Lecture Notes: Non-Standard Approach to J.F. Colombeau's Theory of Generalized Functions

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Todor D. Todorov
Mathematics Department
California Polytechnic State University
San Luis Obispo, California 93407, USA
(ttodorov@calpoly.edu)

Abstract

In these lecture notes we present an introduction to non-standard analysis especially written for the community of mathematicians, physicists and engineers who do research on J. F. Colombeau' theory of new generalized functions and its applications. The main purpose of our non-standard approach to Colombeau' theory is the improvement of the properties of the scalars of the varieties of spaces of generalized functions: in our non-standard approach the sets of scalars of the functional spaces always form algebraically closed non-archimedean Cantor complete fields. In contrast, the scalars of the functional spaces in Colombeau's theory are rings with zero divisors. The improvement of the scalars leads to other improvements and simplifications of Colombeau's theory such as reducing the number of quantifiers and possibilities for an axiomatization of the theory. Some of the algebras we construct in these notes have already counterparts in Colombeau's theory, other seems to be without counterpart. We present applications of the theory to PDE and mathematical physics. Although our approach is directed mostly to Colombeau's community, the readers who are already familiar with non-standard methods might also find a short and comfortable way to learn about Colombeau's theory: a new branch of functional analysis which naturally generalizes the Schwartz theory of distributions with numerous applications to partial differential equations, differential geometry, relativity theory and other areas of mathematics and physics.

MSC: Functional Analysis (46F30); Generalized Solutions of PDE (35D05).

1 Introduction

This lecture notes are an extended version of the several lectures I gave at the University of Vienna during my visit in the Spring of 2006. My audience consisted mostly of colleagues, graduate and undergraduate students who do research on J.F. Colombeau's non-linear theory of generalized functions (J.F. Colombeau's ([10]-[15]) and its applications to ordinary and partial differential equations, differential geometry, relativity theory and mathematical physics. With very few exceptions the colleagues attended my talks were not familiar with nonstandard analysis. This fact strongly influenced the nature of my lectures and these lecture notes. I do not assume that the reader of these notes is necessarily familiar neither with A. Robonson's non-standard analysis (A. Robonson [73]) nor with A. Robonson's non-standard asymptotic analysis (A. Robinson [74] and A. Robonson and A.H. Lightstone [56]). I have tried to downplay the role of mathematical logic as much as possible. With examples from Colombeau's theory I tried to convince my colleagues that the involvement of the non-standard methods in Colombeau theory has at least the following three advantages:

1. The scalars of the non-standard version of Colombeau's theory are algebraically closed Cantor complete fields. Recall that in Colmbeau's theory the scalars of the functional spaces are rings with zero divisors.

2. The involvement of non-standard analysis in Colombeau's theory leads to simplification of the theory by reducing the number of the quantifiers. This should be not of surprise because non-standard analysis is famous with the so called reduction of quantifiers. For comparison, the familiar definition of a limit of a function in standard analysis involves three (non-commuting) quantifiers. In contrast, its non-standard characterization uses only one quantifier. Another example gives the definition of a compact set in point set topology involves at least two quantifiers. In contrast, there is a free of quantifiers non-standard characterization of the compactness in terms of monads. Since Colombeau' theory is relatively heavy of quantifiers, the reduction of the numbers of quantifiers makes the theory more attractive to colleagues outside the Colombeau's community and in particular to theoretical physicists.

3. In my lectures and in these notes I decided to follow mostly the so called constructive version of the non-standard analysis where the non-standard real number $a \in {}^*\mathbb{R}$ is equivalence class of families (a_i) in the ultrapower $\mathbb{R}^{\mathcal{I}}$ for some infinite set $\mathcal{I}$. Similarly, every non-standard

smooth function $f \in {}^*\mathcal{E}(\Omega)$ is defined as equivalence class of families (f_i) in the ultrapower $\mathcal{E}(\Omega)^{\mathcal{I}}$. Here $\mathcal{E}(\Omega)$ is a (short) notation for $\mathcal{C}^\infty(\Omega)$. The equivalence relation in both $\mathbb{R}^{\mathcal{I}}$ and $\mathcal{E}(\Omega)^{\mathcal{I}}$ is defined in terms of a free ultrafilter $\mathcal{U}$ on $\mathcal{I}$. In our approach the choice of the index set $\mathcal{I}$ and the choice of the ultrafilter $\mathcal{U}$ are borrowed from Colombeau's theory. This approach to non-standard analysis is more directly connected with the standard (real) analysis and allow to involve the non-standard analysis in research with comparatively limited knowledge in the non-standard theory. The non-standard analysis however has also axiomatic version based on two axioms known a Saturation Principle and Transfer Principle. The involvement of non-standard analysis, if based on these two principles, opens the opportunities for axiomatization of Colombeau's theory. I have demonstrated this in the notes by presenting a couple of proofs to several theorems: one using families (nets), and another using these two axioms. The first might be more convincible for beginners to non-standard analysis but the second proofs are more elegant and short because it does not involve the representatives of the generalized numbers and generalized functions.

Let $\mathcal{T}$ stand for the usual topology on $\mathbb{R}^d$. J.F. Colombeau's non-linear theory of generalized functions is based on varieties of families of differential commutative rings $\mathcal{G} \stackrel{\text{def}}{=} \{\mathcal{G}(\Omega)\}_{\Omega\in\mathcal{T}}$ such that: 1) Each $\mathcal{G}$ is a **sheaf** of differential rings (consequently, each $f \in \mathcal{G}(\Omega)$ has a **support** which is a closed set of Ω). 2) Each $\mathcal{G}(\Omega)$ is supplied with a chain of sheaf-preserving embeddings $\mathcal{C}^\infty(\Omega) \subset \mathcal{D}'(\Omega) \subset \mathcal{G}(\Omega)$, where $\mathcal{C}^\infty(\Omega)$ is a **differential subring** of $\mathcal{G}(\Omega)$ and the space of L. Schwartz's distributions $\mathcal{D}'(\Omega)$ is a **differential linear subspace** of $\mathcal{G}(\Omega)$. 3) The ring of the scalars $\widetilde{\mathbb{C}}$ of the family $\mathcal{G}$ (defined as the set of the functions in $\mathcal{G}(\mathbb{R}^d)$ with zero gradient) is a non-Archimedean ring with zero devisors containing a copy of the complex numbers $\mathbb{C}$. Colombeau theory has numerous applications to ordinary and partial differential equations, fluid mechanics, elasticity theory, quantum field theory and more recently to general relativity.

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2 κ-Good Two Valued Measures

I follow the philosophy that every non-standard real number $a \in {}^*\mathbb{R}$ is, roughly speaking, a family (a_i) in the ultrapower $\mathbb{R}^{\mathcal{I}}$ for some infinite set $\mathcal{I}$. Similarly, every nonstandard smooth function $f \in {}^*\mathcal{E}(\Omega)$ is again, roughly speaking, a family (f_i) in the ultrapower $\mathcal{E}(\Omega)^{\mathcal{I}}$. Here $\mathcal{E}(\Omega)$ is a (short) notation for $\mathcal{C}^\infty(\Omega)$.

Definition 2.1 (κ-Good Two Valued Measures) *Let $\mathcal{I}$ be an infinite set of cardinality κ, i.e.* $\mathrm{card}(\mathcal{I}) = \kappa$. *A mapping $p : \mathcal{P}(\mathcal{I}) \to \{0, 1\}$ is a* **κ-good two-valued (probability) measure** *if*

(i) *p is finitely additive, i.e. $p(A \cup B) = p(A) + p(B)$ for disjoint A and B.*

(ii) $p(\mathcal{I}) = 1$.

(iii) *$p(A) = 0$ for finite A.*

(iv) *There exists a sequence of sets $(\mathcal{I}_n)$ such that*

- **(a)** $\mathcal{I} \supset \mathcal{I}_1 \supset \mathcal{I}_2 \supset \dots,$
- **(b)** *$\mathcal{I}_n \setminus \mathcal{I}_{n-1} \neq \varnothing$ for all n,*
- **(c)** $\bigcap_{n=1}^{\infty} \mathcal{I}_n = \varnothing,$
- **(d)** *$p(\mathcal{I}_n) = 1$ for all n.*

(v) *If $\mathcal{I}$ is uncountable, we impose one more property: p should be* **κ-good** *in the sense that for every set $\Gamma \subseteq \mathcal{I}$, with* $\mathrm{card}(\Gamma) \leq \kappa$, *and every* **reversal** *$R : \mathcal{P}_\omega(\Gamma) \to \mathcal{U}$ there exists a* **strict reversal** *$S : \mathcal{P}_\omega(\Gamma) \to \mathcal{U}$ such that $S(X) \subseteq R(X)$ for all $X \in \mathcal{P}_\omega(\Gamma)$. Here $\mathcal{P}_\omega(\Gamma)$ denotes the set of all finite subsets of Γ and $\mathcal{U} = \{A \in \mathcal{P}(\mathcal{I}) \mid P(A) = 1\}$.*

Remark 2.1 (Reversals) *Let $\Gamma \subseteq I$. A function $R : \mathcal{P}_\omega(\Gamma) \to \mathcal{U}$ is called a* **reversal** *if $X \subseteq Y$ implies $R(X) \supseteq R(Y)$ for every $X, Y \in \mathcal{P}_\omega(\Gamma)$. A function $S : \mathcal{P}_\omega(\Gamma) \to \mathcal{U}$ is called a* **strict reversal** *if $S(X \cup Y) = S(X) \cap S(Y)$ for every $X, Y \in \mathcal{P}_\omega(\Gamma)$. It is clear that every strict reversal is a reversal (which justifies the terminology).*

Remark 2.2 (Ignore (v) unless you really need it) *The property (v) in the definition of p is needed only to prove the Saturation Principle (see later) in the full generality, i.e. for family of internal sets $(\mathcal{A}_\gamma)_{\gamma \in \Gamma}$ with card$(\Gamma) \leq$* $\mathrm{card}(\mathcal{I})$. *In the case when Γ is countable or $\mathcal{I}$ is countable, the property (v) is not needed. In particular we do not need property (v) in this Lecture Notes and the reader is advised to ignore it.*

3 Existence of Two Valued κ-Good Measures

Theorem 3.1 (Existence of Two Valued κ-Good Measures) *Let $\mathcal{I}$ be an infinite set and let $(\mathcal{I}_n)$ be a sequence of sets with the properties (a)-(c) (think of Colombeau's theory). Then there exists a two valued κ-good measure $p : \mathcal{P}(\mathcal{I}) \to \{0, 1\}$, where $\kappa = \mathrm{card}(\mathcal{I})$, such that $p(\mathcal{I}_n) = 1$ for all $n \in \mathbb{N}$.*

Remark 3.1 *We should note that for every infinite set $\mathcal{I}$ there exists a sequence $(\mathcal{I}_n)$ with the properties (a)-(c).*

Proof: *Step 1:* **Define** $\mathcal{F}_0 \subset \mathcal{P}(\mathcal{I})$ by

$$\mathcal{F}_0 = \{A \in \mathcal{P}(\mathcal{I}) \mid \mathcal{I}_n \subseteq A \text{ for some } n\}.$$

It is easy to check that $\mathcal{F}_0$ **is a free countably incomplete filter on** $\mathcal{I}$ in the sense that $\mathcal{F}_0$ has the following properties:

(i) $\varnothing \notin \mathcal{F}_0$.

(ii) $\mathcal{F}_0$ is closed under finite intersections.

(iii) $\mathcal{F}_0 \ni A \subseteq B \in \mathcal{P}(\mathcal{I})$ implies $B \in \mathcal{F}_0$.

(iv) $\mathcal{I}_n \in \mathcal{F}_0$ for all $n \in \mathbb{N}$.

Step 2: **We extend $\mathcal{F}_0$ to a ultrafilter** $\mathcal{U}$ on $\mathcal{I}$ by Zorn lemma: Let $\mathcal{L}$ denote the set of all free filter $\mathcal{F}$ on $\mathcal{I}$ containing $\mathcal{I}_n$, i.e.

$$\mathcal{L} = \{\mathcal{F} \subset \mathcal{P}(\mathcal{I}) \mid \mathcal{F} \text{ satisfies (i)-(iv), where } \mathcal{F}_0 \text{ should be replaced by } \mathcal{F}\}.$$

We shall order $\mathcal{L}$ by inclusion $\subset$. Observe that every chain L in $\mathcal{L}$ is bounded from above by $\bigcup_{A \in L} A$ and it is not difficult to show that $\bigcup_{A \in L} A \in \mathcal{L}$. Thus $\mathcal{L}$ has maximal elements $\mathcal{U}$ by Zorn lemma. In what follows we shall keep $\mathcal{U}$ fixed.

Step 3: We shall prove now that $\mathcal{U}$ **has the following (free ultrafilter) properties:**

(1) $\varnothing \notin \mathcal{U}$.

(2) $\mathcal{U}$ is closed under finite intersections.

(3) $\mathcal{U} \ni A \subseteq B \in \mathcal{P}(\mathcal{I})$ implies $B \in \mathcal{U}$.

(4) $\mathcal{I}_n \in \mathcal{U}$ for all $n \in \mathbb{N}$.

(5) $A \cup B \in \mathcal{U}$ implies either $A \in \mathcal{U}$ or $B \in \mathcal{U}$.

Indeed, $\mathcal{U}$ satisfies (1)-(4) by the choice of $\mathcal{U}$ since $\mathcal{U} \in \mathcal{L}$. To show the property (5), suppose (on the contrary) that $A \cup B \in \mathcal{U}$ and $A, B \notin \mathcal{U}$ for some subsets A and B of $\mathcal{I}$. Next, we observe that $\mathcal{F}_A = \{X \in \mathcal{P}(\mathcal{I}) \mid A \cup X \in \mathcal{U}\}$ is also a free filter on $\mathcal{I}$ (i.e. $\mathcal{F}_A$ satisfies the properties (1)-(4)). Next, we observe that $\mathcal{F}_A$ is a proper extension of $\mathcal{U}$ since $B \in \mathcal{F}_A \setminus \mathcal{U}$ by the assumption for B, contradicting the maximality of $\mathcal{U}$.

Step 4: **Define** $p : \mathcal{P}(\mathcal{I}) \to \{0, 1\}$ by $p(A) = 1$ whenever $A \in \mathcal{U}$ and $p(A) = 0$ whenever $A \notin \mathcal{U}$. We have to show now that p is a κ-good two valued measure (Definition 2.1). To check the finite additivity property (i) of p, suppose that $A \cap B = \varnothing$ for some $A, B \in \mathcal{P}(\mathcal{I})$. Suppose, first, that $A \cup B \in \mathcal{U}$, so we have $p(A \cup B) = 1$. On the other hand, by properties (1) and (5), exactly one of the following two statements is true: either (a) $A \in \mathcal{U}$ and $A \notin \mathcal{U}$ or (b) $A \notin \mathcal{U}$ and $A \in \mathcal{U}$. In either case we have $p(A) + p(B) = 1$, as required. Suppose, now, that $A \cup B \notin \mathcal{U}$, so we have $p(A \cup B) = 0$. In this case we have $A \notin \mathcal{U}$ and $B \notin \mathcal{U}$ by property (3). Thus $p(A) + p(B) = 0$. The property (ii): $p(\mathcal{I}) = 1$ holds since $\mathcal{I} \in \mathcal{U}$ by properties (3) and (4) of $\mathcal{U}$. To prove the property (iii), suppose (on the contrary) that $p(A) = 1$ for some finite set $A \subset \mathcal{I}$, i.e. $A \in \mathcal{U}$. It follows that there exists $i \in A$ such that $\{i\} \in \mathcal{U}$ by property (5) of $\mathcal{U}$ since we have $\bigcup_{i \in A} \{i\} = A$. Thus $\{i\} \in \mathcal{I}_n$ for all $n \in \mathcal{N}$ by properties (1), (2) and (4) of $\mathcal{U}$. It follows that $\{i\} \in \bigcap_{n \in \mathbb{N}} \mathcal{I}_n$ contradicting property (c) of the sequence $(\mathcal{I}_n)$. The property (iv) holds by the choice of $\mathcal{U}$ since $\mathcal{I}_n \in \mathcal{F}_0 \subset \mathcal{U}$ thus $p(\mathcal{I}_n) = 1$. For the proof of the property (v) of the measure p we shall refer to C. C. Chang and H. J. Keisler [8] or to T. Lindstrøm [55]. ▲

4 A Non-Standard Analysis: The General Theory

Definition 4.1 (A Non-Standard Extension of a Set) *Let S be a set and $\mathcal{I}$ be and infinite set, and $S^{\mathcal{I}}$ be the corresponding ultrapower.*

- **(i)** *We say that (a_i) and (b_i) are equal almost everywhere in $\mathcal{I}$, in symbol $a_i = b_i$ a.e., if $p(\{i \in \mathcal{I} \mid a_i = b_i \text{ in } S\}) = 1$, or equivalently, if $\{i \in \mathcal{I} \mid a_i = b_i \text{ in } S\} \in \mathcal{U}$, where $\mathcal{U} = \{A \in \mathcal{P}(\mathcal{I}) \mid p(A) = 1\}$. We denote by $\sim$ the corresponding equivalence relation, i.e. $(a_i) \sim (b_i)$ if $a_i = b_i$ a.e..*

- **(ii)** *We denote by $\langle a_i \rangle$ the equivalence class determined by (a_i). The set of all equivalence classes ${}^*S = S^{\mathcal{I}}/_\sim$ is called a* **non-standard extension** *of S.*

(iii) *Let $s \in S$. We define ${}^*s = \langle a_i \rangle$, where $a_i = s$ for all $i \in \mathcal{I}$. We define the canonical embedding $\sigma : S \to {}^*S$ by $\sigma(s) = {}^*s$, and denote by ${}^\sigma S = \{{}^*s \mid s \in S\}$ the range of σ. We shall sometimes treat this embedding as an inclusion, $S \subseteq {}^*S$, by letting $s = {}^*s$ for all $s \in S$.*

(iv) *More generally, if $X \subseteq S$, we define ${}^*X \subseteq {}^*S$ by*

$$ {}^*X = \{\langle x_i \rangle \in {}^*S \mid x_i \in X \;\; a.e.\}. $$

We have $X \subseteq {}^*X$ under the embedding $x \to {}^*x$. We say that ${}^*X$ is the **non-standard extension** *of X.*

Theorem 4.1 (Axiom 1. Extension Principle) *Let S be a set. Then $S \subseteq {}^*S$ and $S = {}^*S$ iff S is a finite set.*

Proof: $S \subseteq {}^*S$ holds in the sense of the embedding σ. Suppose, first, that S is a finite set and let $\langle a_i \rangle \in {}^*S$. We observe that the finite collection of sets $\{i \in \mathcal{I} \mid a_i = s\}$, $s \in S$, are mutually disjoint and $\bigcup_{s \in S}\{i \in \mathcal{I} \mid a_i = s\} = \mathcal{I}$. Thus $\sum_{s \in S} p(\{i \in \mathcal{I} \mid a_i = s\}) = 1$ by the finite additivity of the measure p. It follows that there exists a unique $s_0 \in S$ such that $p(\{i \in \mathcal{I} \mid a_i = s_0\}) = 1$ (and $p(\{i \in \mathcal{I} \mid a_i = s_0\}) = 0$ for all $s \in S, s \neq s_0$). Thus we have $\langle a_i \rangle = {}^*s_0 \in S$, as required. Suppose now, that S is an infinite set. We have to show that ${}^*S \setminus S \neq \varnothing$. Indeed, by axiom of choice, there exists a sequence (s_n) in S such that $s_m \neq s_n$ whenever $m \neq n$. Next, we define $(a_i) \in S^{\mathcal{I}}$ by $a_i = s_n$, where $n = \max\{m \in \mathbb{N} \mid i \in \mathcal{I}_{m-1} \setminus \mathcal{I}_m\}$ and we have let also $\mathcal{I}_0 = \mathcal{I}$. Let $s \in S$. We have to show that the set $\{i \in \mathcal{I} \mid a_i \neq s\}$ is of measure 1. Indeed, if s is not in the range of (s_n), then $\{i \in \mathcal{I} \mid a_i \neq s\} = \mathcal{I}$ and is of measure 1. If s is in the range of (s_n), then $s = s_k$ for exactly one $k \in \mathbb{N}$. We observe that $\mathcal{I}_k \subseteq \{i \in \mathcal{I} \mid a_i \neq s\}$. Now the set $\{i \in \mathcal{I} \mid a_i \neq s\}$ is of measure 1 because $\mathcal{I}_k$ is of measure one, by property (iv)-(c) of p. The proof is complete. Thus $\langle s_i \rangle \in {}^*S \setminus S$ as required.

▲

In what follows $(A_i) \in \mathcal{P}(S)^{\mathcal{I}}$ means that $A_i \subseteq S$ for all $i \in \mathcal{I}$.

Definition 4.2 (Internal Sets) *Let $\mathcal{A} \subseteq {}^*S$. We say that $\mathcal{A}$ is an* **internal set** *of ${}^*S$ if there exists a family $(A_i) \in \mathcal{P}(S)^{\mathcal{I}}$ of subsets of S such that*

$$ \mathcal{A} = \{\langle s_i \rangle \in {}^*S \mid s_i \in A_i \;\; a.e.\}. $$

We say that the family (A_i) generates $\mathcal{A}$ and we write $\mathcal{A} = \langle A_i \rangle$. Let, in the particular, $A_i = A$ for all $i \in \mathcal{I}$ and some $A \subseteq S$. We say that the internal set ${}^*A = \langle A_i \rangle$ is the **non-standard extension** *of A. We denote* *by ${}^*\mathcal{P}(S)$ the set of the internal subsets of ${}^*S$. The sets in ${}^*\mathcal{P}(S) \setminus \mathcal{P}(S)$ are call* **external**.

If $X \subseteq S$, then ${}^*X$ is internal and ${}^*X$ is generated by the constant family $X_i = X$ for all $i \in \mathcal{I}$. In particular ${}^*S$ is an internal set. Let $\langle s_i \rangle \in {}^*S \setminus S$ be the element defined in the proof of Theorem 4.1. Then the singleton $\{\langle s_i \rangle\}$ is an internal set which is not of the form ${}^*X$ for some $X \subseteq S$. This internal set is generated by the singletons $\{s_i\}$, i.e. $\{\langle s_i \rangle\} = \langle \{s_i\} \rangle$. More generally, every finite subset of ${}^*S$ is an internal set. We shall give more examples of infinite internal sets of ${}^*\mathbb{R}$ and ${}^*\mathbb{C}$ in the next section. If $A \subseteq S$, then A is an external set of ${}^*S$.

In what follows we use the notation $\mathbb{N}_0 = \{0, 1, 2, \dots\}$.

Theorem 4.2 (Axiom 2. Sequential Saturation) *${}^*S$ is sequentially saturated in the sense that every sequence $\{\mathcal{A}_n\}_{n \in \mathbb{N}_0}$ of internal sets of ${}^*S$ with the finite intersection property has a non-empty intersection.*

Proof: We have $\bigcap_{n=0}^m \mathcal{A}_n \neq \varnothing$ for all $m \in \mathbb{N}_0$, by assumption. We have to show that $\bigcap_{n=0}^\infty \mathcal{A}_n \neq \varnothing$. The fact that $\mathcal{A}_n$ are internal sets means that $\mathcal{A}_n = \langle \mathbb{A}_{n,i} \rangle$ for some $\mathbb{A}_{n,i} \subseteq \mathbb{C}$, where $n \in \mathbb{N}_0$, $i \in \mathcal{I}$. We have $\langle \bigcap_{n=0}^m \mathbb{A}_{n,i} \rangle = \bigcap_{n=0}^m \langle \mathbb{A}_{n,i} \rangle = \bigcap_{n=0}^m \mathcal{A}_n \neq \varnothing$. Thus for every $m \in \mathbb{N}_0$ we have

$$\Phi_m = \{i \in \mathcal{I} \mid \cap_{n=0}^m \mathbb{A}_{n,i} \neq \varnothing\} \in \mathcal{U}. \tag{1}$$

Without loss of generality we can assume that $\mathbb{A}_{0,i} \neq \varnothing$ for all $i \in \mathcal{I}$ (indeed, if $\Phi_0 \neq \mathcal{I}$, we can choose another representative of $\mathcal{A}_0$ by $\mathbb{A}'_{0,i} = \mathbb{A}_{0,i}$ for $i \in \Phi_0$ and by $\mathbb{A}'_{0,i} = \mathbb{C}$ for $i \in \mathcal{I} \setminus \Phi_0$). Next, we define the function $\mu : \mathcal{I} \to \mathbb{N}_0 \cup \{\infty\}$, by

$$\mu(i) = \max\{m \in \mathbb{N}_0 \mid \cap_{n=0}^m \mathbb{A}_{n,i} \neq \varnothing\}.$$

Notice that μ is well-defined because the set

$$\{m \in \mathbb{N}_0 \mid \cap_{n=0}^m \mathbb{A}_{n,i} \neq \varnothing\},$$

is non-empty for all $i \in \mathcal{I}$ due to our assumption for $\mathbb{A}_{0,i}$. Thus we have $\bigcap_{n=0}^{\mu(i)} \mathbb{A}_{n,i} \neq \varnothing$ for all $i \in \mathcal{I}$. Hence (by Axiom of Choice) there exists $(A_i) \in \mathbb{C}^{\mathcal{I}}$ such that $A_i \in \bigcap_{n=0}^{\mu(i)} \mathbb{A}_{n,i}$ for all $i \in \mathcal{I}$. We intend to show that $\langle A_i \rangle \in \bigcap_{n=0}^\infty \mathcal{A}_n$ or equivalently, to show that for every $m \in \mathbb{N}_0$ we have $\{i \in \mathcal{I} \mid A_i \in \mathbb{A}_{m,i}\} \in \mathcal{U}$. We observe that

$$\Phi_m \subseteq \{i \in \mathcal{I} \mid A_i \in \mathbb{A}_{m,i}\}.$$

Indeed, $i \in \Phi_m$ implies $\bigcap_{n=0}^{m} \mathbb{A}_{n,i} \neq \varnothing$ which implies $0 \leq m \leq \mu(i)$ (by the definition of $\mu(i)$) leading to $A_i \in \mathbb{A}_{m,i}$, by the choice of (A_i). On the other hand, we have $\Phi_m \in \mathcal{U}$, by (1) implying $\{i \in \mathcal{I} \mid A_i \in \mathbb{A}_{m,i}\} \in \mathcal{U}$, as required, by property (3) of $\mathcal{U}$.

▲

In the next theorem we use for the first time the property (v) of the probability measure p (Definition 2.1). Recall that $\kappa = \text{card}(\mathcal{I})$.

Theorem 4.3 (Saturation Principle in $^*\mathbb{C}$: The General Case) *$^*\mathbb{C}$ is* **κ^+-saturated** *in the sense that every family $\{\mathcal{A}_\gamma\}_{\gamma\in\Gamma}$ of internal sets of $^*\mathbb{C}$ with the finite intersection property, and an index set Γ with $\text{card}(\Gamma) \leq \kappa$, has a non-empty intersection.*

Proof: We shall refer to the original source C. C. Chang and H. J. Keisler [8] (or, for a presentation, to T. Lindstrøm [55]).

Definition 4.3 (Superstructure) Let S be an infinite set. The **superstructure** $V(S)$ on S is the union

$$V(S) = \bigcup_{n=0}^{\infty} V_n(S),$$

where the $V_n(S)$ are defined inductively by

$$V_0(S) = S, \quad V_1(S) = S \cup \mathcal{P}(S),$$
$$V_{n+1}(S) = V_n(S) \cup \mathcal{P}(V_n(S)). \tag{2}$$

The members of $V(S)$ are called **entities**. The members of $V(S) \setminus S$ are called the **sets** of the superstructure V(S) and the members of S are called the **individuals** of the superstructure $V(S)$.

Definition 4.4 (The Language $\mathcal{L}(V(S))$) The language $\mathcal{L}(V(S))$ is the usual "language of the analysis" with the following restrictions: All quantifiers are bounded by sets in the superstrucure $V(S)$, i.e. quantifiers appear in the formulae of the language $\mathcal{L}(V(S))$ only in the form

$$(\forall x \in A)P(x) \text{ or } (\exists x \in A)P(x),$$

where $P(x)$ is a predicate in one or more variables and $A \in V(S) \setminus S$. In particular, formulae such as

$$(\forall x)P(x),$$
$$(\exists x)P(x),$$
$$(\forall x \in s)P(x),$$
$$(\exists x \in s)P(x),$$

where $s \in S$, do not belong the the language $\mathcal{L}(V(S))$.

In what follow $V(^*S)$ stands for the supersructure of $^*S$ and $\mathcal{L}(V(^*S))$ stands for the language on $V(^*S)$ which are defined exactly as $V(S)$ and $\mathcal{L}(V(S))$ after replacing S by $^*S$.

Theorem 4.4 (Axiom 3. Transfer Principle) *Let $P(x_1, x_2, \dots x_n)$ be a predicate in $\mathcal{L}(V(S))$ and $A_1, A_2, \dots, A_n \in V(S)$. Then $P(A_1, A_2, \dots A_n)$ is true $\mathcal{L}(V(S))$ iff $P(^*A_1, ^*A_2, \dots ^*A_n)$ is true in $\mathcal{L}(V(^*S))$.*

For examples of application of the Transfer Principle we refer to the first proofs of Lemma 5.1 and Lemma 5.2 later in this text.

5 A. Robinson's Non-Standard Numbers

In this section we apply the non-standard construction in the particular case $S = \mathbb{C}$, where $\mathbb{C}$ is the field of the complex numbers.

Definition 5.1 (Non-Standard Numbers) **(i)** We define the **complex non-standard numbers** as the factor ring $^*\mathbb{C} = \mathbb{C}^{\mathcal{I}}/\sim$, where $(a_i) \sim (b_i)$ if $a_i = b_i$ a.e., i.e. if

$$p\left(\{i \in \mathcal{I} \mid a_i = b_i\}\right) = 1$$

(or, equivalently, if $\{i \in \mathcal{I} \mid a_i = b_i\} \in \mathcal{U}$, where $\mathcal{U} = \{A \in \mathcal{P} \mid p(A) = 1\}$.) We denote by $\langle a_i \rangle \in {}^*\mathbb{C}$ the equivalence class determined by (a_i). The algebraic operations and the absolute value in $^*\mathbb{C}$ is inherited from $\mathbb{C}$. For example, $|\langle x_i \rangle| = \langle |x_i| \rangle$.

(ii) The set of **real non-standard numbers** $^*\mathbb{R}$ is (by definition) the non-standard extension of $\mathbb{R}$, i.e.

$$^*\mathbb{R} = \{\langle x_i \rangle \in {}^*\mathbb{C} \mid x_i \in \mathbb{R} \text{ a.e. }\}.$$

The order relation if $^*\mathbb{R}$ is defined by $\langle a_i \rangle < \langle b_i \rangle$ if $a_i < b_i$ in $\mathbb{R}$ a.e., i.e. if

$$p\left(\{i \in \mathcal{I} \mid a_i < b_i\}\right) = 1.$$

(iii) The mapping $r \to {}^*r$ defines an embeddings $\mathbb{C} \subset {}^*\mathbb{C}$ and $\mathbb{R} \subset {}^*\mathbb{R}$ by the constant nets, i.e. $^*r = \langle a_i \rangle$, where $a_i = r$ for all $i \in \mathcal{I}$.

Theorem 5.1 (Basic Properties) **(i)** *$^*\mathbb{C}$ is an algebraically closed non-archimedean field.*

(ii) ${}^*\mathbb{R}$ *is a real closed (totally ordered) non-archimedean field.*

Proof: We shall separate the proof of the above theorem in several small lemmas and prove some of them. We shall present also two proofs to each of the lemmas; one of them based on the Saturation Principle (Theorem 4.4) and the other on the properties of the measure p. The content of the next lemma is a small (but typical) part of the statement that both ${}^*\mathbb{C}$ and ${}^*\mathbb{R}$ are fields.

Lemma 5.1 (No Zero Divisors) ${}^*\mathbb{C}$ *is free of zero divisors.*

Proof 1: The statement

$$(\forall x, y \in \mathbb{C})(xy = 0 \Rightarrow x = 0 \vee y = 0),$$

is true because $\mathbb{C}$ is free of zero divisors. Thus

$$(\forall x, y \in {}^*\mathbb{C})(xy = 0 \Rightarrow x = 0 \vee y = 0),$$

is true (as required) by Transfer Principle (Theorem 4.4).
▲
Proof 2: Suppose $\langle a_i \rangle \langle b_i \rangle = 0$ in ${}^*\mathbb{C}$ for some $\langle a_i \rangle$, $\langle b_i \rangle \in {}^*\mathbb{C}$. Thus $\langle a_i b_i \rangle = 0$ implying $p(\{i \in \mathcal{I} \mid a_i b_i = 0\}) = 1$. On the other hand,

$$\{i \in \mathcal{I} \mid a_i b_i = 0\} = \{i \in \mathcal{I} \mid a_i = 0\} \cup \{i \in \mathcal{I} \mid b_i = 0\},$$

because $\mathbb{C}$ is free of zero divisors. It follows that

$$p(\{i \in \mathcal{I} \mid a_i = 0\}) + p(\{i \in \mathcal{I} \mid b_i = 0\}) \geq 1,$$

by the additivity of p. Since the range of p is $\{0, 1\}$, it follows that ether $p(\{i \in \mathcal{I} \mid a_i = 0\}) = 1$ or $p(\{i \in \mathcal{I} \mid b_i = 0\}) = 1$, i.e. either $\langle a_i \rangle = 0$ or $\langle b_i \rangle = 0$, as required. ▲

Lemma 5.2 (Trichotomy) *Let $a, b \in {}^*\mathbb{R}$. Then either $a < b$ or $a = b$ or $a > b$.*

Proof 1: The statement

$$(\forall x, y \in \mathbb{R})(x \neq y \Rightarrow x < y \vee x > y),$$

is true because $\mathbb{R}$ is a totally ordered set. Thus

$$(\forall x, y \in {}^*\mathbb{R})(x \neq y \Rightarrow x < y \vee x > y),$$

is true (as required) by Transfer Principle (Theorem 4.4).

▲

Proof 2: Suppose that $\langle a_i\rangle, \langle b_i\rangle \in {}^*\mathbb{R}$. We observe that the sets

$$A = \{i \in \mathcal{I} \mid a_i < b_i\}, \quad B = \{i \in \mathcal{I} \mid a_i = b_i\}, \quad C = \{i \in \mathcal{I} \mid a_i > b_i\},$$

are mutually disjoint and $A \cup B \cup C = \mathcal{I}$ because $\mathbb{R}$ is a totally ordered set. Thus $p(A) + p(B) + p(C) = 1$ by the additivity of the measure p. It follows that exactly one of the following is true: $p(A) = 1$ or $p(B) = 1$ or $p(C) = 1$, since the range of p is $\{0, 1\}$. Thus exactly one of the following is true: $\langle a_i\rangle < \langle b_i\rangle$, $\langle a_i\rangle = \langle b_i\rangle$, and $\langle a_i\rangle > \langle b_i\rangle$.

▲

The rest of the proof of Theorem 5.1 can be done in a similar manner and we leave it to to the reader. ▲

6 Infinitesimals, Finite and Infinitely Large Numbers

Definition 6.1 **(i)** We define the sets of **infinitesimal**, **finite**, and **infinitely large** numbers as follows:

$$\mathcal{I}({}^*\mathbb{C}) = \{x \in {}^*\mathbb{C} : |x| < 1/n \text{ for all } n \in \mathbb{N}\},$$

$$\mathcal{F}({}^*\mathbb{C}) = \{x \in {}^*\mathbb{C} : |x| < n \text{ for some } n \in \mathbb{N}\},$$

$$\mathcal{L}({}^*\mathbb{C}) = \{x \in {}^*\mathbb{C} : |x| > n \text{ for all } n \in \mathbb{N}\},$$

(ii) Let $x, y \in {}^*\mathbb{C}$. We say x and y are infinitely close, in symbol $x \approx y$, if $x - y \in \mathcal{I}({}^*\mathbb{C})$. The relation $\approx$ is called **infinitesimal relation** on ${}^*\mathbb{C}$.

(iii) Let $x \in {}^*\mathbb{C}$ and $r \in \mathbb{C}$. We write $x \rightsquigarrow y$ if $x - r \in \mathcal{I}({}^*\mathbb{C})$. We shall often refer to $\rightsquigarrow$ an **asymptotic expansion** of x.

Proposition 6.1 (Basic Properties)

$$ {}^*\mathbb{C} = \mathcal{F}({}^*\mathbb{C}) \cup \mathcal{L}({}^*\mathbb{C}), \tag{3}$$

$$\mathcal{F}({}^*\mathbb{C}) \cap \mathcal{L}({}^*\mathbb{C}) = \varnothing, \tag{4}$$

$$\mathcal{I}({}^*\mathbb{C}) \subset \mathcal{F}({}^*\mathbb{C}), \tag{5}$$

$$\mathcal{I}({}^*\mathbb{C}) \cap \mathbb{C} = \{0\}, \tag{6}$$

and similarly for ${}^*\mathbb{R}$.

Proof: These results follow directly from the definitions of infinitesimal, finite and infinitely large numbers and the fact that ${}^*\mathbb{R}$ is a totally ordered field. ▲

Example 6.1 (Infinitesimals) *Let* $\nu = \langle a_i \rangle$*, where* $(a_i) \in \mathbb{C}^{\mathcal{I}}$*,* $a_i = n$*,* $n = \max\{m \in \mathbb{N} \mid i \in \mathcal{I}_{m-1} \setminus \mathcal{I}_m\}$*. The non-standard number* ν *is an infinitely large natural number in the sense that* $\nu \in {}^*\mathbb{N}$ *and* $(\forall \varepsilon \in \mathbb{R}_+)(\varepsilon < \nu)$*. Indeed, we choose* $n \in \mathbb{N}$ *such that* $\varepsilon \leq n$ *and observe that* $\mathcal{I}_n \subset \{i \in \mathcal{I} \mid a_i > n \geq \varepsilon\}$*. Thus* $p(\{i \in \mathcal{I} \mid a_i > \varepsilon\}) = 1$ *since* $p(\mathcal{I}_n) = 1$*. Among other things this example show that* ${}^*\mathbb{R}$ *and* ${}^*\mathbb{C}$ *are proper extensions of* $\mathbb{R}$ *and* $\mathbb{C}$*, respectively.The numbers* ν^n*,* $\sqrt[n]{\nu}$*,* $\ln \nu$*,* e^{ν} *are infinitely large numbers in* ${}^*\mathbb{R}$*. In contrast, the numbers* $1/\nu^n$*,* $1/\sqrt[n]{\nu}$*,* $1/\ln \nu$*,* $e^{-\nu}$ *are non-zero infinitesimals in* ${}^*\mathbb{R}$*. If* $r \in \mathbb{R}$*, then* $r + 1/\nu^n$ *is a finite (but not standard) number in* ${}^*\mathbb{R}$*. Also* $e^{i\nu}$ *is a finite number in* ${}^*\mathbb{C}$ *and* $e^{i\nu}\nu^2 + i \ln \nu + 5 + 3i$ *is an infinitely large number in* ${}^*\mathbb{C}$*.*

Our next goal is to define and study a ring homomorphism st from the ring of finite numbers $\mathcal{F}({}^*\mathbb{C})$ to $\mathbb{C}$, called *standard part mapping*. The standard part mapping is, in a sense, a counterpart of the concept of *limit* in the usual (standard) analysis. In contrast to limit, however, the standard part mapping is applied to non-standard numbers rather than to sequences of standard numbers or functions.

Definition 6.2 (Standard Part Mapping) **(i)** The **standard part mapping** $\mathrm{st} : \mathcal{F}({}^*\mathbb{R}) \to \mathbb{R}$ is defined by the formula:

$$\mathrm{st}(x) = \sup\{r \in \mathbb{R} \mid r < x\}. \tag{7}$$

If $x \in \mathcal{F}({}^*\mathbb{R})$, then $\mathrm{st}(x)$ is called the **standard part** of x.

The standard part mappings defined in (ii) and (iii) below are extensions of the standard part mapping just defined; we shall keep the same notation, st, for all.

(ii) The **standard part mapping** $\mathrm{st} : \mathcal{F}({}^*\mathbb{C}) \to \mathbb{C}$ is defined by the formula $\mathrm{st}(x + y\,i) = \mathrm{st}(x) + \mathrm{st}(y)\,i$.

(iii) The mapping $\mathrm{st} : {}^*\mathbb{R} \to \mathbb{R} \cup \{\pm\infty\}$ is defined by (i) and by $\mathrm{st}(x) = \pm\infty$ for $x \in \mathcal{L}({}^*\mathbb{R}_\pm)$, respectively.

Theorem 6.1 (Standard Part Mapping on Finite Numbers) **(i)** *Every finite non-standard number* $x \in \mathcal{F}({}^*\mathbb{C})$ *has a unique asymptotic expansion*

$$x = \mathrm{st}(x) + dx. \tag{8}$$

where $dx \in \mathcal{I}(^*\mathbb{C})$. Consequently, if $x \in {}^*\mathbb{C}$, then $x \in \mathcal{F}(^*\mathbb{C})$ iff *$x = c + dx$ for some $c \in \mathbb{C}$ and some $dx \in \mathcal{I}(^*\mathbb{C})$.*

(ii) *The standard part mapping is a ring homomorphism from $\mathcal{F}(^*\mathbb{C})$ onto $\mathbb{C}$, i.e. for every $x, y \in \mathcal{F}(^*\mathbb{C})$ we have:*

$$\begin{aligned} \mathrm{st}(x \pm y) &= \mathrm{st}(x) \pm \mathrm{st}(y), \\ \mathrm{st}(x\,y) &= \mathrm{st}(x)\,\mathrm{st}(y), \\ \mathrm{st}(x/y) &= \mathrm{st}(x)/\mathrm{st}(y), \quad \textit{whenever } \ \mathrm{st}(y) \neq 0. \end{aligned} \tag{9}$$

(iii) *$\mathbb{C}$ consists exactly of the* **fixed points** *of* st *in $^*\mathbb{C}$, in symbol,*

$$\mathbb{C} = \{x \in {}^*\mathbb{C} \mid \mathrm{st}(x) = x\}. \tag{10}$$

Consequently, $\mathrm{st} \circ \mathrm{st} = \mathrm{st}$*, where $\circ$ denotes "composition".*

(iv) $x \in \mathcal{I}(^*\mathbb{R})$ iff $\mathrm{st}(x) = 0$.

(v) *The standard part mapping* st *is an order preserving ring homomorphism from $\mathcal{F}(^*\mathbb{R})$ onto $\mathbb{R}$, where "order preserving" means that if $x, y \in \mathcal{F}(^*\mathbb{R})$, then $x < y$ implies* $\mathrm{st}(x) \leq \mathrm{st}(y)$ *(hence, $x \leq y$ implies* $\mathrm{st}(x) \leq \mathrm{st}(y)$*).*

Proof: (i) Suppose, first, that $x \in \mathcal{F}(^*\mathbb{R})$. We have to show that $x - \mathrm{st}(x)$ is infinitesimal. Suppose (on the contrary) that $1/n < |x - \mathrm{st}(x)|$ for some n. In the case $x > \mathrm{st}(x)$, it follows $1/n < x - \mathrm{st}(x)$, contradicting (7). In the case $x < \mathrm{st}(x)$, it follows $1/n < \mathrm{st}(x) - x$, again contradicting (7). To show the uniqueness of (8), suppose that $r + dx = s + dy$ for some $r, s \in \mathbb{R}$ and some $dx, dy \in \mathcal{I}(^*\mathbb{R})$. It follows that $r - s$ is infinitesimal, hence, $r = s$, since the zero is the only infinitesimal in $\mathbb{R}$. The result extends to $\mathcal{F}(^*\mathbb{C})$ directly by the formula in part (ii) of Definition 6.2.

(ii) follows immediately from (i).

(iii) follows immediately from (i) by letting $dx = 0$.

(iv) follows directly from the definition of st.

(v) If $x \approx y$, then it follows $\mathrm{st}(x) = \mathrm{st}(y)$ (regardless whether $x < y$, $x = y$ or $x > y$). Suppose $x < y$ and $x \not\approx y$. It follows $\mathrm{st}(x) + dx < \mathrm{st}(y) + dy$. We have to show that $\mathrm{st}(x) \leq \mathrm{st}(y)$. Suppose (on the contrary) that $\mathrm{st}(x) > \mathrm{st}(y)$. It follows $0 < \mathrm{st}(x) - \mathrm{st}(y) < dy - dx$ implying $\mathrm{st}(x) - \mathrm{st}(y) \approx 0$, hence, $\mathrm{st}(x) = \mathrm{st}(y)$, a contradiction. ▲

Corollary 6.1 (An Isomorphism) **(i)** *$\mathcal{F}(^*\mathbb{R})/\mathcal{I}(^*\mathbb{R})$ is ordered field isomorphic to $\mathbb{R}$ under the mapping $q(x) \to \mathrm{st}(x)$, where $q : \mathcal{F}(^*\mathbb{R}) \to \mathcal{F}(^*\mathbb{R})/\mathcal{I}(^*\mathbb{R})$ is the quotient mapping.*

(ii) *$\mathcal{F}(^*\mathbb{C})/\mathcal{I}(^*\mathbb{C})$ is field isomorphic to $\mathbb{C}$ under the mapping $Q(x) \to \mathrm{st}(x)$, where $Q : \mathcal{F}(^*\mathbb{C}) \to \mathcal{F}(^*\mathbb{C})/\mathcal{I}(^*\mathbb{C})$ is the quotient mapping.*

(iii) *The isomorphism described in (ii) is an extension of the isomorphism described in (i).*

We leave the proof to the reader.

Example 6.2 *Let $c \in \mathbb{C}$ and let $dx \in \mathcal{I}(^*\mathbb{C})$ be a non-zero infinitesimal. Then we have:*

$$\begin{aligned}
&\mathrm{st}(c + dx^n) = c,\\
&\mathrm{st}(dx/|dx|) = \pm 1,\\
&\mathrm{st}\left(\frac{c\,dx + 7dx^2 + dx^3}{dx}\right) = \mathrm{st}(c + 7dx + dx^2) = c,\\
&\mathrm{st}\left(\frac{-3 + 4dx}{dx}\right) = \mathrm{st}(1/dx) \times \mathrm{st}(-3 + 4dx) = (\pm\infty) \times (-3) = \mp\infty,
\end{aligned}$$

where the choice of the sign $\pm$ depends on whether dx is positive or negative, respectively.

Definition 6.3 (Standard Part of a Set) *If $\mathcal{A} \subseteq {}^*\mathbb{C}$, we define the* **standard part** *of $\mathcal{A}$ by*

$$\mathrm{st}[\mathcal{A}] = \{\mathrm{st}(x) \mid x \in \mathcal{A} \cap \mathcal{F}(^*\mathbb{C})\}. \tag{11}$$

Lemma 6.1 *If $\mathcal{A} \subseteq {}^*\mathbb{C}$, then $\mathcal{A} \cap \mathbb{C} \subseteq \mathrm{st}[\mathcal{A}]$. (A proper inclusion might occur; see the example below.). In particular, we have $\mathrm{st}[^*\mathbb{R}] = \mathbb{R}$ and $\mathrm{st}[^*\mathbb{C}] = \mathbb{C}$.*

Proof: The inclusion $\mathcal{A} \cap \mathbb{C} \subseteq \mathrm{st}[\mathcal{A}]$ follows directly from part (iii) of Theorem 6.1.

Example 6.3 *Consider the set $\mathcal{A} = \{x \in {}^*\mathbb{R} \mid 0 < x < 1\}$. We have $\mathcal{A} \cap \mathbb{C} = \{x \in \mathbb{R} \mid 0 < x < 1\}$. On the other hand, $\mathrm{st}[\mathcal{A}] = \{x \in \mathbb{R} \mid 0 \leq x \leq 1\}$. Indeed, if ϵ is a positive infinitesimal in $^*\mathbb{R}$, then ϵ, $1 - \epsilon \in \mathcal{A}$ and $\mathrm{st}(\epsilon) = 0$, and $\mathrm{st}(1 - \epsilon) = 1$.*

7 NSA and the Usual Topology on $\mathbb{R}^d$

In what follows we let ${}^*\mathbb{R}^d = {}^*\mathbb{R} \times {}^*\mathbb{R} \times \cdots \times {}^*\mathbb{R}$ (d times). If $x \in {}^*\mathbb{R}^d$, then $x \approx 0$ means that $||x||$ is infinitesimal.

Definition 7.1 (Monads) *If $\mathbb{X} \subseteq \mathbb{R}^d$, then*

$$\mu(\mathbb{X}) = \{r + dx \mid r \in \mathbb{X},\ dx \in {}^*\mathbb{R}^d, ||dx|| \approx 0\}.$$

is called the **monad** *of $\mathbb{X}$ in ${}^*\mathbb{R}^d$. If $r \in \mathbb{R}^d$, we shall write simply $\mu(r)$ instead of the more precise $\mu(\{r\})$, i.e.*

$$\mu(r) = \{r + dx \mid dx \in \mathcal{I}(\mathbb{R}^d)\}.$$

We observe that $\mu(\mathbb{X}) = \bigcup_{r \in \mathbb{X}} \mu(r)$.

In what follows $\mathcal{T}$ stands for the usual topology on $\mathbb{R}^d$.

Theorem 7.1 (Boolean Properties) *The mapping $\mu : \mathcal{T} \to \mathcal{P}({}^*\mathbb{R}^d)$ is a Boolean homomorphism. Also μ preserves the arbitrary unions in the sense that $\mu\left(\bigcup_{\lambda \in \Lambda} \Omega_\lambda\right) = \bigcup_{\lambda \in \Lambda} \mu(\Omega_\lambda)$ for any set Λ and any family of open sets $\{\Omega_\lambda\}_{\lambda \in \Lambda}$.*

Theorem 7.2 (The Usual Topology on $\mathbb{R}^d$) *Let $\mathbb{X} \subseteq \mathbb{R}^d$. Then:*

(i) *A set $\mathbb{X}$ is* **open** *in $\mathbb{R}^d$* iff *$\mu(\mathbb{X}) \subseteq {}^*\mathbb{X}$.*

(ii) *$\mathbb{X}$ is* **compact** *in $\mathbb{R}^d$* iff *${}^*\mathbb{X} \subseteq \mu(\mathbb{X})$.*

8 Non-Standard Smooth Functions

Definition 8.1 (Non-Standard Smooth Functions) Let Ω is an open set of $\mathbb{R}^d$. Then:

(i) The ring (algebra) of the **non-standard smooth functions** is defined the factor ring

$$^*\mathcal{E}(\Omega) = \mathcal{E}(\Omega)^{\mathcal{I}} / \sim,$$

where $(f_i) \sim (g_i)$ if $f_i = g_i$ in $\mathcal{E}(\Omega)$ **for almost all** i in the sense that

$$p\left(\{i \mid f_i = g_i\}\right) = 1.$$

We denote by $\langle f_i \rangle \in {}^*\mathcal{E}(\Omega)$ the equivalence class determined by (f_i).

(ii) The algebraic operations and partial differentiation in ${}^*\mathcal{E}(\Omega)$ is inherited from ${}^*\mathcal{E}(\Omega)$. For example, $\partial^\alpha \langle f_i \rangle = \langle \partial^\alpha f_i \rangle$.

(iii) The mapping $f \to {}^*f$ defines an embedding $\mathcal{E}(\Omega) \hookrightarrow {}^*\mathcal{E}(\Omega)$ by the constant families, i.e. $f_i = f$ for all $i \in \mathcal{I}$. We say that ${}^*f$ is the **non-standard extension** of f.

(iv) Every $\langle f_i \rangle \in {}^*\mathcal{E}(\Omega)$ is a **pointwise mapping** of the form $\langle f_i \rangle : {}^*\Omega \to {}^*\mathbb{C}$, where $\langle f_i \rangle(\langle x_i \rangle) = \langle f_i(x_i) \rangle$ and

$${}^*\Omega = \{\langle x_i \rangle \in {}^*\mathbb{R}^d \mid x_i \in \Omega \ \text{ a.e. }\},$$

is the **non-standard extension** of Ω.

(v) Let $X \subseteq \mathcal{E}$. The non-standard extension ${}^*X$ of X is defined by

$${}^*X = \{\langle f_i \rangle \in {}^*\mathcal{E}(\Omega) \mid f_i \in X \text{ a.e.}\}.$$

In particular,

$${}^*\mathcal{D}(\Omega) = \{\langle f_i \rangle \in {}^*\mathcal{E}(\Omega) \mid f_i \in \mathcal{D}(\Omega) \text{ a.e.}\}.$$

Proposition 8.1 *${}^*\mathcal{E}(\Omega)$ is a differential algebra over the field ${}^*\mathbb{C}$.*

Definition 8.2 (Sup and Support) Let $\langle f_i \rangle \in {}^*\mathcal{E}(\Omega)$ and let $K \subset\subset \Omega$. Then

(i) $\sup_{x \in {}^*K} |\langle f_i \rangle(x)| = \langle \sup_{x \in K} |f_i(x)| \rangle$.

(ii) $\mathrm{supp}\langle f_i \rangle = \langle \mathrm{supp}(f_i) \rangle$.

We shall refer to these as **internal sup** and **internal support** of $\langle f_i \rangle$, respectively.

Proposition 8.2 *Let $f \in {}^*\mathcal{E}(\Omega)$. Then:*

(i) $(\forall K \subset\subset \Omega)(\sup_{x \in {}^*K} f(x) \in {}^*\mathbb{R})$.

(ii) $\mathrm{supp}(f)$ *is a closed set of* ${}^*\mathbb{R}$ *in the interval topology of* ${}^*\mathbb{R}$.

Lemma 8.1 (Characterizations) *Let $f \in {}^*\mathcal{E}(\Omega)$ and* $\mathrm{supp}(f)$ *denote the (internal) support of f in ${}^*\Omega$. Then* **the following are equivalent:**

(i) $\mathrm{supp}(f) \subset \mu(\Omega)$.

(ii) $\exists K \subset\subset \Omega$ *such that* $\mathrm{supp}(f) \subseteq {}^*K$.

(iii) *There exists an open relatively compact subset $\mathcal{O}$ of Ω such that $f \in {}^*\mathcal{D}(\mathcal{O})$ (The latter implies $f(x) = 0$ for all $x \in {}^*(\Omega \setminus \mathcal{O})$.)*

Definition 8.3 (Compact Support) Let $\mathcal{X} \subseteq {}^*\mathcal{E}(\Omega)$. We denote

$$\mathcal{X}_c = \{f \in \mathcal{X} \mid \mathrm{supp}(f) \subset \mu(\Omega)\}.$$

In particular, we have:

$$ {}^*\mathcal{D}_c(\Omega) = \{f \in {}^*\mathcal{D}(\Omega) \mid \mathrm{supp}(f) \subset \mu(\Omega)\}, \tag{12}$$

$$ \mathcal{X}_c = {}^*\mathcal{D}_c(\Omega) \cap \mathcal{X}, \tag{13}$$

$$ {}^*\mathcal{D}_c(\Omega) = {}^*\mathcal{E}_c(\Omega) = \{f \in {}^*\mathcal{E}(\Omega) \mid \mathrm{supp}(f) \subset \mu(\Omega)\}. \tag{14}$$

Lemma 8.2 (Characterizations) *Let $f \in {}^*\mathcal{E}(\Omega)$. Then the following are equivalent:*

(i) $(\forall x \in \mu(\Omega))\,[f(x) \in \mathcal{M}_\rho({}^*\mathbb{C})]$.

(ii) $(\forall K \subset\subset \Omega)(\exists n \in \mathbb{N})(\sup_{x \in {}^*K} |f(x)| \leq \rho^{-n})$.

(iii) $(\forall K \subset\subset \Omega)(\forall n \in {}^*\mathbb{N} \setminus \mathbb{N})(\sup_{x \in {}^*K} |f(x)| \leq \rho^{-n})$.

Lemma 8.3 (Characterizations) *Let $f \in {}^*\mathcal{E}(\Omega)$. Then the following are equivalent:*

(i) $(\forall x \in \mu(\Omega))\,[f(x) \in \mathcal{N}_\rho({}^*\mathbb{C})]$.

(ii) $(\forall K \subset\subset \Omega)(\forall n \in \mathbb{N})(\sup_{x \in {}^*K} |f(x)| \leq \rho^{n})$.

(iii) $(\forall K \subset\subset \Omega)(\exists n \in {}^*\mathbb{N} \setminus \mathbb{N})(\sup_{x \in {}^*K} |f(x)| \leq \rho^{n})$.

9 Local Properties of ${}^*\mathcal{E}(\Omega)$

In what follows $\mathcal{T}$ stands for the usual topology on $\mathbb{R}^d$ and ${}^*\mathcal{T}$ stands for order topology of ${}^*\mathbb{R}^d$ (more precisely, ${}^*\mathcal{T}$ stands for the product topology on ${}^*\mathbb{R}^d$ generated by the order topology on ${}^*\mathbb{R}$).

Theorem 9.1 (A Non-Standard Sheaf) *The collection $\{{}^*\mathcal{E}(\Omega)\}_{\Omega\in{}^*\mathcal{T}}$ is a sheaf of differential rings on ${}^*\mathbb{R}^d$ in the sense that $f \in {}^*\mathcal{E}(\Omega)$ and $\mathcal{O} \subseteq \Omega$ implies $f|\mathcal{O} \in {}^*\mathcal{E}(\mathcal{O})$ for every $\Omega, \mathcal{O} \in {}^*\mathcal{T}$.*

Proof: From the (standard) functional analysis we know that the collection $\{\mathcal{E}(\Omega)\}_{\Omega\in\mathcal{T}}$ is a sheaf of differential rings on $\mathbb{R}^d$ in the sense that $f \in \mathcal{E}(\Omega)$ and $\mathcal{O} \subseteq \Omega$ implies $f|\mathcal{O} \in \mathcal{E}(\mathcal{O})$ for every $\Omega, \mathcal{O} \in \mathcal{T}$. Thus our result follows directly from the Transfer Principle (Theorem 4.4). ▲

Corollary 9.1 (Non-Standard Support) *Let $f \in {}^*\mathcal{E}(\Omega)$ and* $\mathrm{supp}(f)$ *be the support of f (Definition 8.2). Then* $\mathrm{supp}(f)$ *is a closed set of ${}^*\Omega$ in the topology ${}^*\mathcal{T}$ on ${}^*\mathbb{R}^d$.*

Proof: The result follows (also) by Transfer Principle (or directly from the above theorem). ▲

Remark 9.1 (A Counter Example) *The next example shows that the collection $\{{}^*\mathcal{E}(\Omega)\}_{\Omega\in\mathcal{T}}$* **is not a sheaf** *of differential rings on $\mathbb{R}^d$ under the restriction $f \upharpoonright \mathcal{O} = f|{}^*\mathcal{O}$. Indeed, let $\Omega = \mathbb{R}_+$ and $\Omega_n = (0, n)$ for $n \in \mathbb{N}$. Let $\varphi \in \mathcal{D}(\mathbb{R}_+), \varphi \neq 0$, and let ν be an infinitely large number in ${}^*\mathbb{R}_+$ (see Example 6.1). We define $f(x) = {}^*\varphi(x - \nu)$ for all $x \in {}^*\mathbb{R}_+$. It is clear that $\bigcup_{n\in\mathbb{N}}(0, n) = \mathbb{R}_+$ and $f \upharpoonright (0, n) = f|{}^*(0, n) = 0$ for all n. Yet, $f \upharpoonright \mathbb{R}_+ = f|{}^*\mathbb{R}_+ = f \neq 0$.*

Our conclusion is that in order to convert the non-standard smooth functions ${}^*\mathcal{E}(\Omega)$ into an algebra of generalized functions, **we have to perform a factorization of the space** ${}^*\mathcal{E}(\Omega)$. A general method for such factorization will be presented in the next section.

10 $\mathcal{F}$-Asymptotic Numbers

In this section we describe a *variety of algebraically closed fields* $\widehat{\mathcal{F}}$ in terms of a given convex subring $\mathcal{F}$ of ${}^*\mathbb{C}$. We call these fields **$\mathcal{F}$-asymptotic hulls** and their elements **$\mathcal{F}$-asymptotic numbers**. The fields $\widehat{\mathcal{F}}$ are non-archimedean fields whenever $\mathcal{F}$ is a non-archimedean ring. We construct an embedding $\widehat{\mathcal{F}} \hookrightarrow {}^*\mathbb{C}$ and a ring homomorphism $\widehat{\mathrm{st}} : \mathcal{F} \to {}^*\mathbb{C}$ which we call a *quasi-standard part mapping.* The quasi-standard part mapping reduces to the familiar standard part mapping $\mathrm{st} : \mathcal{F}({}^*\mathbb{C}) \to {}^*\mathbb{C}$ in particular case when $\mathcal{F}$ is the ring $\mathcal{F}({}^*\mathbb{C})$ of finite numbers in ${}^*\mathbb{C}$. Our *asymptotic hull construction* can be viewed as a generalization of A. Robinson's theory of asymptotic numbers (A. H. Lightstone and A. Robinson [56]). We also generalize some more recent results in (T. Todorov and R. Wolf [95]) on the A. Robinson field ${}^\rho\mathbb{R}$. Non-archimedean fields isomorphic to the fields of the form $\widehat{\mathcal{F}}$ are studied in model theory (D. Marker, M. Messmer, A. Pillay [61]) although the fields in model theory are rarely constructed in the framework of ${}^*\mathbb{R}$ or ${}^*\mathbb{C}$ as here (see the discussion below). A construction similar to the presented here appears in the H. Vernaeve Ph.D. Thesis [98] (for a comparison see the equivalence relation $\sim$ defined on p. 87, Sec. 3.6, altered by the additional condition used in Lemma 3.32 on p. 89).

We believe that every algebraically closed non-archimedean field in mathematics is either isomorphic to some asymptotic hull $\widehat{\mathcal{F}}$ (for a suitable choice of ${}^*\mathbb{C}$ and $\mathcal{F}$), or it is isomorphic to a subfield of some $\widehat{\mathcal{F}}$. For example, the field $\mathbb{C}\langle t\rangle$ of Levi-Civita power series with complex coefficients is isomorphic to a subfield of A. Robinson's field ${}^\rho\mathbb{C}$ of asymptotic numbers (A. H. Lightstone and A. Robinson [56]). On the other hand, we show that ${}^\rho\mathbb{C}$ is of the form $\widehat{\mathcal{F}}$ (Example 10.2). For that reason we hope that our asymptotic hull construction might facilitate the communication between the mathematicians working in non-standard analysis and its applications on one side and those working in model theory of fields on the other (A. Macintyre, Lou van den Dries and [**?**]). Our immediate purpose however is to support the theory of **$\mathcal{F}$-asymptotic functions** $\widehat{\mathcal{E}}_{\mathcal{F}}(\Omega)$ presented in the next section: each $\widehat{\mathcal{E}}_{\mathcal{F}}(\Omega)$ is an algebra of generalized functions of Colombeau type with a field of scalars $\widehat{\mathcal{F}}$.

In what follows ${}^*\mathbb{C}$ stands for a non-standard extension of the field of the complex numbers $\mathbb{C}$. Here is the **summary** of our basic definitions (the justification and the detail follow later in this section):

1. Let $\mathcal{F}$ be a **convex subring** in ${}^*\mathbb{C}$, i.e. $\mathcal{F}$ is a subring of ${}^*\mathbb{C}$ such that

$$(\forall z \in {}^*\mathbb{C})(\forall \zeta \in \mathcal{F})(|z| \leq |\zeta| \Rightarrow z \in \mathcal{F}). \tag{15}$$

We denote by $\mathcal{F}_0$ the set of all **non-invertible elements** of $\mathcal{F}$, i.e.

$$\mathcal{F}_0 = \{z \in \mathcal{F} \mid z = 0 \text{ or } 1/z \notin \mathcal{F}\}. \tag{16}$$

We also define the **real part** $\Re(\mathcal{F})$ of $\mathcal{F}$ by

$$\Re(\mathcal{F}) = \{\pm|z| \,:\, z \in \mathcal{F}\}.$$

We also denote by $\mathcal{F}_+$ the set of the **positive elements** of $\mathcal{F}$, i.e.

$$\mathcal{F}_+ = \{|z| : z \in \mathcal{F}, z \neq 0\}. \tag{17}$$

2. The $\mathcal{F}$**-asymptotic hull** is the factor ring $\widehat{\mathcal{F}} = \mathcal{F}/\mathcal{F}_0$. The elements of $\widehat{\mathcal{F}}$ are the **complex** $\mathcal{F}$**-asymptotic numbers** (or simply *asymptotic numbers* if no confusion could arise). Let $q : \mathcal{F} \to \widehat{\mathcal{F}}$ stand for the corresponding **quotient mapping**. If $z \in \mathcal{F}$, we shall often write $\widehat{z}$ instead of $q(z)$ when no confusion could arise. Similarly, if $S \subseteq {}^*\mathbb{C}$, we let $\widehat{S} = q[S \cap \mathcal{F}]$. In the particular case $S \subseteq \mathbb{C}$ we shall often write simply S instead of the more precise $\widehat{S}$. We also define the **real part** $\Re(\widehat{\mathcal{F}})$ of $\widehat{\mathcal{F}}$ by

$$\Re(\widehat{\mathcal{F}}) = \{\pm|z| \,:\, z \in \widehat{\mathcal{F}}\}, \tag{18}$$

and observe that $\widehat{\Re(\mathcal{F})} = \Re(\widehat{\mathcal{F}})$. The elements of $\Re(\widehat{\mathcal{F}})$ are the **real** $\mathcal{F}$**-asymptotic numbers** (or simply *real asymptotic numbers* if no confusion could arise). Also, $\widehat{\mathcal{F}}_+$ stands for the set of the **positive elements** of $\widehat{\mathcal{F}}$, i.e.

$$\widehat{\mathcal{F}}_+ = \{|z| : z \in \widehat{\mathcal{F}}, z \neq 0\}. \tag{19}$$

3. We define the embeddings $\mathbb{C} \hookrightarrow \widehat{\mathcal{F}}$ and $\mathbb{R} \hookrightarrow \Re(\widehat{\mathcal{F}})$ by the mapping $z \to \widehat{z}$. We shall often identify z with its image $\widehat{z}$ writing simply $\mathbb{C} \subset \widehat{\mathcal{F}}$ and $\mathbb{R} \subset \Re(\widehat{\mathcal{F}})$, respectively.

4. We denote by $\mathcal{I}(\widehat{\mathcal{F}})$, $\mathcal{F}(\widehat{\mathcal{F}})$ and $\mathcal{L}(\widehat{\mathcal{F}})$ the sets of the infinitesimal, finite and infinitely large elements of $\widehat{\mathcal{F}}$, respectively. We write $x \approx 0$ whenever $x \in \mathcal{I}(\widehat{\mathcal{F}})$.

5. Let us denote

$$\mathcal{F}^d = \mathcal{F} \times \mathcal{F} \times \cdots \mathcal{F},$$
$$\mathcal{F}_0^d = \mathcal{F}_0 \times \mathcal{F}_0 \times \cdots \mathcal{F}_0,$$
$$\widehat{\mathcal{F}}^d = \widehat{\mathcal{F}} \times \widehat{\mathcal{F}} \times \cdots \widehat{\mathcal{F}}.$$

(d times). We denote by $||\cdot||$ the usual Euclidean norm in either $\mathcal{F}^d$ or $\widehat{\mathcal{F}}^d$. If $z = (z_1, z_2, \cdots, z_d) \in \mathcal{F}$, we shall write $\widehat{z} = (\widehat{z}_1, \widehat{z}_2, \cdots, \widehat{z}_d) \in \widehat{\mathcal{F}}$. Let $z \in \mathbb{C}^d$. We observe that $z \in \mathcal{F}^d \quad iff \quad ||z|| \in \mathcal{F}$. Also $z \in \mathcal{F}_0^d$ $iff \quad ||z|| \in \mathcal{F}_0$. Notice that $\widehat{\mathcal{F}}^d$ is a **vector space over the field** $\widehat{\mathcal{F}}$.

6. Similarly, let $\Re(\mathcal{F})$ be the real part of $\mathcal{F}$ and $\Re(\widehat{\mathcal{F}})$ be the real part of of $\widehat{\mathcal{F}}$. We define the real parts of $\mathcal{F}^d$ and $\widehat{\mathcal{F}}$ by

$$\Re(\mathcal{F}^d) = \Re(\mathcal{F}) \times \Re(\mathcal{F}) \times \cdots \Re(\mathcal{F}) \ (d \text{ times}),$$
$$\Re(\widehat{\mathcal{F}}^d) = \Re(\widehat{\mathcal{F}}) \times \Re(\widehat{\mathcal{F}}) \times \cdots \Re(\widehat{\mathcal{F}}) \ (d \text{ times}),$$

respectively. Notice that $\Re(\widehat{\mathcal{F}}^d)$ is a **vector space over the field** $\Re(\widehat{\mathcal{F}})$.

7. We define the embeddings $\mathbb{C}^d \hookrightarrow \widehat{\mathcal{F}}^d$ and $\mathbb{R}^d \hookrightarrow \Re(\widehat{\mathcal{F}}^d)$ by the mapping $z \to \widehat{z}$. We shall often identify z with its image $\widehat{z}$ writing simply $\mathbb{C}^d \subset \widehat{\mathcal{F}}^d$ and $\mathbb{R}^d \subset \Re(\widehat{\mathcal{F}}^d)$, respectively.

8. If $X \subseteq \mathbb{R}^d$, then the $\mathcal{F}$-**monad** of X is the set $\mu_{\mathcal{F}}(X) \subset \Re(\widehat{\mathcal{F}}^d)$,

$$\mu_{\mathcal{F}}(X) = \{r + dx \mid r \in X,\, dx \in \Re(\widehat{\mathcal{F}}^d),\ ||dx|| \approx 0\}. \tag{20}$$

We certainly have $X \subset \mu_{\mathcal{F}}(X)$. Notice that $\widehat{x} \in \mu_{\mathcal{F}}(X) \quad iff$ $x \in \mu(X)$, where $\mu(X)$ is the usual monad of X in ${}^*\mathbb{R}^d$ (Section 7).

It is not immediately clear that the set $\mathcal{F}_0$ defined above is an ideal in $\mathcal{F}$ (let alone a convex maximal ideal). To show this we need some preparation.

Theorem 10.1 (Convex Rings) *Let $\mathcal{F}$ be a convex subring of ${}^*\mathbb{C}$. Then $\mathcal{F}$ contains a copy of the ring $\mathcal{F}({}^*\mathbb{C})$ of the finite elements of ${}^*\mathbb{C}$. Consequently, $\mathcal{F}$ contains a copy $\mathbb{C}$. We summarize these as $\mathbb{C} \subset \mathcal{F}({}^*\mathbb{C}) \subseteq \mathcal{F} \subseteq {}^*\mathbb{C}$.*

Proof: Suppose $z \in \mathcal{F}({}^*\mathbb{C})$. We have $|z| < n$ for some $n \in \mathbb{Z}$ by the definition of $\mathcal{F}({}^*\mathbb{C})$. To show that $z \in \mathcal{F}$, it is sufficient to show that $\mathbb{Z} \subset \mathcal{F}$. Indeed, we observe that $\{\pm|z| : z \in \mathcal{F}\}$ is a subring of $\mathcal{F}$. This follows from the fact that $z \in \mathcal{F}$ implies $\pm|z| \in \mathcal{F}$ by the convexity of $\mathcal{F}$ (since, obviously, $|\pm|z|| \leq |z|$) and also the inequalities $||z| \pm |\zeta|| \leq \max\{|2z|, |2\zeta|\}$ and $|z||\zeta| \leq \max\{|z^2|, |\zeta^2|\}$ combined, again, with the convexity of $\mathcal{F}$. Thus $\{\pm|z| : z \in \mathcal{F}\}$ is a totally ordered ring as a subring of ${}^*\mathbb{R}$. This proves that $\{\pm|z| : z \in \mathcal{F}\}$ contains a copy of $\mathbb{Z}$ which implies that $\mathcal{F}$ contains a copy of $\mathbb{Z}$. Now, $|x| < n$ and $n \in \mathcal{F}$ implies $x \in \mathcal{F}$ (as required) by the convexity of $\mathcal{F}$. ▲

Definition 10.1 (Maximal Fields) Let $\mathcal{F}$ be (as before) a convex subring in $^*\mathbb{C}$. A subfield $\mathbb{M}$ of $^*\mathbb{C}$ is called **maximal in** $\mathcal{F}$ if $\mathbb{M}$ is a subring of $\mathcal{F}$ and there is no a subfield $\mathbb{F}$ of $^*\mathbb{C}$ such that: **(a)** $\mathbb{F}$ is also a subring of $\mathcal{F}$; **(b)** $\mathbb{F}$ is a proper field extension of $\mathbb{M}$. We denote by $\mathcal{M}ax(\mathcal{F})$ the **set of all maximal fields** in $\mathcal{F}$.

Lemma 10.1 (Some Properties of $\mathcal{M}ax(\mathcal{F})$) *Let $\mathcal{F}$ be (as before) a convex subring in $^*\mathbb{C}$. Then:*

- **(i)** $\mathcal{M}ax(\mathcal{F}) \neq \varnothing$.
- **(ii)** *If $\mathbb{M} \in \mathcal{M}ax(\mathcal{F})$, then $\mathbb{M} \cap \mathcal{F}_0 = \{0\}$.*
- **(iii)** *Let $\mathbb{F}$ be a field which is a subring of $\mathcal{F}$. Then $\mathbb{F}$ can be extended to a maximal field, i.e. there exists $\mathbb{M} \in \mathcal{M}ax(\mathcal{F})$ such that $\mathbb{F} \subseteq \mathbb{M}$.*
- **(iv)** *Every $\mathbb{M} \in \mathcal{M}ax(\mathcal{F})$ is an algebraically closed field.*
- **(v)** *If $\mathbb{M} \in \mathcal{M}ax(\mathcal{F})$, then $\Re(\mathbb{M})$ is a real closed field.*
- **(vi)** *Let $\mathbb{M} \in \mathcal{M}ax(\mathcal{F})$. Then $\mathbb{M}^d$ is a vector space over $\mathbb{M}$ and $\Re(\mathbb{M}^d) \overset{def}{=} \Re(\mathbb{M})^d$ is a vector space over $\Re(\mathbb{M})$.*

Proof: (i) Let $\mathcal{L}$ denote the set of all subfields $\mathbb{L}$ of $^*\mathbb{C}$ which are subrings of $\mathcal{F}$ and we order $\mathcal{L}$ by inclusion. We have $\mathcal{L} \neq \varnothing$, since $\mathbb{C} \in \mathcal{L}$ by Theorem 10.1. Also, we observe that if S is a totally ordered subset of $\mathcal{L}$ under the inclusion $\subset$, then $\bigcup_{\mathbb{L}\in S} \mathbb{L} \in \mathcal{L}$. Thus $\mathcal{L}$ has maximal elements $\mathbb{M}$, as required, by Zorn's lemma.

(ii) Suppose (on the contrary) that there exists $z \in \mathbb{M} \cap \mathcal{F}_0$ such that $x \neq 0$. It follows that $1/x \in \mathbb{M} \cap (^*\mathbb{C} \setminus \mathcal{F})$ contradicting $\mathbb{M} \subset \mathcal{F}$.

(iii) follows with almost the same arguments as in (i): The set $\mathcal{L}$ should be replaced by the set $\mathcal{L}_{\mathbb{F}}$ of all subfields $\mathbb{L}$ of $^*\mathbb{C}$ such that $\mathbb{F} \subseteq \mathbb{L} \subset \mathcal{F}$.

(iv) Let $\mathrm{cl}(\mathbb{M})$ denote the algebraically closure of $\mathbb{M}$ in $^*\mathbb{C}$. Since $^*\mathbb{C}$ is an algebraically closed field, it suffices to show that $\mathbb{M} = \mathrm{cl}(\mathbb{M})$. We show first that $\mathrm{cl}(\mathbb{M}) \subset \mathcal{F}$. For suppose $\gamma \in \mathrm{cl}(\mathbb{M})$. Notice that γ is algebraic over $\mathbb{M}$ which means that γ is a solution of some polynomial equation: $\gamma^n + a_1\,\gamma^{n-1} + \cdots + a_n = 0$ with coefficients a_k in $\mathbb{M}$. Thus the estimation $|\gamma| \leq 1 + |a_1| + \cdots + |a_n|$ implies that $\gamma \in \mathcal{F}$, as desired, by the convexity of $\mathcal{F}$. Now, $\mathbb{M} = \mathrm{cl}(\mathbb{M})$ follows from the maximality of $\mathbb{M}$ (D. Marker, M. Messmer, A. Pillay [61]).

(v) follows directly from (iv) (see again D. Marker, M. Messmer, A. Pillay [61]).

(vi) follows directly from (iv) and (v).

▲

The next result shows, among other things, that $\mathcal{F}$ and $\mathcal{F}_0$ are exactly the sets of the finite and infinitesimal numbers in ${}^*\mathbb{C}$, respectively, *relative to a given maximal field* $\mathbb{M}$. In what follows, $\mathbb{M}_+$ stands for the set of the positive elements of $\mathbb{M}$, i.e.

$$\mathbb{M}_+ = \{|z| : z \in \mathbb{M}, z \neq 0\}. \tag{21}$$

Theorem 10.2 (Characterization) *Let $\mathcal{F}$ be (as before) a convex subring in ${}^*\mathbb{C}$.*

(i) *If $\mathbb{M} \in \mathcal{M}ax(\mathcal{F})$ (Definition 10.1), then*

$$\mathcal{F} = \{z \in {}^*\mathbb{C} \mid (\exists \varepsilon \in \mathbb{M}_+)(|z| \leq \varepsilon\}, \tag{22}$$

$$\mathcal{F}_0 = \{z \in {}^*\mathbb{C} \mid (\forall \varepsilon \in \mathbb{M}_+)(|z| < \varepsilon\}. \tag{23}$$

(ii) *The sets $\mathcal{F}_0$, $\mathcal{F} \setminus \mathcal{F}_0$ and ${}^*\mathbb{C} \setminus \mathcal{F}$ are* ***disconnected*** *in the sense that*

$$(\forall z_1 \in \mathcal{F}_0)(\forall z_2 \in \mathcal{F} \setminus \mathcal{F}_0)(\forall z_3 \in {}^*\mathbb{C} \setminus \mathcal{F})(|z_1| < |z_2| < |z_3|).$$

(iii) *$\mathcal{F}_0$ consists of infinitesimals only, i.e. $\mathcal{F}_0 \subseteq \mathcal{F}({}^*\mathbb{C})$.*

(iv) *$\mathcal{F}_0$ is a convex maximal ideal in $\mathcal{F}$. Consequently, the factor ring $\widehat{\mathcal{F}} = \mathcal{F}/\mathcal{F}_0$ is a field.*

(v) *$\widehat{\mathcal{F}}$ is an archimedean field* iff *$\mathcal{F} = \mathcal{F}({}^*\mathbb{C})$.*

Proof: (i) Let $\gamma \in \mathcal{F}$ and suppose (on the contrary) that $(\forall \varepsilon \in \mathbb{M}_+)(|\gamma| > \varepsilon)$. We observe that γ is transcendental over $\mathbb{M}$ since $\mathbb{M}$ is an algebraically closed field by part (iii) of Lemma 10.1. Thus $\mathbb{M}(\gamma)$ is a proper field extension of $\mathbb{M}$ within $\mathcal{F}$, contradicting the maximality of $\mathbb{M}$. This proves the formula (22) about $\mathcal{F}$. Let $\gamma \in \mathcal{F}_0$. If $\gamma = 0$, there is nothing to prove. If $\gamma \neq 0$, we have $1/\gamma \notin \mathcal{F}$ by the definition of $\mathcal{F}_0$. Next, suppose (on the contrary) that $|\gamma| \geq \varepsilon$ for some $\varepsilon \in \mathbb{M}_+$. It follows that $|1/\gamma| \leq 1/\varepsilon$ implying $1/\gamma \in \mathcal{F}$ by formula (22), a contradiction. Conversely, suppose that $|\gamma| < \varepsilon$ for all $\varepsilon \in \mathbb{M}_+$ and some $\gamma \in {}^*\mathbb{C}$. It follows that $1/\varepsilon < |1/\gamma|$ for all $\varepsilon \in \mathbb{M}_+$ implying $1/\gamma \notin \mathcal{F}$ by the formula (22). Thus $\gamma \in \mathcal{F}_0$ which proves the formula (23).

(ii) follows immediately from (i).

(iii) The inclusion $\mathcal{F}_0 \subseteq \mathcal{F}({}^*\mathbb{C})$ follows from the formula (23) and the fact that $\mathbb{Q} \subset \mathbb{M}$.

(iv) The proof that $\mathcal{F}_0$ is a convex maximal ideal in $\mathcal{F}$ is almost identical to the proof that the set of infinitesimals $\mathcal{I}({}^*\mathbb{C})$ is a convex maximal ideal in the ring of the finite numbers $\mathcal{F}({}^*\mathbb{C})$ of ${}^*\mathbb{C}$ and we leave the detail to the reader.

(v) Suppose that $\widehat{\mathcal{F}}$ is an archimedean field. In view of the inclusion $\mathcal{F}({}^*\mathbb{C}) \subseteq \mathcal{F}$ (Theorem 10.1) it suffices to show that $\mathcal{F} \subseteq \mathcal{F}({}^*\mathbb{C})$. Indeed, $z \in \mathcal{F}$ implies that $\widehat{z}$ is finite (since $\widehat{\mathcal{F}}$ is archimedean by assumption) thus z is finite. Conversely, $\mathcal{F} = \mathcal{F}({}^*\mathbb{C})$ implies that $\widehat{\mathcal{F}}$ is archimedean as a factor ring of an archimedean ring.

▲

Our next goal is to study the factor ring $\widehat{\mathcal{F}}$.

Lemma 10.2 (Isomorphic Fields) *Let $\mathbb{F}$ be a field which is a subring of $\mathcal{F}$ and $\widehat{\mathbb{F}} = q[\mathbb{F}]$. Then the fields $\mathbb{F}$ and $\widehat{\mathbb{F}}$ are isomorphic under the mapping $q\,|\,\mathbb{F}$ from $\mathbb{F}$ to $\widehat{\mathbb{F}}$ (or, alternatively, under the mapping $(q\,|\,\mathbb{F})^{-1}$ from $\widehat{\mathbb{F}}$ to $\mathbb{F}$). In particular, $\mathbb{M}$ and $\widehat{\mathbb{M}}$ are isomorphic fields for every $\mathbb{M} \in \mathcal{M}ax(\mathcal{F})$ (Definition 10.1).*

Proof: We have $\mathbb{F} \subseteq \mathcal{F}$ by assumption. Notice that there exists a maximal field $\mathbb{M}$ in $\mathcal{F}$ such that $\mathbb{F} \subseteq \mathbb{M}$ by part (ii) of Lemma 10.1. It follows that $\mathbb{F} \cap \mathcal{F}_0 = \{0\}$ by Lemma 10.2. Thus $\mathbb{F}$ and $\widehat{\mathbb{F}}$ are isomorphic.

▲

Our next goal is to prove that $\widehat{\mathcal{F}}$ is an algebraically closed field by showing that $\widehat{\mathcal{F}}$ and $\widehat{\mathbb{M}}$ are, actually, the same (that is to say that $\mathbb{M}$ is a *field of representatives* for $\widehat{\mathcal{F}}$).

Lemma 10.3 (Remote Points) *Let $\mathcal{F}$ be (as before) a convex subring in ${}^*\mathbb{C}$ and let $\mathbb{M} \in \mathcal{M}ax(\mathcal{F})$ (Definition 10.1). Let $\gamma \in \mathcal{F}$ be a point such that $\gamma - r \notin \mathcal{F}_0$ for all $r \in \mathbb{M}$. Then $P(\gamma) \notin \mathcal{F}_0$ for all polynomials $P \in \mathbb{M}[x]$, $P \neq 0$.*

Proof: Suppose (on the contrary) that $P(\gamma) \in \mathcal{F}_0$ for some $P \in \mathbb{M}[x], P \neq 0$. It follows that $\widehat{P(\gamma)} = 0$ implying $\widehat{P}(\widehat{\gamma}) = 0$ in $\widehat{\mathcal{F}}$, where $\widehat{P}$ denotes the polynomial in $\widehat{\mathbb{M}}[x]$, obtained from P by replacing the coefficients a_k in P by $\widehat{a_k}$. Observe, now, that $\widehat{\mathbb{M}}$ is an algebraically closed field, by part (iii) of Lemma 10.1, as a field isomorphic to $\mathbb{M}$ (Lemma 10.2). Hence, it follows $\widehat{\gamma} \in \widehat{\mathbb{M}}$ meaning $\gamma - r \in \mathcal{F}_0$ for some $r \in \mathbb{M}$, a contradiction. ▲

Theorem 10.3 (Embeddings) *Let $\mathcal{F}$ be (as before) a convex subring in ${}^*\mathbb{C}$ and $\mathbb{M} \in \mathcal{M}ax(\mathcal{F})$ (Definition 10.1). Then:*

(i) *We have* $\mathcal{F} = \mathbb{M} \oplus \mathcal{F}_0$ *in the sense that every* $z \in \mathcal{F}$ *has a unique asymptotic expansion* $z = r + dz$*, where* $r \in \mathbb{M}$ *and* $dz \in \mathcal{F}_0$*. Consequently,* $\mathbb{M}$ *is a* ***field of representatives*** *for* $\widehat{\mathcal{F}}$ *in the sense that* $\widehat{\mathcal{F}} = \widehat{\mathbb{M}}$.

(ii) *The fields* $\mathbb{M}$ *and* $\widehat{\mathcal{F}}$ *are isomorphic under the mapping* $q \,|\, \mathbb{M}$ *from* $\mathbb{M}$ *to* $\widehat{\mathcal{F}}$ *(or, alternatively, under the mapping* $(q \,|\, \mathbb{M})^{-1}$ *from* $\widehat{\mathcal{F}}$ *to* $\mathbb{M}$*). Consequently:*

- **(a)** *The field* $\widehat{\mathcal{F}}$ *(of the complex* $\mathcal{F}$*-asymptotic numbers) is an* ***algebraically closed field****.*
- **(b)** *The field* $\Re(\widehat{\mathcal{F}})$ *(of the real* $\mathcal{F}$*-asymptotic numbers) is a* ***real closed field****.*

(iii) *The mapping* $\sigma_{\mathbb{M}} : \widehat{\mathcal{F}} \to {}^*\mathbb{C}$*, defined by* $\sigma_{\mathbb{M}} = (q \,|\, \mathbb{M})^{-1}$*, is a* ***field embedding***

$$\widehat{\mathcal{F}} \hookrightarrow {}^*\mathbb{C}, \tag{24}$$

of $\widehat{\mathcal{F}}$ *into* ${}^*\mathbb{C}$*. The situation just described can be summarized in the following commutative diagram:*

$$\begin{array}{ccc}
\mathcal{F} & \xrightarrow{\;\;q\;\;} & \widehat{\mathcal{F}} \\
{\scriptstyle id}\uparrow & & \downarrow{\scriptstyle id} \\
\mathbb{M} & \xrightarrow{\;q|\mathbb{M}\;} & \widehat{\mathbb{M}}.
\end{array}$$

(iv) *The mapping* $\mathrm{st}_{\mathbb{M}} : \mathcal{F} \to {}^*\mathbb{C}$*, defined by* $\mathrm{st}_{\mathbb{M}}(r + dz) = r$*, is a ring homomorphism with range* $\mathrm{st}_{\mathbb{M}}[\mathcal{F}] = \mathbb{M}$*. Also* $\mathrm{st}_{\mathbb{M}}$ *is an extension of the standard part mapping* $\mathrm{st} : \mathcal{F}({}^*\mathbb{C}) \to \mathbb{C}$*, i.e.* $\mathrm{st}_{\mathbb{M}} \,|\, \mathcal{F}({}^*\mathbb{C}) = \mathrm{st}$*. We say that* $\mathrm{st}_{\mathbb{M}}$ *is a* $\mathbb{M}$***-standard part mapping*** *(see the remark below). Consequently, for every* $z \in \mathcal{F}$ *we have*

$$z = \mathrm{st}_{\mathbb{M}}(z) + dz,$$

where $dz \in \mathcal{F}_0$.

(v) *The mapping* $\sigma : \mathbb{C} \to \widehat{\mathcal{F}}$*, defined by* $\sigma(z) = \widehat{z}$*, is a* ***field embedding*** *of* $\mathbb{C}$ *into* $\widehat{\mathcal{F}}$ *and we have the formula* $\sigma_{\mathbb{M}} \,|\, \mathcal{F}({}^*\mathbb{C}) = \sigma \circ \mathrm{st}$.

Proof: (i) Suppose (on the contrary) that $\gamma \in \mathcal{F}$ and $\gamma - r \notin \mathcal{F}_0$ for all $r \in \mathbb{M}$ (see Lemma 10.3). We have $\mathbb{M}(\gamma) \subset \mathcal{F}$, contradicting the maximality of $\mathbb{M}$, since $\mathbb{M}(\gamma)$ is a proper field extension of $\mathbb{M}$.

(ii) The isomorphism between $\mathbb{M}$ and $\widehat{\mathcal{F}}$ follows directly from the asymptotic expansion $z = r + dz$. Consequently, $\widehat{\mathcal{F}}$ is an algebraically closed field since $\mathbb{M}$ is an algebraically closed field and $\Re(\widehat{\mathcal{F}})$ is a real closed field since $\Re(\mathbb{M})$ is a real closed field by Lemma 10.1.

(iii) follows directly from (ii) because $\mathbb{M}$ and $\widehat{\mathbb{M}}$ are isomorphic by Lemma 10.2, and because $\widehat{\mathcal{F}} = \widehat{\mathbb{M}}$ by what was just proved in part (i).

(iv) follows directly from (i).

(v) We have $\mathcal{F}(^*\mathbb{C}) \subseteq \mathcal{F}$ by Theorem 10.1 and $\mathcal{F}_0 \subseteq \mathcal{I}(^*\mathbb{C})$ by Theorem 10.2. The latter implies the formula $\sigma_{\mathbb{M}}|\mathcal{F}(^*\mathbb{C}) = \sigma \circ \mathrm{st}$ and the statement about σ follows from (iii). ▲

Remark 10.1 (Quasi-Standard Part Mapping) *According to the above theorem, every maximal field* $\mathbb{M}$ *determines a unique field embedding* $\sigma_{\mathbb{M}}$ *(24). Conversely, every field embedding* $\sigma_{\mathbb{M}}$ *of* $\widehat{\mathcal{F}}$ *into* $^*\mathbb{C}$ *determines a maximal field* $\mathbb{M} \subset \mathcal{F}$ *by* $\sigma_{\mathbb{M}}[\widehat{\mathcal{F}}] = \mathbb{M}$. *On the ground of the isomorphism between* $\mathbb{M}$ *and* $\widehat{\mathcal{F}}$ *we shall sometimes identify* $\mathbb{M}$ *with* $\widehat{\mathcal{F}}$ *by simply letting* $\mathbb{M} = \widehat{\mathcal{F}}$. *That means nothing but to "pick up and fix" a particular maximal field* $\mathbb{M}$ *within* $\mathcal{F}$, *to replace the embedding* $\widehat{\mathcal{F}} \hookrightarrow {}^*\mathbb{C}$ *(24) by the simple inclusion* $\widehat{\mathcal{F}} \subset {}^*\mathbb{C}$. *In this environment* $\mathrm{st}_{\mathbb{M}}$ *reduces to the quotient mapping* $q : \mathcal{F} \to \widehat{\mathcal{F}}$. *We shall write simply* $\widehat{\mathrm{st}} : \mathcal{F} \to {}^*\mathbb{C}$ *instead of the more precise* $\mathrm{st}_{\mathbb{M}} : \mathcal{F} \to \mathbb{M}$ *and call* $\widehat{\mathrm{st}}$ *a* ***quasi-standard part mapping*** *associated with the asymptotic hull* $\widehat{\mathcal{F}}$ *and its particular embedding* $\widehat{\mathcal{F}} \hookrightarrow {}^*\mathbb{C}$ *(24). We summarize all these as:*

$$\widehat{\mathrm{st}} : \mathcal{F} \to \widehat{\mathcal{F}} \subseteq {}^*\mathbb{C},$$
$$\widehat{\mathrm{st}}(z) = q(z) \textit{ for all } z \in \mathcal{F}.$$

"Quasi" stands to distinguish $\widehat{\mathrm{st}}$ *from the "genuine standard part mapping* $\mathrm{st} : \mathcal{F}(^*\mathbb{C}) \to {}^*\mathbb{C}$ *with range* $\mathrm{st}[\mathcal{F}(^*\mathbb{C})] = \mathbb{C}$. *Recall that* $\widehat{\mathrm{st}}$ *is an extension of* st, *i.e.* $\widehat{\mathrm{st}}\,|\,\mathcal{F}(^*\mathbb{C}) = \mathrm{st}$.

Example 10.1 (Archimedean Hull) *Let* $\mathcal{F} = \mathcal{F}(^*\mathbb{C})$. *In this case we have* $\mathcal{F}_0 = \mathcal{I}(^*\mathbb{C})$ *and* $\widehat{\mathcal{F}} = \mathbb{C}$ *(see part (iv) of Theorem 10.2). Also* $\widehat{\mathrm{st}}(z) = \mathrm{st}(z)$ *for all* $z \in \mathcal{F}(^*\mathbb{C})$. *In this particular case the spilling principles presented earlier in Theorem 11.1 reduce the the familiar spilling principles in non-standard analysis.*

Here are **several example** of non-archimedean asymptotic hulls, i.e. examples of non-archimedean algebraically closed fields of the form $\widehat{\mathcal{F}}$.

Example 10.2 (A. Robinson's Asymptotic Numbers) *Let ρ be a positive infinitesimal in ${}^*\mathbb{R}$ and let $\mathcal{F} = \mathcal{M}_\rho({}^*\mathbb{C})$, where*

$$\mathcal{M}_\rho({}^*\mathbb{C}) = \{z \in {}^*\mathbb{C} :\ |z| \le \rho^{-n} \ \text{ for some } n \in \mathbb{N}\}, \tag{25}$$

*is the ring of the ρ-**moderate numbers** in ${}^*\mathbb{C}$. In this case $\mathcal{F}_0 = \mathcal{N}_\rho({}^*\mathbb{C})$, where*

$$\mathcal{N}_\rho({}^*\mathbb{C}) = \{z \in {}^*\mathbb{C} :\ |z| \le \rho^{n} \ \text{ for all } n \in \mathbb{N}\}, \tag{26}$$

*is the ideal of the ρ-**negligible numbers** in ${}^*\mathbb{C}$. The elements of the non-standard hull $\widehat{\mathcal{F}} = \mathcal{M}_\rho({}^*\mathbb{C})/\mathcal{N}_\rho({}^*\mathbb{C}) \stackrel{def}{=} {}^\rho\mathbb{C}$ are called **complex ρ-asymptotic numbers**. The field of the **real ρ-asymptotic numbers** ${}^\rho\mathbb{R} = \Re({}^\rho\mathbb{C}) = \mathcal{M}_\rho({}^*\mathbb{R})/\mathcal{N}_\rho({}^*\mathbb{R})$ is introduced by A. Robinson [74] and is intimately connected with the asymptotic expansions of standard functions (A.H. Lightstone and A. Robinson [56]). The field ${}^\rho\mathbb{C}$ is also known as **A. Robinson's valuation field** because it is endowed with a non-archimedean valuation $v : {}^\rho\mathbb{C} \to \mathbb{R} \cup \{\infty\}$ defined by*

$$v(z) = \sup\{r \in \mathbb{Q} \mid \frac{z}{q(\rho^r)} \approx 0\}, \quad z \ne 0,$$

and $v(0) = \infty$. We also have the following formula for the valuation:

$$v(q(z)) = \mathrm{st}(\log_\rho |z|), \qquad z \in \mathcal{M}_\rho({}^*\mathbb{C}) \setminus \mathcal{N}_\rho({}^*\mathbb{C}),$$

and $v(q(z)) = \infty$ for $z \in \mathcal{N}_\rho({}^*\mathbb{C})$. The valuation metric $d_v : {}^\rho\mathbb{C} \times {}^\rho\mathbb{C} \to \mathbb{R}$ is defined by $d_v(z, \zeta) = e^{-v(z-\zeta)}$ under the convention that $e^{-\infty} = 0$. We should note that the valuation topology and the order topology on ${}^\rho\mathbb{C}$ are the same. For more recent results on ${}^\rho\mathbb{R}$ we refer to (T. Todorov and R. Wolf [95]).

Example 10.3 (ρ-Finite Constants) *Let ρ be (as before) a positive infinitesimal in ${}^*\mathbb{R}$ and let $\mathcal{F} = \mathcal{F}_\rho({}^*\mathbb{C})$, where*

$$\mathcal{F}_\rho({}^*\mathbb{C}) = \{z \in {}^*\mathbb{C} :\ |z| < 1/\sqrt[n]{\rho} \ \text{ for all } \ n \in \mathbb{N}\}, \tag{27}$$

*is the set of the ρ-**finite numbers** in ${}^*\mathbb{C}$. In this case $\mathcal{F}_0 = \mathcal{I}_\rho({}^*\mathbb{C})$, where*

$$\mathcal{I}_\rho({}^*\mathbb{C}) = \{z \in {}^*\mathbb{C} :\ |z| \le \sqrt[n]{\rho} \ \text{ for some } \ n \in \mathbb{N}\}, \tag{28}$$

*is the set of the ρ-**infinitesimal numbers** in ${}^*\mathbb{C}$. We denote $\widehat{\mathcal{F}} = \mathcal{F}_\rho({}^*\mathbb{C})/\mathcal{I}_\rho({}^*\mathbb{C}) \stackrel{def}{=} \mathcal{C}$ and the elements of $\mathcal{C}$ will be often called ρ-**finite constants**.*

Example 10.4 (Logarithmic Field) *Let ρ be (as before) a positive infinitesimal in ${}^*\mathbb{R}$ and let*

$$\mathcal{F} = \{z \in {}^*\mathbb{C} : \ |z| < l_n(\rho) \ \text{ for all } \ n \in \mathbb{N}\},$$

where $l_1(\rho) = |\ln \rho| \stackrel{def}{=} |{}^*\ln \rho|$ is the non-standard extension of the usual logarithmic function $\ln x$ evaluated at ρ, $l_2(\rho) = \ln(|\ln \rho|)$, $\dots$, $l_{n+1}(\rho) = \ln(l_n(\rho))$ for $n = 1, 2, \dots$. Notice that $(l_n(\rho))$ is a strictly decreasing sequence of infinitely large positive numbers in ${}^*\mathbb{R}$. In this case we have

$$\mathcal{F}_0 = \{z \in {}^*\mathbb{C} : \ |z| \leq 1/l_n(\rho) \ \text{ for some } \ n \in \mathbb{N}\},$$

The corresponding non-standard hull $\widehat{\mathcal{F}} = \mathcal{F}/\mathcal{F}_0$ is a non-archimedean field.

Example 10.5 (Exponential Field) *Let ρ be (as before) a positive infinitesimal in ${}^*\mathbb{R}$ and let*

$$\mathcal{F} = \{z \in {}^*\mathbb{C} : \ |z| \leq \exp_n(\rho) \ \text{ for some } n \in \mathbb{N}\},$$

where $\exp_1(\rho) = \exp(\rho) = {}^*e^\rho$ is the non-standard extension of the usual real exponential function e^x evaluated at ρ, $\exp_2(\rho) = \exp(\exp(\rho))$, $\exp_{n+1}(\rho) = \exp(\exp_n(\rho))$. Notice that $(\exp_n(\rho))$ is a increasing sequence of infinitely large positive numbers in ${}^*\mathbb{R}$. In this case we have

$$\mathcal{F}_0 = \{z \in {}^*\mathbb{C} : \ |z| < 1/\exp_n(\rho) \ \text{ for all } n \in \mathbb{N}\}.$$

We shall call $\widehat{\mathcal{F}} = \mathcal{F}/\mathcal{F}_0 \stackrel{def}{=} \mathbb{E}$ ***exponential field*** *and the elements of $\mathbb{E}$ will be sometimes called* ***exponential constants*** *since both e^ρ and $e^{1/\rho}$ are in $\mathcal{F} \setminus \mathcal{F}_0$.*

Example 10.6 (The Case $\mathcal{F} = {}^*\mathbb{C}$) *Let $\mathcal{F} = {}^*\mathbb{C}$. In this case $\mathcal{F}_0 = \{0\}$ and $\widehat{\mathcal{F}} = {}^*\mathbb{C}$. In this case $\widehat{\mathrm{st}}$ reduces the the identity function in ${}^*\mathbb{C}$, i.e. $\widehat{\mathrm{st}}(x) = x$ for all $x \in {}^*\mathbb{C}$. This asymptotic hull, although somewhat trivial, plays part of the construction of one particular algebra of generalized functions (see Section* **??** *of Chapter* **??***)*

Remark 10.2 (Real Asymptotic Hulls) *We presented the complex version of the asymptotic hull construction because it better fits to our immediate needs in the next section. Some readers however might prefer a real version of the same construction. They should consider a convex subring $\mathcal{F}$ of ${}^*\mathbb{R}$ (not of ${}^*\mathbb{C}$) instead, i.e. $\mathcal{F}$ is a subring of ${}^*\mathbb{R}$ such that*

$$(\forall x \in {}^*\mathbb{R})(\forall y \in \mathcal{F})(|x| \leq |y| \Rightarrow x \in \mathcal{F}). \tag{29}$$

All results presented in this section remain valid if the phrase "algebraically closed field" is replaced everywhere by "real closed field". In particular, the fields of the form $\widehat{\mathcal{F}}$ *(and the fields* $\mathbb{M}$ *and* $\widehat{\mathbb{M}}$*) will be real closed (not algebraically closed) fields. We shall summarize this by simply saying that* $\widehat{\mathcal{F}}$ *is a* ***real asymptotic hull****. For example, if* $\mathcal{F}$ *is a convex subring of* ${}^*\mathbb{C}$*, then* $\Re(\mathcal{F})$ *is a convex subring of* ${}^*\mathbb{R}$*. Consequently,* $\Re(\widehat{\mathcal{F}})$ *is a real asymptotic hull because* $\Re(\widehat{\mathcal{F}}) = \widehat{\Re(\mathcal{F})}$*. Here is another example of a convex subring of* ${}^*\mathbb{R}$ *(compare with Example 10.2):*

$$\mathcal{M}_\rho({}^*\mathbb{R}) = \{x \in {}^*\mathbb{R} : \ |x| \le \rho^{-n} \ \text{ for some } n \in \mathbb{N}\}. \tag{30}$$

The corresponding real asymptotic hull coincides with A. Robinson's field of real asymptotic numbers ${}^\rho\mathbb{R}$*. We shall leave the detail of the "real case" to the reader.*

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11 Spilling Principles

In this section we present several **spilling principles** in terms of a given convex subring $\mathcal{F}$ of ${}^*\mathbb{C}$ (Section 10). These principles play role in our theory similar, say, to the Cantor principle in real analysis or to the Hahn-Banach theorem in functional analysis. We should note that the spilling principles presented below are more general than the more familiar **underflow and overflow principles** in non-standard analysis. Actually the latter follow as a particular case for $\mathcal{F} = \mathcal{F}({}^*\mathbb{C})$. We are unaware of any counterparts of the spilling principles presented here in J.F. Colombeau's theory.

Theorem 11.1 (Spilling Principles) *Let $\mathcal{F}$ be a convex subring of ${}^*\mathbb{C}$ (Section 10) and $\mathcal{A} \subseteq {}^*\mathbb{C}$ be an internal set (Definition 4.2). Then:*

(i) Overflow of $\mathcal{F}$: *If $\mathcal{A}$ contains arbitrarily large numbers in $\mathcal{F}$, then $\mathcal{A}$ contains arbitrarily small numbers in ${}^*\mathbb{C} \setminus \mathcal{F}$. Consequently,*

$$\mathcal{F} \setminus \mathcal{F}_0 \subset \mathcal{A} \Rightarrow \mathcal{A} \cap ({}^*\mathbb{C} \setminus \mathcal{F}) \neq \varnothing.$$

(ii) Underflow of $\mathcal{F} \setminus \mathcal{F}_0$: *If $\mathcal{A}$ contains arbitrarily small numbers in $\mathcal{F}\setminus\mathcal{F}_0$, then $\mathcal{A}$ contains arbitrarily large numbers in $\mathcal{F}_0$. Consequently,*

$$\mathcal{F} \setminus \mathcal{F}_0 \subset \mathcal{A} \Rightarrow \mathcal{A} \cap \mathcal{F}_0 \neq \varnothing.$$

(iii) Overflow of $\mathcal{F}_0$: *If $\mathcal{A}$ contains arbitrarily large numbers in $\mathcal{F}_0$, then $\mathcal{A}$ contains arbitrarily small numbers in $\mathcal{F}\setminus\mathcal{F}_0$. Consequently,*

$$\mathcal{F}_0 \subset \mathcal{A} \Rightarrow \mathcal{A} \cap (\mathcal{F} \setminus \mathcal{F}_0) \neq \varnothing.$$

(iv) Underflow of ${}^*\mathbb{C} \setminus \mathcal{F}$: *If $\mathcal{A}$ contains arbitrarily small numbers in ${}^*\mathbb{C}\setminus\mathcal{F}$, then $\mathcal{A}$ contains arbitrarily large numbers in $\mathcal{F}$. Consequently,*

$${}^*\mathbb{C} \setminus \mathcal{F} \subset \mathcal{A} \Rightarrow \mathcal{A} \cap (\mathcal{F} \setminus \mathcal{F}_0) \neq \varnothing.$$

Proof: (i) If $\mathcal{A}$ is unbounded in ${}^*\mathbb{C}$, there is nothing to prove. If $\mathcal{A}$ is bounded in ${}^*\mathbb{C}$, then $\sup(|\mathcal{A}|) = x$ exists in ${}^*\mathbb{R}$, where $|\mathcal{A}| = \{|z| : z \in \mathcal{A}\}$. Notice that $x \notin \mathcal{F}$. To show this, suppose (on the contrary) that $x \in \mathcal{F}$ and let $\mathbb{M}$ be a maximal field within $\mathcal{F}$ (Definition 10.1). We have $x \leq \varepsilon$ for some $\varepsilon \in \mathbb{M}_+$ by Theorem 10.2, contradicting the assumptions for $\mathcal{A}$ since $\mathbb{M}_+ \subset \mathcal{F}$. Next, there exists $z \in \mathcal{A}$ such that $x/2 < |z| < x$ by the choice of x and we have $z \notin \mathcal{F}$ since $x/2 \notin \mathcal{F}$. We just proved that $\mathcal{A} \cap ({}^*\mathbb{C} \setminus \mathcal{F}) \neq \varnothing$.

It remains to show that $\mathcal{A} \cap ({}^*\mathbb{C} \setminus \mathcal{F})$ does not have a lower upper bound in ${}^*\mathbb{C} \setminus \mathcal{F}$. Suppose (on the contrary) that that there exists $\lambda \in {}^*\mathbb{C} \setminus \mathcal{F}$ such that $\lambda \leq |z|$ for all $z \in \mathcal{A} \cap ({}^*\mathbb{C} \setminus \mathcal{F})$. The set $\mathcal{A}_\lambda = \{z \in \mathcal{A} : |z| < \lambda\}$ is internal and we have $\mathcal{A} \cap \mathcal{F} \subseteq \mathcal{A}_\lambda$ by the choice of λ. It follows that $\mathcal{A}_\lambda$ has (just like $\mathcal{A}$) arbitrarily large elements in $\mathcal{F}$ and we conclude that $\mathcal{A}_\lambda \cap ({}^*\mathbb{C} \setminus \mathcal{F}) \neq \varnothing$ by what was proved above. Thus there exists $z \in \mathcal{A} \cap ({}^*\mathbb{C} \setminus \mathcal{F})$ such that $|z| < \lambda$, a contradiction.

(ii) follows immediately from (i) and the fact that $z \in \mathcal{F} \setminus \mathcal{F}_0$ implies $1/z \in \mathcal{F} \setminus \mathcal{F}_0$ and also that $z \in {}^*\mathbb{C} \setminus \mathcal{F}$ implies $1/z \in \mathcal{F}_0$.

The proof of (iii) is similar to the proof of (i) and we leave it to the reader.

(iv) follows immediately from (iii) and the fact that $z \in \mathcal{F}_0 \setminus \{0\}$ implies $1/z \in {}^*\mathbb{C} \setminus \mathcal{F}$ and also that $z \in \mathcal{F} \setminus \mathcal{F}_0$ implies $1/z \in \mathcal{F} \setminus \mathcal{F}_0$.

▲

Recall that $\mathcal{F}({}^*\mathbb{C})$, $\mathcal{I}({}^*\mathbb{C})$ and $\mathcal{L}({}^*\mathbb{C})$ denote the sets of the finite, infinitesimal and infinitely large numbers in ${}^*\mathbb{C}$, respectively, and $\mathcal{L}({}^*\mathbb{C}) = \mathcal{F}({}^*\mathbb{C}) \setminus \mathcal{I}({}^*\mathbb{C})$ (Section 6). Here is the more familiar spilling (underflow and overflow) principles about $\mathcal{F}({}^*\mathbb{C})$, $\mathcal{I}({}^*\mathbb{C})$ and $\mathcal{L}({}^*\mathbb{C})$.

Corollary 11.1 (The Usual Spilling Principles) *Let $\mathcal{A} \subseteq {}^*\mathbb{C}$ be an internal set. Then:*

(i) Overflow of $\mathcal{F}({}^*\mathbb{C})$: *If $\mathcal{A}$ contains arbitrarily large finite numbers, then $\mathcal{A}$ contains arbitrarily small infinitely large numbers. Consequently,*

$$\mathcal{F}({}^*\mathbb{C}) \setminus \mathcal{I}({}^*\mathbb{C}) \subset \mathcal{A} \Rightarrow \mathcal{A} \cap \mathcal{L}({}^*\mathbb{C}) \neq \varnothing.$$

(ii) Underflow of $\mathcal{F}({}^*\mathbb{C}) \setminus \mathcal{I}({}^*\mathbb{C})$: *If $\mathcal{A}$ contains arbitrarily small finite non-infinitesimals, then $\mathcal{A}$ contains arbitrarily large infinitesimals. Consequently,*

$$\mathcal{F}({}^*\mathbb{C}) \setminus \mathcal{I}({}^*\mathbb{C}) \subset \mathcal{A} \Rightarrow \mathcal{A} \cap \mathcal{I}({}^*\mathbb{C}) \neq \varnothing.$$

(iii) Overflow of $\mathcal{I}({}^*\mathbb{C})$: *If $\mathcal{A}$ contains arbitrarily large infinitesimals, then $\mathcal{A}$ contains arbitrarily small finite non-infinitesimals. Consequently,*

$$\mathcal{I}({}^*\mathbb{C}) \subset \mathcal{A} \Rightarrow \mathcal{A} \cap (\mathcal{F}({}^*\mathbb{C}) \setminus \mathcal{I}({}^*\mathbb{C})) \neq \varnothing.$$

(iv) Underflow of $\mathcal{L}({}^*\mathbb{C})$: *If $\mathcal{A}$ contains arbitrarily small infinnitely large numbers, then $\mathcal{A}$ contains arbitrarily large finite numbers. Consequently,*

$$\mathcal{L}({}^*\mathbb{C}) \subset \mathcal{A} \Rightarrow \mathcal{A} \cap (\mathcal{F}({}^*\mathbb{C}) \setminus \mathcal{I}({}^*\mathbb{C})) \neq \varnothing.$$

Proof: The result follows directly from the previous theorem in the particular case of $\mathcal{F} = \mathcal{F}(^*\mathbb{C})$ taking into account that in this case $\mathcal{F}_0 = \mathcal{I}(^*\mathbb{C})$ and $\mathcal{F} \setminus \mathcal{F}_0 = \mathcal{L}(^*\mathbb{C})$ (Example 10.1). ▲

12 $\mathcal{F}$-Asymptotic Functions

In this section we describe a variety of differential rings $\widehat{\mathcal{E}_{\mathcal{F}}}(\Omega)$ of generalized functions on an open set Ω in terms of a given convex subring $\mathcal{F}$ of ${}^*\mathbb{C}$ (Section 10). The elements of $\widehat{\mathcal{E}_{\mathcal{F}}}(\Omega)$ are named **$\mathcal{F}$-asymptotic functions** because their values are in the field $\widehat{\mathcal{F}}$ of the $\mathcal{F}$-asymptotic numbers and because, more importantly, each $\widehat{\mathcal{E}_{\mathcal{F}}}(\Omega)$ is an algebra over the field $\widehat{\mathcal{F}}$ (Section 10). We intend to convert some of $\widehat{\mathcal{E}_{\mathcal{F}}}(\Omega)$ into **algebras of Colombeau's type** by supplying $\widehat{\mathcal{E}_{\mathcal{F}}}(\Omega)$ with a copy of the space of Schwartz distributions $\mathcal{D}'(\Omega)$ in one of the next sections. In this section we generalize some of the results in (Oberguggenberger and T. Todorov [66]), where the algebra of ρ-asymptotic functions ${}^\rho\mathcal{E}(\Omega)$ is introduced; within our more general theory the algebra ${}^\rho\mathcal{E}(\Omega)$ appears as a particular example (Example 12.2). Similar to some of our results appear in the H. Vernaeve Ph.D. Thesis [98] (for comparison see the definition of $\mathcal{E}_M(\Omega)$ on p. 90, Sec. 3.6).

Here is the **summary** of the basic definitions. The justification of the definitions will be presented later in this section and some of the results will be worked out in detail in some of the next sections.

1. In what follows ${}^*\mathbb{C}$ stands for a non-standard extension of the field of the complex numbers $\mathbb{C}$. Let $\mathcal{F}$ be a convex subring in ${}^*\mathbb{C}$, $\mathcal{F}_0$ be the ideal of the non-invertible elements of $\mathcal{F}$. Let $\widehat{\mathcal{F}}$ be the field of $\mathcal{F}$-asymptotic numbers. Recall $\widehat{\mathcal{F}}$ is an algebraically closed (possibly non-archimedean) field (Section 10). Let Ω be an open set of $\mathbb{R}^d$. In what follows $\mu_{\mathcal{F}}(\Omega)$ denotes the $\mathcal{F}$-monad of Ω (34). Also ${}^*\mathcal{E}(\Omega)$ stands for the ring of internal non-standard smooth functions of the form $f : {}^*\Omega \to {}^*\mathbb{C}$ (Section 8).

2. We define the set of **$\mathcal{F}$-moderate functions** $\mathcal{M}_{\mathcal{F}}(\Omega)$ and the set of the **$\mathcal{F}$-negligible functions** in ${}^*\mathcal{E}(\Omega)$ by
$$\mathcal{M}_{\mathcal{F}}(\Omega) = \{f \in {}^*\mathcal{E}(\Omega) \mid (\forall\alpha \in \mathbb{N}_0^d)(\forall x \in \mu(\Omega)(\partial^\alpha f(x) \in \mathcal{F})\},$$
$$\mathcal{N}_{\mathcal{F}}(\Omega) = \{f \in {}^*\mathcal{E}(\Omega) \mid (\forall\alpha \in \mathbb{N}_0^d)(\forall x \in \mu(\Omega)(\partial^\alpha f(x) \in \mathcal{F}_0)\},$$
respectively. Let $\widehat{\mathcal{E}_{\mathcal{F}}}(\Omega) = \mathcal{M}_{\mathcal{F}}(\Omega)/\mathcal{N}_{\mathcal{F}}(\Omega)$ be the corresponding factor ring. We say that $\widehat{\mathcal{E}_{\mathcal{F}}}(\Omega)$ **is generated by** $\mathcal{F}$. The elements of $\widehat{\mathcal{E}_{\mathcal{F}}}(\Omega)$ are named **$\mathcal{F}$-asymptotic functions on** Ω. We denote by $Q_\Omega : \mathcal{M}_{\mathcal{F}}(\Omega) \to \widehat{\mathcal{E}_{\mathcal{F}}}(\Omega)$ the corresponding quotient mapping. However we shall often $\widehat{f}$ instead of $Q_\Omega(f)$ for the equivalence class of $f \in \mathcal{M}_{\mathcal{F}}(\Omega)$.

3. We define the embedding
$$\mathcal{E}(\Omega) \hookrightarrow \widehat{\mathcal{E}_{\mathcal{F}}(\Omega)}, \tag{31}$$

by $f \to \widehat{{}^*f}$, where ${}^*f$ is the *non-standard extension* of f.

4. Let $\widehat{f} \in \widehat{\mathcal{E}_{\mathcal{F}}}(\Omega)$ and $\widehat{x} \in \mu_{\mathcal{F}}(\Omega)$ (34). We define the **value of** $\widehat{f}$ **at** $\widehat{x}$ by the formula $\widehat{f}(\widehat{x}) = \widehat{f(x)}$. We shall use the same notation, $\widehat{f}$, for the corresponding graph $\widehat{f} : \mu_{\mathcal{F}}(\Omega) \to \widehat{\mathcal{F}}$.

5. Let $\Omega, \mathcal{O}$ be two open sets of $\mathbb{R}^d$ such that $\mathcal{O} \subseteq \Omega$. Let $\widehat{f} \in \widehat{\mathcal{E}}_{\mathcal{F}}(\Omega)$. We define the **restriction** $\widehat{f} \restriction \mathcal{O}$ of $\widehat{f}$ on $\mathcal{O}$ by the formula
$$\widehat{f} \restriction \mathcal{O} = \widehat{f|{}^*\mathcal{O}},$$
where ${}^*\mathcal{O}$ is the non-standard extension of $\mathcal{O}$ and $f|{}^*\mathcal{O}$ is the usual (pointwise) restriction of f on ${}^*\mathcal{O}$.

6. **Simpler Notation:** We shall sometimes **drop $\mathcal{F}$, as a lower-index**, in $\mathcal{M}_{\mathcal{F}}(\Omega)$, $\mathcal{N}_{\mathcal{F}}(\Omega)$, $\widehat{\mathcal{E}}_{\mathcal{F}}(\Omega)$, $\mu_{\mathcal{F}}(\Omega)$, etc. and write simply
$$\mathcal{M}(\Omega),\ \mathcal{N}(\Omega),\ \widehat{\mathcal{E}}(\Omega),\ \mu(\Omega), \dots,$$
instead when no confusion could arise. The elements of $\widehat{\mathcal{E}}(\Omega)$ will be called simply **asymptotic functions on** Ω (meaning $\mathcal{F}$-asymptotic functions for the given specific $\mathcal{F}$).

Theorem 12.1 (Some Basic Results) *Let $\mathcal{F}$ be (as before) a convex subring of ${}^*\mathbb{C}$. Then:*

(i) *$\mathcal{M}_{\mathcal{F}}(\Omega)$ is a differential subring of ${}^*\mathcal{E}(\Omega)$ and $\mathcal{N}_{\mathcal{F}}(\Omega)$ is a differential ideal in $\mathcal{M}_{\mathcal{F}}(\Omega)$. Consequently, $\widehat{\mathcal{E}}_{\mathcal{F}}(\Omega)$ is a differential ring.*

(ii) *$\mathcal{E}(\Omega)$ is a differential subalgebra of $\widehat{\mathcal{E}}_{\mathcal{F}}(\Omega)$ over $\mathbb{C}$ under the embedding $f \to \widehat{{}^*f}$. We shall often write this as an inclusion*
$$\mathcal{E}(\Omega) \subset \widehat{\mathcal{E}}_{\mathcal{F}}(\Omega),$$
instead of (31).

(iii) *Let $\mathcal{T}_d$ stand for the usual topology on $\mathbb{R}^d$. The collection $\widehat{\mathcal{E}}_{\mathcal{F}} \stackrel{def}{=} \{\widehat{\mathcal{E}}_{\mathcal{F}}(\Omega)\}_{\Omega \in \mathcal{T}_d}$ is a* **sheaf of differential rings** *on the topological space $(\mathbb{R}^d, \mathcal{T}_d)$ under the restriction $\restriction$. Consequently, every function $\widehat{f} \in \widehat{\mathcal{E}_{\mathcal{F}}}(\Omega)$ has a support* $\mathrm{supp}(\widehat{f})$ *which is a closed set of* Ω.

(iv) *Each $\widehat{\mathcal{E}}_{\mathcal{F}}(\Omega) \in \widehat{\mathcal{E}}_{\mathcal{F}}$ is a differential ring with ring of scalars $\widehat{\mathcal{F}}$ in the sense that*

$$\widehat{\mathcal{F}} = \left\{ \widehat{f} \in \widehat{\mathcal{E}}_{\mathcal{F}}(\mathbb{R}^d) \mid \nabla \widehat{f} = 0 \right\}, \tag{32}$$

where $\nabla \widehat{f} = \widehat{\nabla f}$ is the gradient $\widehat{f}$ in $\widehat{\mathcal{E}_{\mathcal{F}}}(\mathbb{R}^d)$ and 0 in $\nabla \widehat{f} = 0$ is the zero of the ring $\widehat{\mathcal{E}_{\mathcal{F}}}(\mathbb{R}^d)$. Consequently, each $\widehat{\mathcal{E}}_{\mathcal{F}}(\Omega)$ is a **differential algebra over the field** $\widehat{\mathcal{F}}$ *under the ring operations in $\widehat{\mathcal{E}}_{\mathcal{F}}(\Omega)$.*

(v) *The embedding $\widehat{\mathcal{E}}_{\mathcal{F}}(\Omega) \hookrightarrow \widehat{\mathcal{F}}^{\mu_{\mathcal{F}}(\Omega)}$, defined by the pointwise values of $\widehat{f} \in \widehat{\mathcal{E}_{\mathcal{F}}}(\Omega)$, preserves the addition, multiplication and partial differentiation in $\widehat{\mathcal{E}}_{\mathcal{F}}(\Omega)$.*

Proof: The properties (i), (ii) and (iv) follow easily from the definition of $\widehat{\mathcal{E}_{\mathcal{F}}}(\Omega)$ and we shall leave to the reader to check the detail. The proof of (iii) and (v) is more complicated. We shall proof (iii) in Section **??** and we shall prove (v) in Section **??**.

▲

Here are **several examples** algebras of asymptotic functions.

Example 12.1 (Nothing New) *Let $\mathcal{F} = \mathcal{F}({}^*\mathbb{C})$. In this case we have $\mathcal{F}_0 = \mathcal{I}({}^*\mathbb{C})$ and $\widehat{\mathcal{F}} = \mathbb{C}$ (Example 10.1). For the $\mathcal{F}$-moderate and $\mathcal{F}$-negligible functions we have $\mathcal{M}_{\mathcal{F}}(\Omega) = \mathcal{F}({}^*\mathcal{E}(\Omega))$ and $\mathcal{N}_{\mathcal{F}}(\Omega) = \mathcal{I}({}^*\mathcal{E}(\Omega))$, respectively, where*

$$\mathcal{F}({}^*\mathcal{E}(\Omega)) \overset{def}{=} \{f \in {}^*\mathcal{E}(\mathbb{R}^d) \mid (\forall \alpha \in \mathbb{N}_0^d)(\forall x \in \mu(\Omega)(\partial^\alpha f(x) \in \mathcal{F}({}^*\mathbb{C})),$$
$$\mathcal{I}({}^*\mathcal{E}(\Omega)) \overset{def}{=} \{f \in {}^*\mathcal{E}(\mathbb{R}^d) \mid (\forall \alpha \in \mathbb{N}_0^d)(\forall x \in \mu(\Omega)(\partial^\alpha f(x) \in \mathcal{I}({}^*\mathbb{C}))\},$$

The $\mathcal{F}$-asymptotic functions are the familiar smooth functions, i.e.

$$\widehat{\mathcal{E}}_{\mathcal{F}}(\Omega) = \mathcal{E}(\Omega).$$

Example 12.2 (ρ-Asymptotic Functions) *Let ρ be a positive infinitesimal in ${}^*\mathbb{R}$ and let*

$$\mathcal{F} = \mathcal{M}_\rho({}^*\mathbb{C}) = \{x \in {}^*\mathbb{C} : \ |x| \le \rho^{-n} \ \text{for some}\ n \in \mathbb{N}\},$$

is the ring of the ρ-moderate numbers in ${}^*\mathbb{C}$. In this case we have:

$$\mathcal{F}_0 = \mathcal{N}_\rho({}^*\mathbb{C}) = \{x \in {}^*\mathbb{C} : \ |x| \le \rho^{n} \ \text{for all}\ n \in \mathbb{N}\},$$
$$\widehat{\mathcal{F}} = \mathcal{M}_\rho({}^*\mathbb{C})/\mathcal{N}_\rho({}^*\mathbb{C}) \overset{def}{=} {}^\rho\mathbb{C}, \tag{33}$$

(Example 10.2). For the $\mathcal{F}$-moderate and $\mathcal{F}$-negligible functions we have $\mathcal{M}_{\mathcal{F}}(\Omega) = \mathcal{M}_\rho({}^*\mathcal{E}(\Omega))$ and $\mathcal{N}_{\mathcal{F}}(\Omega) = \mathcal{N}_\rho({}^*\mathcal{E}(\Omega))$, respectively, where

$$\mathcal{M}_\rho({}^*\mathcal{E}(\Omega)) \stackrel{def}{=} \left\{ f \in {}^*\mathcal{E}(\Omega) \mid (\forall \alpha \in \mathbb{N}_0^d)(\forall x \in \mu(\Omega))\, [\partial^\alpha f(x) \in \mathcal{M}_\rho({}^*\mathbb{C})] \right\},$$
$$\mathcal{N}_\rho({}^*\mathcal{E}(\Omega)) \stackrel{def}{=} \left\{ f \in {}^*\mathcal{E}(\Omega) \mid (\forall \alpha \in \mathbb{N}_0^d)(\forall x \in \mu(\Omega))\, [\partial^\alpha f(x) \in \mathcal{N}_\rho({}^*\mathbb{C})] \right\}.$$

The corresponding factor ring

$${}^\rho\mathcal{E}(\Omega) = \mathcal{M}_\rho({}^*\mathcal{E}(\Omega))/\mathcal{N}_\rho({}^*\mathcal{E}(\Omega)),$$

is an algebra over the field of A. Robinson's asymptotic numbers ${}^\rho\mathbb{C}$ (Example **??***). The algebra ${}^\rho\mathcal{E}(\Omega)$ is introduced in (M. Oberguggenberger and T. Todorov[66]) under the name* ρ**-asymptotic functions***. We shall follow this terminology. The reader will find a more detail about ${}^\rho\mathcal{E}(\Omega)$ in Chapter* **??***. The algebra ${}^\rho\mathcal{E}(\Omega)$ is, in a sense, a non-standard counterpart of a* **special Colombeau's algebra** *(J. F. Colombeau [12]) with the important* **improvement of the properties of the scalars:** *The ring of the scalars ${}^\rho\mathbb{C}$ of ${}^\rho\mathcal{E}(\Omega)$ constitutes an algebraically closed Cantor-complete field. In contrast, the ring of the scalars $\widetilde{\mathbb{C}}$ of Colombeau simple algebra $\mathcal{G}^s(\Omega)$ is a ring with zero divisors.*

Example 12.3 (Logarithmic Hull) *Let ρ be (as before) a positive infinitesimal in ${}^*\mathbb{R}$ and let*

$$\mathcal{F} = \mathcal{F}_\rho({}^*\mathbb{C}) = \{x \in {}^*\mathbb{C} : \ |x| < 1/\sqrt[n]{\rho} \ \text{ for all } \ n \in \mathbb{N}\},$$

is the set of the ρ-finite numbers in ${}^*\mathbb{C}$. In this case we have:

$$\mathcal{F}_0 = \mathcal{I}_\rho({}^*\mathbb{C}) = \{x \in {}^*\mathbb{C} : \ |x| \leq \sqrt[n]{\rho} \ \text{ for some } \ n \in \mathbb{N}\},$$
$$\widehat{\mathcal{F}} = \mathcal{F}_\rho({}^*\mathbb{C})/\mathcal{I}_\rho({}^*\mathbb{C}) \stackrel{def}{=} \mathcal{C},$$

(Example 10.4). For the $\mathcal{F}$-moderate and $\mathcal{F}$-negligible functions we have $\mathcal{M}_{\mathcal{F}}(\Omega) = \mathcal{F}_\rho({}^*\mathcal{E}(\Omega))$ and $\mathcal{N}_{\mathcal{F}}(\Omega) = \mathcal{I}_\rho({}^*\mathcal{E}(\Omega))$, respectively, where

$$\mathcal{F}_\rho({}^*\mathcal{E}(\Omega)) \stackrel{def}{=} \left\{ f \in {}^*\mathcal{E}(\Omega) \mid (\forall \alpha \in \mathbb{N}_0^d)(\forall x \in \mu(\Omega))\, [\partial^\alpha f(x) \in \mathcal{F}_\rho({}^*\mathbb{C})] \right\},$$
$$\mathcal{I}_\rho({}^*\mathcal{E}(\Omega)) \stackrel{def}{=} \left\{ f \in {}^*\mathcal{E}(\Omega) \mid (\forall \alpha \in \mathbb{N}_0^d)(\forall x \in \mu(\Omega))\, [\partial^\alpha f(x) \in \mathcal{I}_\rho({}^*\mathbb{C})] \right\}.$$

The corresponding ring of $\mathcal{F}$-asymptotic functions

$$\widehat{\mathcal{E}}_{\mathcal{F}}(\Omega) = \mathcal{F}_\rho({}^*\mathcal{E}(\Omega))/\mathcal{I}_\rho({}^*\mathcal{E}(\Omega)),$$

is an algebra over the field of logarithmic constants $\mathcal{C}$ (Example 10.4).

Example 12.4 (Exponential Asymptotic Functions) *Let ρ be (as before) a positive infinitesimal in ${}^*\mathbb{R}$ and let*

$$\mathcal{F} = \{x \in {}^*\mathbb{C} : \ |x| \leq \exp_n(\rho) \ \textit{ for some } n \in \mathbb{N}\}.$$

In this case we have $\mathcal{F}_0 = \{x \in {}^*\mathbb{C} : \ |x| < 1/\exp_n(\rho) \ \textit{ for all } n \in \mathbb{N}\}$ and $\widehat{\mathcal{F}} = \mathcal{F}/\mathcal{F}_0 \stackrel{def}{=} \mathbb{E}$. The corresponding ring of asymptotic functions $\widehat{\mathcal{E}_{\mathcal{F}}}(\Omega)$ is an algebra over the exponential field $\mathbb{E}$ (Example 10.5).

Example 12.5 (The case $\mathcal{F} = {}^*\mathbb{C}$) *Let $\mathcal{F} = {}^*\mathbb{C}$. In this case $\mathcal{F}_0 = \{0\}$ and $\widehat{\mathcal{F}} = {}^*\mathbb{C}$ (Example 12.5). For the $\mathcal{F}$-moderate and $\mathcal{F}$-negligible functions we have*

$$\begin{aligned}
&\mathcal{M}_{\mathcal{F}}(\Omega) = {}^*\mathcal{E}(\Omega),\\
&\mathcal{N}_{\mathcal{F}}(\Omega) = \{f \in {}^*\mathcal{E}(\mathbb{R}^d) \mid (\forall \alpha \in \mathbb{N}_0^d)(\forall x \in \mu(\Omega)(\partial^\alpha f(x) = 0)\},
\end{aligned}$$

respectively. The ring of $\mathcal{F}$-asymptotic functions

$$\widehat{\mathcal{E}_{\mathcal{F}}}(\Omega) = {}^*\mathcal{E}(\Omega)/\mathcal{N}_{\mathcal{F}}(\Omega) \stackrel{def}{=} \widehat{\mathcal{E}}(\Omega).$$

is an algebra over the field ${}^*\mathbb{C}$. The algebra $\widehat{\mathcal{E}}(\Omega)$ is, in a sense, a **non-standard counterpart of Egorov algebra** *(Yu. V. Egorov [20]-[21]) with the important improvement of the properties of the scalars: The ring of the scalars ${}^*\mathbb{C}$ of $\widehat{\mathcal{E}}(\Omega)$ constitutes an algebraically closed saturated field. In contrast, the the scalars of Egorov's algebra are a ring with zero divisors. The algebra $\widehat{\mathcal{E}}(\Omega)$ will be studied in detail in Chapter* **??**.

empty

13 $\mathcal{F}$-Moderate and $\mathcal{F}$-Negligible Functions

In this section we present **several characterizations** of the $\mathcal{F}$-moderate and $\mathcal{F}$-negligible functions (Section 12).

Through out this section $\mathcal{F}$ stands for a convex subring of ${}^*\mathbb{C}$ (Section 10) and $\mathbb{M} \in \mathcal{M}ax(\mathcal{F})$ stands for a maximal field within $\mathcal{F}$ (Definition 10.1).

Theorem 13.1 *Let $f \in {}^*\mathcal{E}(\Omega)$. Then the following are equivalent:*

(i) $(\forall x \in \mu(\Omega))(f(x) \in \mathcal{F})$.

(ii) $(\forall x \in \mu(\Omega))(\exists M \in \mathbb{M}_+)(|f(x)| \leq M)$.

(iii) $(\forall K \subset\subset \Omega)(\exists M \in \mathbb{M}_+)(\sup_{x\in {}^*K} |f(x)| \leq M)$.

(iv) $(\forall x \in \mu(\Omega))(\exists A \in \mathcal{F} \setminus \mathcal{F}_0)(|f(x)| \leq A)$.

(v) $(\forall K \subset\subset \Omega)(\exists A \in \mathcal{F} \setminus \mathcal{F}_0)(\sup_{x\in {}^*K} |f(x)| \leq A)$.

(vi) $(\forall x \in \mu(\Omega))(\forall B \in {}^*\mathbb{R}_+ \setminus \mathcal{F})(|f(x)| < B)$.

(vii) $(\forall K \subset\subset \Omega)(\forall B \in {}^*\mathbb{R}_+ \setminus \mathcal{F})(\sup_{x\in {}^*K} |f(x)| < B)$.

Remark 13.1 We should note that the above theorem remains true even if the maximal field $\mathbb{M}$ is replaced by a set $S \subseteq \mathcal{F} \setminus \mathcal{F}_0$ such that S contains arbitrarily large numbers.

Proof: (i)$\Leftrightarrow$(ii) follows immediately by part (i) of Theorem 10.2.

(ii)$\Rightarrow$(iii): Let $K \subset\subset \Omega$ and recall that ${}^*K \subset \mu(\Omega)$ by Theorem 7.2. We observe that $\sup_{\xi\in {}^*K} |f(\xi)| \in \mathcal{F}$. Indeed, suppose (on the contrary) that $\gamma =: \sup_{\xi\in {}^*K} |f(\xi)| \notin \mathcal{F}$ which implies also $\gamma/2 \notin \mathcal{F}$. There exists $y \in {}^*K$ such that $\gamma/2 < |f(y)| < \gamma$ by the choice of γ. It follows $f(y) \notin \mathcal{F}$ which contradicts to (i) (hence it contradicts to (ii)) since $y \in \mu(\Omega)$. On the other hand, $\sup_{\xi\in {}^*K} |f(\xi)| \in \mathcal{F}$ implies that the internal set

$$\mathcal{A} = \{a \in {}^*\mathbb{R}_+ : \sup_{\xi\in {}^*K} |f(\xi)| \leq a\},$$

contains ${}^*\mathbb{R}_+ \setminus \mathcal{F}$ by by part (ii) of Theorem 10.2. Thus $\mathcal{A}$ contains arbitrarily small numbers in ${}^*\mathbb{C} \setminus \mathcal{F}$. It follows that $\mathcal{A} \cap (\mathcal{F} \setminus \mathcal{F}_0) \neq \varnothing$ by the Underflow of ${}^*\mathbb{C} \setminus \mathcal{F}$ (Theorem 11.1). Thus $\sup_{x\in {}^*K} |f(x)| \leq A$ holds for any $A \in \mathcal{A} \cap (\mathcal{F} \setminus \mathcal{F}_0)$. Also there exists $M_1 \in \mathbb{M}$ such that $A - M_1 \in \mathcal{F}_0$ by part (i) of Theorem 10.3. Let $H \in \mathbb{M}_+$. Then (iii) holds for $M = M_1 + H$.

(iii)⇒(iv): Suppose that $x \in \mu(\Omega)$ and observe that $\mathrm{st}(x) \in \Omega$ by the definition of $\mu(\Omega)$. Since Ω is an open set, there exists $\varepsilon \in \mathbb{R}_+$ such that $K \subset\subset \Omega$, where $K = \{r \in \Omega : |r - \mathrm{st}(x)| \leq \varepsilon\}$. There exists $M \in \mathbb{M}_+$ such that $\sup_{\xi \in {}^*K} |f(\xi)| \leq M$ by assumption which implies (iv) for $A = M$ since $x \in {}^*K$ and $M \in \mathbb{M}_+ \subset \mathcal{F} \setminus \mathcal{F}_0$.

The proof of (iv)⇒(v) is almost identical to the proof of (ii)⇒(iii) and we leave it to the reader.

(v)⇒(vi) follows immediately by part (ii) of Theorem 10.2.

(vi)⇒(vii): Suppose (on the contrary) that $\gamma =: \sup_{\xi \in {}^*K} |f(\xi)| \geq B$ for some $K \subset\subset \Omega$ and some $B \in {}^*\mathbb{R}_+ \setminus \mathcal{F}$. We have $B/2 \leq |f(y)| < \gamma$ for some $y \in {}^*K$ by the choice of γ. This contradicts (vi) since $y \in \mu(\Omega)$ and $B/2 \in {}^*\mathbb{R}_+ \setminus \mathcal{F}$.

(vii)⇒(i): Suppose that $x \in \mu(\Omega)$ and observe that $\mathrm{st}(x) \in \Omega$ by the definition of $\mu(\Omega)$. As before there exists $K \subset\subset \Omega$ such that $x \in {}^*K$. As before the internal set $\mathcal{A}$ contains ${}^*\mathbb{R}_+ \setminus \mathcal{F}$. Thus (as before) $\mathcal{A} \cap (\mathcal{F} \setminus \mathcal{F}_0) \neq \varnothing$ by the Underflow for ${}^*\mathbb{C} \setminus \mathcal{F}$ (Theorem 11.1). Thus $\sup_{\xi \in {}^*K} |f(\xi)| < A$ for any $A \in \mathcal{A} \cap (\mathcal{F} \setminus \mathcal{F}_0)$. It follows that $|f(x)| < A$ since $x \in {}^*K$ by the choice of K. Thus $f(x) \in \mathcal{F}$ (as required) by the convexity of $\mathcal{F}$.

▲

Here is a **list of characterizations of the $\mathcal{F}$-moderate functions**.

Corollary 13.1 ($\mathcal{F}$-Moderate Functions) *Let $f \in {}^*\mathcal{E}(\Omega)$. Then the following are equivalent:*

- **(i)** $f \in \mathcal{M}_{\mathcal{F}}(\Omega)$.
- **(ii)** $(\forall \alpha \in \mathbb{N}_0^d)(\forall x \in \mu(\Omega))(\exists M \in \mathbb{M}_+)(|\partial^\alpha f(x)| \leq M)$.
- **(iii)** $(\forall \alpha \in \mathbb{N}_0^d)(\forall K \subset\subset \Omega)(\exists M \in \mathbb{M}_+)(\sup_{x \in {}^*K} |\partial^\alpha f(x)| \leq M)$.
- **(iv)** $(\forall \alpha \in \mathbb{N}_0^d)(\forall x \in \mu(\Omega))(\exists A \in \mathcal{F} \setminus \mathcal{F}_0)(|\partial^\alpha f(x)| \leq A)$.
- **(v)** $(\forall \alpha \in \mathbb{N}_0^d)(\forall K \subset\subset \Omega)(\exists A \in \mathcal{F} \setminus \mathcal{F}_0)(\sup_{x \in {}^*K} |\partial^\alpha f(x)| \leq A)$.
- **(vi)** $(\forall \alpha \in \mathbb{N}_0^d)(\forall x \in \mu(\Omega))(\forall B \in {}^*\mathbb{R}_+ \setminus \mathcal{F})(|\partial^\alpha f(x)| < B)$.
- **(vii)** $(\forall \alpha \in \mathbb{N}_0^d)(\forall K \subset\subset \Omega)(\forall B \in {}^*\mathbb{R}_+ \setminus \mathcal{F})(\sup_{x \in {}^*K} |\partial^\alpha f(x)| < B)$.

Remark 13.2 We should note that the above corollary remains true even if the maximal field $\mathbb{M}$ is replaced by a set $S \subseteq \mathcal{F} \setminus \mathcal{F}_0$ such that S contains arbitrarily large numbers.

Proof: An immediate after replacing f by $\partial^\alpha f$ in Theorem 13.1.

▲

We turn to the $\mathcal{F}$-negligible functions.

Theorem 13.2 *Let $f \in {}^*\mathcal{E}(\Omega)$. Then the following are equivalent:*

(i) $(\forall x \in \mu(\Omega))(f(x) \in \mathcal{F}_0)$.

(ii) $(\forall x \in \mu(\Omega))(\forall M \in \mathbb{M}_+)(|f(x)| < M)$.

(iii) $(\forall K \subset\subset \Omega)(\forall M \in \mathbb{M}_+)(\sup_{x\in {}^*K} |f(x)| < M)$.

(iv) $(\forall x \in \mu(\Omega))(\exists A \in \mathcal{F}_0)(|f(x)| \leq A)$.

(v) $(\forall K \subset\subset \Omega)(\exists A \in \mathcal{F}_0)(\sup_{x\in {}^*K} |f(x)| \leq A)$.

(vi) $(\forall x \in \mu(\Omega))(\forall B \in \mathcal{F} \setminus \mathcal{F}_0)(|f(x)| < |B|)$.

(vii) $(\forall K \subset\subset \Omega)(\forall B \in \mathcal{F} \setminus \mathcal{F}_0)(\sup_{x\in {}^*K} |f(x)| < |B|)$.

Remark 13.3 We should note that the above theorem remains true even if the maximal field $\mathbb{M}$ is replaced by a set $S \subseteq \mathcal{F} \setminus \mathcal{F}_0$ such that S contains arbitrarily small numbers.

Proof: We shall prove the equivalence of (i) and (v) only and leave the rest of the proof to the reader (who might decide to adapt the arguments used in the proof of the previous lemma).

(i)$\Rightarrow$(v) Suppose that K is a compact subset of Ω and recall that ${}^*K \subset \mu(\Omega)$ by Theorem 7.2. Notice that $\sup_{x\in {}^*K} |f(x)| \in \mathcal{F}_0$. Indeed, suppose (on the contrary) that $\gamma =: \sup_{x\in {}^*K} |f(x)| \notin \mathcal{F}_0$ which implies $\gamma/2 \notin \mathcal{F}_0$. Also there exists $y \in {}^*K$ such that $\gamma/2 < |f(y)| < \gamma$ by the choice of γ. Thus $|f(y)| \notin \mathcal{F}_0$ contradicting to our assumption (i) since $y \in \mu(\Omega)$. On the other hand, $\sup_{x\in {}^*K} |f(x)| \in \mathcal{F}_0$ implies that the internal set

$$\mathcal{A} = \{c \in {}^*\mathbb{C} : \sup_{x\in {}^*K} |f(x)| \leq |c|\, \},$$

contains $\mathcal{F} \setminus \mathcal{F}_0$ by by part (ii) of Theorem 10.2. It follows that $\mathcal{A} \cap \mathcal{F}_0 \neq \varnothing$ by the Underflow of $\mathcal{F} \setminus \mathcal{F}_0$ (Theorem 11.1). Thus $\sup_{x\in {}^*K} |f(x)| \leq A$ holds (as required) for any $c \in \mathcal{A} \cap \mathcal{F}_0$ and $A = |c|$.

(i)$\Leftarrow$(v): Suppose that $x \in \mu(\Omega)$. As in the previous lemma, there exists $\varepsilon \in \mathbb{R}_+$ such that $K = \{r \in \Omega : |r - \mathrm{st}(x)| \leq \varepsilon\} \subset\subset \Omega$. Observe that there exists $A \in \mathcal{F}_0$ such that $\sup_{\xi\in {}^*K} |f(\xi)| \leq A$ by assumption. Thus $f(\xi) \in \mathcal{F}_0$ for all $\xi \in {}^*K$ (as required) by the convexity of $\mathcal{F}_0$.

▲

Here is a **list of characterizations of the $\mathcal{F}$-negligible functions**.

Corollary 13.2 ($\mathcal{F}$-Negligible Functions) *Let $f \in {}^*\mathcal{E}(\Omega)$. Then the following are equivalent:*

(i) $f \in \mathcal{N}_{\mathcal{F}}(\Omega)$.

(ii) $(\forall \alpha \in \mathbb{N}_0^d)(\forall x \in \mu(\Omega))(\forall M \in \mathbb{M}_+)(|f(x)| < M)$.

(iii) $(\forall \alpha \in \mathbb{N}_0^d)(\forall K \subset\subset \Omega)(\forall M \in \mathbb{M}_+)(\sup_{x \in {}^*K} |f(x)| < M)$.

(iv) $(\forall \alpha \in \mathbb{N}_0^d)(\forall x \in \mu(\Omega))(\exists A \in \mathcal{F}_0)(|f(x)| \leq A)$.

(v) $(\forall \alpha \in \mathbb{N}_0^d)(\forall K \subset\subset \Omega)(\exists A \in \mathcal{F}_0)(\sup_{x \in {}^*K} |f(x)| \leq A)$.

(vi) $(\forall \alpha \in \mathbb{N}_0^d)(\forall x \in \mu(\Omega))(\forall B \in \mathcal{F} \setminus \mathcal{F}_0)(|f(x)| < |B|)$.

(vii) $(\forall \alpha \in \mathbb{N}_0^d)(\forall K \subset\subset \Omega)(\forall B \in \mathcal{F} \setminus \mathcal{F}_0)(\sup_{x \in {}^*K} |f(x)| < |B|)$.

Remark 13.4 We should note that the above corollary remains true even if the maximal field $\mathbb{M}$ is replaced by a set $S \subseteq \mathcal{F} \setminus \mathcal{F}_0$ such that S contains arbitrarily small numbers.

Proof: An immediate after replacing f by $\partial^\alpha f$ in Theorem 13.2.
▲

In the next theorem we present several more characterizations of the $\mathcal{F}$-negligible functions (in addition to the presented above), where the quantifier $\forall \alpha \in \mathbb{N}_0^d$ is replaced simply by $\alpha = 0$.

Theorem 13.3 (A Simplification) *Let $f \in \mathcal{M}_{\mathcal{F}}(\Omega)$. Then $f \in \mathcal{N}_{\mathcal{F}}(\Omega)$ iff $f(x) \in \mathcal{F}_0$ for all $x \in \mu(\Omega)$. Consequently, we have the following several formulas for $\mathcal{N}_{\mathcal{F}}(\Omega)$:*

$$\mathcal{N}_{\mathcal{F}}(\Omega) = \{f \in \mathcal{M}_{\mathcal{F}}(\Omega) \mid (\forall x \in \mu(\Omega)(f(x) \in \mathcal{F}_0)\},$$
$$\mathcal{N}_{\mathcal{F}}(\Omega) = \{f \in \mathcal{M}_{\mathcal{F}}(\Omega) \mid (\forall x \in \mu(\Omega))(\forall M \in \mathbb{M}_+)(|f(x)| < M)\},$$
$$\mathcal{N}_{\mathcal{F}}(\Omega) = \{f \in \mathcal{M}_{\mathcal{F}}(\Omega) \mid (\forall K \subset\subset \Omega)(\forall M \in \mathbb{M}_+)(\sup_{x \in {}^*K} |f(x)| < M)\},$$
$$\mathcal{N}_{\mathcal{F}}(\Omega) = \{f \in \mathcal{M}_{\mathcal{F}}(\Omega) \mid (\forall x \in \mu(\Omega))(\exists A \in \mathcal{F}_0)(|f(x)| \leq A)\},$$
$$\mathcal{N}_{\mathcal{F}}(\Omega) = \{f \in \mathcal{M}_{\mathcal{F}}(\Omega) \mid (\forall K \subset\subset \Omega)(\exists A \in \mathcal{F}_0)(\sup_{x \in {}^*K} |f(x)| \leq A)\},$$
$$\mathcal{N}_{\mathcal{F}}(\Omega) = \{f \in \mathcal{M}_{\mathcal{F}}(\Omega) \mid (\forall x \in \mu(\Omega))(\forall B \in \mathcal{F} \setminus \mathcal{F}_0)(|f(x)| < |B|)\},$$
$$\mathcal{N}_{\mathcal{F}}(\Omega) = \{f \in \mathcal{M}_{\mathcal{F}}(\Omega) \mid (\forall K \subset\subset \Omega)(\forall B \in \mathcal{F} \setminus \mathcal{F}_0)(\sup_{x \in {}^*K} |f(x)| < |B|)\}.$$

Proof: ($\Rightarrow$) follows immediately after letting $\alpha = 0$.

($\Leftarrow$) Suppose that $x \in \mu(\Omega)$. We have to show that $\partial^\alpha f(x) \in \mathcal{F}_0$ for all multi-indexes $\alpha \in \mathbb{N}_0^d$, $|\alpha| \geq 1$. We start with $|\alpha| = 1$. If $\nabla f(x) = 0$, there is nothing to prove. Suppose that $\nabla f(x) \neq 0$ and let $\varepsilon \in \mathbb{M}_+$. It suffices to show that $||\nabla f(x)|| < \varepsilon$ in view of Theorem 10.2. Since Ω is an open set, there exists an open relatively compact set $\mathcal{O}$ of Ω such that $\mathrm{st}(x) \in \mathcal{O} \subset\subset \Omega$. Now $f \in \mathcal{M}_{\mathcal{F}}(\Omega)$ implies $\left|\sum_{|\alpha|=2} \partial^\alpha f(\xi)\right| < \delta$ for some $\delta \in \mathbb{M}_+$ and all $\xi \in {}^*\mathcal{O}$ by Corollary 13.1 since ${}^*\mathcal{O} \subset \mu(\Omega)$. Let $h \in \mathcal{I}(\mathbb{M}^d)$ be an infinitesimal vector with the direction of $\nabla f(x)$ and of length $||h|| < \varepsilon/\delta$. Notice that $||h|| \in \mathbb{M}_+$ thus $||h|| \in \mathcal{F} \setminus \mathcal{F}_0$ which is important for what follows. We have $|f(x+h) - f(x)| < \delta ||h||^2/2$ by part (vi) of Theorem 10.2 since $f(x+h) - f(x) \in \mathcal{F}_0$ by assumption and $x + h \in \mu(\Omega)$. Next we observe that the Taylor formula:

$$\nabla f(x) \cdot h = f(x+h) - f(x) - \frac{1}{2} \sum_{|\alpha|=2} \partial^\alpha f(x + \theta h)\, h^\alpha.$$

holds for some $\theta \in {}^*\mathbb{R}$, $0 < \theta < 1$, by Transfer Principle (Theorem 4.4). Thus $x + \theta h \approx x \approx \mathrm{st}(x)$ implying $x + \theta h \in {}^*\mathcal{O}$. We have

$$|\nabla f(x) \cdot h| < \delta ||h||^2/2 + \delta ||h||^2/2 < \delta ||h||^2.$$

Also we have $|\nabla f(x) \cdot h| = ||\nabla f(x)||\, ||h||$ by the choice of the direction of h. It follows $||\nabla f(x)|| = \delta ||h|| < \varepsilon$ as required. We generalize this result for $|\alpha| = 2, 3, \ldots$ by induction. The different formulas for $\mathcal{N}_{\mathcal{F}}(\Omega)$ follow immediately by Theorem 13.2. ▲

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14 Pointwise Values and Fundamental Theorem in $\widehat{\mathcal{E}_{\mathcal{F}}}(\Omega)$

In this section we show that every asymptotic function $\widehat{f} \in \widehat{\mathcal{E}_{\mathcal{F}}}(\Omega)$ can be characterized with its pointwise values in the field $\widehat{\mathcal{F}}$ (Section 10). We also prove a fundamental theorem of calculus in $\widehat{\mathcal{E}_{\mathcal{F}}}(\Omega)$ (Section 12). In this section we generalize some of the results in Todor Todorov [93] which deals with the particular case $\mathcal{F} = \mathcal{M}_\rho({}^*\mathbb{C})$ (Example 10.2) only. The closest counterpart in J.F. Colombeau's theory of generalized functions can be found in M. Kunzinger and M. Oberguggenberger's article [45], where a characterization of Colombeau's generalized functions in $\mathcal{G}(\Omega)$ in the ring of generalized scalars $\widetilde{\mathbb{C}}$ is established.

Recall that every non-standard smooth function $f \in {}^*\mathcal{E}(\Omega)$ is a pointwise function of the form $f : {}^*\Omega \to {}^*\mathbb{C}$, i.e. ${}^*\mathcal{E}(\Omega) \subset {}^*\mathbb{C}^{\,{}^*\Omega}$ (Section 8). We shall use the notation introduced in the first several pages in (Section 10) and (Section 12). In particular, let $\mathcal{F}$ be a convex subring of ${}^*\mathbb{C}$ and $\Omega \subseteq \mathbb{R}^d$ be an open set of $\mathbb{R}^d$. Then

$$\mu_{\mathcal{F}}(\Omega) = \{r + dx \mid r \in \Omega,\ dx \in \Re(\widehat{\mathcal{F}}^d),\ ||dx|| \approx 0\},$$

is the $\mathcal{F}$-monad of Ω. Here $\Re(\widehat{\mathcal{F}}^d)$ stands for the real part of the vector space $\widehat{\mathcal{F}}^d$ (Section 10, # 8). We denote by $\widehat{\mathcal{F}}^{\,\mu_{\mathcal{F}}(\Omega)}$ the ring of the functions F of the form $F : \mu_{\mathcal{F}}(\Omega) \to \widehat{\mathcal{F}}$. For convenience of the reader we shall recall the definition pointwise values presented in (Section 10).

Definition 14.1 (Pointwise Values) *Let $\widehat{f} \in \widehat{\mathcal{E}_{\mathcal{F}}}(\Omega)$ be a $\mathcal{F}$-asymptotic function (Section 12) and $\widehat{x} \in \mu_{\mathcal{F}}(\Omega)$ be a $\mathcal{F}$-asymptotic point. We define the* **value of $\widehat{f}$ at** $\widehat{x}$ *by the formula*

$$\widehat{f}(\widehat{x}) = \widehat{f(x)}.$$

We shall use the same notation, $\widehat{f}$, for the asymptotic function $\widehat{f} \in \widehat{\mathcal{E}_{\mathcal{F}}}(\Omega)$ and its graph $\widehat{f} \in \widehat{\mathcal{F}}^{\mu_{\mathcal{F}}(\Omega)}$ given by the mapping $\widehat{f} : \mu_{\mathcal{F}}(\Omega) \to \widehat{\mathcal{F}}$.

The correctness of the above definition is justified by the following result.

Lemma 14.1 (Correctness) *Let $x, y \in \mu(\Omega)$ and $f, g \in \mathcal{M}_{\mathcal{F}}(\Omega)$. Then $x - y \in \mathcal{F}_0$ and $f - g \in \mathcal{N}_{\mathcal{F}}(\Omega)$ implies $f(x) - g(y) \in \mathcal{F}_0$.*

Proof: We have $f(x) - f(y) = \nabla f(t) \cdot (x - y)$ by Transfer Principle (Theorem 4.4) for some $t \in {}^*\mathbb{R}^d$ between x and y (in the sense that $t = x + \theta(y - x)$

for some $\theta \in {}^*\mathbb{R},\ 0 < \theta < 1$). Also

$$|f(x) - g(y)| = |f(x) - f(y) + f(y) - g(y)| \leq |f(x) - f(y)| + |f(y) - g(y)| \leq$$
$$\leq ||\nabla f(t)||\,||x - y|| + |f(y) - g(y)|.$$

Observe that $x - y \in \mathcal{F}_0$ implies $x - y \approx 0$ by part (iii) of Theorem 10.2) implying $\mathrm{st}(x) = \mathrm{st}(y) = \mathrm{st}(t)$. It follows $t \in \mu(\Omega)$ since $x, y \in \mu(\Omega)$ by assumption. Thus $f \in \mathcal{M}_{\mathcal{F}}(\Omega)$ implies $||\nabla f(t)|| \in \mathcal{F}$. For the first term we have $||\nabla f(t)||\,||x - y|| \in \mathcal{F}_0$ since $||x - y|| \in \mathcal{F}_0$ by assumption and $\mathcal{F}_0$ is an ideal in $\mathcal{F}$. Also $f - g \in \mathcal{N}_{\mathcal{F}}(\Omega)$ implies $|f(y) - g(y)| \in \mathcal{F}_0$ since $y \in \mu(\Omega)$ by assumption. Thus $|f(x) - g(y)| \in \mathcal{F}_0$ as required.
▲

Recall that we have the embedding $\mathcal{E}(\Omega) \hookrightarrow \widehat{\mathcal{E}}(\Omega)$ under the mapping $f \to \widehat{{}^*f}$ (Section 12). The next result shows that the evaluation in $\widehat{\mathcal{E}_{\mathcal{F}}}(\Omega)$ reduces to the usual evaluation in $\mathcal{E}(\Omega)$. Recall that

Proposition 14.1 (The Usual Evaluation) *Let $f \in \mathcal{E}(\Omega)$. Then $\widehat{{}^*f}$ is an extension of f, i.e. $\widehat{{}^*f}\,|\,\Omega = f$.*

Proof: $\widehat{{}^*f}(\widehat{x}) = \widehat{{}^*f(x)} = \widehat{f(x)} = f(x)$ since ${}^*f$ is an extension of f. We also have $x = \widehat{x}$ for all $x \in \Omega$ by the identification Ω with its image in $\Re(\mathcal{F}^d)$ (# 14, Section 10). Thus $\widehat{{}^*f}(x) = f(x)$ as required. ▲

Theorem 14.1 (Ring Homomorphism) *The mapping*

$$\widehat{\mathcal{E}_{\mathcal{F}}}(\Omega) \ni \widehat{f} \to \widehat{f} \in \widehat{\mathcal{F}}^{\,\mu_{\mathcal{F}}(\Omega)},$$

from $\widehat{\mathcal{E}_{\mathcal{F}}}(\Omega)$ into $\widehat{\mathcal{F}}^{\,\mu_{\mathcal{F}}(\Omega)}$ is a ring homorphism.

Proof: To show that the mapping is injective, observe that $\widehat{f}(\widehat{x}) = 0$ for all $\widehat{x} \in \mu_{\mathcal{F}}(\Omega)$ is equivalent to $f(x) \in \mathcal{F}_0$ for all $\forall x \in \mu(\Omega)$. The latter implies $f \in \mathcal{N}_{\mathcal{F}}(\Omega))$ by Theorem 13.3. Thus $\widehat{f} = 0$ as required. The mapping preserves the addition because $(\widehat{f} + \widehat{g})(\widehat{x}) = \widehat{f}(\widehat{x}) + \widehat{g}(\widehat{x}) = f(\widehat{x) + g}(x)$ and similarly for the multiplcation. ▲.

Theorem 14.2 (Fundamental Theorem) *Let Ω be an arcwise connected open set of $\mathbb{R}^d$ and let $f \in \mathcal{M}_{\mathcal{F}}(\Omega)$. Then the followinng are equivalent:*

- **(i)** $(\exists\, \widehat{c} \in \widehat{\mathcal{F}})(\forall\, \widehat{x} \in \mu_{\mathcal{F}}(\Omega))(\widehat{f}(\widehat{x}) = \widehat{c})$.
- **(ii)** $(\exists\, c \in \mathcal{F})(\forall\, x \in \mu(\Omega))(f(x) - c \in \mathcal{F}_0)$.

(iii) $(\forall\, x \in \mu(\Omega))(||\nabla f(x)|| \in \mathcal{F}_0)$.

(iv) $(\forall\, \widehat{x} \in \mu_{\mathcal{F}}(\Omega))(\nabla \widehat{f}(\widehat{x}) = 0)$.

(v) $\nabla \widehat{f} = 0$ *in* $\widehat{\mathcal{E}_{\mathcal{F}}}(\Omega)$.

Proof: (i)$\Leftrightarrow$(ii), (iii)$\Leftrightarrow$(iv) and (iv)$\Leftrightarrow$(v) follow directly from Theorem 14.1.

(ii)$\Rightarrow$(iii): Suppose that $x \in \mu(\Omega)$. If $\nabla f(x) = 0$, there is nothing to prove. Suppose that $\nabla f(x) \neq 0$ and let $h \in \mathcal{I}(\mathbb{M}^d)$ be an infinitesimal vector in the direction of $\nabla f(x)$. By the Mean Value Theorem applied by Transfer Principle (Theorem 4.4), we have

$$\nabla f(x) \cdot h = f(x+h) - f(x) - \frac{1}{2} \sum_{|\alpha|=2} \partial^\alpha f(x + \theta h)\, h^\alpha,$$

for some $\theta \in {}^*\mathbb{R}$, $0 < \theta < 1$. We have $\left| \frac{1}{2} \sum_{|\alpha|=2} \partial^\alpha f(x+\theta h) \right| \leq \delta$ for some $\delta \in \mathbb{M}_+$ by Theorem 10.2 since $x + \theta h \in \mu(\Omega)$ and $f \in \mathcal{M}_{\mathcal{F}}(\Omega)$ by assumption. Also $|\nabla f(x) \cdot h| = ||\nabla f(x)||\, ||h||$ by the choice of the direction of h. Thus

$$||\nabla f(x)|| \leq \left(\frac{f(x+h) - f(x)}{||h||^2} + \delta \right) ||h||,$$

Observe that $f(x+h) - f(x) \in \mathcal{F}_0$ by assumption since $x + h \in \mu(\Omega)$. Thus $\frac{f(x+h)-f(x)}{||h||^2} + \delta \in \mathbb{M}_+$. Consequently, there exists $M \in \mathbb{M}_+$ such that the internal set

$$\mathcal{A} = \left\{ ||h|| \ : \ h \in {}^*\mathbb{R}^d,\ \frac{\nabla f(x)}{||\nabla f(x)||} = \frac{h}{||h||},\ ||\nabla f(x)|| \leq M\, ||h|| \right\},$$

contains $\mathcal{I}(\mathbb{M}_+)$. Thus $\mathcal{A}$ contains arbitrarily small numbers in $\mathcal{F} \setminus \mathcal{F}_0$ since $\mathbb{M}_+ \subset \mathcal{F} \setminus \mathcal{F}_0$. It follows that $\mathcal{A}$ contains arbitrarily large numbers $\mathcal{F}_0$ by the Underflow of $\mathcal{F} \setminus \mathcal{F}_0$ (Theorem 11.1). Thus there exists $h \in {}^*\mathbb{R}^d$ such that $||\nabla f(x)|| \leq M\, ||h||$ and $||h|| \in \mathcal{F}_0$. It follows that $||\nabla f(x)|| \in \mathcal{F}_0$ (as required) since $\mathcal{F}_0$ is an ideal in $\mathcal{F}$.

(ii)$\Leftarrow$(iii): Suppose that $x, y \in \mu(\Omega)$. Since Ω is arcwise connected by assumption, it follows that ${}^*\Omega$ is $*$-arcwise connected by Transfer Principle (Theorem 4.4). Thus there exists a $*$-continuous curve $L \subset \mu(\Omega)$ which connects x and y. We have

$$f(x) - f(y) = \int_L \nabla f(t) \cdot dl,$$

(again, by Transfer Principle). It follows that

$$f(x) - f(y) = \nabla f(t) \cdot (x - y),$$

for some $t \in L$ by the Mean Value Theorem (and Transfer Principle). Thus $|f(x) - f(y)| \leq ||\nabla f(t)||\,||x - y|| \in \mathcal{F}_0$, since (as before) $\mathcal{F}_0$ is an ideal in $\mathcal{F}$ and we have $||\nabla f(t)|| \in \mathcal{F}_0$ by assumption and $||x - y|| \in \mathcal{F}({}^*\mathbb{R}) \subset \mathcal{F}$. Let $c = f(y)$ for some (any) $y \in \mu(\Omega)$. The result is $f(x) - c \in \mathcal{F}_0$ for all $x \in \mu(\Omega)$ as required. ▲

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15 Convolution in Non-Standard Setting

Definition 15.1 (Convolution) **(i)** *Let $T \in \mathcal{D}'(\Omega)$ and let $T : \mathcal{D}(\Omega) \to \mathbb{C}$ be the corresponding mapping. We define the* **non-standard extension** $^*T : {}^*\mathcal{D}(\Omega) \to {}^*\mathbb{C}$ *of T by the formula*

$$\langle {}^*T, \langle \varphi_i \rangle \rangle = \langle \langle T, \varphi_i \rangle \rangle \,,$$

where $\langle \varphi_i \rangle \in {}^*\mathcal{D}(\Omega)$.

(ii) *Let $T \in \mathcal{E}'(\Omega)$ and $\langle D_i \rangle \in {}^*\mathcal{D}(\mathbb{R}^d)$. We define the* **convolution** *between $^*T$ and $\langle D_i \rangle$ by the formula*

$$^*T \star \langle D_i \rangle = \langle T \star D_i \rangle,$$

where $T \star D_i$ is the usual convolution between T and D_i in the sense of distribution theory (i.e. $\langle T(\xi), D_i(x - \xi) \rangle$ for every $x \in \Omega$ and every $i \in \mathcal{I}$).

Lemma 15.1 *For every $T \in \mathcal{E}'(\Omega)$ and every $D \in {}^*\mathcal{D}(\mathbb{R}^d)$ we have $^*T \star D \in {}^*\mathcal{E}(\Omega)$.*

16 Schwartz Distributions in $^\rho\mathcal{E}(\Omega)$

If $f \in \mathcal{L}^1_{loc}(\Omega)$, we denote by $T_f \in \mathcal{D}'(\Omega)$ the Schwartz distribution with kernel f, i.e.

$$\langle T_f, \varphi \rangle = \int_\Omega f(x)\varphi(x)\, dx,$$

for all $\varphi \in \mathcal{D}(\Omega)$. Recall that $\mathcal{E}(\Omega)$ is a differential subring of $^\rho\mathcal{E}(\Omega)$ under the embedding

$$\mathcal{E}(\Omega) \hookrightarrow {}^\rho\mathcal{E}(\Omega),$$

defined by the mapping $f \to \widehat{{}^*f}$, where $^*f$ is the non-standard extension of f (i.e. $^*f = \langle f_i \rangle$, $f_i = f$ for all $i \in \mathcal{I}$) and $\widehat{{}^*f}$ stands for the corresponding equivalence class (see the beginning of Section **??**).

Theorem 16.1 (Existence of an Embedding) *There exists an embedding $\Sigma_\Omega : \mathcal{D}'(\Omega) \to {}^\rho\mathcal{E}(\Omega)$ which preserves the sheaf-properties and the linear operations in $\mathcal{D}'(\Omega)$ (including partial differentiation) and such that $\Sigma_\Omega(T_f) = \Sigma_\Omega({}^*f)$ for every $f \in \mathcal{E}(\Omega)$. Consequently, the multiplication in $^\rho\mathcal{E}(\Omega)$ reduces to the usual pointwise multiplication on $\mathcal{E}(\Omega)$. We summarize this in:*

$$\mathcal{E}(\Omega) \hookrightarrow \mathcal{D}'(\Omega) \hookrightarrow {}^\rho\mathcal{E}(\Omega)$$

Proof: We shall separate the proof in numerous definitions and lemmas:

Definition 16.1 (ρ-Delta Function) *$D \in {}^*\mathcal{E}(\mathbb{R}^d)$ is called a* **ρ-delta function** *if:*

1. *$||x|| \not\approx 0$ implies $D(x) = 0$. (***Lemma:** *There exists a* **positive infinitesimal***, say ρ, such that $||x|| \leq \rho$ implies $D(x) = 0$).*

   The next conditions on D depend on the choice of ρ:

2. *$\int_{||x|| \leq \rho} D(x)\, dx - 1 \in \mathcal{N}_\rho({}^*\mathbb{C})$.*

3. *$\int_{||x|| \leq \rho} D(x)\, x^\alpha\, dx \in \mathcal{N}_\rho({}^*\mathbb{C})$ for all $|\alpha| \neq 0$.*

4. *$D \in \mathcal{M}_\rho({}^*\mathcal{E}(\mathbb{R}^d))$, i.e.*

$$(\forall \alpha \in \mathbb{N}_0^d)(\forall x \in \mu(\mathbb{R}^d))\,(\partial^\alpha D(x) \in \mathcal{M}_\rho({}^*\mathbb{C}))\,.$$

Theorem 16.2 *There exists a ρ-delta function D.*

Proof: : For the original proof we refer to (M. Oberguggenberger and T. Todorov [66]). Here is a **summary of this result**:

Step 1) For every $n \in \mathbb{N}$, we define the set of test-functions:

$$\begin{aligned}
(34) \qquad \mathcal{B}_n = \{\varphi \in \mathcal{D}(\mathbb{R}^d) :& \\
&\int_{\mathbb{R}^d} \varphi(x)\, dx = 1, \\
&\int_{\mathbb{R}^d} x^\alpha \varphi(x)\, dx = 0 \ \text{ for all } \ \alpha \in \mathbb{N}_0^d,\ 1 \leq |\alpha| \leq n, \\
&||x|| \geq 1/n \Rightarrow \varphi(x) = 0, \\
&1 \leq \int_{\mathbb{R}^d} |\varphi(x)|\, dx < 1 + \frac{1}{n}\,\}.
\end{aligned}$$

Lemma 16.1 (Properties of $\mathcal{B}_n$) (B_1) *$\mathcal{B}_n \neq \varnothing$ for all n.*
(B_2) *$\mathcal{D}(\mathbb{R}^d) = \mathcal{B}_0 \supset \mathcal{B}_1 \supset \mathcal{B}_2 \supset \mathcal{B}_3 \supset \ldots$. (Thus $\mathcal{B}_n \cap \mathcal{B}_n = \mathcal{B}_{\max(m,n)}$).*
(B_3) *$\cap_n \mathcal{B}_n = \varnothing$.*

Step 2) Find the **non-standard extension** of $\mathcal{B}_n$:

$$
(35) \qquad {}^*\mathcal{B}_n = \{\varphi \in {}^*\mathcal{D}(\mathbb{R}^d) : \\
\int_{^*\mathbb{R}^d} \varphi(x)\, dx = 1, \\
\int_{^*\mathbb{R}^d} x^\alpha \varphi(x)\, dx = 0 \text{ for all } \alpha \in \mathbb{N}_0^d,\ 1 \leq |\alpha| \leq n, \\
||x|| \geq 1/n \Rightarrow \varphi(x) = 0, \\
1 \leq \int_{^*\mathbb{R}^d} |\varphi(x)|\, dx < 1 + \frac{1}{n}\,\}.
$$

Step 3) Let M be an infinitely large positive number in $\mathcal{F}_\rho(^*\mathbb{R})$. For example, $M = |\ln \rho|$ will do. Define the internal sets:

$$
\mathcal{A}_n = \{\varphi \in {}^*\mathcal{B}_n : \quad {}^*\sup_{||x|| \leq 1/n} |\partial^\alpha \varphi(x)| < \frac{M}{n} \text{ for all } |\alpha| \leq n\},
$$

We observe that (trivially) ${}^*\mathcal{D}(\mathbb{R}^d) \supset \mathcal{A}_1 \supset \mathcal{A}_2 \supset \dots$. Also, $\mathcal{A}_n \neq \varnothing$ for all n. Indeed, $\varphi \in \mathcal{B}_n$ implies ${}^*\varphi \in \mathcal{A}_n$ since

$$
{}^*\sup_{||x|| \leq 1/n} |\partial^\alpha ({}^*\varphi(x))| = \sup_{||x|| \leq 1/n} |\partial^\alpha \varphi(x)| < \frac{M}{n},
$$

and $\sup_{||x|| \leq 1/n} |\partial^\alpha \varphi(x)|$ is a real number and M/n is an infinitely large positive number for any $n \in \mathbb{N}$. Thus there exists

$$
\Theta \in \bigcap_{n=1}^{\infty} \mathcal{A}_n \neq \varnothing,
$$

by Saturation Principle (Theorem 4.2). Notice that Θ **satisfies all properties (1)-(4)** of the definition of ρ-delta function **except (possibly) the property (5)**.

Step 3) The non-standard function $D \in {}^*\mathcal{D}(\mathbb{R}^d)$, defined by the formula

$$
D(x) = \rho^{-d}\Theta(x/\rho),
$$

is the ρ-delta function we are looking for.

Definition 16.2 *The mapping $T \to Q_\Omega\,({}^*T \star D)$ from $\mathcal{E}'(\Omega)$ to ${}^\rho\mathcal{E}(\Omega)$ is the embedding of the space of distributions with compact support in Ω.*

Step 4)

Definition 16.3 (ρ-Cut-Off Function) $\Pi_\Omega \in {}^*\mathcal{D}(\Omega)$ *is called a* ρ-**cut-off function** *for the open set* $\Omega \subseteq \mathbb{R}^d$ *if*

(a) $\Pi_\Omega(x) = 0$ *for all* $x \in \mu(\Omega)$.

(b) $\operatorname{supp}(\Pi_\Omega) \subseteq \{x \in {}^*\Omega \mid {}^*d(x, \partial\Omega) \geq \rho\}$

Lemma 16.2 *There exists a ρ-cut-off-function.*

Proof: Let $\Omega_\rho = \{x \in {}^*\Omega \mid {}^*d(x, \partial\Omega) \geq 2\rho, ||x|| < 1/\rho\}$ and let χ be the characteristic function of Ω_ρ. The function $\Pi_\Omega = \chi \star D$ is the ρ-cut-off function we are looking for. ▲

Definition 16.4 *The mapping* $T \to Q_\Omega\left({}^*T\Pi_\Omega\right) \star D)$ *from* $\mathcal{D}'(\Omega)$ *to* ${}^\rho\mathcal{E}(\Omega)$ *is the embedding the existence of which was stated in Theorem 16.1.*

The proof of Theorem 16.1 is complete. ▲

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