Chern-Simons invariants and flat extensions

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- This talk reports on joint work with K. Flood and T. Mettler (Brig), see arXiv:2409.12811
- Building on classical work of Chern-Simons, we define global invariants for certain types of connection forms with values in a Lie algebra $\mathfrak{g}$ on principal H-bundles over compact oriented 3-manifolds, which admit global smooth sections.
- This needs a non-degenerate invariant bilinear form on $\mathfrak{g}$ and depending on this form and on H, the invariants can have values in $\mathbb{R}$ or in $\mathbb{R}/\mathbb{Z}$.
- In the case of principal connections, we introduce a concept of flat extension, which is then shown to either imply vanishing of the invariants or integrality of an $\mathbb{R}$-valued invariant.
- Geometric interpretations of flat extensions are provided in several cases.

Contents

Chern-Simons forms and the resulting invariants can be defined in the general setting of Chern-Weil theory. Here we restrict to the simplest version of the Chern-Simons 3-form, which leads to invariants in dimension 3.

Consider a Lie algebra $(\mathfrak{g}, [\ ,\ ])$ endowed with a non-degenerate, invariant, symmetric bilinear form $\langle\ ,\ \rangle$ and $\Omega^*(N, \mathfrak{g})$ for a manifold N. For $\alpha \in \Omega^k(N, \mathfrak{g})$ and $\beta \in \Omega^\ell(N, \mathfrak{g})$ we then obtain $[\alpha, \beta] \in \Omega^{k+\ell}(N, \mathfrak{g})$ and $\langle \alpha, \beta \rangle \in \Omega^{k+\ell}(N)$ which are nicely compatible with the exterior derivative.

To $\theta \in \Omega^1(N, \mathfrak{g})$ one associates $\Theta := d\theta + \frac{1}{2}[\theta, \theta] \in \Omega^2(N, \mathfrak{g})$ and then considers $\langle \Theta, \Theta \rangle \in \Omega^4(N)$. If N is the total space of a principal G-bundle and θ is a principal connection form, then Θ is its curvature and $\langle \Theta, \Theta \rangle$ is closed, horizontal and equivariant. Hence it determines a cohomology class on the base, which generalizes the first Pontryagin class.

The starting point of Chern-Simons theory is that for $CS(\theta) := \langle\theta, d\theta\rangle + \frac{1}{3}\langle\theta, [\theta,\theta]\rangle$ one gets $\langle\Theta,\Theta\rangle = dCS(\theta)$ (but $CS(\theta)$ does not descend to the base). Note that if θ is flat, $CS(\theta)$ is closed and hence determines a cohomology class on N.

Let (M, g) be a closed, oriented, Riemannian 3-manifold, $N \to M$ its orthonormal frame bundle, and $\theta \in \Omega^1(N, \mathfrak{o}(n))$ the Levi-Civita connection. Since M is parallelizable, there is a global section $\sigma : M \to N$ and one defines $c_\sigma := \int_M \sigma^* CS(\theta) \in \mathbb{R}$. Normalizing $\langle\ ,\ \rangle$ appropriately, one obtains for any other section $\hat{\sigma}$, $c_{\hat{\sigma}} - c_\sigma \in \mathbb{Z}$, and hence an invariant in $\mathbb{R}/\mathbb{Z}$. Chern-Simons proved that this is conformally invariant and vanishes if M admits an isometric immersion into $\mathbb{R}^4$.

D. Burns and C. Epstein used $CS(\theta)$ for the canonical Cartan connection of a compact, oriented CR 3-manifold with trivial Cartan bundle to similarly define a global invariant. Here one can show that $c_{\hat{\sigma}} = c_\sigma$ so the invariant is $\mathbb{R}$-valued.

General definition of the invariants

We fix $(\mathfrak{g}, [\ ,\ ], \langle\ ,\ \rangle)$ and consider a subgroup H of a Lie group G with Lie algebra $\mathfrak{g}$, so $\mathfrak{h} \subset \mathfrak{g}$. For a principal H-bundle $\pi : P \to M$ let $R : P \times H \to P$ be the principal action and consider the "partial maps" $R_h : P \to P$ for $h \in H$ and $i_u : H \to P$ for $u \in P$.

Definition

$\theta \in \Omega^1(P, \mathfrak{g})$ is called a $\mathfrak{g}$-connection form if $R_h^*\theta = \mathrm{Ad}(h^{-1}) \circ \theta$ and $i_u^*\theta = \mu_H$, the Maurer-Cartan form of H.

Observe that $\mu_H \in \Omega^1(H, \mathfrak{h}) \subset \Omega^1(H, \mathfrak{g})$ and using the latter interpretation, we can form $CS^{\mathfrak{g}}(\mu_H) \in \Omega^3(H)$, which is closed since μ_H satisfies the Maurer-Cartan equation. This also implies that $CS^{\mathfrak{g}}(\mu_H)$ is the left invariant form associated to $(X, Y, Z) \mapsto -\frac{1}{6}\langle X, [Y, Z]\rangle$. If $\mathfrak{h}$ is simple, this is a multiple of the Cartan 3-form, which generates $H^3(H, \mathbb{Z})$.

Fix a principal H-bundle $\pi : P \to M$ over a closed oriented 3-manifold that admits a global section σ. For a $\mathfrak{g}$-connection form $\theta \in \Omega^1(P, \mathfrak{g})$ consider $c_\sigma := \int_M \sigma^* CS(\theta) \in \mathbb{R}$.

Proposition

(1) If $CS^{\mathfrak{g}}(\mu_H)$ is exact, then $c_\sigma \in \mathbb{R}$ is independent of σ and hence an invariant of θ.
(2) If $[CS^{\mathfrak{g}}(\mu_H)] \in H^3(H, \mathbb{Z}) \subset H^3(H, \mathbb{R})$, then $c_\sigma + \mathbb{Z} \in \mathbb{R}/\mathbb{Z}$ is independent of σ and hence an invariant of θ.

Sketch of proof: For a section $\hat{\sigma}$, we get $\hat{\sigma}(x) = R(\sigma(x), h(x))$ for some smooth function $h : M \to H$. A direct computation shows that $R^* CS(\theta) = CS(\theta) + CS^{\mathfrak{g}}(\mu_H) + d\varphi$ for some $\varphi \in \Omega^3(P)$. This easily implies that $c_{\hat{\sigma}} = c_\sigma + \int_M h^* CS^{\mathfrak{g}}(\mu_H)$.
Note: The restriction of $\langle\ ,\ \rangle$ to $\mathfrak{h}$ may be degenerate.

Examples on compact, oriented 3-manifolds

1. $G = H = SO(3)$, $\langle\ ,\ \rangle$ normalized such that $\int_H CS(\mu_H) = \pm 1$: classical $\mathbb{R}/\mathbb{Z}$-valued invariant for Riemannian manifolds
2. $G = H = SO_0(2,1)$: $\mathbb{R}$-valued invariant for Lorentzian manifolds admitting a global orthonormal frame
3. $G = H = SL(3,\mathbb{R})$,$\langle\ ,\ \rangle$ normalized as for $SO(3)$: $\mathbb{R}/\mathbb{Z}$-valued invariant for volume preserving affine connections
4. $G = PSU(2,1) \supset H$ stabilizer of isotropic line: $\mathbb{R}$-valued Burns-Epstein invariant for CR manifolds admitting a global CR vector field; Here $\langle\ ,\ \rangle$ is degenerate on $\mathfrak{h}$.
5. Similar $\mathbb{R}$-valued invariants for Legendrean contact structures (or equivalently path geometries or 2nd order ODE) respectively contact projective structures. In both cases, a condition ensuring triviality of the Cartan bundle has to be imposed.

These provide a systematic way to construct sufficient conditions for vanishing of Chern-Simons invariants. Here we realize $\mathfrak{g}$ as a Lie subalgebra of a bigger Lie algebra $\tilde{\mathfrak{g}}$ and consider $\langle\ ,\ \rangle$ on $\tilde{\mathfrak{g}}$ such that the restriction to $\mathfrak{g}$ is non-degenerate.

This implies that $\tilde{\mathfrak{g}} = \mathfrak{g} \oplus \mathfrak{g}^{\perp}$ and this decomposition is $\mathfrak{g}$-invariant. Hence $\tilde{\mathfrak{g}}$-valued forms decompose as $\alpha = \alpha^{\top} + \alpha^{\perp}$ according to their values and G-equivariancy properties are preserved. Note that if in addition $[\mathfrak{g}^{\perp}, \mathfrak{g}^{\perp}] \subset \mathfrak{g}$, then $(\tilde{\mathfrak{g}}, \mathfrak{g})$ is a symmetric pair.

Using $\langle\ ,\ \rangle$ to define CS for both $\mathfrak{g}$-valued and $\tilde{\mathfrak{g}}$-valued forms, a computation leads to the following key lemma

Lemma

For $\theta \in \Omega^1(N, \tilde{\mathfrak{g}})$, $\theta = \theta^{\top} + \theta^{\perp}$ with curvature $\Theta = \Theta^{\top} + \Theta^{\perp}$, we get $CS(\theta) = CS(\theta^{\top}) + \langle \theta^{\perp}, \Theta^{\perp} \rangle$. In particular, if θ satisfies the Maurer-Cartan equation, then $CS(\theta) = CS(\theta^{\top})$.

Starting from $G \subset \tilde{G}$, this first implies that if $\langle\ ,\ \rangle$ is chosen such that $[CS(\mu_{\tilde{G}})] \in H^3(\tilde{G}, \mathbb{Z})$, then $[CS(\mu_G)] \in H^3(G, \mathbb{Z})$.

For Lie subalgebras of $\mathfrak{gl}(n, \mathbb{R})$, one can obtain invariant bilinear forms from the trace-form on $\mathfrak{gl}(n, \mathbb{R})$. In particular, this provides $\langle\ ,\ \rangle$ for $\mathfrak{so}(n) \subset \mathfrak{sl}(n, \mathbb{R})$ as well as for $\mathfrak{so}(n) \subset \mathfrak{so}(n+1)$ and $\mathfrak{sl}(n, \mathbb{R}) \subset \mathfrak{sl}(n+1, \mathbb{R})$. For this choice, one obtains the familiar expression $CS(\theta) = \operatorname{tr}(\theta \wedge d\theta + \frac{2}{3}\theta \wedge \theta \wedge \theta)$.

Now let $p : P \to M$ be a principal P-bundle and let $\theta \in \Omega^1(P, \mathfrak{g})$ be a principal connection. Then a *flat extension* of type $(G, \tilde{G})$ is a G-equivariant smooth map $F : P \to \tilde{G}$ such that $\theta = F^*(\mu_{\tilde{G}}^{\top})$.

Theorem

Suppose that $\theta \in \Omega^1(P, \mathfrak{g})$ as above admits a flat extension F of type $(G, \tilde{G})$ such that $[F^*(\mu_{\tilde{G}}^{\perp}), F^*(\mu_{\tilde{G}}^{\perp})] \in \Omega^2(P, \mathfrak{g})$. If $CS(\mu_{\tilde{G}})$ is exact then $c_\sigma = 0$ and $[CS(\mu_{\tilde{G}})] \in H^3(\tilde{G}, \mathbb{Z})$ implies $c_\sigma(\theta) \in \mathbb{Z}$.

The basic examples of flat extensions are obtained from lifting the Gauss map of a flat immersion to a frame bundle. In the Riemannian case, $G = SO(3)$ and $\tilde{G} = SO(4)$ and we use an isometric immersion $f : M \to \mathbb{R}^4$. Viewing a point $u \in P$, the ON-frame bundle, as $u : \mathbb{R}^3 \to T_xM$, we can add the oriented unit normal to $T_xf \circ u$ to obtain an orthogonal map $\mathbb{R}^4 \to \mathbb{R}^4$. This defines $F : P \to SO(4)$ and since $(\tilde{\mathfrak{g}}, \mathfrak{g})$ is a symmetric pair, the theorem implies vanishing of the Chern-Simons invariant.

In the Lorentzian case, there are two cases with $\tilde{G} = SO_0(3,1)$ and $\tilde{G} = SO_0(2,2)$, respectively. As above, flat extensions are obtained from isometric immersions into $\mathbb{R}^{3,1}$ respectively into $\mathbb{R}^{2,2}$ and $(\tilde{\mathfrak{g}}, \mathfrak{g})$ is a symmetric pair in both cases. In the first case, the theorem implies integrality, in the second case vanishing of the Chern-Simons invariant (which is $\mathbb{R}$-valued here).

Both in the Riemannian and the Lorentzian case, $F^*(\mu^{\perp}_{\tilde{G}})$ equivalently encodes the second fundamental form.

The case $G = SL(3,\mathbb{R})$ of volume preserving connections with $\tilde{G} = SL(4,\mathbb{R})$ is a bit more difficult, but ties in nicely with the classical notion of an *equiaffine immersion* of (M,∇). In addition to an immersion $f : M \to \mathbb{R}^4$ one has to choose $\ell : M \to \mathbb{R}P^3$, such that $\ell(x)$ is transversal to $T_xf(T_xM)$ for any $x \in M$.

Given (f,ℓ) and $x \in M$, we can decompose $\mathbb{R}^4 = T_{f(x)}\mathbb{R}^4$ as $T_xf(T_xM) \oplus \ell(x)$. Hence we can decompose the restriction of the flat connection $\tilde{\nabla}$ into a tangential and a transversal component. The immersion is called equiaffine iff $f^*(\tilde{\nabla}^\top) = \nabla$.

The pair (f,ℓ) determines a lift of the Gauss map to a map F from the volume preserving frame bundle of M to $SO(4)$, which then defines a flat extension. Again, $F^*(\mu^\perp_{\tilde{G}})$ admits an interpretation as the second fundamental form and the shape operator (which are independent objects here). Using that ∇ is volume preserving, one proves that $[F^*(\mu^\perp_{\tilde{G}}), F^*(\mu^\perp_{\tilde{G}})]$ is $\mathfrak{g}$-valued and the theorem implies vanishing of the Chern-Simons invariant.