euclidean} \quad (4,0): & \quad \frac{\text{Poincaré group}}{\text{Lorentz group}} \times \frac{SU(2)}{U(1)} \times \frac{SU(2)}{U(1)} \\ \textit{Lorentzian} \quad (3,1): & \quad \frac{\text{Poincaré group}}{\text{Lorentz group}} \times \frac{SL(2, \mathbb{C})}{L(1, \mathbb{C})} \\ \textit{Kleinian} \quad (2,2): & \quad \frac{\text{Poincaré group}}{\text{Lorentz group}} \times \frac{SL(2, \mathbb{R})}{SO(2)} \times \frac{SL(2, \mathbb{R})}{SO(2)}\end{aligned}$$

To thus harmonise the entire Lorentz group, we need to introduce harmonics with both dotted and undotted indices: $u_{\dot{\alpha}}^+, u_{\dot{\alpha}}^-$ and $v_{\alpha}^{\oplus}, v_{\alpha}^{\ominus}$, satisfying the constraints

$$u^{+\dot{\alpha}} u_{\dot{\alpha}}^- = 1, \quad v^{\oplus\alpha} v_{\alpha}^{\ominus} = 1$$

and having the hermiticity condition $\overline{u_{\dot{\alpha}}^+} = v^{\ominus\alpha}$ for the Lorentzian signature. Now in harmonic space with coordinates $u_{\dot{\alpha}}^{\pm}, u_{\dot{\alpha}}^{\mp}$ and $v_{\alpha}^{\oplus}, v_{\alpha}^{\ominus}$ and

$$\begin{aligned}x^{\pm\oplus} &= x^{\alpha\dot{\alpha}} u_{\dot{\alpha}}^{\pm} v_{\alpha}^{\oplus}, \quad x^{\pm\ominus} = x^{\alpha\dot{\alpha}} u_{\dot{\alpha}}^{\pm} v_{\alpha}^{\ominus}, \\ \vartheta^{i\oplus} &= \vartheta^{i\alpha} v_{\alpha}^{\oplus}, \quad \vartheta^{i\ominus} = \vartheta^{i\alpha} v_{\alpha}^{\ominus}, \quad \bar{\vartheta}_i^{\pm} = \bar{\vartheta}_i^{\dot{\alpha}} u_{\dot{\alpha}}^{\pm};\end{aligned}$$

The superspace constraints (31) are equivalent to the following system of equations in harmonic superspace

$$\begin{aligned}\{\bar{\mathcal{D}}^{+i}, \bar{\mathcal{D}}^{+j}\} &= 0 = \{\mathcal{D}_i^{\oplus}, \mathcal{D}_j^{\oplus}\} \\ \{\bar{\mathcal{D}}^{+j}, \mathcal{D}_i^{\oplus}\} &= 2\nabla^{+\oplus} \\ [D^{++}, \bar{\mathcal{D}}^{+j}] &= 0 = [D^{++}, \mathcal{D}_i^{\oplus}] = [D^{++}, \nabla^{+\oplus}] \\ [D^{\oplus\oplus}, \bar{\mathcal{D}}^{+j}] &= 0 = [D^{\oplus\oplus}, \mathcal{D}_i^{\oplus}] = [D^{\oplus\oplus}, \nabla^{+\oplus}] \\ [D^{++}, D^{\oplus\oplus}] &= 0,\end{aligned}\tag{32}$$

where

$$D^{++} = u^{+\dot{\alpha}} \frac{\partial}{\partial u^{-\dot{\alpha}}}, \quad D^{\oplus\oplus} = v^{\oplus\alpha} \frac{\partial}{\partial v^{\ominus\alpha}}.$$

Now as before, we can go to an ‘analytic frame’ in which the covariant derivatives $(\bar{\mathcal{D}}^{+j}, \mathcal{D}_i^\oplus, \nabla^{+\oplus})$ lose their connection parts and the derivatives $(D^{++}, D^{\oplus\oplus})$ acquire connections $(V^{++}, V^{\oplus\oplus})$ instead. For the latter, (32) are just the generalised Cauchy-Riemann ‘analyticity’ conditions

$$\begin{aligned}\frac{\partial}{\partial \bar{\vartheta}_i^-} V^{++} &= 0 = \frac{\partial}{\partial \bar{\vartheta}_i^-} V^{\oplus\oplus} \\ \frac{\partial}{\partial \bar{\vartheta}^{\oplus 1}} V^{++} &= 0 = \frac{\partial}{\partial \bar{\vartheta}^{\oplus 1}} V^{\oplus\oplus} \\ \frac{\partial}{\partial x^{-\ominus}} V^{++} &= 0 = \frac{\partial}{\partial x^{-\ominus}} V^{\oplus\oplus}\end{aligned}$$

together with the zero-curvature condition

$$D^{++}V^{\oplus\oplus} - D^{\oplus\oplus}V^{++} + [V^{++}, V^{\oplus\oplus}] = 0,$$

which relates the two harmonic space connections $(V^{++}, V^{\oplus\oplus})$. Analytic $(V^{++}, V^{\oplus\oplus})$ satisfying this relationship therefore encode the solution of N=3 super Yang-Mills theory [20].

7. Self-dual gravity and supergravity

Analogously to (7) self-dual gravity may be described by the equation

$$[\mathcal{D}_{\alpha\dot{\alpha}}, \mathcal{D}_{\beta\dot{\beta}}] = \epsilon_{\dot{\alpha}\dot{\beta}} \bar{R}_{\alpha\beta}, \quad (33)$$

where now the covariant derivative contains a vierbein as well as a connection,

$$\mathcal{D}_{\alpha\dot{\alpha}} = E_{\alpha\dot{\alpha}}^{\mu\dot{\beta}} \partial_{\mu\dot{\beta}} + \omega_{\dot{\alpha}\alpha}, \quad (34)$$

so (33) is not only a curvature constraint on the components of the connection, but also a zero-torsion condition on the vierbein. Moreover, since the Riemann tensor has irreducible components

$$\begin{aligned}R_{\alpha\beta} &\equiv C_{(\alpha\beta\gamma\delta)} \Gamma^{\gamma\delta} + R_{(\alpha\beta)(\dot{\gamma}\dot{\delta})} \Gamma^{\dot{\gamma}\dot{\delta}} + \frac{1}{6} R \Gamma_{\alpha\beta}, \\ R_{\dot{\alpha}\dot{\beta}} &\equiv C_{(\dot{\alpha}\dot{\beta}\dot{\gamma}\dot{\delta})} \Gamma^{\dot{\gamma}\dot{\delta}} + R_{(\gamma\delta)(\dot{\alpha}\dot{\beta})} \Gamma^{\gamma\delta} + \frac{1}{6} R \Gamma_{\dot{\alpha}\dot{\beta}},\end{aligned}$$

where $C_{(\dot{\alpha}\dot{\beta}\dot{\gamma}\dot{\delta})}$ ($C_{(\alpha\beta\gamma\delta)}$) are the (anti-) self-dual components of the Weyl tensor, $R_{(\alpha\beta)(\dot{\gamma}\dot{\delta})}$ are the components of the tracefree Ricci tensor, R is the scalar curvature, $(\Gamma^{\gamma\delta}, \Gamma^{\dot{\gamma}\dot{\delta}})$ are generators of the tangent space gauge algebra, self-duality, i.e. the vanishing of $R_{\dot{\alpha}\dot{\beta}}$ clearly implies that the curvature takes values only in one $SU(2)$ algebra with generators $\Gamma^{\gamma\delta}$, so we may work in a ‘self-dual gauge’ in which the connection also takes values only in this $SU(2)$, i.e. only this half of the tangent space group is localised, while the second $SU(2)$ remains rigid. Restricting the holonomy group in this fashion, the curvature part of (33) is automatic, these equations reducing to the zero torsion conditions on the vierbein. Now, since we have to deal with the vanishing of torsions, the harmonic space system equivalent to (33) is rather different to that in the self-dual Yang-Mills case. It takes the form

$$\begin{aligned}[\mathcal{D}_\alpha^+, \mathcal{D}_\beta^+] &= 0 \\ [\mathcal{D}^{++}, \mathcal{D}_\alpha^+] &= 0 \\ [\mathcal{D}_\alpha^+, \mathcal{D}_\beta^-] &= 0 \quad (\text{modulo } R_{\alpha\beta}) \\ [\mathcal{D}^{++}, \mathcal{D}_\alpha^-] &= \mathcal{D}_\alpha^+.\end{aligned} \quad (35)$$

Now, going to Frobenius coordinates

$$x^{\mu a} \rightarrow x_h^{\mu\pm} = x_h^{\mu\pm}(x^{\mu a} u_a^\pm, u_a^\pm), \quad (36)$$

in which the covariant derivative $\mathcal{D}_\alpha^+ = \partial_\alpha^+$, the partial derivative, all the dynamics gets concentrated in the vielbeins and connection components of

$$\mathcal{D}^{++} = \partial^{++} + H^{++\mu+} \partial_{h\mu}^- + (x_h^{\mu+} + H^{++\mu-}) \partial_{h\mu}^+ + \omega^{++}.$$

These may be solved for [6] in terms of an arbitrary analytic prepotential \mathcal{L}^{+4} and the problem reduces to finding the explicit functions (36) for any specified choice of \mathcal{L}^{+4} . Inverting the transformation (36) the self-dual vierbein then allows itself to be decoded. An explicit illustration of the procedure may be found in [6], where the simplest monomial choice of prepotential, $\mathcal{L}^{+4} = g(x_h^{1+} x_h^{2+})^2$, where g is a dimensionful parameter, is shown to correspond to the self-dual Taub-NUT metric.

Remarkably, the N-extended supersymmetric self-duality equations allow themselves to be expressed in chiral superspace in the same form as (33),

$$[\mathcal{D}_{B\dot{\beta}}, \mathcal{D}_{A\dot{\alpha}}] = \epsilon_{\dot{\alpha}\dot{\beta}} R_{AB}, \quad (37)$$

except that now the indices A, B are ‘superindices’ of the superalgebra $OSp(N|1)$. The explicit construction of the self-dual super-vielbein therefore closely follows that for the non-supersymmetric case. This yields interesting non-trivial supersymmetrisations of hyper-Kähler manifolds. In [9] we construct some explicit examples of super deformations of flat space (with curvature only in the odd directions) and of Taub-NUT space.

8. Open problems

We have discussed a large class of four dimensional integrable systems allowing solution using the harmonic-twistor transform. Our considerations have raised a number of interesting questions. Integrability in two dimensions is known to imply remarkable constraints on the S-matrix yielding factorisation into two-particle amplitudes. Whether the integrability of the four dimensional theories discussed here has any analogous consequences, either for these theories themselves or for their dimensional reductions, remains an open question. This question is especially interesting for the full N=3 Yang-Mills theory, which is known to be an ultraviolet finite field theory. A further intriguing open question is what class of non-self-dual solutions to the usual N=0 Yang-Mills equations can be obtained by reduction of this supersymmetric construction; and whether the existence of two spectral parameters (corresponding to the two sets of harmonics) yields new classes of lower dimensional exactly solvable systems.

The remarkable conjunction of maximal supersymmetrisation, ultraviolet finiteness, and classical integrability in the sense described here, suggests the need to investigate the full (non-self-dual) Poincaré and conformal supergravity theories in this light as well. The corresponding super-twistor construction remains an open problem.

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