On the geometry of chains

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- based on the joint article math.DG/0504469 with V. Žádník (Brno)

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- We analyze this Cartan geometry in terms of the canonical Cartan connection associated to the CR structure
- Apart from other results on the chains, we obtain a conceptual proof of the fact that a chain preserving diffeomorphism must be a CR isomorphism or a CR anti–isomorphism

Structure

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1. CR structures and the canonical Cartan connection
2. Chains and the associated path geometry
3. Relating the two Cartan geometries
4. Applications

Let M be a smooth manifold of dimension $2n+1$.

- *almost CR structure* of hypersurface type: subbundle $H \subset TM$ of corank 1, $J : H \to H$ complex structure on H
- *Levi–bracket*: The tensorial map $\mathcal{L} : H \times H \to TM/H$ induced by the Lie bracket of vector fields. We always assume that $\mathcal{L}$ is non–degenerate

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- *partial integrability*: $\mathcal{L}(J\xi, J\eta) = \mathcal{L}(\xi, \eta)$. Then $\mathcal{L}$ is the imaginary part of a Hermitian form; signature (p, q) of M
- equivalently for $H \otimes \mathbb{C} = H^{1,0} \oplus H^{0,1}$ we have $[H^{0,1}, H^{0,1}] \subset H \otimes \mathbb{C}$.

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- equivalently for $H \otimes \mathbb{C} = H^{1,0} \oplus H^{0,1}$ we have $[H^{0,1}, H^{0,1}] \subset H \otimes \mathbb{C}$.
- *integrability*: $[H^{0,1}, H^{0,1}] \subset H^{0,1}$ or equivalently vanishing of the Nijenhuis tensor $N : H \times H \to H$ induced by

$$N(\xi, \eta) = [\xi, \eta] - [J\xi, J\eta] + J([J\xi, \eta] + [\xi, J\eta]).$$

The homogeneous model

Consider $\mathbb{V} = \mathbb{C}^{n+2}$ with a Hermitian form $\langle\ ,\ \rangle$ of signature $(p+1, q+1)$ with $p+q=n$, let $\mathcal{C} \subset \mathbb{V}$ be the cone of non–zero null–vectors, and let $M \subset \mathcal{P}(\mathbb{V}) = \mathbb{C}P^{n+1}$ be the projectivization of $\mathcal{C}$.

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As a real hypersurface in a complex manifold, M inherits a CR structure of signature (p,q). Let $\pi : \mathcal{C} \to M$ be the projection. Then for $v \in \mathcal{C}$, map $T_v\pi$ induces isomorphisms

$$\{w \in \mathbb{V} : \Re(\langle w, v\rangle) = 0\}/\mathbb{C}v \to T_{\pi(v)}M$$
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The group $G = PSU(\mathbb{V})$ acts on M by CR automorphisms so $M \cong G/P$, where $P \subset G$ is the stabilizer of an isotropic line $\ell_0 \subset \mathbb{V}$. It turns out that $G = Aut_{CR}(M)$.

Choose a basis $\{e_0, \dots, e_{n+1}\}$ for $\mathbb{V}$ such that

$$\langle v, w \rangle = v_0 \bar{w}_{n+1} + v_{n+1} \bar{w}_0 + \sum_{j=1}^{p} v_j \bar{w}_j - \sum_{j=p+1}^{n} v_j \bar{w}_j.$$

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Then the Lie algebra $\mathfrak{g}$ of G has the form

$$\mathfrak{g} = \left\{ \begin{pmatrix} w & Z & iz \\ X & A & -\mathbb{I}Z^* \\ ix & -X^*\mathbb{I} & -\bar{w} \end{pmatrix} : w - \bar{w} + \operatorname{tr}(A) = 0 \right\}$$

with $x, z \in \mathbb{R}$, $w \in \mathbb{C}$, $X \in \mathbb{C}^n$, $Z \in \mathbb{C}^{n*}$, and $A \in \mathfrak{u}(p, q)$, where $\mathbb{I}$ is the diagonal matrix $\operatorname{diag}(1, \dots, 1, -1, \dots, -1)$.

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$$\mathfrak{g} = \mathfrak{g}_{-2} \oplus \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1 \oplus \mathfrak{g}_2$$

such that $[\mathfrak{g}_i, \mathfrak{g}_j] \subset \mathfrak{g}_{i+j}$ and the Lie algebra $\mathfrak{p}$ of P is $\mathfrak{g}_0 \oplus \mathfrak{g}_1 \oplus \mathfrak{g}_2$.

Theorem (Tanaka, Chern–Moser)

Let M be a partially integrable almost CR manifold of signature (p, q). Then there is a canonical principal bundle $p : \mathcal{G} \to M$ with structure group P and a canonical Cartan connection $\omega \in \Omega^1(\mathcal{G}, \mathfrak{g})$ which provide an equivalent description of the structure.

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The correspondence to the underlying structure is as follows: Via ω, we obtain an isomorphism $TM\cong\mathcal{G}\times_P(\mathfrak{g}/\mathfrak{p})$. Now $\mathfrak{g}/\mathfrak{p}$ contains the P–invariant complex subspace $(\mathfrak{g}_{-1}\oplus\mathfrak{p})/\mathfrak{p}$ which under this isomorphism has to induce the CR subbundle $H\subset TM$ (with its complex structure).
The Cartan connection satisfies a normalization condition on its curvature, which makes it unique.

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Chains form a family of distinguished paths (unparametrized curves) in a CR manifold. Given a point $x \in M$ and a line ℓ in T_xM which is not contained in H_x, there is a unique chain c through x such that $T_xc = \ell$.

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In general, chains can be defined as solutions of a certain system of ODE's or via the canonical Cartan connection:

Let $(p : \mathcal{G} \to M, \omega)$ be the canonical Cartan geometry associated to a partially integrable almost CR structure on M. Then the chains on M are the projections of the integral curves of the rank one subbundle $\omega^{-1}(\mathfrak{g}_{-2}) \subset T\mathcal{G}$.

Path geometries

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- Any immersed one–dimensional submanifold $\ell \subset N$ naturally lifts to an immersed submanifold $\hat{\ell} \subset \mathcal{P}TN$.
- Smooth families of paths with one path through each point in each direction are equivalent to rank one foliations of $\mathcal{P}TN$ and hence to appropriate line subbundles in $T\mathcal{P}TN$.
- This notion of *path geometries* immediately generalizes to open subsets of $\mathcal{P}TN$.

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- This notion of *path geometries* immediately generalizes to open subsets of $\mathcal{P}TN$.
- Hence chains in M determine a line subbundle on the space $\mathcal{P}_0TM$ of transverse directions.

Put $\tilde{G} := PGL(k+2, \mathbb{R})$, and let $\tilde{P}$ be the joint stabilizer of the line spanned by the first basis vector and the plane spanned by the first two basis vectors.

- block decomposition of the Lie algebra $\tilde{\mathfrak{g}}$ of the form

$$\begin{pmatrix} \tilde{\mathfrak{g}}_0 & \tilde{\mathfrak{g}}_1^E & \tilde{\mathfrak{g}}_2 \\ \tilde{\mathfrak{g}}_{-1}^E & \tilde{\mathfrak{g}}_0 & \tilde{\mathfrak{g}}_1^V \\ \tilde{\mathfrak{g}}_{-2} & \tilde{\mathfrak{g}}_{-1}^V & \tilde{\mathfrak{g}}_0 \end{pmatrix}$$

with blocks of size 1, 1, and k, such that $\tilde{\mathfrak{p}} = \tilde{\mathfrak{g}}_0 \oplus \tilde{\mathfrak{g}}_1 \oplus \tilde{\mathfrak{g}}_2$.

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  with blocks of size 1, 1, and k, such that $\tilde{\mathfrak{p}} = \tilde{\mathfrak{g}}_0 \oplus \tilde{\mathfrak{g}}_1 \oplus \tilde{\mathfrak{g}}_2$.
- $\tilde{G}/\tilde{P} \cong \mathcal{P}T(\mathbb{R}P^k)$ in such a way that $\tilde{\mathfrak{g}}_{-1}^V$ corresponds to the vertical subbundle, while $\tilde{\mathfrak{g}}_{-1}^E$ corresponds to foliation determined by the lifts of projective lines.

Theorem

Let $U \subset \mathcal{P}TN$ be an open subset and let $E \subset TU$ be a line subbundle which is complementary to the vertical subbundle within the tautological subbundle. Then there is a canonical principal bundle $\tilde{\mathcal{G}} \to U$ with structure group $\tilde{P}$ endowed with a canonical normal Cartan connection $\tilde{\omega} \in \Omega^1(\tilde{\mathcal{G}}, \tilde{\mathfrak{g}})$ which induces the given structure. The pair $(\tilde{\mathcal{G}}, \tilde{\omega})$ is unique up to isomorphism.

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- "inducing the given structure" means that under the isomorphism $TU \cong \tilde{\mathcal{G}} \times_{\tilde{P}} (\tilde{\mathfrak{g}}/\tilde{\mathfrak{p}})$ obtained from $\tilde{\omega}$, the subspaces $\tilde{\mathfrak{g}}^V_{-1}$ and $\tilde{\mathfrak{g}}^E_{-1}$ of $\tilde{\mathfrak{g}}/\tilde{\mathfrak{p}}$ correspond to the vertical subbundle respectively to E.

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- $\tilde{\omega}$ satisfies a normalization condition on its curvature, which makes it unique

Let M be a manifold endowed with a partially integrable almost CR structure, and let $(p : \mathcal{G} \to M, \omega)$ be the canonical Cartan bundle and connection. Via the adjoint action, P acts on $\mathfrak{g}/\mathfrak{p} \cong \mathfrak{g}_{-2} \oplus \mathfrak{g}_{-1}$. Let $Q \subset P$ be the stabilizer of the line $\mathfrak{g}_{-2}$. Then

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- $\mathcal{G} \to N$ is a principal fiber bundle with structure group Q on which $\omega \in \Omega^1(\mathcal{G}, \mathfrak{g})$ defines a Cartan connection
- $\mathfrak{q} = \mathfrak{g}_0 \oplus \mathfrak{g}_2$ and $\mathfrak{g}_{-2}$ defines a Q–invariant subspace in $\mathfrak{g}/\mathfrak{q}$. Under the isomorphism $TN \cong \mathcal{G} \times_Q (\mathfrak{g}/\mathfrak{q})$ obtained via ω this subspace induces the line subbundle defining the chains.

Structure

Basic question: Can we construct the canonical Cartan geometry $(\tilde{\mathcal{G}} \to N, \tilde{\omega})$ on $N = \mathcal{P}_0 TM$ determined by the chains directly (i.e. without prolongation) from the Cartan geometry $(\mathcal{G} \to N, \omega)$.

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To obtain $(\tilde{\mathcal{G}}, \tilde{\omega})$ from $(\mathcal{G}, \omega)$ we need a homomorphism $i : Q \to \tilde{P}$ and a linear map $\alpha : \mathfrak{g} \to \tilde{\mathfrak{g}}$ such that

1. $\alpha \circ \mathrm{Ad}(g) = \mathrm{Ad}(i(g)) \circ \alpha$ for all $g \in Q$
2. $\alpha|_{\mathfrak{q}} = i' : \mathfrak{q} \to \tilde{\mathfrak{p}}$
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Given these data, one obtains an extension functor by putting $\tilde{\mathcal{G}} := \mathcal{G} \times_Q \tilde{P}$, where Q acts on $\tilde{P}$ via i, and letting $\tilde{\omega}$ be the unique form such that $\tilde{\omega}|_{T\mathcal{G}} = \alpha \circ \omega$.

Equivalence: Two pairs (i,α) and $(\hat{i},\hat{\alpha})$ are equivalent iff there is an element $\tilde{g}\in\tilde{P}$ such that $\hat{i}(g)=\tilde{g}^{-1}i(g)\tilde{g}$ and $\hat{\alpha}=\mathrm{Ad}(\tilde{g}^{-1})\circ\alpha$. The extension functors associated to equivalent pairs are naturally isomorphic.

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Proposition

If the extension functors associated to (i, α) and $(\hat{i}, \hat{\alpha})$ produce isomorphic results for one partially integrable almost CR structure, then the two pairs are equivalent.

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Strategy

- Find (i, α) which produces a regular normal Cartan geometry for the homogeneous model G/P
- Then (i, α) is determined up to equivalence, an we can check for which structures the resulting extension functor produces the canonical Cartan geometry

We have to determine the effect of the extension functor associated to (i, α) on curvatures. The curvatures are most easily encoded in the curvature functions

$$\kappa : \mathcal{G} \to L(\Lambda^2(\mathfrak{g}/\mathfrak{p}), \mathfrak{g}) \qquad \tilde{\kappa} : \tilde{\mathcal{G}} \to L(\Lambda^2(\tilde{\mathfrak{g}}/\tilde{\mathfrak{p}}), \tilde{\mathfrak{g}})$$

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Proposition

The restriction of $\tilde{\kappa}$ to $\mathcal{G} \subset \tilde{\mathcal{G}}$ is given by

$$(\tilde{X} + \tilde{\mathfrak{p}}, \tilde{Y} + \tilde{\mathfrak{p}}) \mapsto \alpha(\kappa(X, Y)) + [\alpha(X), \alpha(Y)] - \alpha([X, Y]),$$

where $X \in \mathfrak{g}$ is such that $\underline{\alpha}(X + \mathfrak{q}) = \tilde{X} + \tilde{\mathfrak{p}}$ and likewise for Y. This restriction completely determines $\tilde{\kappa}$.

Thus the problem for the homogeneous model is reduced to the purely algebraic question of finding (i, α) in such a way that the map

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$$i\begin{pmatrix} \varphi & 0 & ia\varphi \\ 0 & \Phi & 0 \\ 0 & 0 & \bar{\varphi}^{-1} \end{pmatrix} := \begin{pmatrix} |\varphi| & -a|\varphi| & 0 & 0 \\ 0 & |\varphi|^{-1} & 0 & 0 \\ 0 & 0 & \Re(\frac{|\varphi|}{\varphi}\Phi) & -\Im(\frac{|\varphi|}{\varphi}\Phi) \\ 0 & 0 & \Im(\frac{|\varphi|}{\varphi}\Phi) & \Re(\frac{|\varphi|}{\varphi}\Phi) \end{pmatrix},$$

$$\alpha\begin{pmatrix} w & Z & iz \\ X & A & -\mathbb{I}Z^* \\ ix & -X^*\mathbb{I} & -\bar{w} \end{pmatrix} := \begin{pmatrix} \Re(w) & -z & \Re(Z) & -\Im(Z) \\ x & -\Re(w) & -\Im(X^*\mathbb{I}) & -\Re(X^*\mathbb{I}) \\ \Re(X) & \Im(\mathbb{I}Z^*) & \Re(A) & -\Im(A)+\Im(w) \\ \Im(X) & -\Re(\mathbb{I}Z^*) & \Im(A)-\Im(w) & \Re(A) \end{pmatrix}$$

The map $\alpha : \mathfrak{g} \to \tilde{\mathfrak{g}}$ is *not* a Lie algebra homomorphism, so even for the homogeneous model we get $\tilde{\kappa} \neq 0$. If $\kappa = 0$, then $\tilde{\kappa}$ is a map $\tilde{\mathfrak{g}}^V_{-1} \otimes \tilde{\mathfrak{g}}_{-2} \to \tilde{\mathfrak{g}}_0$ which can be computed explicitly.

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Theorem

Let M be a partially integrable almost CR manifold, and put $N = \mathcal{P}_0 TM$. Applying the extension functor associated to (i, α) to $(\mathcal{G} \to N, \omega)$, one the canonical Cartan associated to the path geometry of chains if and only if the CR Cartan connection is torsion free and hence iff M is integrable, i.e. CR.

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From the explicit description of $\tilde{\kappa}$, one can compute the *harmonic curvature components* which are the principal curvature quantities. There are two independent components for this geometry whose geometric interpretation is well understood. Using this one proves

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- The chain equation is never locally equivalent to the trivial equation $y'' = 0$. For the homogeneous model, one obtains a non–trivial system of ODE's with large automorphism group.
- The path geometry of chains is torsion free if and only if M is spherical, i.e. locally CR isomorphic to G/P.

From the explicit description of $\tilde{\kappa}$, one can compute the *harmonic curvature components* which are the principal curvature quantities. There are two independent components for this geometry whose geometric interpretation is well understood. Using this one proves

- The chain equation is never locally equivalent to the trivial equation $y'' = 0$. For the homogeneous model, one obtains a non–trivial system of ODE's with large automorphism group.
- The path geometry of chains is torsion free if and only if M is spherical, i.e. locally CR isomorphic to G/P.
- There is no linear connection on M which has the chains among its (unparametrized) geodesics.

- The complete symmetrization of the mapping

$$(\xi, \eta, \zeta) \mapsto \mathcal{L}(\xi, J(\eta))J(\zeta)$$

can be recovered from the harmonic curvature of the path geometry of chains. Hence the almost complex structure J can be reconstructed up to sign, and the signature of M can be reconstructed. This implies

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Corollary

Let M_1 and M_2 be CR manifolds and let $f : M_1 \to M_2$ be a local contact diffeomorphism which maps chains to chains. Then f is either a local CR isomorphism or a local CR anti–isomorphism.