BGG sequences – a Riemannian perspective

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- Bernstein–Gelfand–Gelfand resolutions originally arose in representation theory but it was soon realized that there are relations to certain geometric structures.
- A geometric version of BGG sequences was developed in the early 2000's in the setting of parabolic geometries. The construction uses tools derived from the associated canonical Cartan geometry like tractor bundles and hence is not easily accessible.
- Starting with a (pseudo–)Riemannian manifold, one obtains two parabolic geometries as underlying structures, namely a conformal and a projective structure.
- Relations to applied math recently motivated a simpler version of the construction for domains in $\mathbb{R}^n$ (with the induced flat metric) in low (Sobolev) regularity.
- In the smooth case, this extends to general Riemannian manifolds. In my talk, I'll outline this construction with a focus on the conformal case.

Contents

Natural vector bundles on Riemannian n-manifolds $\leftrightarrow$ representations of $O(n)$. For (M, g) let $\mathcal{O}M$ be the ON-frame bundle and for a representation $\mathbb{W}$ of $O(n)$ form the associated bundle $\mathcal{W}M := \mathcal{O}M \times_{O(n)} \mathbb{W}$. Basic properties:

- Via the principal Levi-Civita connection on $\mathcal{O}M$ one obtains a linear connection on $\mathcal{W}M$. All these are denoted by ∇.
- $O(n)$-equivariant maps induce parallel natural bundle maps between associated bundles. In particular, the infinitesimal representation $\mathfrak{o}(n) \times \mathbb{W}$ to $\mathbb{W}$ induces a bilinear bundle map $\bullet : \mathfrak{o}(TM) \times \mathcal{W}M \to \mathcal{W}M$.
- The Riemann curvature of (M, g) can be viewed as $R \in \Omega^2(M, \mathfrak{o}(TM))$. In these terms, the curvature of the induced connection on $\mathcal{W}M$ is characterized by the fact that for $\xi, \eta \in \mathfrak{X}(M)$ and $\sigma \in \Gamma(\mathcal{W}M)$ one gets $\nabla_\xi \nabla_\eta \sigma - \nabla_\eta \nabla_\xi \sigma - \nabla_{[\xi,\eta]}\sigma = R(\xi, \eta) \bullet \sigma$.

The main input for the construction comes from Lie algebra theory, which produces interesting representations of $O(n)$ with additional structures. For $n \geq 3$, consider $\mathbb{R}^{n+2}$ with basis $e_0, \dots, e_{n+1}$ and the Lorentzian inner product corresponding to the quadratic form $q(x) = 2x^0x^{n+1} + \sum_{i=1}^n (x^i)^2$. Let $G \cong O(n+1,1)$ be the corresponding orthogonal group. Then we can naturally view $O(n)$ as the subgroup of those maps in G that fix both e_0 and e_{n+1}.

With blocks of sizes 1, n, and 1, the Lie algebra $\mathfrak{g}$ of G consists of matrices of the form $\begin{pmatrix} a & Z & 0 \\ X & A & -Z^t \\ 0 & -X^t & -a \end{pmatrix}$ with $a \in \mathbb{R}$, $X \in \mathbb{R}^n$, $Z \in \mathbb{R}^{n*}$, and $A \in \mathfrak{o}(n)$. This can be viewed as a decomposition $\mathfrak{g} = \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1$ with components spanned by X, (a, A), and Z, respectively, so $\mathfrak{o}(n) \subset \mathfrak{g}_0$. One immediately verifies that $[\mathfrak{g}_i, \mathfrak{g}_j] \subset \mathfrak{g}_{i+1}$, where $\mathfrak{g}_\ell = 0$ for $|\ell| \geq 2$. ("|1|-grading of $\mathfrak{g}$")

In particular, the bracket restricts to a map $\mathfrak{g}_0 \times \mathfrak{g}_{-1} \to \mathfrak{g}_{-1}$, which gives the standard action of $\mathfrak{o}(n)$ on $\mathbb{R}^n \cong \mathfrak{g}_{-1}$ together with multiples of the identity, so $\mathfrak{g}_0 \cong \mathfrak{co}(n)$. The action of $\mathfrak{g}_0$ on $\mathfrak{g}_1$ then is just the usual action of $\mathfrak{co}(n)$ on $\mathbb{R}^{n*}$.

The grading of $\mathfrak{g}$ is actually induced by the decomposition of $\mathbb{R}^{n+2} = \mathbb{R} \oplus \mathbb{R}^n \oplus \mathbb{R}$ with components spanned by e_0, by $e_1, \dots, e_n$ and by e_{n+1}, and this extends to general representations of G (assumed to be irreducible for simplicity). Any such representation $\mathbb{V}$ canonically decomposes as $\oplus_{i=0}^N \mathbb{V}_i$ for some $N \in \mathbb{N}$ such that the infinitesimal action of $\mathfrak{g}_i$ maps $\mathbb{V}_j$ to $\mathbb{V}_{i+j}$ for $i \in \{-1, 0, 1\}$ and $j \in \{0, \dots, N\}$.

The representation $\mathbb{V}$ can be restricted to $O(n)$ and then each $\mathbb{V}_i$ is $O(n)$-invariant. Thus for the associated bundles, we get $\mathcal{V}M = \oplus_i \mathcal{V}_i M$. Moreover, we see from above that $\mathcal{O}M \times_{O(n)} \mathfrak{g}_{-1} = TM$ and $\mathcal{O}M \times_{O(n)} \mathfrak{g}_1 = T^*M$.

We can restrict the infinitesimal action of $\mathfrak{g}$ on $\mathbb{V}$ to each of the components $\mathfrak{g}_i$. For $i = 0$, this provides an extension $\bullet : \mathfrak{co}(TM) \times \mathcal{V}M \to \mathcal{V}M$ of the operation of $\mathfrak{o}(TM)$ defined before, and $\mathfrak{co}(TM) \bullet \mathcal{V}_j M \subset \mathcal{V}_j M$.

In addition, we obtain bilinear bundle maps $\bullet : TM \times \mathcal{V}M \to \mathcal{V}M$ and $\bullet : T^*M \times \mathcal{V}M \to \mathcal{V}M$ such that $TM \bullet \mathcal{V}_j M \subset \mathcal{V}_{j-1}M$ and $T^*M \bullet \mathcal{V}_j M \subset \mathcal{V}_{j+1}M$.

We will also denote by $\bullet$ the induced tensorial operations on sections. So for $\eta \in \mathfrak{X}(M)$, $\alpha \in \Omega^1(M)$ and $s_j \in \Gamma(\mathcal{V}_j M)$, we get $\eta \bullet s_j \in \Gamma(\mathcal{V}_{j-1}M)$ and $\alpha \bullet s_j \in \Gamma(\mathcal{V}_{j+1}M)$.
The fact that the bundle maps are parallel reads as $\nabla_\xi(\eta \bullet s_j) = (\nabla_\xi \eta) \bullet s_j + \eta \bullet (\nabla_\xi s_j)$ and so on.

Examples

By construction, the decomposition of the standard representation $\mathbb{V} := \mathbb{R}^{n+2}$ of $\mathfrak{g}$ is $\mathbb{R} \oplus \mathbb{R}^n \oplus \mathbb{R}$. Hence a section of $\mathcal{V}M$ can be viewed as a triple (f, φ, h) with $f, h \in C^\infty(M, \mathbb{R})$ and $\varphi \in \Omega^1(M)$. Our operations are given by $\xi \bullet (f, \varphi, h) = (-\varphi(\xi), h\xi^\flat, 0)$ for $\xi \in \mathfrak{X}(M)$ and $\alpha \bullet (f, \varphi, h) = (0, -f\alpha, g(\alpha, \varphi))$, respectively.

General vector representations of G can be constructed from this, which also leads to explicitly formulae for the operations. For example, $\Lambda^2\mathbb{R}^{n+2}$ is isomorphic to the adjoint representation. Sections of the corresponding bundles are triples (η, Φ, φ) with $\eta \in \mathfrak{X}(M)$, $\Phi \in \mathfrak{co}(TM)$, and $\varphi \in \Omega^1(M)$. Defining $\{\xi, \alpha\}(\zeta) := \alpha(\xi)\zeta + \alpha(\zeta)\xi - g(\xi, \zeta)\alpha^\#$, the operations are given by $\xi \bullet (\eta, \Phi, \varphi) = (-\Phi(\xi), \{\xi, \varphi\}, 0)$ and $\alpha \bullet (\eta, \Phi, \varphi) = (0, -\{\eta, \alpha\}, \alpha \circ \Phi)$, respectively.

The twisted connection

For a metric g on M, there is the *Schouten tensor* P, an equivalent encoding of the Ricci curvature, characterized by $\mathrm{Ric} = (n-2)\mathrm{P} + \mathrm{tr}(\mathrm{P})g$. Using this and the operation $\{\ ,\ \}$ from above, the decomposition of curvature reads as $R(\xi,\eta)(\zeta) = W(\xi,\eta)(\zeta) + \{\xi, \mathrm{P}(\eta)\}(\zeta) - \{\eta, \mathrm{P}(\xi)\}(\zeta)$.

Given a representation $\mathbb{V} = \oplus_i \mathbb{V}_i$ of G we have the Levi-Civita connection ∇ which acts on $\Gamma(\mathcal{V}M)$ component wise. We modify this by tensorial terms as $\nabla^{\mathcal{V}}_\xi s := \nabla_\xi s + \xi \bullet s - \mathrm{P}(\xi) \bullet s$. This mixes components, since $(\nabla^{\mathcal{V}}_\xi s)_i = \nabla_\xi s_i + \xi \bullet s_{i+1} - \mathrm{P}(\xi) \bullet s_{i-1}$.

Now one simply expands the defining equation for the curvature $R^{\mathcal{V}}$ of $\nabla^{\mathcal{V}}$. By construction, $R^{\mathcal{V}}(\xi,\eta)(s)_i$ depends only on s_j for $j \in \{i-2, \dots, i+2\}$. But $\eta \bullet \xi \bullet s = \xi \bullet \eta \bullet s$ and similarly for the P-terms, so there is no dependence on s_{i-2} and s_{i+2}.

Recall: $(\nabla^{\mathcal{V}}_\xi s)_i = \nabla_\xi s_i + \xi \bullet s_{i+1} - \mathrm{P}(\xi) \bullet s_{i-1}$

Next, one verifies that torsion-freeness of ∇ implies vanishing of the part depending on s_{i+1}, so it remains to analyze the parts depending on s_{i-1} and on s_i. For the former one, we obtain $Y(\xi,\eta) \bullet s_{i-1}$, where $Y(\xi,\eta) = \nabla_\xi \mathrm{P}(\eta) - \nabla_\eta \mathrm{P}(\xi) - \mathrm{P}([\xi,\eta])$ is the Cotton-tensor. The operation $\{\ ,\ \}$ we considered above is induced by the restriction $\mathfrak{g}_{-1} \times \mathfrak{g}_1 \to \mathfrak{g}_0$ of the Lie bracket in $\mathfrak{g}$. Thus, acting by $\bullet$, the relation between R and W from above extends to arbitrary associated bundles to $\mathcal{O}M$. But on a representation of $\mathfrak{g}$, the action of $\{\xi, \mathrm{P}(\eta)\}$ on s can be written as the commutator of the actions of ξ and of $\mathrm{P}(\eta)$. Using this, one shows:

Theorem

$R^{\mathcal{V}}(\xi,\eta)(s)_i = W(\xi,\eta) \bullet s_i + Y(\xi,\eta) \bullet s_{i-1}$. In particular, the connection $\nabla^{\mathcal{V}}$ is flat iff the metric g is conformally flat.

Any linear connection can be coupled to the exterior derivative to define the *covariant exterior derivative* on bundle valued differential forms. In particular, we obtain d^∇ and $d^{\nabla^\mathcal{V}}$ acting on $\Omega^*(M,\mathcal{V}M)$. Decomposing $\varphi\in\Omega^k(M,\mathcal{V}M)$ into components $\varphi_i\in\Omega^k(M,\mathcal{V}_iM)$, we get the formula

$(d^{\nabla^\mathcal{V}}\varphi)_i = d^\nabla\varphi_i + \partial\varphi_{i+1} - \partial^{\mathsf{P}}\varphi_{i-1}$, where
$\partial\varphi(\xi_0,\dots,\xi_k) = \sum_i(-1)^i\xi_i\bullet\varphi(\xi_0,\dots,\widehat{\xi_i},\dots,\xi_k)$ and
$\partial^{\mathsf{P}}\varphi(\xi_0,\dots,\xi_k) = \sum_i(-1)^i\mathsf{P}(\xi_i)\bullet\varphi(\xi_0,\dots,\widehat{\xi_i},\dots,\xi_k)$.

The operation ∂ is induced by bundle maps $\Lambda^kT^*M\otimes\mathcal{V}_iM\to\Lambda^{k+1}T^*M\otimes\mathcal{V}_{i-1}M$, which we denote by the same symbol. Since $\partial^2=0$, we obtain natural subbundles $\mathrm{im}(\partial)\subset\ker(\partial)$ in each degree. Define $\mathcal{H}_k:=\ker(\partial)\cap\mathrm{im}(\partial)^\perp$. These correspond to Lie algebra cohomology groups which are known by Kostant's theorem. They form a very small part of $\Omega^k(M,\mathcal{V}M)$.

In each degree, ∂ defines an isomorphism $\ker(\partial)^\perp \to \operatorname{im}(\partial)$ and we define $T : \Lambda^k T^*M \otimes \mathcal{V}_i M \to \Lambda^{k-1} T^*M \otimes \mathcal{V}_{i+1} M$ by $T|_{\operatorname{im}(\partial)} = \partial^{-1}$ and $T|_{\operatorname{im}(\partial)^\perp} = 0$. Then in particular $\partial \circ T \circ \partial = \partial$ and $T \circ \partial \circ T = T$, $T \circ T = 0$, $\ker(T) = \operatorname{im}(\partial)^\perp$ and $\operatorname{im}(T) = \ker(\partial)^\perp$. In examples, T can be easily made explicit.

Theorem

There is a natural differential operator $S : \Gamma(\mathcal{H}_k) \to \Omega^k(M, \mathcal{V}M)$ such that for $\varphi \in \Gamma(\mathcal{H}_k)$, the form $S(\varphi)$ is uniquely characterized by $S(\varphi) - \varphi \in \operatorname{im}(T)$ and $T(d^{\nabla^{\mathcal{V}}} S(\varphi)) = 0$.

Sketch of proof: Uniqueness: The difference of two forms with these properties can be written as $T(\psi)$ for $\psi \in \Omega^{k+1}(M, \mathcal{V}M)$ such that $Td^{\nabla^{\mathcal{V}}} T(\psi) = 0$. Supposing that $T(\psi)_j = 0$ for $j \leq i$ we see that $(d^{\mathcal{V}} T(\psi))_j$ vanishes for $j < i$ and equals $\partial T(\psi)_{i+1}$ for $j = i$. But then $0 = T\partial T(\psi)_{i+1} = T(\psi)_{i+1}$. Since $T(\psi)_0 = 0$ holds by definition, this inductively implies uniqueness.

Sketch of proof of existence: Take $\varphi = \varphi_i \in \mathcal{H}_k$. Then $d^{\nabla^{\mathcal{V}}}\varphi = d^{\nabla}\varphi_i - \partial^{\mathsf{P}}\varphi_i$ and applying T, we see that $(Td^{\nabla^{\mathcal{V}}}\varphi)_j = 0$ for $j \leq i$. Hence $(Td^{\nabla^{\mathcal{V}}})^2\varphi$ has lowest component $T\partial(Td^{\nabla^{\mathcal{V}}}\varphi)_{i+1} = (Td^{\nabla^{\mathcal{V}}}\varphi)_{i+1}$. But this means that $(Td^{\nabla^{\mathcal{V}}}(\varphi - Td^{\nabla^{\mathcal{V}}}\varphi))_j$ vanishes for all $j \leq i+2$. Higher homogeneous components can be removed iteratively in the same way, which also provides a formula for S.

Now for $\varphi \in \mathcal{H}_k$ we have $d^{\nabla^{\mathcal{V}}}S(\varphi) \in \ker(T) = \operatorname{im}(\partial)^{\perp}$ and we let $D(\varphi)$ be its component in $\mathcal{H}_{k+1} = \ker(\partial) \cap \ker(T)$. This defines the *BGG-operator* $D = D_k : \mathcal{H}_k \to \mathcal{H}_{k+1}$.

In particular, D_0 is defined on $\mathcal{H}_0 = \Gamma(\mathcal{V}_0 M)$. It turns out that this can be of any order $r \geq 1$, and $\mathbb{V}_0$ and r determine $\mathbb{V}$. The principal part of D_0 is then obtained by projecting $\operatorname{Symm}(\nabla^r\sigma)$ to the Cartan power in $S^rT^*M \otimes \mathcal{V}_0 M$. This e.g. leads to all types of conformal Killing and conformal Killing–Yano operators.

Applications

If g is conformally flat, then $(\Omega^*(M,\mathcal{V}M), d^{\nabla^{\mathcal{V}}})$ is a complex and a fine resolution of the sheaf of local parallel sections of $\mathcal{V}M$. One easily verifies that here we get $d^{\nabla^{\mathcal{V}}} \circ S = S \circ D$ which implies that $(\mathcal{H}_*, D)$ also is a complex and one shows that it is also a resolution.

In particular, $\sigma \in \Gamma(\mathcal{V}_0 M)$ satisfies $D_0(\sigma) = 0$ iff $S(\sigma) \in \Gamma(\mathcal{V}M)$ is parallel for $\nabla^{\mathcal{V}}$. This provides a rewriting of the first BGG operator in first order closed form. The BGG complex is a compatibility complex for this first operator. For $\mathbb{V} = \mathfrak{g}$, this produces the conformal deformation complex.

For general g, there still is a close relation between $(\Omega^*(M,\mathcal{V}M), d^{\nabla^{\mathcal{V}}})$ and $(\mathcal{H}_*, D)$. In particular, one can still rewrite the first BGG operators in first order closed form using modifications of $\nabla^{\mathcal{V}}$. There also are many applications of BGG to topics like conformally compact metrics.

Remarks on the projective case

BGG sequences in this case come from the Lie algebra $\mathfrak{sl}(n+1,\mathbb{R})$, which admits a grading of the form $\mathbb{R}^n \oplus \mathfrak{gl}(n,\mathbb{R}) \oplus \mathbb{R}^{n*}$. There is a simplified version of the construction in the setting of smooth n-manifolds M endowed with a torsion-free connection ∇ on TM which preserves a volume form on M.

Via the Levi-Civita connection, this applies to Riemannian manifolds. Among the resulting first BGG operators there are the Killing operators on all types of tensors and one obtains a version of the Riemannian deformation sequence. Complexes and resolutions are available here only for projectively flat connections, which in the Riemannian case is equivalent to constant sectional curvature.

Thank you for your attention!