Proseminar “Lie groups”

Andreas Čap
Fall term 2024/25

(1) Prove that the determinant function $\det : M_n(\mathbb{R}) \to \mathbb{R}$ is regular in each point $A \in M_n(\mathbb{R})$ for which $\det(A) \neq 0$. (Hint: Compute $D\det(A)(A)$.) Use this to show that $SL(n, \mathbb{R}) = \{A \in M_n(\mathbb{R}) : \det(A) = 1\}$ is a Lie group and determine the tangent space to $SL(n, \mathbb{R})$ in the unit matrix.

(2) Let $O(n) \subset M_n(\mathbb{R})$ be the set of all orthogonal matrices of size $n \times n$. Show that $O(n)$ is a Lie group. (Hint: Consider $A \mapsto A^tA$ as a function from $M_n(\mathbb{R})$ to the space of symmetric $n \times n$-matrices. Prove that this function is regular in each point $A \in O(n)$.) Determine the dimension of $O(n)$ and show that the tangent space $\mathfrak{o}(n)$ to $O(n)$ in the unit matrix is given by $\mathfrak{o}(n) = \{X \in M_n(\mathbb{R}) : X^t = -X\}$.

(3) Derive an explicit description of the underlying manifold of the Lie group $SL(2, \mathbb{R})$ and of the subgroup $SO(2) = O(2) \cap SL(2, \mathbb{R})$. (Hint: Writing real 2×2-matrices in the form $A = \begin{pmatrix} x+w & -y+z \\ y+z & x-w \end{pmatrix}$, one obtains $\det(A) = x^2 + y^2 - z^2 - w^2$.)

(4) For matrices $X, Y \in M_n(\mathbb{R})$ define the commutator by $[X, Y] := XY - YX$. Prove that this satisfies the Jacobi identity $[X, [Y, Z]] = [[X, Y], Z] + [Y, [X, Z]]$. Further show that for $X, Y \in M_n(\mathbb{R})$ the matrix $[X, Y]$ is always trace-free while for $X, Y \in \mathfrak{o}(n)$ one also has $[X, Y] \in \mathfrak{o}(n)$.

(5) For fixed n consider the *elementary matrices* $E_{ij} \in M_n(\mathbb{R})$ which have an entry 1 in the jth column of the ith row and all other entries equal to zero. Compute $[E_{ij}, E_{k\ell}]$ explicitly. Using this show that for $i < j$ the elements E_{ij}, E_{ji} and $H_{ij} := E_{ii} - E_{jj}$ of $\mathfrak{sl}(n, \mathbb{R})$ span a subspace that is closed under the Lie bracket and isomorphic to $\mathfrak{sl}(2, \mathbb{R})$.

(6) Use the computations in the last exercise to prove that for any Lie algebra $\mathfrak{h}$, a non-zero homomorphism $f : \mathfrak{sl}(n, \mathbb{R}) \to \mathfrak{h}$ of Lie algebras has to be injective.

Hint: Similarly to Example 1.5 of the lecture, use the fact that $f(X) = 0$ implies that $f([X, Y]) = 0$ for any Y. The main step in the proof is to show that f vanishes on one element E_{ij} with $i \neq j$. To achieve this, consider an element X with $f(X) = 0$ and distinguish cases (a bit tedious) depending on where X has non-zero entries.

(7) Consider the space $\mathfrak{o}(3)$ of skew-symmetric 3×3-matrices. Show that for $A \in O(3)$ and $X \in \mathfrak{o}(3)$ one always gets $AXA^{-1} \in \mathfrak{o}(3)$. Show further that $\langle X, Y \rangle := -\frac{1}{2}\operatorname{tr}(XY)$ defines a positive-definite inner product on $\mathfrak{o}(3)$ such that $\langle AXA^{-1}, AYA^{-1} \rangle = \langle X, Y \rangle$.

(8) Find an orthonormal basis for the inner product $\langle\ ,\ \rangle$ on $\mathfrak{o}(3)$ defined in the previous example. Prove that the commutator on $\mathfrak{o}(3)$ coincides with the cross-product on $\mathbb{R}^3$ in the resulting identification.

(9) For $X \in \mathfrak{o}(3)$ consider the map $\operatorname{ad}_X : \mathfrak{o}(3) \to \mathfrak{o}(3)$, which is given by $\operatorname{ad}_X(Y) := [X, Y]$. Determine the matrix representation of ad_X with respect to the orthonormal basis

constructed in the previous example. Verify that $(X, Y) \mapsto \operatorname{tr}(\operatorname{ad}_X \circ \operatorname{ad}_Y)$ is a non-zero multiple of the inner product $\langle\ ,\ \rangle$.

(10) Prove that $b(X, Y) := \operatorname{tr}(XY)$ defines a non-degenerate, symmetric bilinear form on $M_n(\mathbb{R})$ such that $b(AXA^{-1}, AYA^{-1}) = b(X, Y)$ for all $A \in GL(n, \mathbb{R})$ and all $X, Y \in M_n(\mathbb{R})$. Determine the signature of this bilinear form.

(11) Similarly to example (2), prove that $U(n) := \{A \in M_n(\mathbb{C}) : A^*A = \mathbb{I}\}$ and $SU(n) := \{A \in U(n) : \det(A) = 1\}$ are Lie groups. Show that their tangent spaces in the unit matrix $\mathbb{I}$ are given by $\mathfrak{u}(n) = \{A \in M_n(\mathbb{C}) : A^* = -A\}$ and $\mathfrak{su}(n) = \{A \in \mathfrak{u}(n) : \operatorname{tr}(A) = 0\}$, respectively.

(12) Verify directly that the manifold underlying the Lie group $SU(2)$ from the previous example is the three-dimensional sphere S^3.

(13) Show that for $X, Y \in \mathfrak{su}(2)$, one always has $\operatorname{tr}(XY) \in \mathbb{R}$ and that $\langle X, Y\rangle = -\frac{1}{2}\operatorname{tr}(XY)$ defines a positive definite inner product on the vector space $\mathfrak{su}(2)$. Prove that for all $X \in \mathfrak{su}(2)$, the map $\operatorname{ad}_X : \mathfrak{su}(2) \to \mathfrak{su}(2)$ is skew symmetric with respect to this inner product and that $X \mapsto \operatorname{ad}_X$ defines an isomorphism $\mathfrak{su}(2) \to \mathfrak{o}(3)$.

(14) Prove that for each matrix $A \in SU(2)$, the adjoint action $\operatorname{Ad}(A) : \mathfrak{su}(2) \to \mathfrak{su}(2)$ is orthogonal for the inner product constructed in the previous example. Conclude that Ad defines a homomorphism $SU(2) \to SO(3)$ and use that previous example and the fact that $SO(3)$ is connected to show that this homomorphism is onto.

(15) Show that for a connected Lie group G, the kernel of the homomorphism $\operatorname{Ad} : G \to GL(\mathfrak{g})$ coincides with the center $Z(G)$ of G. Prove that for $SU(2)$ one has $Z(SU(2)) = \{\pm\mathbb{I}\}$ and use this to show that the underlying manifold of $SO(3)$ is the real projective space $\mathbb{R}P^3$.

(16) Verify that the 3-dimensional Lie algebra $\mathfrak{g} := \mathfrak{sl}(2, \mathbb{R})$ is not isomorphic to $\mathfrak{o}(3)$. (Hint: Find a matrix $X \in \mathfrak{g}$ for which the map $\operatorname{ad}(X) : \mathfrak{g} \to \mathfrak{g}$ is nilpotent. Show that the existence of an isomorphism $\varphi : \mathfrak{g} \to \mathfrak{o}(3)$ would lead to a contradiction to example (9).)

(17) Let $G \subset GL(2, \mathbb{R})$ be the subgroup of all matrices of the form $\begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix}$ with $a, b \in \mathbb{R}$ and $a \neq 0$. Determine the Lie algebra $\mathfrak{g}$ of G and show that any non-commutative 2-dimensional Lie algebra is isomorphic to $\mathfrak{g}$. (Hint: Given an arbitrary non-commutative 2-dimensional Lie algebra $\mathfrak{h}$, construct a basis $\{X, Y\}$ for $\mathfrak{h}$ such that $[X, Y] = Y$.)

(18) *Complex projective space* $\mathbb{C}P^n$ is defined as the space of all complex lines through 0 in $\mathbb{C}^{n+1}$. Show that $\mathbb{C}P^n$ can be naturally made into a compact smooth manifold by realizing it as a homogeneous space of the Lie groups $GL(n+1, \mathbb{C})$, $U(n+1)$ and $SU(n+1)$.

(19) Show that the complex projective space $\mathbb{C}P^1$ can be identified with the sphere S^2 (Riemannian sphere extending $\mathbb{C}$). Together with the previous example, this shows that S^2 is a homogeneous space of $GL(2, \mathbb{C})$. Show that the chart obtained from this point of view via the proof of Theorem 1.16 of the course defines a diffeomorphism

from $\mathbb{R}^2$ onto the complement of a point in S^2 which leads to the usual inhomogeneous coordinates on $\mathbb{C}P^1$.

(20) Similarly as in the previous example, interpret $S^2 \cong \mathbb{C}P^1$ as a homogeneous space of $SU(2)$. Show that this leads to a smooth map $p : S^3 \to S^2$ such that $p^{-1}(x) \cong S^1$ for each $x \in S^2$ (*"Hopf fibration"*). Prove that for each $y \in S^2$ putting $V := S^2 \setminus y$, one obtains $p^{-1}(V) \cong V \times S^1$. Is it true that $S^3 \cong S^2 \times S^1$?

(21) A *complex structure* on a real vector space V is a linear map $J : V \to V$ such that $J \circ J = -\operatorname{id}_V$. Show that such a structure can be used to make V into a complex vector space by defining scalar multiplication as $(a + ib) \cdot v := av + bJ(v)$. Use this to conclude that a finite dimensional real vector space V admits such a structure if and only if its dimension is even.

(22) Prove that the set $\mathcal{J}_n$ of all complex structures on $\mathbb{R}^{2n}$ can be made into a smooth manifold by identifying it with the homogeneous space $GL(2n, \mathbb{R})/GL(n, \mathbb{C})$.

(23) Let G be a Lie group, $H \subset G$ a closed subgroup and G/H the corresponding homogeneous space. Let $\mathfrak{g}$ and $\mathfrak{h}$ be the Lie algebras of G and H and consider the quotient vector space $\mathfrak{g}/\mathfrak{h}$. Prove that the adjoint action of elements of H on $\mathfrak{g}$ defines an action $\underline{\mathrm{Ad}}$ of H on $\mathfrak{g}/\mathfrak{h}$.

(24) In the setting of the previous exercise consider the canonical projection $p : G \to G/H$. Show that for an element $g \in G$ and a tangent vector $\xi \in T_gG$, the element $T_g\lambda_{g^{-1}} \cdot \xi + \mathfrak{h} \in \mathfrak{g}/\mathfrak{h}$ depends only on $T_gp \cdot \xi \in T_{gH}(G/H)$. Conclude that this construction leads to a linear isomorphism $\varphi_g : T_{gH}(G/H) \to \mathfrak{g}/\mathfrak{h}$ such that $\varphi_{gh} = \underline{\mathrm{Ad}}(h^{-1}) \circ \varphi_g$ holds for all $g \in G$ and $h \in H$.

(25) Use the previous example to show that $\mathfrak{X}(G/H)$ can be identified with the space

$$\{f \in C^\infty(G, \mathfrak{g}/\mathfrak{h}) : f(gh) = \underline{\mathrm{Ad}}(h^{-1})(f(g)) \quad \forall g \in G, h \in H\}.$$

Prove that the group G naturally acts on $\mathfrak{X}(G/H)$ and show that in this picture this action is given by $(g \cdot f)(g') := f(g^{-1}g')$. Conclude that for a G-invariant vector field $\xi \in \mathfrak{X}(G/H)$ corresponding to a function $f : G \to \mathfrak{g}/\mathfrak{h}$ as above, the element $f(e) \in \mathfrak{g}/\mathfrak{h}$ is H–invariant. Conversely show that an H-invariant element of $\mathfrak{g}/\mathfrak{h}$ naturally can be extended to a function corresponding to a G-invariant vector field.

(26) Let V be a finite-dimensional real vector space and let $b : V \times V \to \mathbb{R}$ be a non-degenerate, symmetric bilinear form on V. For a subspace $W \subset V$ put $W^\perp := \{v \in V : \forall w \in W : b(v, w) = 0\}$. Prove that $W^\perp$ is a linear subspace of V of dimension $\dim(V) - \dim(W)$ and that the following conditions are equivalent.
   (a) The restriction of b to W is non-degenerate.
   (b) The restriction of b to $W^\perp$ is non-degenerate.
   (c) $W \cap W^\perp = \{0\}$.
   (d) $V = W \oplus W^\perp$.

(27) Consider $\mathbb{R}^{n+1}$ endowed with the bilinear form $b(x, y) = x^1y^1 + \cdots + x^ny^n - x^{n+1}y^{n+1}$ and the subset $\mathcal{H}^n := \{x \in \mathbb{R}^{n+1} : b(x, x) = -1, x^{n+1} > 0\}$. Show that $\mathcal{H}^n$ is a smooth submanifold of $\mathbb{R}^{n+1}$ and that b induces a (positive definite) Riemannian metric on $\mathcal{H}^n$. The resulting Riemannian manifold is called *hyperbolic space* of dimension n. Show

that there is a natural transitive action of the orthogonal group $O(n,1)$ of b on $\mathcal{H}^n$, which has the property that each of the maps ℓ_g is compatible with the Riemannian metric.

(28) Let $B^n = \{x \in \mathbb{R}^n : |x| < 1\}$ be the n-dimensional unit ball. Define a map $f : B^n \to \mathcal{H}^n$ by $f(x) := \frac{1}{\sqrt{1-|x|^2}}(x,1)$. Prove that f is a diffeomorphism for each n and give a geometric description of f for $n = 2$. Compute the Riemannian metric on B^n that is obtained by pulling back the hyperbolic metric on $\mathcal{H}^n$ by f.

(29) Consider the standard Hermitian inner product $\langle\ ,\ \rangle$ on $\mathbb{C}^n$ and let $\omega : \mathbb{C}^n \times \mathbb{C}^n \to \mathbb{R}$ its imaginary part. Show that ω defines a non-degenerate, skew-symmetric real bilinear form on the real vector space $\mathbb{C}^n$, which in addition satisfies $\omega(iz, iw) = \omega(z, w)$ for all $z, w \in \mathbb{C}^n$.

(30) **The Heisenberg group**: Prove that the set of all real $(n+2) \times (n+2)$-matrices, which have the block form $\begin{pmatrix} 1 & u & a \\ 0 & \mathbb{I} & v \\ 0 & 0 & 1 \end{pmatrix}$, with block sizes 1, n and 1 and $\mathbb{I} \in M_n(\mathbb{R})$ denoting the unit matrix, form a Lie subgroup of $GL(n+2,\mathbb{R})$. Show further that the matrices such that $u = v = 0$ and $a \in \mathbb{Z}$ form a discrete normal subgroup. Define the Heisenberg group $\mathcal{H}_n$ as the quotient group.

(31) In the setting of the previous exercise, show the sending a matrix as above to $(u_1 + iv_1, \dots, u_n + iv_n, e^{2\pi i a})$ defines a diffeomorphism $\mathcal{H}_n \to \mathbb{C}^n \times U(1)$. Describe the multiplication on $\mathcal{H}_n$ in this picture. In particular, show that for elements $(z,\varphi), (w,\psi) \in \mathbb{C}^n \times U(1)$, the commutator in $\mathcal{H}_n$ corresponds to $(0, e^{-2\pi i \omega(z,w)})$, where ω is the skew symmetric bilinear form from exercise (29). Conclude from this that the commutator subgroup of $\mathcal{H}_n$ coincides with the center $Z(\mathcal{H}_n)$ and is isomorphic to $U(1)$.

(32) Similarly to exercise (31) prove that the Lie algebra $\mathfrak{h}_n$ of the Heisenberg group can be identifies with $\mathbb{C}^n \oplus \mathbb{R}$, endowed with the bracket $[(z,a),(w,b)] = (0, -\omega(z,w))$. Compute the exponential map of the Heisenberg group as a map $\mathbb{C}^n \oplus \mathbb{R} \to \mathbb{C}^n \times U(1)$. Show that there is a real basis $\{q_1,\dots,q_n,p_1,\dots,p_n,z\}$ for $\mathfrak{h}_n$, such that $z \in \mathfrak{z}(\mathfrak{h}_n)$, $[q_i,q_j] = [p_i,p_j] = 0$, and $[q_i,p_j] = \delta_{ij} z$.

(33) Consider a matrix with entries u, v and a as in exercise (30). Define an action of such a matrix on a function $f : \mathbb{R}^n \to \mathbb{C}$ by $((u,v,a)\cdot f)(x) := e^{2\pi i(a - \langle v,x\rangle)} f(x-u)$. Prove that this defines a homomorphism from $\mathcal{H}_n$ to the group of invertible linear maps on the vector space of all functions $\mathbb{R}^n \to \mathbb{C}$. Explain why for an L^2-function f, also $(u,v,a)\cdot f$ is an L^2-function. Finally, show that $f \mapsto (u,v,a)\cdot f$ defines a unitary operator on $L^2(\mathbb{R}^n,\mathbb{C})$.

(34) Consider the Lie algebra $\mathfrak{h}_n$ of the Heisenberg group as in exercise (32). For a smooth function $f : \mathbb{R}^n \to \mathbb{R}^n$ with compact support and an element $X \in \mathfrak{h}_n$ prove that $(X\cdot f)(x) := \frac{d}{dt}|_{t=0}(\exp(tX)\cdot f)(x)$ defines a compactly supported smooth function $X \cdot f : \mathbb{R}^n \to \mathbb{R}^n$. Show that $[X,Y]\cdot f = X\cdot(Y\cdot f) - Y\cdot(X\cdot f)$ holds for all $X, Y \in \mathfrak{h}_n$ and $f \in C_c^\infty(\mathbb{R}^n,\mathbb{R}^n)$. Compute the actions of the elements of the basis constructed in exercise (32).