

**q-Epsilon Tensor for Quantum  
and Braided Spaces****Shahn Majid**

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# $q$ -EPSILON TENSOR FOR QUANTUM AND BRAIDED SPACES

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**Abstract** The machinery of braided geometry introduced previously is used now to construct the  $\epsilon$  ‘totally antisymmetric tensor’ on a general braided vector space determined by R-matrices. This includes natural  $q$ -Euclidean and  $q$ -Minkowski spaces. The formalism is completely covariant under the corresponding quantum group such as  $\widetilde{SO}_q(4)$  or  $\widetilde{SO}_q(1,3)$ . The Hodge  $*$  operator and differentials are also constructed in this approach.

## 1 Introduction

In this paper we apply the systematic theory of braided geometry introduced during the last few years by the author[1][2][3][4][5] to the problem of defining the totally antisymmetric tensor  $\epsilon_{ijkl}$  and other antisymmetrisers on quantum spaces of R-matrix type, for the first time in a general way.

Braided-geometry differs from other approaches to  $q$ -deforming physics in that the deformation is put directly into non-commutativity or ‘braid statistics’ of the tensor product of independent systems. Individual algebras also tend to be non-commutative (as in non-commutative geometry) but this is a secondary phenomenon. The theory is modelled on ideas of super-geometry with a braiding  $\Psi$  (typically defined by a parameter  $q$ ) in the role of  $\pm 1$  for usual bose or fermi statistics. It turns out that this point of view is rather powerful and using it a great many problems encountered in other approaches are immediately overcome.

The starting point of braided geometry is that quantum group covariance, unlike usual group covariance, induces braid statistics[1][2]. The quantum group plays the role of  $\mathbb{Z}_2$ -grading in the

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theory of super-symmetry even when the quantum group is very far from discrete (e.g. when it is the  $q$ -deformed Lorentz group). The systematic development of braided geometry has been a matter of going back to basics and re-inventing from scratch the most fundamental concepts in physics on this basis, layer by layer. After covariance, the next layer is that of coaddition on quantum spaces, introduced in [3]. Once one can add vectors on braided or  $q$ -deformed vector spaces, the next layer is differentiation, introduced in [4] as an infinitesimal coaddition:

$$\partial^i f(\mathbf{x}) = \text{coeff of } a_i \text{ in } f(\mathbf{a} + \mathbf{x}) \quad (1)$$

where the addition is a braided addition (so  $\mathbf{a}$  and  $\mathbf{x}$  braided-commute). Following this, there is also translation-invariant integration[6]. Braided matrices, traces etc were also introduced in [1][2]. The approach also links up with the more usual approach based on quantum forms and non-commutative geometry by pushing the arguments of [7] backwards (from partial derivatives  $\partial^i$  to exterior derivative  $d$ ). This is essentially known though some details are included in the present paper for completeness. It provides a constructive approach to  $d$ .

The antisymmetric tensor by contrast needs a conceptually new point of view in order to be able to apply this existing braided geometry. Here we present a novel and, we believe, powerful point of view about it. This point of view is useful even when  $q = 1$  where it corresponds to the view that the exterior algebra of forms can and should be viewed as a super-space with co-ordinates  $\theta_i$ , say. Usually, one realises super-spaces using exterior algebras, but our point of view is the reverse of this. The braided geometry applies just as well to super-spaces and their  $q$ -deformations as to bosonic-spaces and their deformations, so we can apply it at once to the exterior algebra without effort. In particular it is natural for us to define

$$\epsilon^{i_1 i_2 \dots i_n} = \frac{\partial}{\partial \theta_{i_1}} \frac{\partial}{\partial \theta_{i_2}} \dots \frac{\partial}{\partial \theta_{i_n}} \theta_1 \theta_2 \dots \theta_n \quad (2)$$

on any reasonable  $n$ -dimensional braided space with top form  $\theta_1 \theta_2 \dots \theta_n$ . We also construct the Hodge  $*$ -operator and interior products on forms in this setting.

Finally, important examples such as  $q$ -Euclidean and  $q$ -Minkowski spaces are also known in this framework of braided geometry[8][9], which examples are compatible too with the earlier ideas of [10][11][12] based on spinors. Hence our results apply at once to these important braided spaces.

During the preparation of this paper there appeared [13] in which the  $q$ -epsilon tensor in the case of  $q$ -Minkowski space was found directly by computer and used to develop Hodge theory and scalar electrodynamics. Our general formulation in Section 3 is motivated in part by this. We would also like to mention [14] where  $q$ -epsilon tensors for  $SO_q(n)$ -covariant Euclidean spaces were considered, again rather explicitly. The tensor for  $GL_q$ -covariant quantum planes is even more well known. By contrast with such specific examples, we present here a uniform R-matrix approach.

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### Preliminaries on braided vector spaces

Here we recall the formulation in [3] of braided vector and covector spaces, and strengthen their construction slightly for our purposes. The position co-ordinates  $\mathbf{x} = \{x_i\}$  form a braided-covector space, while their differentials  $\partial^i$  form a braided vector space[4]. Throughout this paper, we treat only spaces of this type, i.e. braided versions of  $\mathbb{R}^n$ .

The input data for these constructions are a pair of R-matrices  $R, R' \in M_n \otimes M_n$  such that[3]

$$R'_{12}R_{13}R_{23} = R_{23}R_{13}R'_{12}, \quad R'_{23}R_{13}R_{12} = R_{12}R_{13}R'_{23}$$

$$R_{12}R_{13}R_{23} = R_{23}R_{13}R_{12}, \quad (PR + 1)(PR - 1) = 0, \quad R_{21}R' = R'_{21}R$$

where  $P$  is the usual permutation matrix. These are enough to ensure that there are braided vector and covector spaces

$$V^{\vee}(R', R) = \{x_i\} : \mathbf{x}_1\mathbf{x}_2 = \mathbf{x}_2\mathbf{x}_1R', \quad V(R', R) = \{v^i\} : \mathbf{v}_1\mathbf{v}_2 = R'\mathbf{v}_2\mathbf{v}_1$$

with braided coaddition  $\mathbf{x}'' = \mathbf{x} + \mathbf{x}'$ ,  $\mathbf{v}'' = \mathbf{v} + \mathbf{v}'$  where these obey the same relations provided  $\mathbf{x}, \mathbf{v}$  and their identical primed copies have braid statistics

$$\mathbf{x}'_1\mathbf{x}_2 = \mathbf{x}_2\mathbf{x}'_1R, \quad \mathbf{v}'_1\mathbf{v}_2 = R\mathbf{v}_2\mathbf{v}'_1.$$

There are also braid statistics between  $\mathbf{x}$  and  $\mathbf{v}$  etc. Mathematically, they form braided-Hopf algebras in the braided category of  $A(R)$ -comodules where  $A(R)$  is the usual quantum group or bialgebra associated to  $R$ . For regular R-matrices they also live in the braided category of  $\tilde{A}$ -comodules, where  $A$  is a Hopf algebra quotient of  $A(R)$  and  $\tilde{A}$  is its dilatonic extension[3]. The transformation laws are  $\mathbf{x} \rightarrow \mathbf{x}\mathbf{t}\zeta$  and  $\mathbf{v} \rightarrow \zeta^{-1}\mathbf{t}^{-1}\mathbf{v}$  where  $\mathbf{t}$  is the quantum matrix generator and  $\zeta$  the dilaton.

To this framework, we now add the additional conditions

$$R_{12}R'_{13}R'_{23} = R'_{23}R'_{13}R_{12}, \quad R_{23}R'_{13}R'_{12} = R'_{12}R'_{13}R_{23}$$

$$R'_{12}R'_{13}R'_{23} = R'_{23}R'_{13}R'_{12}$$

so that there is a certain symmetry between  $R$  and  $R'$ . More precisely, we have a symmetry

$$R \leftrightarrow -R'$$

and can thereby define

$$\Lambda(R', R) \equiv V^*(-R, -R') = \{\theta_i\}, \quad \Lambda^*(R', R) \equiv V(-R, -R') = \{\phi^i\}$$

which we call respectively the braided covector space of antisymmetric tensors or *forms*  $\Lambda$  and braided vector space of *coforms*  $\Lambda^*$ . As a braided-Hopf algebra, the latter is the dual of the former. In our geometrical application, the differentials  $\theta_i = dx_i$  obey the algebra of forms, while the operators  $\frac{\partial}{\partial \theta_i}$  necessarily obey the algebra of coforms. The forms and coforms are covariant under the transformation  $\theta \rightarrow \theta\mathbf{t}\zeta$  and  $\phi \rightarrow \zeta^{-1}\mathbf{t}^{-1}\phi$  of a quantum group obtained from  $A(-R') = A(R')$ . We assume for convenience that this is the same as the quantum group obtained from  $A(R)$ . This is true in some generality, for example whenever  $PR' = f(PR)$  for some function  $f$ . It is also true for our  $q$ -Euclidean and  $q$ -Minkowski examples. See [8] for the latter.

## 2 Symmetric and antisymmetric tensors by differentiation

In [4] was introduced a general theory of partial differentiation  $\partial^i$  on braided spaces of the type above. This recovered all known cases and, moreover, works generally. If  $\{x_i\}$  are the position

co-ordinates, then  $\partial^i$  are given explicitly as the operators[4]

$$\frac{\partial}{\partial x_i}(\mathbf{x}_1 \cdots \mathbf{x}_m) = \mathbf{e}^i_1 \mathbf{x}_2 \cdots \mathbf{x}_m [m; R]_{1 \dots m} \quad (3)$$

where  $\mathbf{e}^i$  is a basis covector  $(\mathbf{e}^i)_j = \delta^i_j$ , i.e.

$$\frac{\partial}{\partial x_i}(x_{i_1} \cdots x_{i_m}) = \delta^i_{j_1} x_{j_2} \cdots x_{j_m} [m; R]_{i_1 \dots i_m}^{j_1 \dots j_m}.$$

Here

$$[m; R] = 1 + (PR)_{12} + (PR)_{12}(PR)_{23} + \cdots + (PR)_{12} \cdots (PR)_{m-1, m} \quad (4)$$

are the *braided integers* which we introduced for this purpose. One of the main theorems in [4] is that these differentiation operators on  $\{x_i\}$  obey the vector relations as for the  $\{v^i\}$ . One can say that the partial-derivatives  $R'$ -commute. They also obey a braided-Leibniz rule with braiding  $R$ [4].

Moreover, since the result is quite general, it holds just as well for the partial derivatives  $\frac{\partial}{\partial \theta_i}$ ,

$$\frac{\partial}{\partial \theta_i}(\theta_1 \cdots \theta_m) = \mathbf{e}^i_1 \theta_2 \cdots \theta_m [m; -R']_{1 \dots m} \quad (5)$$

on the algebra of forms  $\{\theta_i\}$ . We deduce that these differential operators obey the relations of the coforms  $\Lambda^*$ . This means that they  $-R$ -commute and obey a braided Leibniz rule with braiding  $-R'$ .

These theorems about the partial-derivatives are quite powerful, and we use them now. In particular we can differentiate any function  $f$  and will know that

$$v^{i_1 i_2 \dots i_m} = \frac{\partial}{\partial x_{i_1}} \cdots \frac{\partial}{\partial x_{i_m}} f$$

is an  $R'$ -symmetric tensor of rank  $m$ , in the sense

$$R_{ab}^{i_p i_{p+1}} v^{i_1 \dots i_{p-1} b a i_{p+2} \dots i_m} = v^{i_1 \dots i_m}, \quad \forall p = 1, \dots, m-1. \quad (6)$$

If  $f$  is a scalar function (quantum group covariant) then, because all our constructions in [4] are covariant, we will know that this tensor is likewise invariant. The same applies in the  $\theta$  space, in which case the tensors must be manifestly  $-R$ -symmetric i.e.,  $R$ -antisymmetric in the sense

$$R_{ab}^{i_p i_{p+1}} \phi^{i_1 \dots i_{p-1} b a i_{p+2} \dots i_m} = -\phi^{i_1 \dots i_m}, \quad \forall p = 1, \dots, m-1. \quad (7)$$

For a simple example of this idea, we suppose that the co-ordinate algebra  $\{x_i\}$  has a *radius function*  $r^2$  which is quantum-group invariant (a scalar under the transformation). Then

$$\eta^{ij} = \frac{\partial}{\partial x_i} \frac{\partial}{\partial x_j} r^2$$

is an  $R'$ -symmetric invariant tensor, which we call the *metric* associated to the radius function. If  $r^2$  is quadratic then  $\eta$  is an ordinary  $\mathbb{C}$ -number matrix. In nice cases it will be invertible. Moreover, invariance implies at once the first half of the identities

$$\eta_{ia}\eta_{jb}R^a{}_m{}^b{}_n = R^a{}_i{}^b{}_j\eta_{am}\eta_{bn}, \quad \eta_{ia}\eta_{jb}R'^a{}_m{}^b{}_n = R'^a{}_i{}^b{}_j\eta_{am}\eta_{bn} \quad (8)$$

We adopt the second half too in order to keep the symmetry between  $R$  and  $-R'$ . They have the meaning that then[8] the algebra of vectors and covectors are isomorphic via

$$x_i = \eta_{ia}v^a, \quad v^i = x_a\eta^{ai}, \quad \eta_{ja}\eta^{ia} = \delta^i{}_j = \eta_{aj}\eta^{ai}$$

so that the metric can be used to raise and lower indices for any operators behaving like the vectors and covectors. It clearly does the same job for raising and lowering indices of the forms and coforms by the symmetry.

We now use the same idea in the deformed super-space of forms. We say that the braided space has *form dimension*  $n$  if the algebra of forms has (up to normalisation) a unique element of highest degree  $n$ , which we call the *top form*  $\omega$ . In nice cases the form dimension will be the same as the number  $n$  of our co-ordinate generators and indeed, the top form will be  $\omega = \theta_1 \cdots \theta_n$ . We then define

$$\epsilon^{i_1 i_2 \cdots i_n} = \frac{\partial}{\partial \theta_{i_1}} \cdots \frac{\partial}{\partial \theta_{i_n}} \omega = \frac{\partial}{\partial \theta_{i_1}} \cdots \frac{\partial}{\partial \theta_{i_n}} \theta_1 \cdots \theta_n.$$

By the reasoning above, it will be  $R$ -antisymmetric.

If we want to have tensors with lower indices, we can obtain them also by differentiation of monomials in the co-ordinates. Thus an  $R'$ -symmetric tensor of rank  $m$  with lower indices means

$$x_{i_1 \cdots i_{p-1} b a i_{p+2} \cdots i_m} R'^a{}_i{}_p{}^b{}_i_{p+1} = x_{i_1 \cdots i_m}, \quad \forall p = 1, \cdots, m-1 \quad (9)$$

and an  $R$ -antisymmetric tensor with lower indices means

$$\theta_{i_1 \cdots i_{p-1} b a i_{p+2} \cdots i_m} R^a{}_i{}_p{}^b{}_i_{p+1} = -\theta_{i_1 \cdots i_m}, \quad \forall p = 1, \cdots, m-1. \quad (10)$$

The first of these can be obtained by applying any  $m$ -th order differential operator built from  $\frac{\partial}{\partial x_i}$  to monomials  $x_{i_1} \cdots x_{i_m}$ . Likewise, we can follow the same idea in form-space and obtain an  $R$ -antisymmetric tensor by applying any  $m$ -th order operator built from  $\frac{\partial}{\partial \theta_i}$  to  $\theta_{i_1} \cdots \theta_{i_m}$ . For example, we define

$$\epsilon_{i_1 \cdots i_n} = \frac{\partial}{\partial \theta_n} \cdots \frac{\partial}{\partial \theta_1} \theta_{i_1} \cdots \theta_{i_n}.$$

Its total  $R$ -antisymmetry is inherited this time from antisymmetry of the  $\theta_i$  co-ordinates in form-space.

**Proposition 2.1** *If the top form is  $\omega = \theta_1 \cdots \theta_n$  say, we have an explicit formula:*

$$\epsilon^{i_1 \cdots i_n} = ([n; -R']!)_{1_2 \cdots n}^{i_n \cdots i_1}, \quad \epsilon_{i_1 \cdots i_n} = ([n; -R']!)_{i_1 \cdots i_n}^{1_2 \cdots n}$$

where

$$\begin{aligned} [n; -R']! &= (\text{id} \otimes [2; -R']) (\text{id} \otimes [3; -R']) \cdots [n; -R'] \\ &= (1 - PR'_{n-1 \ n}) (1 - PR'_{n-2 \ n-1} + PR'_{n-2 \ n-1} PR'_{n-1 \ n}) \cdots \\ &\cdots (1 - PR'_{12} + PR'_{12} PR'_{23} - \cdots + (-1)^{n-1} PR'_{12} PR'_{23} \cdots PR'_{n-1 \ n}) \end{aligned}$$

is built from braided-integers (4).

**Proof** This follows directly from the above definitions by carefully iterating the formula (5) for braided-differentiation on the  $\theta$  co-ordinates.  $\square$

For example, in four dimensions, the braided 4-factorial matrix is

$$([4; -R']!)_{j_1 \cdots j_4}^{i_1 \cdots i_4} = [2, -R']_{b_3 b_4}^{i_3 i_4} [3, -R']_{a_2 a_3 a_4}^{i_2 b_3 b_4} [4; -R']_{j_1 j_2 j_3 j_4}^{i_1 a_2 a_3 a_4}$$

and is totally  $R$ -antisymmetric in its lower indices and in its upper-indices.

If one wants totally antisymmetric tensors of lower rank, these are provided by the lower braided-factorials  $[m, -R']!$  in the same way. For example

$$\frac{\partial}{\partial \theta_{i_1}} \frac{\partial}{\partial \theta_{i_2}} \theta_{j_1} \theta_{j_2} = [2; -R']_{j_1 j_2}^{i_2 i_1}, \quad [2; -R'] = 1 - PR'$$

$$\frac{\partial}{\partial \theta_{i_1}} \frac{\partial}{\partial \theta_{i_2}} \frac{\partial}{\partial \theta_{i_3}} \theta_{j_1} \theta_{j_2} \theta_{j_3} = ([3; -R']!)_{j_1 j_2 j_3}^{i_3 i_2 i_1}, \quad [3; -R']! = (1 - PR'_{23})(1 - PR'_{12} + PR'_{12} PR'_{23})$$

etc. One can also take different numbers of  $\theta$  derivatives and co-ordinates, giving tensors with different numbers of lower and upper indices, but again totally  $R$ -antisymmetric among the lower and among the upper.

If one wants tensors with totally  $R'$ -symmetric inputs and outputs, these are provided by braided-factorials  $[n : R]!$  and other braided-integers, with  $R$  in place of  $-R'$ . The symmetric and antisymmetric theory here are just the same construction, with a different choice of  $R$ -matrix. In this context, there is already proven a *braided-binomial theorem* in [4] for ‘counting’ such ‘braided permutations’.

### 3 Application to Hodge \*-operator

One can obtain still more tensors with symmetric or antisymmetric inputs and outputs by contraction along the lines of [13]. For example, given our  $\epsilon$  tensors it is natural to consider the contractions of  $n - m$  indices,

$$\mathcal{P}_{i_1 \dots i_m}^{j_1 \dots j_m} = \epsilon_{i_1 \dots i_m a_{m+1} \dots a_n} \epsilon^{j_1 \dots j_m a_n \dots a_{m+1}}. \quad (11)$$

These are typically proportional to projection operators, i.e.

$$\mathcal{P}_{i_1 \dots i_m}^{a_n \dots a_{m+1}} \mathcal{P}_{a_{m+1} \dots a_n}^{j_m \dots j_1} = d_m \mathcal{P}_{i_1 \dots i_m}^{j_m \dots j_1}$$

for some constants  $\{d_m\}$ . This is verified in examples, where also these constants can be determined. On the other hand, it appears to be a rather general feature which can be expected for any suitably nice  $R, R'$ -matrices. These  $\mathcal{P}$  project onto tensors with totally  $R$ -antisymmetric indices.

**Proposition 3.1** *There is a well-defined operator on forms given by*

$$\mathcal{P} : \Lambda \rightarrow \Lambda, \quad \mathcal{P}(\theta_{i_1} \dots \theta_{i_m}) = \mathcal{P}_{i_1 \dots i_m}^{a_m \dots a_1} \theta_{a_1} \dots \theta_{a_m} = d_m \theta_{i_1} \dots \theta_{i_m}$$

**Proof** One can expect the diagonal form in view of the above remarks since the products  $\theta_{i_1} \dots \theta_{i_m}$  are already  $R$ -antisymmetric. Here we check that  $\mathcal{P}$  as an operator is indeed well-defined. Indeed, the relations of  $\Lambda$  are respected as

$$\mathcal{P}(\theta_{i_1} \dots \theta_b \theta_a \dots \theta_{i_m}) R_{i_p}^{a b} = \epsilon_{i_1 \dots b a \dots i_m a_{m+1} \dots a_n} \epsilon^{b_1 \dots b_m a_n \dots a_{m+1}} \theta_{b_1} \dots \theta_{b_m} = -\mathcal{P}(\theta_{i_1} \dots \theta_{i_m})$$

for all  $p$  due to  $R$ -antisymmetry of  $\epsilon$ . We give this in detail because this and a similar consideration for the output of  $\mathcal{P}$  dictates the ordering of the indices in the action of  $\mathcal{P}$ .  $\square$

As another immediate application of our epsilon tensor one can write a general R-matrix formula for the quantum-determinant of the symmetry quantum group of our theory, namely

$$\det(\mathbf{t}) = d_0^{-1} \epsilon_{i_1 \dots i_n} t^{i_1 j_1} \dots t^{i_n j_n} \epsilon^{j_1 \dots j_n} = ([n; -R']!)_{i_1 \dots i_n}^{1 \dots n} t^{i_1 j_1} \dots t^{i_n j_n} ([n; -R']!)_{1 \dots n}^{j_1 \dots j_n}.$$

There is no metric needed here, but if one exists then it is easy to see from (8) that it can be used to turn any  $R'$ -symmetric or  $R$ -antisymmetric tensor with upper-indices to one with lower indices. In our setting with a unique top form, one can also expect that a totally antisymmetric tensor with  $n$  indices is unique up to a scale. In this case one has

$$\epsilon_{i_1 i_2 \dots i_n} = \lambda \eta_{i_1 a_1} \dots \eta_{i_n a_n} \epsilon^{a_1 \dots a_n} = \lambda \epsilon^{a_1 \dots a_n} \eta_{a_1 i_1} \dots \eta_{a_n i_n} \quad (12)$$

where  $\lambda$  is a constant depending on the metric.

Finally, one can use the  $\epsilon$  tensor in the usual way to define a Hodge  $*$ -operator, along the lines in [13] for  $q$ -Minkowski space, where  $\epsilon$  was found by hand. In the present formulation we have

**Proposition 3.2** *There is a well-defined operator on forms given by*

$$* : \Lambda \rightarrow \Lambda, \quad (\theta_{i_1} \dots \theta_{i_m})^* = \epsilon^{a_1 \dots a_m b_1 \dots b_{m+1}} \eta_{a_1 i_1} \dots \eta_{a_m i_m} \theta_{b_{m+1}} \dots \theta_{b_n} = H_{i_1 \dots i_m}^{a_1 \dots a_{m+1}} \theta_{a_{m+1}} \dots \theta_{a_n}. \quad (13)$$

**Proof** This time consistency with the relations of  $\Lambda$  follows using (8) after which we can then use antisymmetry of  $\epsilon$  as in the preceding proposition.  $\square$

The appropriate tensor  $H$  here has square which is typically proportional to the projectors in (11),

$$H_{i_1 \dots i_m}^{a_1 \dots a_{m+1}} H_{a_{m+1} \dots a_n}^{j_m \dots j_1} \propto \mathcal{P}_{i_1 \dots i_m}^{j_m \dots j_1}.$$

This too is verified in examples, where one also learns the constants of proportionality. It holds in some generality and means that  $*^2 \propto \text{id}$  on forms of each degree. This is analogous to the classical situation. Motivated by this one can also define the interior product of forms by a form  $\theta_i$  as the adjoint under  $*$  of multiplication by  $\theta_i$  in the exterior algebra. It obeys a graded  $\mathbb{Z}_2$ -Leibniz rule, as checked for  $q$ -Minkowski case in [13].

It is possible also to make a much more radical formulation of the interior product and Hodge  $*$  operations, based on the idea of differentiation on form space and not directly on  $\epsilon$ . Thus we

define the braided-interior product  $i$  and braided-Hodge operator  $\circ$  in the algebra of forms  $\{\theta_i\}$  by

$$i_{f(\theta)}g(\theta) = f(\partial)g(\theta), \quad f(\theta)^\circ = i_{f(\theta)}\theta_1 \cdots \theta_n$$

where  $f(\partial)$  consists of relacing  $\theta_i$  by the operators  $\partial_i = \eta_{ia} \frac{\partial}{\partial \theta_a}$ . For  $\circ$  we use  $\theta_1 \cdots \theta_n$  (say) as the top form. We have explicitly,

$$\begin{aligned} (\theta_{i_1} \cdots \theta_{i_m})^\circ &= \eta_{i_1 a_1} \cdots \eta_{i_m a_m} \frac{\partial}{\partial \theta_{a_1}} \cdots \frac{\partial}{\partial \theta_{a_m}} \theta_1 \cdots \theta_n \\ &= \eta_{i_1 a_1} \cdots \eta_{i_m a_m} ((\text{id} \otimes [n - m + 1; -R']) \cdots [n; -R'])_{12 \cdots n}^{a_m \cdots a_1 b_{m+1} \cdots b_n} \theta_{b_{m+1}} \cdots \theta_{b_n} \end{aligned} \quad (14)$$

For example, in four dimensions, the formulae are

$$\begin{aligned} (\theta_{i_1} \theta_{i_2} \theta_{i_3} \theta_{i_4})^\circ &= \lambda^{-1} \epsilon_{i_1 \cdots i_4} \\ (\theta_{i_1} \theta_{i_2} \theta_{i_3})^\circ &= \eta_{i_1 a_1} \eta_{i_2 a_2} \eta_{i_3 a_3} \epsilon^{b a_1 a_2 a_3} \theta_b \\ (\theta_{i_1} \theta_{i_2})^\circ &= \eta_{i_1 a_1} \eta_{i_2 a_2} (\text{id} \otimes [3; -R']) [4; -R']_{1234}^{a_2 a_1 b_1 b_2} \theta_{b_1} \theta_{b_2} \\ (\theta_i)^\circ &= \eta_{ia} [4; -R']_{1234}^{a b_1 b_2 b_3} \theta_{b_1} \theta_{b_2} \theta_{b_3}, \quad 1^\circ = \theta_1 \theta_2 \theta_3 \theta_4. \end{aligned}$$

Note that only the Hodge operations on  $n$  and  $n - 1$  degree forms involve the braided-factorial or  $\epsilon$  tensor directly. The other degrees usually involving  $\epsilon$  and normalisation integers  $\frac{1}{(n-m)!}$  etc. are obtained now by differentiation.

This second approach to the Hodge operation is different from the first one, though agrees when  $q = 1$  after suitable normalisations at each degree. In general we do not have that  $i$  is a graded derivation and we also do not have that  $\circ^2 \propto \text{id}$  on forms of a given degree. On the other hand, this second approach is conceptually quite simple and can be thought of in fact as a kind of ‘Fourier transform’ in form-space  $\{\theta_i\}$ . This is suggested by the interior product  $i_f$  appearing as braided-differentiation in form-space. Moreover, from this point of view one would expect  $\circ^2$  to be something like the braided-antipode  $S$  on the braided-Hopf algebra  $\{\theta_i\}$ , which is not simply  $\pm 1$  in the braided case. The technology for braided Fourier transform is in [6].

## 4 Differential forms

Until now we have considered the algebra of forms  $\theta_i$  in isolation, as some  $q$ -deformed superspace. For completeness we now consider both the co-ordinates  $\mathcal{M} = \{x_i\}$  and the forms  $\theta_i$  together

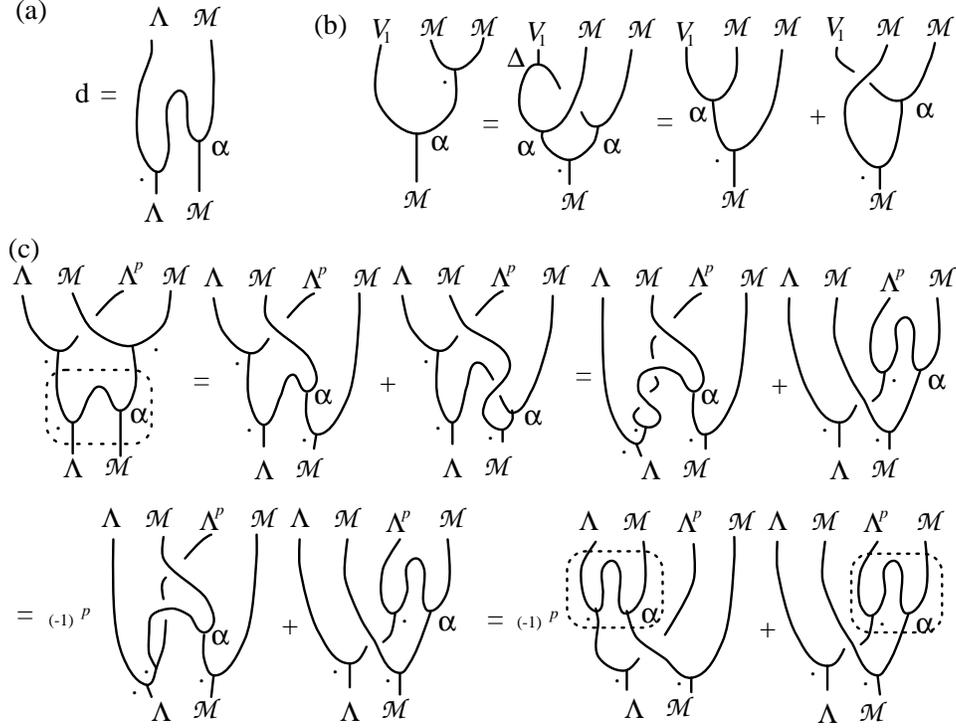


Figure 1: (a) definition of exterior differential  $d$  in braided approach (b) braided-Leibniz rule for action of partial derivatives and (c) proof of usual graded-Leibniz rule for  $d$

with  $\theta_i = dx_i$ . Thus we consider the *exterior algebra*

$$\Omega = \underline{\Lambda} \otimes \mathcal{M}, \quad \mathbf{x}_1 \theta_2 = \theta_2 \mathbf{x}_1 R \quad (15)$$

where the product is the braided tensor product with the cross-relations as stated. The essence of the braided tensor product here is that it keeps all constructions covariant, hence  $\Omega$  remains a comodule algebra under our background quantum group  $\tilde{A}$  (I would like to thank A. Sudbery for this remark[15]).

Next we consider  $\Omega$  as bi-graded with components

$$\Omega^{p|q} = \text{span}\{\theta_{i_1} \cdots \theta_{i_p} x_{j_1} \cdots x_{j_q}\}, \quad \Omega^p = \bigoplus_{q=0}^{\infty} \Omega^{p|q}, \quad \Omega^{p|q} = \bigoplus_{p=0}^{\infty} \Omega^{p|q}$$

where  $\Omega^p$  are the usual  $p$ -forms in differential geometry. Actually, one can proceed quite symmetrically with  $\Omega^{p|q}$  the ‘differential forms in super-space’ generated by  $x_i = d\theta_i$ .

**Proposition 4.1** *We define the exterior derivative  $d$  as*

$$d : \Omega^p \rightarrow \Omega^{p+1}, \quad d(\theta_{i_1} \cdots \theta_{i_p} f(\mathbf{x})) = \theta_{i_1} \cdots \theta_{i_p} \theta_i \frac{\partial}{\partial x_i} f(\mathbf{x}).$$

*It obeys a right-handed  $\mathbb{Z}_2$ -graded-Leibniz rule*

$$d(fg) = (-1)^p(df)g + fdg, \quad \forall f \in \Omega, \quad g \in \Omega^p.$$

**Proof** This is well-defined because the partial derivatives  $\frac{\partial}{\partial x_i}$  are well-defined as operators on  $\mathcal{M} = V^*[4]$ . I.e. our  $d$  is built up from well-defined operations. It is also covariant under the quantum group  $\tilde{A}$  for the same reasons. Here the element  $\frown = \theta_i \otimes v^i$  in  $\Lambda \otimes V$  (where  $V$  is the vector algebra) is invariant under the transformation law in [2] and hence behaves bosonically (i.e. with trivial braiding). We consider  $d$  as the action of the  $V$  part of this element on  $\mathcal{M}$  by the action  $\alpha$  defined by differentiation[4], followed by the product in  $\Lambda$ . This is shown diagrammatically in Figure 1 (a). Part (b) recalls the module-algebra property of the action  $\alpha$ [4] which comes out as the braided-Leibniz rule for the differentials  $\frac{\partial}{\partial x_i} = \alpha_{v^i}$  because the coproduct  $\Delta$  in the degree one part  $V_1$  of the vector algebra is just the linear one  $v^i \otimes 1 + 1 \otimes v^i$ . Using these facts about partial derivatives from [4] we can easily prove the Leibniz rule for  $d$ , which we do in part (c). On the left is the braided tensor product[16] in  $\Lambda \underline{\otimes} \mathcal{M}$ , followed by  $d$  in the box. We then use the braided-Leibniz rule from part (b) for the first equality and functoriality (rearrangement of braids) for the second, as well as associativity of the products. In these diagrams we work in the braided category of  $\tilde{A}$ -comodules in which the braiding  $\Psi = \bowtie$  is given by  $R$ . But the commutation relations of  $\Lambda$  are also given by  $-R$  so we have

$$\cdot \circ \Psi(\theta_i \otimes \theta_{i_1} \cdots \theta_{i_p}) = (-1)^p \theta_i \theta_{i_1} \cdots \theta_{i_p}$$

which we use for the third equality. Note that we do not make use of the coaddition on  $\Lambda$ , which would require a different braiding (based on  $-R'$ ) from the one we use here. Finally we use functoriality and associativity again to recognise the result. Conceptually, the element  $\theta_i \otimes \frac{\partial}{\partial x_i}$  is bosonic (invariant) and hence the resulting derivation property is the usual  $\mathbb{Z}_2$ -graded one (albeit coming out from the right in our present conventions) and not a braided one.  $\square$

This is the construction of the exterior differential calculus on a quantum or braided vector space coming out of braided geometry. The resulting R-matrix formulae

$$d\mathbf{x}_1 d\mathbf{x}_2 = -d\mathbf{x}_2 d\mathbf{x}_1 R, \quad \mathbf{x}_1 d\mathbf{x}_2 = d\mathbf{x}_2 \mathbf{x}_1 R \tag{16}$$

here are essentially the same as in [7] but the difference is that we begin with our well-defined partial differential operators  $\frac{\partial}{\partial x_i}$  and define  $d$  from them in a well-defined way, rather than arguing backwards by consistency requirements within the axiomatic framework of Woronowicz. In the braided approach the starting point is the braided addition law[3], which then defines partial derivatives[4], which in turn define  $d$  as above.

In nice cases, one will also have that  $d^2 = 0$ . Using the above definition it is clear that the essential requirement for this is the identity

$$\theta_1 \theta_2 \partial_2 \partial_1 = 0 \tag{17}$$

which in turn is immediate at least when  $PR' = f(PR)$  for some function  $f$  such that  $f(-1) \neq 1$ . For then  $\theta_1 \theta_2 \partial_2 \partial_1 = \theta_1 \theta_2 (PR') \partial_2 \partial_1 = \theta_1 \theta_2 f(PR) \partial_2 \partial_1 = f(-1) \theta_1 \theta_2 \partial_2 \partial_1$ . This includes the Hecke case where  $R' \propto R$  but also the more general construction[3] where  $R'$  is built from the minimal polynomial of  $R$ . In cases where  $PR'$  is not given explicitly in terms of  $PR$ , we can check (17) directly using the same strategy. The same remarks apply for the proof of covariance of the vector addition law in [3].

Using these ingredients one can immediately write down a Laplacian  $L = d\delta + \delta d$  where  $\delta = *d*$  and develop a general theory of wave-equations and electromagnetism on general braided spaces associated to  $R$ -matrices. In our setting we consider the gauge potential as  $A \in \Omega^1$  and let  $F = dA$ . The gauge theory is well-defined because  $d^2 = 0$ . It is compatible with the Maxwell equations  $\delta F = j$  because  $*^2 \propto \text{id}$ . In other words, the properties needed have been covered in our theory above and hold in some generality. More specific applications will be studied elsewhere, but see [13] where these concepts are developed directly for  $q$ -Minkowski with some interesting results.

## 5 Examples

A simple example is the quantum plane  $\mathbb{C}_q^n$  associated to the usual  $GL_q(n)$   $R$ -matrix. In this case our general formula for the  $\epsilon$  tensor recovers the usual one as in [17]. There is no invariant metric so we do not have a Hodge  $*$ -operator. The differential calculus recovers the one of [18][7]. This is clear already from the comparison of the corresponding partial derivatives in [4].

On the other hand, many other important algebras in  $q$ -deformed physics are in fact braided spaces with a coaddition law, so at once amenable to our machinery. Note that once the additive braid statistics  $R$  is known, we do not need to do any more work for the differential calculus: we just write down (16) with *the same*  $R$  as used in the braid statistics for the addition law on the quantum space.

For example, we know from [5] that the quantum matrices  $A(R) = \{t^i_j\}$  have an addition law, at least when  $R$  is Hecke. We can put it into the form of a braided space by

$$\begin{aligned} R\mathbf{t}_1\mathbf{t}_2 &= \mathbf{t}_2\mathbf{t}_1R & \Leftrightarrow t_I t_J &= t_B t_A \mathbf{R}'^A_{I^B}{}^B_J, & \mathbf{R}'^I_{J^K}{}^K_L &= R^{-1j_0}_{i_0}{}^{l_0}_{k_0} R^{i_1}_{j_1}{}^{k_1}_{l_1} \\ \mathbf{t}'_1\mathbf{t}_2 &= R_{21}\mathbf{t}_2\mathbf{t}'_1R & \Leftrightarrow t'_I t_J &= t_B t'_A \mathbf{R}^A_{I^B}{}^B_J, & \mathbf{R}^I_{J^K}{}^K_L &= R^{l_0}_{k_0}{}^{j_0}_{i_0} R^{i_1}_{j_1}{}^{k_1}_{l_1} \end{aligned}$$

where  $t_I = t^{i_0}_{i_1}$ . We recover at once the vector algebra[5] and bicovariant differential calculus[19]

$$R\partial_2\partial_1 = \partial_1\partial_2R, \quad d\mathbf{t}_1d\mathbf{t}_2 = -R_{21}d\mathbf{t}_2d\mathbf{t}_1R, \quad \mathbf{t}_1d\mathbf{t}_2 = R_{21}d\mathbf{t}_2\mathbf{t}_1R \quad (18)$$

on  $A(R)$  where  $\partial^I = \partial^{i_1}_{i_0} = \frac{\partial}{\partial t^{i_0}_{i_1}}$  is written as a matrix. This includes the usual results for  $M_q(n)$  and multiparametric  $M_{p,q}(n)$  etc. That  $d^2 = 0$  for this class is known so we omit the direct check of (17) which is needed in our constructive approach of the last section. It is very similar to the proof for  $\bar{A}(R)$  below. The construction is covariant under  $\tilde{A}$  built from  $A(R)^{\text{cop}} \otimes A(R)$  as explained in[5], which corresponds to bicovariance under  $GL_q(n)$  etc. in the usual approach. The dilatonic extension here is needed if one wants to go to quotient quantum groups rather than working at the  $GL_q$  level. That  $\mathbf{R}'$  obeys the QYBE etc. is easily checked and means that the theory in Section 2 applies and gives us a  $q$ -epsilon tensor on  $A(R)$  to go with this differential calculus.

For a second class of examples, we have the algebras  $\bar{A}(R) = \{x^i_j\}$  introduced in [9] as a variant of  $A(R)$  and again with a braided addition law when  $R$  is Hecke. It forms a braided space with[9]

$$\begin{aligned} R_{21}\mathbf{x}_1\mathbf{x}_2 &= \mathbf{x}_2\mathbf{x}_1R & \Leftrightarrow x_I x_J &= x_B x_A \mathbf{R}'^A_{I^B}{}^B_J, & \mathbf{R}'^I_{J^K}{}^K_L &= R^{-1l_0}_{k_0}{}^{j_0}_{i_0} R^{i_1}_{j_1}{}^{k_1}_{l_1} \\ \mathbf{x}'_1\mathbf{x}_2 &= R\mathbf{x}_2\mathbf{x}'_1R & \Leftrightarrow x'_I x_J &= x_B x'_A \mathbf{R}^A_{I^B}{}^B_J, & \mathbf{R}^I_{J^K}{}^K_L &= R^{j_0}_{i_0}{}^{l_0}_{k_0} R^{i_1}_{j_1}{}^{k_1}_{l_1} \end{aligned}$$

which gives at once the vector algebra and quantum differential calculus

$$R\partial_2\partial_1 = \partial_1\partial_2R_{21}, \quad d\mathbf{x}_1d\mathbf{x}_2 = -Rd\mathbf{x}_2d\mathbf{x}_1R, \quad \mathbf{x}_1d\mathbf{x}_2 = Rd\mathbf{x}_2\mathbf{x}_1R \quad (19)$$

on  $\bar{A}(R)$ . Since this is a new differential calculus, we formally verify (17) as

$$\begin{aligned} X &= \text{Tr } d\mathbf{x}_1d\mathbf{x}_2\partial_2\partial_1 = \text{Tr } d\mathbf{x}_1d\mathbf{x}_2R^{-1}R\partial_2\partial_1 = -\text{Tr } Rd\mathbf{x}_2d\mathbf{x}_1\partial_1\partial_2R_{21} = -\text{Tr } d\mathbf{x}_2d\mathbf{x}_1\partial_1\partial_2(1 + \lambda PR) \\ 2X &= -\lambda\text{Tr } d\mathbf{x}_2d\mathbf{x}_1R_{21}^{-1}R_{21}\partial_1\partial_2PR = -\lambda\text{Tr } d\mathbf{x}_2d\mathbf{x}_1R_{21}^{-1}\partial_2\partial_1P(1 + \lambda PR) \\ &= \lambda\text{Tr } R_{21}d\mathbf{x}_1d\mathbf{x}_2\partial_2\partial_1P - \lambda^2\text{Tr } d\mathbf{x}_2d\mathbf{x}_1\partial_1\partial_2 = -2X - \lambda^2X \end{aligned}$$

where  $\lambda = q - q^{-1}$  and  $\text{Tr} = \text{Tr}_1\text{Tr}_2$  over the two sets of matrix indices. Hence  $X = 0$  provided  $4 + \lambda^2 \neq 0$ , i.e. provided  $q^2 \neq -1$ . We just used the relations (19) many times, cyclicity of the trace and the Hecke assumption  $R_{21}R = 1 + \lambda PR$ .

It is also easy to see that  $\mathbf{R}'$  obeys the QYBE etc. so that the theory above applies and gives us a  $q$ -epsilon tensor on  $\bar{A}(R)$  to go with this differential calculus. Our constructions here are covariant under  $\tilde{A}$  obtained from  $A(R) \otimes A(R)$ . The simplest case with  $R$  the standard  $4 \times 4$   $SU_q(2)$  R-matrix gives  $q$ -Euclidean space[9] and is studied in detail in Section 5.1. It is covariant under  $SU_q(2) \widetilde{\otimes} SU_q(2)$ , i.e. the  $q$ -Euclidean rotation group with dilatonic extension.

For a third immediate class of examples, we know from [8] that the braided matrices  $B(R) = \{u^i_j\}$  introduced in [1] also have a braided addition law when  $R$  is Hecke. It appears in the form of a braided space with[1][8]

$$R_{21}\mathbf{u}_1R\mathbf{u}_2 = \mathbf{u}_2R_{21}\mathbf{u}_1R \Leftrightarrow u_Iu_J = u_Bu_A\mathbf{R}'^A{}_I{}^B{}_J, \quad \mathbf{R}'^I{}_J{}^K{}_L = R^{-1d}{}_{k_0}{}^{j_0}{}_a R^{k_1}{}_b{}^a{}_{i_0} R^{i_1}{}_c{}^b{}_{l_1} \tilde{R}^c{}_{j_1}{}^{l_0}{}_d$$

$$R^{-1}\mathbf{u}'_1R\mathbf{u}_2 = \mathbf{u}_2R_{21}\mathbf{u}'_1R \Leftrightarrow u'_Iu_J = u_Bu'_A\mathbf{R}^A{}_I{}^B{}_J, \quad \mathbf{R}^I{}_J{}^K{}_L = R^{j_0}{}_a{}^d{}_{k_0} R^{k_1}{}_b{}^a{}_{i_0} R^{i_1}{}_c{}^b{}_{l_1} \tilde{R}^c{}_{j_1}{}^{l_0}{}_d$$

where  $\tilde{R}$  is given by transposition in the second two indices, inversion and transposition again. The first multi-index R-matrix was introduced by the author in [1] (as well as another one for multiplicative braid statistics), while the second was introduced by Meyer in [8]. The algebra here is an important one and appears in other contexts also as explained in [20]. For this algebra we have at once the vector algebra and differential calculus

$$R\partial_2\tilde{R}\partial_1 = \partial_1\tilde{R}_{21}\partial_2R_{21}, \quad R^{-1}d\mathbf{u}_1Rd\mathbf{u}_2 = -d\mathbf{u}_2R_{21}d\mathbf{u}_1R, \quad R^{-1}\mathbf{u}_1Rd\mathbf{u}_2 = d\mathbf{u}_2R_{21}\mathbf{u}_1R \quad (20)$$

as recently studied by several authors[21][22] and references therein. We would like to stress that these relations themselves are just (and necessarily) the same form as Meyer's additive braid statistics and hence could not be considered as new. On the other hand, [21] contains an interesting new result that  $\Omega$  itself can have a braided addition law in this and other cases. Meanwhile [22] contains an interesting observation about its braided-comodules. For completeness, we need to add from our constructive point of view the formal proof of (17) and hence of  $d^2 = 0$  as

$$\begin{aligned}
X &= \text{Tr } d\mathbf{u}_1 d\mathbf{u}_2 \partial_2 \partial_1 = \text{Tr } d\mathbf{u}_1 R d\mathbf{u}_2 \partial_2 \tilde{R} \partial_1 = \text{Tr } d\mathbf{u}_1 R d\mathbf{u}_2 R^{-1} R \partial_2 \tilde{R} \partial_1 \\
&= -\text{Tr } R d\mathbf{u}_2 R_{21} d\mathbf{u}_1 \partial_1 \widetilde{R_{21}} \partial_2 R_{21} = -\text{Tr } d\mathbf{u}_2 R_{21} d\mathbf{u}_1 \partial_1 \widetilde{R_{21}} \partial_2 (1 + \lambda P R) \\
2X &= -\lambda \text{Tr } d\mathbf{u}_2 R_{21} d\mathbf{u}_1 R_{21}^{-1} R_{21} \partial_1 \widetilde{R_{21}} \partial_2 P R = -\lambda \text{Tr } d\mathbf{u}_2 R_{21} d\mathbf{u}_1 R_{21}^{-1} \partial_2 \tilde{R} \partial_1 P (1 + \lambda P R) \\
&= \lambda \text{Tr } R_{21} d\mathbf{u}_1 R d\mathbf{u}_2 \partial_2 \tilde{R} \partial_1 P - \lambda^2 \text{Tr } d\mathbf{u}_2 R_{21} d\mathbf{u}_1 \partial_1 \widetilde{R_{21}} \partial_2 = -2X - \lambda^2 X
\end{aligned}$$

which implies  $X = 0$  provided  $q^2 \neq -1$ . In fact, there is a mathematically precise equivalence between this proof and the one for  $A(R)$  and its variant above for  $\bar{A}(R)$ , provided respectively by the theory of transmutation[1] or twisting[9]. This expresses products of the  $\mathbf{u}$  in terms of products of the  $\mathbf{t}$  or  $\mathbf{x}$  in a precise way and in a corresponding way for the partial differentials  $\partial$ .

It is also known that  $\mathbf{R}'$  here obeys the QYBE[23], while the mixed relations involving  $\mathbf{R}, \mathbf{R}'$  are also easily checked in the same way and reduce to the QYBE for  $R$ . Hence we are in the symmetrical situation needed for our theory of the  $\epsilon$  tensor. Moreover, our construction is manifestly covariant under a quantum group  $\tilde{A}$  obtained from  $A(R) \bowtie A(R)$ , where  $\bowtie$  is the double cross product construction from [24]. See also [3, Sec. 4]. The standard  $4 \times 4$  R-matrix gives  $q$ -Minkowski space studied in detail in Section 5.2. Here the covariance is under  $SU_q(2) \widetilde{\bowtie} SU_q(2)$  which is the  $q$ -Lorentz group of [12][10] with dilatonic extension.

In both cases here the dilatonic extension is needed for the braiding to be given by our categorical constructions with the correct normalisation[3]. One could try to leave it out by adjusting the normalisations in (16) etc. by hand but in this case one can expect an inconsistency at some other point where both the braiding and the determinant or other non-quadratic relations of the background quantum group are needed. An alternative way out is to allow the  $q$ -Lorentz

group to be treated with anyonic or  $q$ -statistics[25]. This will be explained elsewhere. We note also that the relation between  $\bar{A}(R)$  and braided matrices  $B(R)$  by a twisting construction for comodule algebras[9] becomes a *quantum Wick rotation* in the Euclidean/Minkowski case. We know also from their original definition in [1] that the braided matrices  $B(R)$  are strictly related to  $A(R)$  by transmutation. This is already reflected in the similarity of the proofs above and it would be interesting to formalise it further as a theory of twisting and transmutation for differential calculi and  $\epsilon$  tensors.

We conclude with the simplest cases of the  $\bar{A}(R)$  and  $B(R)$  constructions computed in detail using the symbol manipulation package REDUCE. This is needed to determine the normalisations  $d_m$  etc. concerning the projectors and Hodge operators. Direct R-matrix methods like those above are not yet known for these properties, but they are verified in both of these examples as well as in similar ones based on other well-known R-matrices.

## 5.1 q-Euclidean space

For  $q$ -Euclidean space, we use the definition in [9] as twisting  $\bar{M}_q(2)$  of the usual  $2 \times 2$  quantum matrices. This is the simplest example of the  $\bar{A}(R)$  construction above. We have generators  $\mathbf{x} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$  and relations

$$\begin{aligned} ba = qab, \quad ca = q^{-1}ac, \quad da = ad, \quad db = q^{-1}bd \quad dc = qcd \\ bc = cb + (q - q^{-1})ad. \end{aligned} \tag{21}$$

This is actually isomorphic to the usual  $M_q(2)$  by a permutation of the generators, so one can regard the following as results on this with its additive structure as introduced in [5].

The vector algebra of derivatives is

$$\begin{aligned} \frac{\partial}{\partial d} \frac{\partial}{\partial c} = q^{-1} \frac{\partial}{\partial c} \frac{\partial}{\partial d}, \quad \frac{\partial}{\partial d} \frac{\partial}{\partial b} = \frac{\partial}{\partial b} \frac{\partial}{\partial d} q, \quad \frac{\partial}{\partial d} \frac{\partial}{\partial a} = \frac{\partial}{\partial a} \frac{\partial}{\partial d} \\ \frac{\partial}{\partial c} \frac{\partial}{\partial b} = \frac{\partial}{\partial b} \frac{\partial}{\partial c} + (q - q^{-1}) \frac{\partial}{\partial a} \frac{\partial}{\partial d}, \quad \frac{\partial}{\partial c} \frac{\partial}{\partial a} = \frac{\partial}{\partial a} \frac{\partial}{\partial c} q, \quad \frac{\partial}{\partial b} \frac{\partial}{\partial a} = q^{-1} \frac{\partial}{\partial a} \frac{\partial}{\partial b} \end{aligned}$$

The metric is [9]

$$\eta^{IJ} = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & -q & 0 \\ 0 & -q^{-1} & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} = \frac{\partial}{\partial x_I} \frac{\partial}{\partial x_J} (ad - qcb). \tag{22}$$

It has determinant  $\det(\eta) = 1$  and is already correctly normalised, so  $\lambda = 1$  in (12). Here

$$x_J x_I \eta^{IJ} = (1 + q^{-2})(ad - qcb)$$

is a natural ‘radius function’ according to the construction in [9].

The algebra of forms is

$$\begin{aligned} dada &= 0, & dbdb &= 0, & dc dc &= 0, & dddd &= 0 \\ dbda &= -q^{-1}dadb, & dcda &= -dadecq, & dddb &= -dbddq \\ dcdb &= -dbdc, & dddc &= -q^{-1}dcdd, & ddda &= -(q - q^{-1})dbdc - dadd \end{aligned}$$

We have the  $q$ -epsilon tensor as:

$$\begin{aligned} \epsilon_{abcd} &= -\epsilon_{acbd} = \epsilon_{adbc} = -\epsilon_{adcb} = \epsilon_{bcad} = -\epsilon_{bcda} = 1 \\ -\epsilon_{cbad} &= \epsilon_{cbda} = -\epsilon_{dabc} = \epsilon_{dacb} = -\epsilon_{dbca} = \epsilon_{dcba} = 1 \\ \epsilon_{acdb} &= -\epsilon_{cdba} = -\epsilon_{dcab} = q, & -\epsilon_{abdc} &= -\epsilon_{bacd} = \epsilon_{bdca} = \epsilon_{dbac} = q^{-1} \\ -\epsilon_{cadb} &= \epsilon_{cdab} = q^2, & \epsilon_{badc} &= -\epsilon_{bdac} = q^{-2} \\ -\epsilon_{adad} &= \epsilon_{dada} = (q - q^{-1}) \end{aligned}$$

The resulting raw (un-normalised) antisymmetriser projectors  $\mathcal{P}$  have associated constants

$$d_0 = 2q^4[2]^2[3], \quad d_1 = -2q^2[3], \quad d_2 = q^2[2]^2, \quad d_3 = -2q^2[3], \quad d_4 = 2q^4[2]^2[3]$$

where  $[m] \equiv [m, q^{-2}] = \frac{1-q^{-2m}}{1-q^{-2}}$ . In each case, the projections are on the space of totally  $R$ -antisymmetric tensors and have the same ranks as classically.

The Hodge  $*$ -operator for this metric is:

$$\begin{aligned} (dabdcdd)^* &= 1, & (dadbde)^* &= da, & (dadbdd)^* &= db \\ (dadcedd)^* &= -dc, & (dbdcdd)^* &= -dd \\ (dadb)^* &= -q[2]dadb, & (dadc)^* &= q[2]dadc, & (dadd)^* &= 2dbdc - (q - q^{-1})dadd \\ (dbdc)^* &= 2dadd + (q - q^{-1})dbdc, & (dbdd)^* &= q[2]dbdd, & (dcdd)^* &= -q[2]dcdd \end{aligned}$$

$$(da)^* = 2q^2[3]dadbdc, \quad (db)^* = 2q^2[3]dadbdd, \quad (dc)^* = -2q^2[3]dadcedd, \quad (dd)^* = -2q^2[3]dbdcdd$$

$$1^* = 2q^4[2]^2[3]$$

One can check that the square of this Hodge  $*$  operator is

$$*^2 = (-1)^m P = (-1)^m d_m$$

on forms of degree  $m$ . The Hodge  $*$ -operator on 2-forms is a  $6 \times 6$  matrix with eigenvalues  $\pm q[2]$  with multiplicity 3. The self-dual and antiself dual 2-forms with respect to it are characterised by

$$F^* = q[2]F, \quad (\text{self - dual form}); \quad F^* = -2[q]F, \quad (\text{anti - self - dual form}).$$

Of course, one may adjust the normalisation of  $*$  to have the more usual limiting form.

Note that our computations here have been for a matrix basis where the metric  $\eta$  has the signature (2,2) in the  $q = 1$  limit. The  $\epsilon$  tensor and value of  $*^2$  are as one would expect for this after bearing in mind the ordering of the indices in our conventions (there is a reversal in (13)). There is a complex transformation with real determinant which maps the matrix basis to the usual space-time basis  $t, x, y, z$  with Euclidean signature, so in this basis we still have  $*^2$  positive. Again this is the right classical result for our index conventions. The same remarks apply for the quantum case with  $q$  real. The noncommutative matrix generators transform to self-adjoint or ‘real’ ones by a complex linear transformation[9]. The  $\epsilon$  computed in the new basis is not just tensorially related to the one in the matrix basis because the top form  $dt dx dy dz$  is different from  $dadbdcdd$  that we differentiated before. But these top forms are proportional up to a real constant and  $\epsilon$  transforms tensorially up to this.

By way of contrast, we include also the second Hodge  $\circ$ -operator:

$$(dadbdcdd)^\circ = 1, \quad (dadbbdc)^\circ = -da, \quad (dadbdd)^\circ = -db$$

$$(dadcedd)^\circ = dc, \quad (dbdcdd)^\circ = dd$$

$$(dadb)^\circ = -qdadb, \quad (dadce)^\circ = q^{-1}dadce, \quad (dadd)^\circ = dbdc - (q - q^{-1})dadd$$

$$(dbdc)^\circ = dadd, \quad (dbdd)^\circ = q^{-1}dbdd, \quad (dcdd)^\circ = -qdcdd$$

$$(da)^\circ = -dadbbc, \quad (db)^\circ = -dadbdd, \quad (dc)^\circ = dadcdd, \quad (dd)^\circ = dbdcdd$$

On two-forms it has eigenvalues  $q^{-1}, -q$  each with multiplicity 3. Hence with respect to  $\circ$  we have

$$F^\circ = q^{-1}F, \quad (\text{self - dual form}); \quad F^\circ = -qF, \quad (\text{anti - self - dual form})$$

## 5.2 $q$ -Minkowski space

We use for  $q$ -Minkowski space the  $2 \times 2$  braided-hermitian matrices introduced in [1]. It is the simplest example of the  $B(R)$  construction above. The covector algebra of position co-ordinates

$\mathbf{u} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$  is:

$$\begin{aligned} ba &= q^2 ab, & ca &= q^{-2} ac, & da &= ad, & bc &= cb + (1 - q^{-2})a(d - a) \\ db &= bd + (1 - q^{-2})ab, & cd &= dc + (1 - q^{-2})ca \end{aligned} \quad (23)$$

This maps onto a braided tensor product of two copies of the quantum plane and is thereby compatible with the approach of [10][11] also. The additive structure we need is from [8].

The vector algebra of differentiation operators is:

$$\begin{aligned} \frac{\partial}{\partial d} \frac{\partial}{\partial c} &= q^{-2} \frac{\partial}{\partial c} \frac{\partial}{\partial d}, & \frac{\partial}{\partial d} \frac{\partial}{\partial b} &= \frac{\partial}{\partial b} \frac{\partial}{\partial d} q^2 \\ \frac{\partial}{\partial d} \frac{\partial}{\partial a} &= \frac{\partial}{\partial a} \frac{\partial}{\partial d}, & \frac{\partial}{\partial b} \frac{\partial}{\partial a} &= \frac{\partial}{\partial a} \frac{\partial}{\partial b} + \frac{\partial}{\partial b} \frac{\partial}{\partial d} (q^2 - 1) \\ \frac{\partial}{\partial c} \frac{\partial}{\partial a} &= \frac{\partial}{\partial a} \frac{\partial}{\partial c} + \frac{\partial}{\partial c} \frac{\partial}{\partial d} (q^{-2} - 1), & \frac{\partial}{\partial c} \frac{\partial}{\partial b} &= \frac{\partial}{\partial b} \frac{\partial}{\partial c} + \frac{\partial}{\partial d} \frac{\partial}{\partial d} (q^{-2} - 1) + \frac{\partial}{\partial a} \frac{\partial}{\partial d} (q^2 - 1) \end{aligned}$$

The metric is [8]:

$$\eta^{IJ} = \frac{\partial}{\partial u_I} \frac{\partial}{\partial u_J} (ad - q^2 cb) = \begin{pmatrix} q^{-2} - 1 & 0 & 0 & 1 \\ 0 & 0 & -q^2 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}. \quad (24)$$

It has  $\det(\eta) = q^2 = \lambda$  as the required normalisation constant in (12). We get back the ‘radius function’ from the metric as

$$u_J u_I \eta^{IJ} = (1 + q^{-2})(ad - q^2 cb).$$

The algebra of forms is

$$dcdc = 0, \quad dada = 0, \quad dbdb = 0, \quad dbda = -dadb$$

$$\begin{aligned}
dcda &= -dadc, & dcdb &= -dbdc, & dddd &= dbdc(1 - q^{-2}) \\
dddc &= -dcddq^{-2} + dadc(1 - q^{-2}), & dddb &= -dbddq^2 - dadb(q^2 - 1) \\
ddda &= -dbdc(q^2 - 1) - dadd
\end{aligned}$$

The dimensions in each degree are the usual ones: 1:4:6:4:1 and we can take a basis  $dx_{i_1} \cdots dx_{i_m}$  with  $i_1 < i_2 < \cdots < i_m$  with top form  $dadbdcdd$ .

We have the  $q$ -epsilon tensor as:

$$\begin{aligned}
\epsilon_{add} &= -\epsilon_{bcd} = -\epsilon_{dadd} = \epsilon_{dbdc} = \epsilon_{ddad} = -\epsilon_{ddda} = 1 - q^{-2} \\
-\epsilon_{adad} &= -\epsilon_{cdbl} = \epsilon_{dada} = \epsilon_{dcdb} = q^2 - 1 \\
\epsilon_{abcd} &= -\epsilon_{acbd} = \epsilon_{adbc} = -\epsilon_{adcb} = -\epsilon_{bacd} = \epsilon_{bcad} = -\epsilon_{bcda} = \epsilon_{cabd} = 1 \\
-\epsilon_{cbad} &= \epsilon_{cbda} = -\epsilon_{dabc} = \epsilon_{dacb} = \epsilon_{dbac} = -\epsilon_{dbca} = -\epsilon_{dcab} = \epsilon_{dcba} = 1 \\
\epsilon_{acdb} &= -\epsilon_{cadb} = \epsilon_{cdab} = -\epsilon_{cdba} = q^2 \\
-\epsilon_{abdc} &= \epsilon_{badc} = -\epsilon_{bdac} = \epsilon_{bdca} = q^{-2}.
\end{aligned}$$

The resulting raw (un-normalised) antisymmetriser projectors have associated constants

$$d_0 = 2q^4[2]^2[3], \quad d_1 = -2q^2[3], \quad d_2 = q^2[2]^2, \quad d_3 = -2q^2[3], \quad d_4 = 2q^4[2]^2[3]$$

as in the Euclidean case. The corresponding projections are on the space of totally  $R$ -antisymmetric tensors and have the same ranks as classically.

The Hodge  $*$ -operator for this metric is:

$$\begin{aligned}
(dadbdcdd)^* &= q^{-2}, & (dadbdc)^* &= q^{-2}da, & (dadbdd)^* &= q^{-2}db \\
(dadcdd)^* &= -dc, & (dbdcdd)^* &= q^{-2}(1 - q^{-2})da - q^{-2}dd \\
(dadb)^* &= -[2]dadb, & (dadc)^* &= [2]dadc, & (dadd)^* &= 2dbdc - (1 - q^{-2})dadd \\
(dbdc)^* &= 2q^{-2}dadd + (1 - q^{-2})dbdc, & (dbdd)^* &= [2](dbdd + 2(1 - q^{-2})dadb), & (dcdd)^* &= -[2]dcdd \\
(da)^* &= 2q^2[3]dadbdc, & (db)^* &= 2q^2[3]dadbdd, \\
(dc)^* &= -2q^2[3]dadcdd, & (dd)^* &= -2q^2[3](dbdcdd - (1 - q^{-2})dadbdc), & 1^* &= 2q^4[2]^2[3]
\end{aligned}$$

One can check that the square of this Hodge  $*$  operator is

$$*^2 = (-1)^m q^{-2} \mathcal{P} = (-1)^m q^{-2} d_m$$

on forms of degree  $m$ . The Hodge  $*$ -operator on 2-forms has eigenvalues  $\pm[2]$  with multiplicity

3. The self-dual and antiself dual 2-forms with respect to it are characterised by

$$F^* = [2]F, \quad (\text{self - dual form}); \quad F^* = -[2]F, \quad (\text{anti - self - dual form}).$$

As before, one can adjust the normalisation of  $*$  to have the more usual limit when  $q = 1$ .

Also, the same remarks apply as in the Euclidean case to the effect that there is a natural  $*$ -structure and a complex transformation from our matrix basis to self-adjoint or ‘real’ space-time bases  $t, x, y, z$ [26]. This time the top form,  $\epsilon$  and  $*$  change by an imaginary factor. This again brings our results here in line with the classical situation for our indexing conventions.

Finally, by way of contrast our alternative Hodge  $\circ$ -operator is

$$(dadbdcdd)^\circ = q^{-2}, \quad (dadbdc)^\circ = -daq^{-2}$$

$$(dadbdd)^\circ = -dbq^{-2}, \quad (dadccd)^\circ = dc, \quad (dbdcdd)^\circ = ddq^{-2} - daq^{-2}(1 - q^{-2})$$

$$(dadb)^\circ = -dadb, \quad (dadc)^\circ = dadcq^{-2}, \quad (dadd)^\circ = dbdc + dadd(q^{-2} - 1), \quad (dbda)^\circ = dadb$$

$$(dbdc)^\circ = daddq^{-2}, \quad (dbdd)^\circ = dbddq^{-2} + dadb(1 - q^{-4}), \quad (dcdd)^\circ = -dcdd$$

$$(da)^\circ = -dadbdc, \quad (db)^\circ = -dadbdd, \quad (dc)^\circ = dadcddq^{-2}, \quad (dd)^\circ = dbdcdd - dadbdc(1 - q^{-2})$$

On two-forms it has eigenvalues  $q^{-1}, -1$  each with multiplicity 3. Hence with respect to  $\circ$  we have

$$F^\circ = q^{-1}F, \quad (\text{self - dual form}); \quad F^\circ = -F, \quad (\text{anti - self - dual form})$$

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