of this is of course $\xi \mapsto T\pi\cdot(\xi, 0)$. This implies that, viewing S^{2n+1} as the unit sphere for the standard Hermitian inner product on $\mathbb{C}^{n+1}$, the subbundle $H_z \subset T_z S^{2n+1}$ is again the complex orthocomplement of z sitting inside its real orthocomplement. But identifying $\mathbb{C}^{n+1}$ with $\mathbb{R}^{2n+2}$, the real part of the Hermitian inner product is the standard inner product, while its imaginary part is a nondegenerate skew symmetric bilinear form. Hence, we obtain exactly the same subbundle H as in 1.1.4, so in particular we again have obtained a contact structure on S^{2n+1}.

Moreover, as in 1.1.4 we obtain a contact form α, such that the restriction of $d\alpha$ to H is just the imaginary part of a positive definite Hermitian form. In the language of almost CR–structures, this means that the structure is strictly pseudoconvex and partially integrable. In fact, viewing S^{2n+1} as the unit sphere in $\mathbb{C}^{n+1}$, the subspace H_x is the maximal complex subspace of $T_x S^{2n+1} \subset \mathbb{C}^{n+1}$. Hence, the almost CR–structure is embeddable and thus integrable, so it is a CR–structure. This, however, is not important for our purposes.

As in the previous examples, it follows from the existence of a canonical Cartan connection for CR–structures that the actions of elements of $SU(n+1, 1)$ are exactly the CR–diffeomorphisms of the CR–structure on S^{2n+1}. Here a CR–diffeomorphism is a diffeomorphism which preserves the contact structure H and further has the property that the restriction of the tangent map to the contact subbundle is complex linear. The curved analogs of this homogeneous space are strictly pseudoconvex partially integrable almost CR–manifolds of dimension $2n + 1$. These are smooth manifolds M of dimension $2n + 1$ endowed with a rank n complex subbundle $H \subset TM$ which defines a contact structure on M. In addition, one has to require existence of local contact forms α such that the restriction of $d\alpha$ to $H \times H$ is the imaginary part of a definite Hermitian form.

1.2. Some background from differential geometry

In this section, we review some basic facts on differential geometry and analysis on manifolds which will be necessary for further development. Our main purpose here is to fix the notation and conventions used in the sequel, as well as to give a more detailed collection of prerequisites for the further text. The basic reference for this section is [**KMS**]. At the same time, we stress the basic concepts of frames, natural bundles, and the role of the symmetry groups in the properties of geometric objects. This will remain one of the main features of our exposition in the entire book.

1.2.1. Smooth manifolds. Unless otherwise stated, all manifolds we consider are finite dimensional and second countable and we assume that all connected components have the same dimension. Any manifold comes equipped with a maximal *atlas*, i.e. a maximal collection of open subsets $U_\alpha \subset M$ together with homeomorphisms $u_\alpha : U_\alpha \to u_\alpha(U_\alpha)$ onto open subsets of $\mathbb{R}^n$, such that for all α, β with $U_{\alpha\beta} := U_\alpha \cap U_\beta \neq \emptyset$ the subset $u_\alpha(U_{\alpha\beta})$ is open and the composition $u_{\alpha\beta} := u_\alpha \circ u_\beta^{-1} : u_\beta(U_{\alpha\beta}) \to u_\alpha(U_{\alpha\beta})$ is smooth (C^∞). A *chart* on M is any element (U_α, u_α) of this maximal atlas. Such a chart gives rise to local coordinates $u^i_\alpha : U_\alpha \to \mathbb{R}$ on M.

A map $f : M \to N$ between smooth manifolds is smooth, if and only if its expression in one (or equivalently any) local coordinate system around any point in M is smooth. A *diffeomorphism* is a bijective smooth map, whose inverse is smooth, too. A *local diffeomorphism* $f : M \to N$ is a smooth map such that for each point $x \in M$ there is an open neighborhood U of x in M such that $f(U) \subset N$ is open and the restriction $f|_U : U \to f(U)$ is a diffeomorphism.

The space $C^\infty(M, \mathbb{R})$ of smooth real–valued functions on M forms an algebra under the pointwise operations. For $f \in C^\infty(M, \mathbb{R})$ the *support* $\operatorname{supp}(f)$ of f is the closure of the set of all $x \in M$ such that $f(x) \neq 0$. The concept of support generalizes in an obvious way to smooth functions with values in a vector space and to smooth sections of vector bundles; see 1.2.6 below.

A fundamental result is that any open covering of a smooth manifold M admits a subordinate *partition of unity.* This means that if $\{V_i : i \in I\}$ is a family of open subsets of M such that $M = \bigcup_{i \in I} V_i$, then there exists a family $\{f_\alpha\}$ of smooth functions on M with values in $[0, 1] \subset \mathbb{R}$ such that for any α there exists an $i \in I$ with $\operatorname{supp}(f_\alpha) \subset V_i$, any point $x \in M$ has a neighborhood which meets only finitely many of the sets $\operatorname{supp}(f_\alpha)$, and $\sum_\alpha f_\alpha(x) = 1$ for all $x \in M$.

For a point $x \in M$, the *tangent space* T_xM to M in the point x is defined to be the space of all linear maps $\xi_x : C^\infty(M, \mathbb{R}) \to \mathbb{R}$ which are derivations at x, i.e. which satisfy the Leibniz rule $\xi_x(\phi\psi) = \xi_x(\phi)\psi(x) + \phi(x)\xi_x(\psi)$. These derivations form a vector space whose dimension equals the dimension of the manifold. Let $c : I \to M$ be a smoothly parametrized curve defined on an open interval $I \subset \mathbb{R}$. Then for $t \in I$ and $x := c(t)$, one obtains a tangent vector in T_xM by mapping $f \in C^\infty(M, \mathbb{R})$ to $(f \circ c)'(t) \in \mathbb{R}$. This tangent vector will be denoted by $c'(t)$. It turns out that any tangent vector can be obtained in this way.

If $f : M \to N$ is a smooth map between smooth manifolds, then for $x \in M$ and $\xi_x \in T_xM$, we define $T_xf \cdot \xi_x : C^\infty(N, \mathbb{R}) \to \mathbb{R}$ by $(T_xf \cdot \xi_x)(\phi) := \xi_x(\phi \circ f)$. One immediately verifies that $T_xf \cdot \xi_x \in T_{f(x)}N$ and this defines a linear map $T_xf : T_xM \to T_{f(x)}N$, the *tangent map* of f at x. The tangent map T_xf is bijective if and only if there is an open neighborhood U of x in M such that $f(U) \subset N$ is open and f restricts to a diffeomorphism from U to $f(U)$. In particular, $f : M \to N$ is a local diffeomorphism if and only if all tangent maps T_xf are linear isomorphisms. A smooth map f is called an *immersion* if all of its tangent maps are injective and a *submersion* if all of its tangent maps are surjective. The images of injective immersions are called *immersed submanifolds.*

A k–dimensional *submanifold* $N \subset M$ in a smooth manifold M of dimension n is a subset such that for each $x \in N$ there is a chart (U, u) for M with $x \in U$ such that $u(U \cap N) = u(U) \cap \mathbb{R}^k \subset \mathbb{R}^n$. Such a chart is called a *submanifold chart.* Restricting submanifold charts to N and their images to $\mathbb{R}^k$ one obtains an atlas for N, so N itself is a smooth manifold. The inclusion of N into M is not only an injective immersion but also an embedding, i.e. a homeomorphism onto its image. In view of this fact the name *embedded submanifold* is also used in this situation.

The union TM of all tangent spaces is called the *tangent bundle* of the manifold M. For each smooth map $f : M \to N$ we get the *tangent map* $Tf : TM \to TN$ of f by putting together the tangent maps at the individual points of M. The tangent bundle is endowed with the unique smooth structure such that the obvious projection $p : TM \to M$ and all tangent maps Tf become smooth maps. In this picture, the chain rule just states that T is a covariant functor on the category of smooth manifolds, i.e. $T(g \circ f) = Tg \circ Tf$. The individual tangent spaces T_xM are vector spaces and each point $x \in M$ has an open neighborhood U in M such that $p^{-1}(U) \subset TM$ is diffeomorphic to $U \times \mathbb{R}^m$ in a way compatible with the natural projections to U. Thus, TM is naturally a vector bundle over M and the tangent map to any smooth mapping is a vector bundle homomorphism; see 1.2.6 below for the terminology of bundles.

A *vector field* is a smooth section $\xi : M \to TM$ of the projection p, i.e. a smooth map such that $\xi(x) \in T_xM$ for all $x \in M$. The space of all smooth vector fields on M will be denoted by $\mathfrak{X}(M)$. It is a vector space and a module over $C^\infty(M, \mathbb{R})$ under pointwise operations. For $\xi \in \mathfrak{X}(M)$ and $f \in C^\infty(M, \mathbb{R})$, one defines a function $\xi \cdot f : M \to \mathbb{R}$ by $(\xi \cdot f)(x) = (\xi(x))(f)$. Smoothness of ξ easily implies that $\xi \cdot f \in C^\infty(M, \mathbb{R})$, while the fact that any $\xi(x)$ is a derivation at x immediately implies that $\xi \cdot (fg) = (\xi \cdot f)g + f(\xi \cdot g)$. Thus, ξ defines a derivation $C^\infty(M, \mathbb{R}) \to C^\infty(M, \mathbb{R})$ and one shows that this induces a bijection between $\mathfrak{X}(M)$ and the space of all derivations. Since the commutator of two derivations is again a derivation, we may associate to two vector fields $\xi, \eta \in \mathfrak{X}(M)$ a vector field $[\xi, \eta] \in \mathfrak{X}(M)$, which is called the *Lie bracket* of ξ and η and characterized by $[\xi, \eta] \cdot f = \xi \cdot (\eta \cdot f) - \eta \cdot (\xi \cdot f)$.

Any local diffeomorphism $f : M \to N$ induces a *pullback* operator $f^* : \mathfrak{X}(N) \to \mathfrak{X}(M)$ defined by $f^*\xi(x) = (T_xf)^{-1}(\xi(f(x)))$. This pullback is compatible with the Lie bracket, i.e. for $\xi, \eta \in \mathfrak{X}(N)$ we obtain $f^*[\xi, \eta] = [f^*\xi, f^*\eta]$. If P is another manifold and $g : N \to P$ another local diffeomorphism, then $(g \circ f)^* = f^* \circ g^* : \mathfrak{X}(P) \to \mathfrak{X}(M)$. For a diffeomorphism $f : M \to N$, there is also a covariant action on vector fields, i.e. an operator $f_* : \mathfrak{X}(M) \to \mathfrak{X}(N)$, which may be simply defined by $f_* := (f^{-1})^*$.

Given a vector field ξ on M, we may ask for integral curves, i.e. smooth curves $c : I \to M$ defined on open intervals in $\mathbb{R}$ such that $c'(t) = \xi(c(t))$ for all $t \in I$. The theorem on existence and uniqueness of solutions of ordinary differential equations implies that for each point $x \in M$ there are a unique maximal interval $I_x \subset \mathbb{R}$ and maximal integral curve $c_x : I_x \to M$ such that $c_x(0) = x$. A slightly finer analysis also using the smooth dependence of the solution on the initial conditions implies that the union of all I_x forms an open neighborhood $\mathcal{D}(\xi)$ of $\{0\} \times M$ in $\mathbb{R} \times M$, and $\mathrm{Fl}^\xi_t(x) := c_x(t)$ defines a smooth mapping $\mathrm{Fl}^\xi : \mathcal{D}(\xi) \to M$ called the *flow* of the vector field ξ. For $t, s \in \mathbb{R}$ and $x \in M$ one has the basic equation $\mathrm{Fl}^\xi_t(\mathrm{Fl}^\xi_s(x)) = \mathrm{Fl}^\xi_{t+s}(x)$, which is usually referred to as the flow property or the one–parameter group property. It is also known that under additional assumptions existence of one side of the equation implies existence of the other side. More precisely, if (s, x) and $(t, \mathrm{Fl}^\xi_s(x))$ lie in $\mathcal{D}(\xi)$, then $(t+s, x) \in \mathcal{D}(\xi)$ and the opposite implication also holds provided that t and s have the same sign. Finally, it turns out that for any point $x \in M$ and any $t_0 \in I_x$ there is a neighborhood U of x in M such that the restriction of Fl^ξ_t to U is a diffeomorphism onto its image for all $0 \le t \le t_0$. A vector field is called *complete* if $\mathcal{D}(\xi) = \mathbb{R} \times M$, i.e. if its flow is defined for all times. On a compact manifold, any vector field is automatically complete. Let us notice the obvious relation between flows and pullbacks, namely for a local diffeomorphism $f : M \to N$ and $\xi \in \mathfrak{X}(N)$ we have $\mathrm{Fl}^\xi_t \circ f = f \circ \mathrm{Fl}^{f^*\xi}_t$.

The dual bundle to $TM \to M$ is the *cotangent bundle* $T^*M \to M$, so for $x \in M$ the *cotangent space* T^*_xM is the space of all linear maps $T_xM \to \mathbb{R}$. In contrast to the tangent functor, T^* only has functorial properties for local diffeomorphism, and it can be viewed either as a contravariant or as a covariant functor. The smooth sections of the cotangent bundle are called *one–forms*. We write $\Omega^1(M) = \Omega^1(M, \mathbb{R})$ for the space of all smooth real one–forms on M. The pointwise operations make $\Omega^1(M)$ into a vector space and a module over $C^\infty(M, \mathbb{R})$ and for any smooth map

$f : M \to N$ we obtain the *pullback* $f^* : \Omega^1(N) \to \Omega^1(M)$ defined by

$$(f^*\phi)(x)(\xi_x) = \phi(f(x))(T_xf \cdot \xi_x).$$

A simple example of a one–form is the differential df of a real–valued function f, defined by $df(x)(\xi_x) := \xi_x(f)$. One may easily generalize this and consider one–forms with values in a (finite–dimensional) vector space V. These are smooth maps ϕ which associate to each point $x \in M$ a linear map $T_xM \to V$. The space of all such forms is denoted by $\Omega^1(M, V)$. For $f \in C^\infty(M, V)$ one obtains as above $df \in \Omega^1(M, V)$.

The antisymmetric k–linear maps $\alpha : \Lambda^k T_xM \to \mathbb{R}$ are the elements of the kth exterior power $\Lambda^k T^*M$ of the cotangent bundle and the sections of this bundle are called *k–forms* on M. The space of all k–forms on M, which again is a vector space and $C^\infty(M, \mathbb{R})$–module under pointwise operations, is denoted by $\Omega^k(M)$. By convention, $\Omega^0(M) = C^\infty(M, \mathbb{R})$. Differential forms can be pulled back along arbitrary smooth functions, by defining

$$(f^*\alpha)(x)(\xi_1, \dots, \xi_k) = \alpha(f(x))(T_xf \cdot \xi_1, \dots, T_xf \cdot \xi_k).$$

In particular, for $h \in \Omega^0(M) = C^\infty(M, \mathbb{R})$ one has $f^*h = h \circ f$. Inserting the values of vector fields $\xi_1, \dots, \xi_k$ into a k–form α one obtains a smooth function $\alpha(\xi_1, \dots, \xi_k) \in C^\infty(M, \mathbb{R})$, so α gives rise to a k–linear, alternating map $\mathfrak{X}(M)^k \to C^\infty(M, \mathbb{R})$. One shows that such a mapping is induced by a k–form if and only if it is linear over $C^\infty(M, \mathbb{R})$ in one (and thus any) variable.

The differential of functions, $d : C^\infty(M, \mathbb{R}) \to \Omega^1(M)$, is a special case of the *exterior derivative*. In general, the exterior derivative $d : \Omega^k(M) \to \Omega^{k+1}(M)$ is given by the formula

$$\begin{aligned} d\omega(\xi_0, \dots, \xi_k) = &\sum_{i=0}^{k} (-1)^i \xi_i \cdot (\omega(\xi_0, \dots, \widehat{\xi_i}, \dots, \xi_k)) \\ &+ \sum_{i<j} (-1)^{i+j} \omega([\xi_i, \xi_j], \xi_0, \dots, \widehat{\xi_i}, \dots, \widehat{\xi_j} \dots, \xi_k) \end{aligned}$$

for all $\xi_i \in \mathfrak{X}(M)$, where the hats denote omission of an argument. The same formula applies for differential forms with values in any finite–dimensional vector space V.

The exterior derivative d is the only linear differential operator which is invariantly defined on all manifolds; see [**KMS**, Theorem 34.2]. Here invariance means commuting with the action of local diffeomorphisms, i.e. $\phi^*(d\omega) = d(\phi^*\omega)$. One of the goals of this book is to develop general tools for the study of such basic operators in the realm of more specific geometric structures on manifolds.

1.2.2. Distributions and foliations. A *distribution* $\mathcal{D}$ on a manifold M is a subset $\mathcal{D} \subset TM$ such that for each $x \in M$ the subset $\mathcal{D}_x = \mathcal{D} \cap T_xM$ is a vector subspace in T_xM. By elementary linear algebra, each distribution can be defined as the kernel of a (not necessarily continuous) one–form ω with values in a suitable vector space V. The dimension of V is at least $\dim M - \max_{x \in M}\{\dim \mathcal{D}_x\}$. The distribution $\mathcal{D}$ is said to be of *constant rank* k if $\dim \mathcal{D}_x = k$ is constant, and $\mathcal{D}$ is *smooth* if it can be defined by a smooth form ω. Equivalently, locally there must be smooth vector fields which span the distribution in each point. A distribution is called *regular*, if it is of constant rank and is smooth. An *integral manifold* N of a distribution $\mathcal{D}$ is an immersed submanifold such that $T_xN \subset \mathcal{D}_x$ for all $x \in N$. A *maximal integral submanifold* is an integral manifold N whose dimension is equal to the rank of $\mathcal{D}$ in all points $x \in N$. A distribution $\mathcal{D}$ is called *integrable* if there is a maximal integral submanifold through each point $x \in M$.

Each vector field ξ on M spans a one–dimensional distribution whose maximal integral submanifolds are the (unparametrized) flow lines $\mathrm{Fl}_t^\xi(x)$. In general, there are no maximal integral submanifolds and the obstruction to their existence is given by the Lie bracket of vector fields lying in $\mathcal{D}$. The distribution $\mathcal{D}$ is called *involutive* if for all vector fields ξ and $\eta \in \mathfrak{X}(M)$ with $\xi(x) \in \mathcal{D}$, $\eta(x) \in \mathcal{D}$ for all $x \in M$, also $[\xi, \eta](x) \in \mathcal{D}$ for all $x \in M$. The following theorem (called the *Frobenius theorem*) appears in all standard textbooks on differential geometry. For a general version for distributions of nonconstant rank see e.g. [**KMS**, section 3].

THEOREM 1.2.2. *A regular distribution $\mathcal{D} \subset TM$ is integrable if and only if it is involutive.*

Given an integrable regular distribution of rank k on M, the maximal integral submanifolds decompose M into a union of k–dimensional immersed submanifolds. This is called the *regular foliation* on M defined by the distribution and the maximal integral submanifolds are referred to as the *leaves* of this foliation. On the other hand, smooth foliations define the associated distributions (defined by the tangent spaces to the leaves) which are integrable by construction.

An immediate and very useful consequence of the last Theorem and the above formula for the exterior differential reformulates the involutivity of $\mathcal{D}$ in terms of the defining one–form ω.

COROLLARY 1.2.2. *Let $\omega \in \Omega^1(M, V)$ be a smooth V–valued one–form and assume that the dimension of $\ker \omega(x)$ is constant for all $x \in M$. Then the distribution $\mathcal{D} = \ker \omega$ is integrable if and only if $d\omega(x)(X, Y) = 0$ for all X, $Y \in \mathcal{D}_x$.*

Equivalently, the condition of the corollary can be stated as follows: Representing $\mathcal{D}$ as the intersection of the kernels of $\dim V$ many one–forms $\omega_i \in \Omega^1(M)$, the exterior derivatives $d\omega_i$ belong to the ideal in $\Omega^*(M)$ generated by the forms ω_i.

1.2.3. Lie groups and their Lie algebras. A *Lie group* G is a smooth manifold endowed with a smooth mapping $\mu : G \times G \to G$, the *multiplication*, which defines a group structure on G. Using the implicit function theorem one then shows that the inversion mapping $\nu : G \to G$ is smooth, too. Given an element $g \in G$, we define the *left translation* $\lambda_g : G \to G$ by $\lambda_g(h) = \mu(g, h) = gh$, and the *right translation* $\rho^g : G \to G$ by $\rho^g(h) = hg$. Both λ_g and ρ^g are diffeomorphisms of G with inverses $\lambda_{g^{-1}}$ and $\rho^{g^{-1}}$, respectively. Further, one clearly has $\lambda_g \circ \lambda_h = \lambda_{gh}$ and $\rho^g \circ \rho^h = \rho^{hg}$, which also explains the use of subscripts and superscripts.

Let G be a Lie group and let $\xi \in \mathfrak{X}(G)$ be a smooth vector field on G. Then ξ is called *left invariant*, if and only if $(\lambda_g)^*\xi = \xi$ for all $g \in G$, or equivalently $\xi(gh) = T_h\lambda_g \cdot \xi(h)$ for all $g, h \in G$. The latter equation shows that any left invariant vector field ξ is uniquely determined by its value $\xi(e) \in T_eG$ in the unit element e of G. Conversely, it is easy to see that any $X \in T_eG$ extends to a left invariant vector field L_X on G. Consequently, there is a linear isomorphism between the space $\mathfrak{X}_L(G)$ of left invariant vector fields on G and the tangent space T_eG of G at the unit element. Since the pullback along a diffeomorphism is compatible with the Lie bracket of vector fields, the subspace $\mathfrak{X}_L(G) \subset \mathfrak{X}(G)$ is a Lie subalgebra. Via the linear isomorphism, this gives rise to a Lie bracket on the tangent space T_eG, which is explicitly given by $[X,Y] = [L_X, L_Y](e)$ for $X, Y \in T_eG$. The space T_eG together with this Lie bracket is called the *Lie algebra* $\mathfrak{g}$ of the Lie group G.

Similarly, as for left invariant vector fields, one has the Lie subalgebra $\mathfrak{X}_R(G)$ of *right invariant* vector fields on G. Any right invariant vector field on G is uniquely determined by its value in e, and any $X \in \mathfrak{g}$ extends uniquely to $R_X \in \mathfrak{X}_R(G)$. It is easy to see that $R_X = \nu^*(L_{-X})$, where ν is the inversion on G, which, in particular, implies that $[R_X, R_Y] = R_{-[X,Y]}$. Another basic result is that right invariant vector fields commute with left invariant vector fields, i.e. $[L_X, R_Y] = 0$ for all $X, Y \in \mathfrak{g}$.

It follows from the construction of the Lie algebra of a Lie group that if G and H are Lie groups with Lie algebras $\mathfrak{g}$ and $\mathfrak{h}$, and $\phi : G \to H$ is a *homomorphism*, i.e. a smooth map compatible with the group structures, then the tangent map $\phi' = T_e\phi : \mathfrak{g} \to \mathfrak{h}$ is a homomorphism of Lie algebras. For example, if G is a Lie group and $g \in G$ is any element, then the conjugation $h \mapsto ghg^{-1}$ by g defines an automorphism of G, so the tangent map at zero defines an automorphism $\mathrm{Ad}(g)$ of the Lie algebra $\mathfrak{g}$. This is called the *adjoint action* of $g \in G$ on $\mathfrak{g}$.

The simplest example of a Lie group is the group $GL(n, \mathbb{K})$ of linear automorphisms of $\mathbb{K}^n$, where $\mathbb{K}$ is $\mathbb{R}$ or $\mathbb{C}$, which may be also viewed as the group of invertible $n \times n$–matrices with entries from $\mathbb{K}$. This is a smooth manifold, since it is an open subset of the vector space $M_n(\mathbb{K})$ of all $n \times n$–matrices with entries from $\mathbb{K}$, and clearly matrix multiplication is smooth. From this it follows that the Lie algebra $\mathfrak{gl}(n, \mathbb{K})$ of $GL(n, \mathbb{K})$ equals $M_n(\mathbb{K})$, and one easily shows that the adjoint representation is given by the conjugation of matrices, while the Lie bracket is given by the commutator of matrices.

If G is an arbitrary Lie group, then a real or complex (finite–dimensional) *representation* of G is a homomorphism ϕ from G to some $GL(n, \mathbb{R})$, respectively $GL(n, \mathbb{C})$. For such a representation ϕ, the tangent map at $e \in G$ is a homomorphism $\phi' : \mathfrak{g} \to \mathfrak{gl}(n, \mathbb{K})$ of Lie algebras, i.e. a representation of $\mathfrak{g}$, called the *infinitesimal representation* corresponding to ϕ. Equivalently, one may describe a representation of G as a smooth map $\hat{\phi} : G \times \mathbb{K}^n \to \mathbb{K}^n$ which is linear in the second argument and has the property that $\hat{\phi}(gh, v) = \hat{\phi}(g, \hat{\phi}(h, v))$ for all $g, h \in G$ and $v \in \mathbb{K}^n$. Slightly more generally, one may consider representations $G \to GL(V)$ for any finite–dimensional real or complex vector space V. The corresponding infinitesimal representation then has values in the space $L(V, V)$ of all linear mappings. For both group and Lie algebra representations, if there is no risk of confusion, we will often omit the name of the representation and simply use the notation $(g, v) \mapsto g \cdot v$ or gv. The adjoint action associates to any element $g \in G$ an automorphism $\mathrm{Ad}(g) : \mathfrak{g} \to \mathfrak{g}$, which, in particular, is a linear isomorphism, so this defines a map $\mathrm{Ad} : G \to GL(\mathfrak{g})$. Since mapping g to the conjugation by g is a homomorphism, Ad is a group homomorphism and it is easy to see that it is smooth, so it defines a representation of G, the *adjoint representation*. The infinitesimal representation $\mathrm{ad} = \mathrm{Ad}' : \mathfrak{g} \to L(\mathfrak{g}, \mathfrak{g})$ turns out to be given by $\mathrm{ad}(X)(Y) = [X, Y]$ for $X, Y \in \mathfrak{g}$.

For each element X in the Lie algebra $\mathfrak{g}$ of a Lie group G, we have the corresponding left invariant vector field L_X on G. The invariance of L_X easily implies that the vector field L_X is complete, i.e. that its flow $\mathrm{Fl}_t^{L_X}$ is defined for all times t. In particular, we can define the *exponential mapping* $\exp : \mathfrak{g} \to G$ by

$$\exp(X) := \mathrm{Fl}_1^{L_X}(e).$$

One readily verifies that $\exp : \mathfrak{g} \to G$ is a smooth map, whose tangent map at $0 \in \mathfrak{g}$ is the identity, so exp restricts to a diffeomorphism from an open neighborhood of $0 \in \mathfrak{g}$ to an open neighborhood of $e \in G$. Further, the flows of left invariant vector fields are given by $\mathrm{Fl}_t^{L_X}(g) = g\exp(tX)$ and for right invariant vector fields one gets $\mathrm{Fl}_t^{R_X}(g) = \exp(tX)g$. In particular, the flows through e of L_X and R_X coincide. By the flow property, this flow defines a *one–parameter subgroup* of G, i.e. a smooth homomorphism from the additive group $\mathbb{R}$ to G. Conversely, any such one–parameter subgroup of G is determined by its derivative at e, so $t \mapsto \exp(tX)$ is the unique curve $\alpha : \mathbb{R} \to G$ such that $\alpha(t+s) = \alpha(t)\alpha(s)$ and such that $\alpha'(0) = X$. For the group $GL(n, \mathbb{K})$ of invertible $n \times n$–matrices, the exponential mapping is given by the usual exponential of matrices, i.e. $\exp(A) = e^A = \sum_{n=0}^{\infty} \frac{1}{n!}A^n$.

If $\phi : G \to H$ is a homomorphism of Lie groups with Lie algebras $\mathfrak{g}$ and $\mathfrak{h}$ and exponential mappings $\exp^G$ and $\exp^H$, then the description of the exponential map as the solution of an ordinary differential equation easily implies that $\phi \circ \exp^G = \exp^H \circ \phi'$. In particular, the values of ϕ on the image of $\exp^G$ are completely determined by the Lie algebra homomorphism ϕ'. From the fact that the image of $\exp^G$ contains an open neighborhood of the unit $e \in G$, one next concludes that the subgroup generated by this image is exactly the connected component G_0 of e in G, so the restriction of ϕ to G_0 is determined by ϕ'. In particular, if G is connected, then any homomorphism from G to some Lie group is determined by its tangent map at e. We may apply this to representations of G. For any representation $\phi : G \to GL(n, \mathbb{K})$ and any $X \in \mathfrak{g}$, we get

$$\phi(\exp(X)) = e^{\phi'(X)} = \sum_{n=0}^{\infty} \frac{1}{n!}\phi'(X)^n.$$

In particular, $\phi'(X) = \frac{d}{dt}|_0 \phi(\exp(tX))$, and if G is connected, then any representation is determined by the corresponding infinitesimal representation.

1.2.4. The Maurer–Cartan form. The left invariant vector fields lead to a trivialization of the tangent bundle of any Lie group. More precisely, the mapping $G \times \mathfrak{g} \to TG$ which is given by $(g, X) \mapsto L_X(g)$ is an isomorphism of vector bundles. The inverse of this isomorphism can be conveniently encoded as a one–form $\omega \in \Omega^1(G, \mathfrak{g})$ on G with values in the Lie algebra $\mathfrak{g}$, which is defined by $\omega(g)(\xi) := T_g\lambda_{g^{-1}} \cdot \xi$. This one–form is called the (left) *Maurer–Cartan form* on G. From the definition of ω it is obvious that $\omega(L_X) = X$, $\lambda_g^*\omega = \omega$, and for each $g \in G$ the map $\omega(g) : T_gG \to \mathfrak{g}$ is a linear isomorphism. Moreover, by definition, $(\rho^g)^*\omega(h)(\xi) = \omega(hg)(T\rho^g \cdot \xi) = T\lambda_{g^{-1}h^{-1}}T\rho^g \cdot \xi$. Since left multiplications commute with right multiplications and $\lambda_{g^{-1}} \circ \rho^g$ is the conjugation with g^{-1}, we conclude that this equals $\mathrm{Ad}(g^{-1})(\omega(h)(\xi))$, and thus we get $(\rho^g)^*\omega = \mathrm{Ad}(g^{-1}) \circ \omega$. Finally, consider the exterior derivative $d\omega$ of the Maurer–Cartan form. Since ω is constant on left invariant vector fields, the standard formula for the exterior derivative (see 1.2.1) implies $d\omega(L_X, L_Y) = -\omega([L_X, L_Y])$ for $X, Y \in \mathfrak{g}$, which equals $-[X, Y]$ by definition of the Lie bracket on $\mathfrak{g}$. Since in each point the values of left invariant vector fields exhaust the whole tangent space, this implies the *Maurer–Cartan equation* $0 = d\omega(\xi, \eta) + [\omega(\xi), \omega(\eta)]$ for all ξ, $\eta \in TG$.

The Maurer–Cartan form leads to a notion of differentiation of functions with values in a Lie group. Indeed, if G is a Lie group with Lie algebra $\mathfrak{g}$, M is an arbitrary smooth manifold, and $f : M \to G$ is an arbitrary smooth map, then we define the *left logarithmic derivative* $\delta f : TM \to \mathfrak{g}$ by $\delta f(\xi_x) = \omega(f(x))(T_x f \cdot \xi_x)$. This means that δf is obtained by composing the tangent map $Tf : TM \to TG$ with the trivialization of TG provided by ω and projecting out the second component. Another way to express this is $\delta f(x) = f^*\omega(x) = T\lambda_{f(x)^{-1}} \circ T_x f$ and, in particular, the Maurer–Cartan form itself equals $\omega = \delta\,\mathrm{id}_G$.

Let us look at some examples. If G is the additive real line $G = (\mathbb{R}, +)$ with standard coordinate x, then $\omega_G = dx \in \Omega^1(G, \mathbb{R})$ and the left logarithmic derivative is just the usual differential. The isomorphic example with the multiplicative positive real line $G = (\mathbb{R}^+, \cdot)$ leads to $\omega = \frac{1}{x}dx$ and the usual logarithmic derivative of real functions, $f \mapsto \frac{f'}{f}dx$. For the general linear group G we easily compute $\omega_G \in \Omega^1(GL(n,\mathbb{R}), \mathfrak{gl}(n,\mathbb{R}))$ in the usual matrix component coordinates $g = (x_{ij}) \in GL(n,\mathbb{R})$, $\omega_G(g) = g^{-1}(dx_{ij})$. For example, in dimension two,

$$\omega_{GL(2,R)} = \begin{pmatrix} x_{22}/\det(g) & -x_{12}/\det(g) \\ -x_{21}/\det(g) & x_{11}/\det(g) \end{pmatrix} \begin{pmatrix} dx_{11} & dx_{12} \\ dx_{21} & dx_{22} \end{pmatrix}.$$

The Maurer–Cartan form provides the infinitesimal information on the multiplication while the Maurer–Cartan equation gives the only local obstruction to its integrability. The explicit local formulation is contained in the following theorem; see [**Sh97**, Chapter 3] for more details.

THEOREM 1.2.4. *Let G be a Lie group with Lie algebra $\mathfrak{g}$ and Maurer–Cartan form ω_G. Let M be a smooth manifold, and let $\omega \in \Omega^1(M, \mathfrak{g})$ be a $\mathfrak{g}$–valued one–form. Then for any $x \in M$ there exist an open neighborhood U of x in M and a function $f : U \to G$ such that $\delta f = f^*\omega_G = \omega$ if and only if ω satisfies $d\omega(\xi, \eta) + [\omega(\xi), \omega(\eta)] = 0$ for all $\xi, \eta \in \mathfrak{X}(M)$.*

If M is connected and $f_1, f_2 : M \to G$ have the property that $\delta f_1 = \delta f_2$, then there is a unique element $c \in G$ (integration constant) such that $f_2(x) = c \cdot f_1(x)$ for all $x \in M$.

PROOF. A straightforward computation establishes the formulae for the actions of the multiplication μ and inversion ν on the Maurer–Cartan form ω_G; cf. [**Sh97**, page 113]. For $g, h \in G$, $\xi \in T_g G$ and $\eta \in T_h G$ one has

$$(\mu^*\omega_G)(\xi, \eta) = \mathrm{Ad}(h^{-1})(\omega_G(\xi)) + \omega_G(\eta),$$
$$(\nu^*\omega_G)(\xi) = -\mathrm{Ad}(g)(\omega_G(\xi)).$$

We start by proving the last statement of the theorem. Consider two functions $f_1, f_2 : M \to G$ such that $f_1^*\omega_G = f_2^*\omega_G$ and define $h : M \to G$ by $h(x) = f_2(x)f_1(x)^{-1}$. We have to show that h is constant, for which it suffices to show that $h^*\omega_G = 0$, since this implies that $\omega_G \circ Th$ and thus Th is identically zero. But by definition, we have $h = \mu \circ (\mathrm{id}, \nu) \circ (f_2, f_1) \circ \Delta$, where $\Delta(x) = (x, x)$. Using the above formulae we thus compute that for $\xi \in T_x M$ we have

$$\begin{aligned}(h^*\omega_G)(\xi) &= (\mu^*\omega_G)(T_x f_2 \cdot \xi, T_x(\nu \circ f_1) \cdot \xi) \\ &= \mathrm{Ad}(f_1(x))(\omega_G(T_x f_2 \cdot \xi)) + (\nu^*\omega_G)(T_x f_1 \cdot \xi) \\ &= \mathrm{Ad}(f_1(x))(\delta f_2(x)(\xi) - \delta f_1(x)(\xi)) = 0.\end{aligned}$$

Concerning existence, $d(f^*\omega_G)(\xi, \eta) + [f^*\omega_G(\xi), f^*\omega_G(\eta)]$ clearly vanishes because of the Maurer–Cartan equation. Thus, it suffices to prove that for $\omega \in \Omega^1(M, \mathfrak{g})$ satisfying the equation we can find a function $f : M \to G$ such that $\omega = \delta f$. To do this, we (locally) construct the graph of f as a leaf of an integrable distribution on $M \times G$. Consider $\Omega = \pi_M^*\omega - \pi_G^*\omega_G$, where $\pi_M : M \times G \to M$ and $\pi_G : M \times G \to G$ are the natural projections. Identifying $T(M \times G)$ with $TM \times TG$, the kernel of Ω is given by the set of all (ξ, η) such that $\omega(\xi) = \omega_G(\eta)$. Since ω_G restricts to a linear isomorphism in each tangent space, there is a unique solution η for this equation for any chosen tangent vector ξ, so the distribution $\ker(\Omega)$ is regular, its rank equals the dimension of M, and $T\pi_M$ restricts to a linear isomorphism on each fiber of $\ker(\Omega)$. To check involutivity, we note that by construction

$$\begin{aligned} d\Omega((\xi_1, \eta_1), (\xi_2, \eta_2)) &= d\omega(\xi_1, \xi_2) - d\omega_G(\eta_1, \eta_2) \\ &= -[\omega(\xi_1), \omega(\xi_2)] + [\omega_G(\eta_1), \omega_G(\eta_2)], \end{aligned}$$

and this obviously vanishes if both (ξ_i, η_i) lie in the kernel of Ω.

By Corollary 1.2.2 this implies integrability of the distribution $\ker(\Omega)$. Given $x \in M$ and $g \in G$, there is a submanifold $N \subset M \times G$ containing (x, g) whose tangent spaces are the fibers of $\ker(\Omega)$. We have observed above that $T\pi_M$ restricts to a linear isomorphism on each of these spaces, so $\pi_M : N \to M$ is a local diffeomorphism. Hence, we can find a neighborhood U of $x \in M$ and a local inverse $j : U \to N$ of this projection. Defining $f := \pi_G \circ j$, we obtain a smooth function $f : U \to G$, and for $\xi \in T_xM$ we get $T_xj \cdot \xi = (\xi, \omega_G^{-1}(\omega(\xi)))$, and thus $\omega_G(T_xf \cdot \xi) = \omega(\xi)$, which means $\omega = f^*\omega_G$. □

If we add the requirement that $\omega : T_xM \to \mathfrak{g}$ is a linear isomorphism, then the theorem implies that there is a unique group structure locally around $x \in M$ which is locally isomorphic to that of G via f, and has Maurer–Cartan form $\omega = f^*\omega_G$. A global version of this theorem works for connected and simply connected manifolds M, or under suitable conditions on the so–called monodromy representation; see again [**Sh97**, Chapter 3] for more details.

From this theorem we may easily conclude that Lie algebra homomorphisms integrate to local group homomorphisms. If G and H are Lie groups with Lie algebras $\mathfrak{g}$ and $\mathfrak{h}$ and $\phi : \mathfrak{g} \to \mathfrak{h}$ is a Lie algebra homomorphism, then consider $\omega := \phi \circ \omega_G \in \Omega^1(G, \mathfrak{h})$. Clearly, we get $d\omega = \phi \circ d\omega_G$, which together with the Maurer–Cartan equation for ω_G and the fact that ϕ is a homomorphism immediately implies that $d\omega(\xi, \eta) + [\omega(\xi), \omega(\eta)] = 0$ for arbitrary tangent vectors ξ and η. By the theorem, we find an open neighborhood U of e in G and a smooth function $f : U \to H$ such that $f(e) = e$ and $\omega = f^*\omega_H$. But then $\omega_H(T_ef \cdot X) = \omega(X) = \phi(X)$, so $\phi = T_ef$. Moreover, we claim that f is a local group homomorphism. For $g_0 \in U$ consider the function $f \circ \lambda_{g_0}$, which is defined locally around e. One immediately verifies that this function also pulls back ω_H to $\omega = \phi \circ \omega_G$, which implies that it coincides with f up to a left multiplication by a fixed element of H. Looking at the values in e, we see that we must have $f = \lambda_{f(g_0)^{-1}} \circ f \circ \lambda_{g_0}$, which implies that $f(g_0g) = f(g_0)f(g)$ if g and g_0g lie in U. The global version of the theorem, in particular, implies that if G is simply connected, then there is a globally defined homomorphism $f : G \to H$ such that $T_ef = \phi$.

Note that in the special case that M is an interval in $\mathbb{R}$, all two–forms on M are automatically zero, so the Maurer–Cartan equation is satisfied for any $\omega \in \Omega^1(M, \mathfrak{g})$. In this case it is also straightforward to deduce global existence of f from local existence. In the special case $G = (\mathbb{R}, +)$, one obtains the theorem on existence of an antiderivative of any smooth function and uniqueness up to an additive constant. Therefore, the whole theorem is referred to as the fundamental theorem of calculus in [**Sh97**].

1.2.5. Lie subgroups, homogeneous spaces, and actions. An embedded *Lie subgroup* H of a Lie group G is a submanifold, which at the same time is a subgroup. We shall omit the adjective embedded in the sequel. A Lie subgroup is automatically a closed subset of G and, conversely, it can be shown (see [**KMS**, Theorem 5.5]) that any closed subgroup of a Lie group G is an (embedded) Lie subgroup. For a Lie subgroup $H \subset G$, the tangent space $\mathfrak{h} = T_eH \subset T_eG = \mathfrak{g}$ is a Lie subalgebra, i.e. the Lie bracket of two elements of $\mathfrak{h}$ again lies in $\mathfrak{h}$. A connected Lie subgroup $H \subset G$ is normal if and only if the Lie algebra $\mathfrak{h} \subset \mathfrak{g}$ is an *ideal*, i.e. $[X, Y] \in \mathfrak{h}$ for any $X \in \mathfrak{h}$ and $Y \in \mathfrak{g}$.

Given a Lie group G with Lie algebra $\mathfrak{g}$, there may exist Lie subalgebras $\mathfrak{h} \subset \mathfrak{g}$ for which there is no Lie subgroup $H \subset G$ having $\mathfrak{h}$ as the tangent space at the identity. This can be seen from the case of the real torus $\mathbb{T}^2 = S^1 \times S^1$. The Lie algebra of this is $\mathbb{R}^2$ with the trivial Lie bracket, so a subalgebra is just a linear subspace. Now if one takes a line of irrational slope, then it is easy to see that any subgroup $H \subset G$, which contains a small submanifold around the unit element that is tangent to the given line, must be dense in G, so it cannot be a Lie subgroup. This generalizes to one–parametric subgroups $\exp tX : \mathbb{R} \to G$, for $X \in \mathfrak{g}$ in general Lie groups. Either the image is topologically a circle or it is a line. A circle is a (embedded) subgroup, while lines are only immersed in general.

To avoid this problem, one defines a *virtual Lie subgroup* of a Lie group G to be a Lie group H together with a homomorphism $i : H \to G$ which is an injective immersion. The derivative $i' : \mathfrak{h} \to \mathfrak{g}$ is then the inclusion of a Lie subalgebra. Using the global version of the Frobenius theorem, one shows that for any Lie subalgebra $\mathfrak{h} \leq \mathfrak{g}$, there is a virtual Lie subgroup $i : H \to G$ with Lie algebra $\mathfrak{h}$; see [**KMS**, Theorem 5.2]. The latter result can also be used to prove that for any finite–dimensional Lie algebra $\mathfrak{g}$ there is a Lie group G with Lie algebra $\mathfrak{g}$. For this one uses the theorem of Ado that asserts that $\mathfrak{g}$ admits a finite–dimensional faithful representation and hence is isomorphic to a Lie subalgebra of $\mathfrak{gl}(N, \mathbb{R})$ for some N. For a short proof of the Ado theorem see [**Ne03**].

For any subgroup H in a Lie group G, one may consider the set G/H of cosets gH with $g \in G$. In order that the topology of G/H induced by the canonical projection $p : G \to G/H$ is Hausdorff, it is necessary that H is a closed subgroup, so from above we know that H is even a Lie subgroup of G. In this case, one shows that G/H is a smooth manifold and the structure is uniquely determined by requiring that p is a smooth surjective submersion; see [**KMS**, 5.11]. In particular, for any manifold M smooth maps from G/H to M are exactly the smooth maps from G to M which are constant on each coset.

Lie groups appear often as the symmetry groups on some manifolds, i.e. as groups of transformations on these manifolds. More explicitly, a *left action* of a Lie group G on a manifold M is a smooth mapping $\ell : G \times M \to M$, such that $\ell(e, x) = x$ and $\ell(g, \ell(h, x)) = \ell(gh, x)$. If there is no risk of confusion, we simply write $(g, x) \mapsto g \cdot x$ for the action, so that the defining properties become $e \cdot x = x$ and $g \cdot (h \cdot x) = (gh) \cdot x$. Otherwise put, the action associates to $g \in G$ a smooth map $\ell_g : M \to M$, defined by $\ell_g(x) = \ell(g, x)$ such that $\ell_e = \mathrm{id}_M$ and $\ell_g \circ \ell_h = \ell_{gh}$. In particular, each ℓ_g is a diffeomorphism with inverse $\ell_{g^{-1}}$, so we can view the action as a homomorphism from G into the group of diffeomorphisms of M. In the special case of a finite–dimensional vector space V, a representation of G on V as defined in 1.2.3 is exactly a left action such that all the maps ℓ_g are linear.

Similarly, a right action is a smooth map $r : M \times G \to M$ such that $r(x,e) = x$ and $r(r(x,g),h) = r(x,gh)$, or writing the action as a dot, $x \cdot e = x$ and $x \cdot (gh) = (x \cdot g) \cdot h$. As above, this can be interpreted as associating to any $g \in G$ a diffeomorphism r^g of M such that $r^{gh} = r^h \circ r^g$, so we get an anti–homomorphism from G to the diffeomorphism group.

Given a left (or right) action of G on M and a point $x \in M$ there are two canonical objects associated to x. First, there is the *orbit* $G \cdot x = \{g \cdot x : g \in G\}$ through x, and second there is the *isotropy subgroup* $G_x = \{g \in G : g \cdot x = x\}$, also called the *stabilizer* of x. By definition, G_x is a closed subgroup and thus a Lie subgroup in G. The map $\ell^x : G \to M$, $\ell^x(g) := \ell(g,x)$ then induces a smooth bijection $G/G_x \to G \cdot x$, so any orbit looks like a coset space. Clearly, two orbits are either disjoint or equal, so M is the disjoint union of all G–orbits. The set of all orbits is denoted by M/G. Note that for $y = g \cdot x \in G \cdot x$, one has $G_y = \{ghg^{-1} : h \in G_x\}$, so along an orbit all isotropy subgroups are conjugate.

An action is called *transitive* if there is just one orbit, or equivalently if for arbitrary elements $x, y \in M$ there is an element $g \in G$ such that $g \cdot x = y$. An action is called *effective* if only the neutral element $e \in G$ acts as the identity of M, or equivalently if the intersection of all isotropy subgroups consists of e only. If all the isotropy subgroups are trivial, then the action is called *free*.

The coset spaces G/H are related to actions in two ways. First, right multiplication by elements of $H \subset G$ defines a free right action of H on G, and by definition G/H is exactly the space of orbits for this action. On the other hand, the left multiplication of G on itself induces a smooth left action of G on G/H defined by $g \cdot (g'H) = (gg')H$. Clearly, this action is transitive. In view of this, the coset space G/H is called the *homogeneous space* of G corresponding to the subgroup H.

Each left action ℓ of a Lie group G on a manifold M defines the so–called *fundamental vector fields* by the formula $\zeta_X(x) = \frac{d}{dt}|_0 \ell_{\exp tX}(x)$, for all $x \in M$ and X in the Lie algebra $\mathfrak{g}$ of G. Similarly, we obtain fundamental vector fields for right actions. These fundamental vector fields provide infinitesimal versions of the Lie group actions. In particular, the left–invariant vector fields on the Lie group G itself are obtained as the fundamental vector fields with respect to the right multiplication by elements in G. The fundamental field mapping for right actions yields a Lie algebra homomorphism $\mathfrak{g} \to \mathfrak{X}(M)$, while $\zeta_{[X,Y]} = -[\zeta_X, \zeta_Y]$ for a left action. A simple computation yields $T_x\ell_g \cdot \zeta_X(x) = \zeta_{\mathrm{Ad}(g)X}(g \cdot x)$, i.e. $\ell_g^*\zeta_X = \zeta_{\mathrm{Ad}(g^{-1})\cdot X}$, for left actions. For each point $x \in M$, we also define the isotropy subalgebra $\mathfrak{g}_x \subset \mathfrak{g}$ of elements X with $\zeta_X(x) = 0$. This isotropy Lie algebra by construction is exactly the Lie algebra of the isotropy subgroup G_x.

1.2.6. Fiber bundles, vector bundles and principal bundles. A *fibered manifold* is a surjective submersion $p : Y \to M$, a trivial fibered manifold with fiber S is $\pi_M : M \times S \to M$. A *section* of a fibered manifold $p : Y \to M$ is a smooth map $\sigma : M \to Y$ such that $p \circ \sigma = \mathrm{id}_M$. The space of all smooth sections is denoted by $\Gamma(Y)$. *Fibered morphisms* $\phi : Y \to Y'$ are smooth mappings between fibered manifolds which cover a smooth mapping $\phi_0 : M \to M'$ between the base manifolds, i.e. $p' \circ \phi = \phi_0 \circ p$. A *fiber bundle* with base M and standard fiber S is a fibered manifold $Y \to M$ which is locally isomorphic (via fibered morphisms) to a trivial fibered manifold. Otherwise put, one must have a fiber bundle atlas $\{(U_\alpha, \phi_\alpha)\}$, i.e. an open covering $\{U_\alpha\}$ of M and diffeomorphisms

$\phi_\alpha : p^{-1}(U_\alpha) \to U_\alpha \times S$ which are fibered morphisms. Each of the pairs (U_α, ϕ_α) is called a fiber bundle chart. For two fiber bundle charts (U_α, ϕ_α) and (U_β, ϕ_β) such that $U_{\alpha\beta} := U_\alpha \cap U_\beta \neq \emptyset$, one has the *transition function* $\phi_{\alpha\beta} : U_{\alpha\beta} \times S \to S$ defined by $\phi_\alpha(\phi_\beta^{-1}(x, y)) = (x, \phi_{\alpha\beta}(x, y))$.

Assume next, that we have given a fiber bundle $p : Y \to M$ whose standard fiber is a finite–dimensional vector space V. Then two fiber bundle charts are called *compatible* if the corresponding transition function is linear in the second variable. A fiber bundle atlas consisting of pairwise compatible fiber bundle charts is then called a *vector bundle atlas*. There is an obvious notion of equivalence of vector bundle atlases and a *vector bundle* is a fiber bundle with standard fiber a vector space endowed with an equivalence class of vector bundle atlases. In this case, each of the fibers $p^{-1}(x)$ is canonically a vector space, and one may interpret the transition functions as smooth functions $\phi_{\alpha\beta} : U_{\alpha\beta} \to GL(V)$. A *vector bundle homomorphism* is a fibered morphism between two vector bundles such that the restriction to each fiber is a linear map.

The simplest example of a vector bundle is the tangent bundle TM. Its sections are the smooth vector fields and for a smooth map $f : M \to N$ the tangent map $Tf : TM \to TN$ is a vector bundle homomorphism. Given a vector bundle $E \to M$, there is the notion of an E–valued differential form. An E–valued k–form ϕ is a smooth function which associates to each $x \in M$ a k–linear alternating map $(T_xM)^k \to E_x$, where $E_x = p^{-1}(x)$ is the fiber of E over x. The space of E–valued k–forms is denoted by $\Omega^k(M, E)$. For $k = 0$, one obtains the space $\Gamma(E)$ of smooth sections of E.

One of the motivating examples for a principal fiber bundle is the projection $p : G \to G/H$ onto a homogeneous space. Since this is a surjective submersion, it admits local smooth sections, so any point $x \in G/H$ admits an open neighborhood U such that there is a smooth function $\sigma : U \to G$ with $\sigma(y)H = y$ for all $y \in U$. Such a section immediately gives rise to a fiber bundle chart $p^{-1}(U) \to U \times H$ by mapping $g \in p^{-1}(U)$ to $(p(g), \sigma(p(g))^{-1}g)$ with inverse given by $(x, h) \mapsto \sigma(x)h$. The corresponding transition functions are given by $(x, h) \mapsto \sigma_{\alpha\beta}(x)h$, where $\sigma_{\alpha\beta} : U_{\alpha\beta} \to H$ is given by $\sigma_{\alpha\beta}(x) = \sigma_\alpha(x)^{-1}\sigma_\beta(x)$.

Given a general fiber bundle $p : \mathcal{P} \to M$ with standard fiber a Lie group H, we define a *principal bundle atlas* to consist of charts which are compatible in the sense that the transition functions are given by $(x, h) \mapsto \phi_{\alpha\beta}(x)h$ for smooth functions $\phi_{\alpha\beta} : U_{\alpha\beta} \to H$. A *principal bundle* is then defined as a fiber bundle $p : \mathcal{P} \to M$ with standard fiber a Lie group H which is endowed with an equivalence class (in the obvious sense) of principal bundle atlases. The group H is referred to as the *structure group* of the principal bundle and principal bundles with structure group H are also called principal H–bundles. Multiplication from the right in charts defines a smooth right action of the structure group H on the total space $\mathcal{P}$ of the principal bundle. This is called the *principal right action*. It is by construction free, and its orbits are exactly the fibers of $p : \mathcal{P} \to M$. Conversely, given a smooth map $p : \mathcal{P} \to M$ and a right H–action on $\mathcal{P}$ which is free and transitive on each fiber, then this is a principal H–bundle if and only if p admits local smooth sections. A *morphism of principal bundles* is a fibred morphism commuting with the principal actions, i.e. $\phi(u \cdot h) = \phi(u) \cdot h$. There is a more general notion of morphisms between principal bundles with different structure groups. Fixing a homomorphism ψ between the structure groups, one imposes the equivariancy condition $\phi(u \cdot h) = \phi(u) \cdot \psi(h)$.

As in the case of homogeneous spaces, a local section $\sigma : U \to \mathcal{P}$ of a principal bundle $p : \mathcal{P} \to M$ defines a principal bundle chart $\phi : p^{-1}(U) \to U \times H$ whose inverse is given by $(x, h) \mapsto \sigma(x) \cdot h$. In particular, a principal H–bundle is trivial, i.e. isomorphic to $M \times H$, if and only if it admits a global smooth section.

Consider a principal bundle atlas $\{(U_\alpha, \phi_\alpha)\}$ for a principal H–bundle $p : \mathcal{P} \to M$ with transition functions $\phi_{\alpha\beta} : U_{\alpha\beta} \to H$. Then clearly $\phi_{\beta\alpha}(x) = \phi_{\alpha\beta}(x)^{-1}$ for all $x \in U_{\alpha\beta}$ and for three indices α, β, γ such that $U_{\alpha\beta\gamma} := U_\alpha \cap U_\beta \cap U_\gamma \neq \emptyset$, one has the *cocycle identity* $\phi_{\alpha\beta}(x)\phi_{\beta\gamma}(x) = \phi_{\alpha\gamma}(x)$ for all $x \in U_{\alpha\beta\gamma}$. Conversely, given an open covering $\{U_\alpha\}$ of M, a family $\phi_{\alpha\beta} : U_{\alpha\beta} \to H$ of smooth functions satisfying these two conditions is called a *cocycle of transition functions*. From such a family, one constructs a principal H–bundle as an appropriate quotient of the disjoint union of the sets $U_\alpha \times H$, which has the given cocycle as transition functions. It is easy to see that the transition functions determine the bundle up to isomorphism.

An important special case of the general concept of morphisms of principal bundles is provided by reductions of structure group. A *reduction* of the principal H–bundle $p : \mathcal{P} \to M$ to the structure group K, where $K \subset H$ is a Lie subgroup, is given by a principal bundle $\mathcal{R} \to M$ with structure group K, together with a principal bundle morphism $\iota : \mathcal{R} \to \mathcal{P}$ with respect to the inclusion $i : K \to H$, which covers the identity on M. The question of whether given $\mathcal{P}$ and K there exists a reduction of structure group is difficult in general, but it can be reduced to the question of existence of a smooth section of a certain bundle. Indeed, restricting the principal action to K, we obtain a free right action of K on $\mathcal{P}$, and one easily shows that the space $\mathcal{P}/K$ of orbits of this action is a smooth manifold and a fiber bundle over M with fiber H/K. A simple argument based on the cocycles of transition functions shows the following fact:

LEMMA 1.2.6. *Let $\mathcal{P} \to M$ be a principal bundle with structure group H and let $K \subset H$ be a subgroup. Then reductions of $\mathcal{P}$ to the structure group K are in bijective correspondence with the set of global smooth sections of the fiber bundle $\mathcal{P}/K \to M$.*

In contrast to the case of vector bundles, the individual fibers of a principal bundle $p : \mathcal{P} \to M$ do *not* carry the structure of a Lie group, since left multiplications are not group homomorphisms. The fibers should rather be thought of as the Lie group analog of affine spaces. Indeed, the simplest example of a principal bundle $\mathcal{P} \to \mathrm{pt}$ (with a one–point base manifold) is the space of all bases of an m–dimensional vector space V, which in turn may be identified with the set of all linear isomorphisms between $\mathbb{R}^m$ and V. Clearly, once we fix one basis (or one isomorphism), we may identify $\mathcal{P}$ with $GL(m, \mathbb{R})$, but there is no canonical choice like that.

An important example of a principal bundle is the *linear frame bundle* $\mathcal{P}^1M \to M$ of a smooth manifold M. Its fiber over $x \in M$ is the set of all bases of the tangent space T_xM. The structure group is $GL(n, \mathbb{R})$ where n is the dimension of M. We may equivalently view the fiber $\mathcal{P}^1_xM$ over $x \in M$ as the space of all linear isomorphisms between $\mathbb{R}^n$ and T_xM. In analogy to this example, we shall often call all elements in principal bundles *frames*. More generally, there is the linear frame bundle $\mathcal{GL}(\mathcal{V})$ of a vector bundle $\mathcal{V} \to M$ with structure group $GL(V)$, where V is the standard fiber of $\mathcal{V}$.

1.2.7. Associated bundles. Recovering a vector bundle from its linear frame bundle is a special case of forming an associated bundle. The idea of this process is rather simple to understand in the toy example of all bases of a vector space V viewed as a principal bundle over a point. Any element $v \in V$ can be described by its coordinates in any of the bases of V, and since none of the bases is preferred, we should view v as an equivalence class of basis–coordinates–pairs. The correct equivalence relation is easily seen in the picture of isomorphisms $\phi : \mathbb{R}^m \to V$. The coordinate vector of v in the basis corresponding to ϕ is $\phi^{-1}(v)$, and the principal right action of $A \in GL(m, \mathbb{R})$ is given by composition from the right. This implies that the pair (ϕ, x) with $x \in \mathbb{R}^n$ must be considered as equivalent to $(\phi \cdot A, A^{-1}(x))$.

Now assume that $p : \mathcal{P} \to M$ is an H–principal bundle and S is a smooth manifold endowed with a left action $H \times S \to S$. Then we define a right action of H on the product $\mathcal{P} \times S$ by $(u, s) \cdot h := (u \cdot h, h^{-1} \cdot s)$. The space $\mathcal{P} \times_H S := (\mathcal{P} \times S)/H$ is called the *associated bundle* to the principal bundle $\mathcal{P}$ with standard fiber S. From a principal bundle atlas for $\mathcal{P}$ one constructs a fiber bundle atlas for $\mathcal{P} \times_H S$ showing that it is indeed a fiber bundle with standard fiber S. Moreover, the obvious projection $\mathcal{P} \times S \to \mathcal{P} \times_H S$ is an H–principal bundle. For $(u, s) \in \mathcal{P} \times S$, we write $[\![u, s]\!] \in \mathcal{P} \times_H S$ for the orbit of (u, s). We will sometimes write $\mathcal{P} \times_\ell S$ to emphasize the role of the left action ℓ. If the left action is a linear representation of the structure group on a vector space V, then the associated bundle $\mathcal{P} \times_H V$ is canonically a vector bundle.

Each principal bundle morphism $\phi : \mathcal{P} \to \mathcal{P}'$ between bundles with structure group H defines the fibered morphisms $\mathcal{P} \times_H S \to \mathcal{P}' \times_H S$ between associated bundles, which is characterized by $[\![u, s]\!] \mapsto [\![\phi(u), s]\!]$. Thus, the construction of associated bundles corresponding to a fixed left action is functorial. Of course, for linear actions this functorial construction has values in vector bundles and vector bundle homomorphisms. On the other hand, any smooth mapping $f : S \to S'$ commuting with given left actions defines the fibered morphism $\mathcal{P} \times_H S \to \mathcal{P} \times_H S'$, given by $[\![u, s]\!] \mapsto [\![u, f(s)]\!]$, which covers the identity on the base manifold M.

Let us consider a few examples in the case of the frame bundle $\mathcal{P}^1 M \to M$ of an n–dimensional smooth manifold M. The trivial representation $\mathbb{R}$ of $H = GL(n, \mathbb{R})$ provides the trivial associated bundle $\mathcal{P}^1 M \times_H \mathbb{R} = M \times \mathbb{R}$, whose sections are the smooth functions on M. For the standard representation of H on $\mathbb{R}^n$ we see from above that the associated bundle $\mathcal{P}^1 M \times_H \mathbb{R}^n$ may be identified with the tangent bundle TM by mapping $[\![u, x]\!]$ to the tangent vector with coordinates x in the frame u. Forming associated vector bundles is compatible with natural constructions on vector spaces. In particular, for the dual $\mathbb{R}^{n*}$ of the standard representation the associated bundle is the cotangent bundle T^*M, forming tensor powers of the standard representation and its dual, one obtains all tensor bundles, and so on.

Proposition 1.2.7. *Let $p : \mathcal{P} \to M$ be a principal H–bundle, and S a smooth manifold endowed with a left H–action. Then there is a natural bijective correspondence between the set $\Gamma(\mathcal{P} \times_H S)$ of all smooth sections s of the associated bundle and the set $C^\infty(\mathcal{P}, S)^H$ of all smooth maps $f : \mathcal{P} \to S$, which are H–equivariant, i.e. satisfy $f(u \cdot h) = h^{-1} \cdot f(u)$. Explicitly, the correspondence is given by $s(p(u)) = [\![u, f(u)]\!]$.*

PROOF. Starting from an equivariant smooth function f, equivariancy implies that $[\![u, f(u)]\!]$ depends only on $p(u)$, so we can use this expression to define $s : M \to \mathcal{P} \times_H S$. Choosing a local smooth section σ of $\mathcal{P}$, we get $s(x) = [\![\sigma(x), f(\sigma(x))]\!]$, which immediately implies smoothness of s.

Conversely, any element in the fiber over $p(u)$ may be uniquely written in the form $[\![u, y]\!]$, so given s, the equation $s(p(u)) = [\![u, f(u)]\!]$ can be used to define f. Smoothness of f follows easily by writing this in terms of a local smooth section of $\mathcal{P}$, while equivariancy is an immediate consequence of $[\![u\cdot h, f(u\cdot h)]\!] = [\![u, f(u)]\!]$. □

In the case of associated vector bundles, we can generalize this result to a description of differential forms with values in an associated bundle. Given a fibered manifold $p : Y \to M$ and a point $y \in Y$ a tangent vector $\xi \in T_yY$ is called *vertical* if $T_yp \cdot \xi = 0$. The vertical tangent vectors form a subbundle $VY \subset TY$, called the *vertical tangent bundle.* This leads to the notion of a vertical vector field on Y. Now if ϕ is a differential form on Y (with values in $\mathbb{R}$, a vector space, or a vector bundle), then ϕ is called *horizontal* if it vanishes upon insertion of one vertical vector field. Suppose further that $\mathcal{P} \to M$ is a principal bundle with structure group H. Then for any $h \in H$ we have the principal right action $r^h : \mathcal{P} \to \mathcal{P}$, and we can use this to pull back differential forms with values in $\mathbb{R}$ or a vector space.

COROLLARY 1.2.7. *Let $p : \mathcal{P} \to M$ be a principal fiber bundle with structure group H and let $\rho : H \to GL(V)$ be a representation of H on a vector space V. Then for each k, the space $\Omega^k(M, \mathcal{P} \times_H V)$ of k–forms with values in the associated bundle is in bijective correspondence with the space of all $\phi \in \Omega^k(\mathcal{P}, V)$ which are horizontal and equivariant in the sense that $(r^h)^*\phi = \rho(h^{-1}) \circ \phi$ for all $h \in H$.*

PROOF. Consider a form $\alpha \in \Omega^k(M, \mathcal{P} \times_H V)$. For $u \in \mathcal{P}$, $x = p(u)$, and tangent vectors $\xi_1, \dots, \xi_k \in T_u\mathcal{P}$, there is a unique element $\phi(u)(\xi_1, \dots, \xi_k) \in V$ such that

$$\alpha(p(u))(T_up \cdot \xi_1, \dots, T_up \cdot \xi_k) = [\![u, \phi(u)(\xi_1, \dots, \xi_k)]\!]. \tag{1.1}$$

This defines a k–linear alternating map $\phi(u) : (T_u\mathcal{P})^k \to V$, which evidently vanishes if one entry is a vertical tangent vector. One easily verifies that $\phi(u)$ depends smoothly on u, so we have constructed a horizontal V–valued k–form on $\mathcal{P}$. For $h \in H$ we get $p \circ r^h = p$, and hence $Tp \cdot Tr^h \cdot \xi_i = Tp \cdot \xi_i$ for each i. This shows that

$$[\![u, \phi(u)(\xi_1, \dots, \xi_k)]\!] = [\![u \cdot h, \phi(u \cdot h)(Tr^h \cdot \xi_1, \dots, Tr^h \cdot \xi_k)]\!],$$

from which equivariancy follows immediately.

Conversely, suppose we have given a horizontal, equivariant form ϕ. Then for each $x \in M$ we can choose $u \in \mathcal{P}$ such that $p(u) = x$, and any tangent vector at x can be written as $T_up \cdot \xi$. Fixing u we can use equation (1.1) to define $\alpha(x)$. This does not depend on the choice of the lifts of the tangent vectors since ϕ is horizontal. It does not depend on the choice of u either by equivariancy of ϕ. Finally, it is easily verified that smoothness of ϕ implies smoothness of α. □

1.2.8. Natural bundles and jets. A *natural bundle* F on the category $\mathcal{M}_n$ of n–dimensional manifolds and local diffeomorphisms is a functor assigning to any n–manifold N a fiber bundle $p_N : F(N) \to N$ and to any local diffeomorphism $f : N_1 \to N_2$ a bundle map $F(f) : F(N_1) \to F(N_2)$ with base map f, i.e. such that $p_{N_2} \circ F(f) = f \circ p_{N_1}$. Furthermore, F has to be regular, i.e. if M is any smooth manifold and $f : M \times N_1 \to N_2$ is smooth and such that for each $x \in P$

the map $f_x : N_1 \to N_2$ defined by $f_x(y) := f(x, y)$ is a local diffeomorphism, then we assume that the map $M \times F(N_1) \to F(N_2)$ defined by $(x, a) \mapsto F(f_x)(a)$ is smooth, too. Thus, regularity means that smoothly parametrized families of local diffeomorphisms are transformed into smoothly parametrized families. Usually, one assumes in addition that F is local, i.e. that for any inclusion $i : U \hookrightarrow N$ of an open subset, $F(i)$ is the inclusion $p_N^{-1}(U) \hookrightarrow F(N)$.

Developing the general theory of natural bundles has been one of the main aims of [**KMS**]. It turns out that regularity follows from functoriality and locality, and one obtains an explicit description of all natural bundles as associated bundles. In order to state the result, we have to recall another basic concept of differential geometry.

Let M, N be smooth manifolds, and $x \in M$ a point. Two smooth mappings f, $g : M \to N$ are said to have the same *jet of order* r (briefly r–jet) at x if $f(x) = g(x)$ and their partial derivatives at x up to order r in some local charts around x and $f(x)$ coincide. (Then this is true in all charts around these points by the chain rule.) This defines an equivalence relation, whose classes are called r–jets at x and denoted by $j^r_x f$. The point x is called the *source* while $f(x)$ is called the *target* of the jet $j^r_x f$. The space of all r–jets with source in M and target in N is denoted by $J^r(M, N)$. Similarly, we write $J^r_x(M, N)$, $J^r(M, N)_y$, or $J^r_x(M, N)_y$ if the source, target or both are fixed. For $s < r$, we may send an r–jet to the underlying s–jet, thus obtaining a canonical map $\pi^r_s : J^r(M, N) \to J^s(M, N)$. Putting $s = 0$, the source and target map define $\pi^r_0 : J^r(M, N) \to M \times N$.

The chain rule immediately implies that the composition of jets in $J^r(M, N)_y$ and $J^r_y(N, Q)$ is well defined by the formula $(j^r_y g) \circ (j^r_x f) = j^r_x(g \circ f)$. A jet $j^r_x f \in J^r_x(M, M)$ is called invertible if there is a jet $j^r_{f(x)} g \in J^r_{f(x)}(M, M)$ such that $j^r_x(g \circ f) = j^r_x \operatorname{id}_M$ and $j^r_{f(x)}(f \circ g) = j^r_{f(x)} \operatorname{id}_M$.

Using the canonical charts on $\mathbb{R}^n$ and $\mathbb{R}^m$ and the translations, we obtain an identification $J^r(\mathbb{R}^m, \mathbb{R}^n) \cong \mathbb{R}^m \times \mathbb{R}^n \times J^r_0(\mathbb{R}^m, \mathbb{R}^n)_0$, and the Taylor coefficients yield canonical coordinates on $J^r_0(\mathbb{R}^m, \mathbb{R}^n)_0$. This, of course, works similarly for open subsets in $\mathbb{R}^m$ and $\mathbb{R}^n$. The construction of the jet spaces $J^r(M, N)$ is functorial in both arguments, and via this, arbitrary charts on M and N give rise to charts on $J^r(M, N)$. Hence, each $J^r(M, N)$ is a smooth manifold and by construction the natural maps π^r_s for $0 \leq s < r$ from above are smooth.

For any fibered manifold $p : E \to M$ we write $J^r(E \to M)$, or briefly $J^r(E)$, for the subset of $J^r(M, E)$ consisting of all jets of local sections of p. This turns out to be a smooth submanifold and a fibered manifold over M. Clearly, $J^r(\ )$ is a functor acting on locally invertible fibered morphisms, the *jet prolongation* functor. There is a universal rth order differential operator j^r which maps sections of $E \to M$ to sections of $J^r(E) \to M$ and is defined by $s \mapsto (x \mapsto j^r_x s)$. For every operator $D : \Gamma(E \to M) \to \Gamma(E' \to M)$, which is an rth order differential operator in local coordinates, there exists a fibered morphism $\tilde{D} : J^r(E) \to E'$ such that $D = \tilde{D} \circ j^r$.

Jets lead to alternative descriptions of many of the basic geometric objects on smooth manifolds. For example, the tangent bundle TM can be naturally identified with the space $J^1_0(\mathbb{R}, M)$ of first order jets of curves in M at $0 \in \mathbb{R}$. Similarly, the cotangent bundle T^*M can be naturally identified with the space $J^1(M, \mathbb{R})_0$. Let us note that in this kinematic approach one naturally obtains a vector bundle structure on the cotangent bundle. Consider the set $J^1_0(\mathbb{R}^n, \mathbb{R}^n)^{\text{inv}}_0$ of invertible one–jets on $\mathbb{R}^n$ with source and target zero. Of course, this is a group under jet composition and it can be naturally identified with the group $GL(n,\mathbb{R})$. This leads to an interpretation of the linear frame bundle $\mathcal{P}^1M$ of an n–dimensional manifold M as $J^1_0(\mathbb{R}^n,M)^{\text{inv}}$ of invertible one–jets from $\mathbb{R}^n$ to M with source 0. In this picture, the principal action of $GL(n,\mathbb{R})\cong J^1_0(\mathbb{R}^n,\mathbb{R}^n)^{\text{inv}}_0$ is given by jet composition from the right.

The jet interpretation leads to higher order generalizations of all the above bundles. The *rth order frame bundle* $\mathcal{P}^rM$ is defined as $J^r_0(\mathbb{R}^n,M)^{\text{inv}}$, so it consists of r–jets of local charts. The structure group of this bundle is the rth *differential group* $G^r_n := J^r_0(\mathbb{R}^n,\mathbb{R}^n)^{\text{inv}}_0$ and the principal action is given by the jet composition from the right. The rth order analog of the tangent bundle is the bundle $T^r_1M = J^r_0(\mathbb{R},M)$. This is the associated bundle to $\mathcal{P}^rM$ with respect to the obvious left action of G^r_n on $S=J^r_0(\mathbb{R},\mathbb{R}^n)$. Now the description of natural bundles is as follows.

THEOREM 1.2.8 ([**KMS**, Theorem 22.1]). *Any local natural bundle on n–dimensional manifolds can be obtained as an associated bundle to some $\mathcal{P}^rM$ with respect to a left action of the differential group G^r_n on a finite–dimensional manifold.*

The lowest possible choice for r in the theorem is called the *order of the natural bundle*. Notice that the composition of the jet prolongation functor with a kth order natural bundle F is the $(k+r)$th order natural bundle J^rF. The theorem was first proved assuming regularity by Palais and Terng (see [**PT77**]), and then in full generality by Epstein and Thurston (see [**ET79**]). Sharp estimates for the orders depending on the dimensions of the base and fiber were obtained by Zajtz. See [**KMS**] for further results and more bibliographic details.

1.2.9. Lie derivatives. Natural bundles also provide the right framework for defining Lie derivatives. Let us first observe that local diffeomorphisms act on the sections of any natural bundle F. For a section $s\in\Gamma(FN)$ and a local diffeomorphism $f:M\to N$, one obtains $f^*s\in\Gamma(FM)$ locally as $F\phi\circ s\circ f$, where ϕ is a local inverse to f. The section f^*s is called the *pullback* of s along f. In particular, this can be applied to the local flow of a vector field $\xi\in\mathfrak{X}(M)$. Fixing $x\in M$, we obtain a curve $t\mapsto(\mathrm{Fl}^\xi_t)^*s(x)$ in the fiber F_xM of FM over x, which is defined for sufficiently small t. Regularity of F implies that this curve is smooth, so we may consider its derivative at $t=0$. If F is a natural vector bundle, then this derivative may be interpreted as an element of the fiber F_xM itself, while for a general natural fiber bundle it has to be viewed as an element of the vertical tangent space $V_{s(x)}FM$. Regularity of F again implies that this element depends smoothly on x, so we obtain the *Lie derivative* $\mathcal{L}_\xi s$ of s along ξ defined by

$$\mathcal{L}_\xi s = \tfrac{d}{dt}|_0(\mathrm{Fl}^\xi_t)^*s = \tfrac{d}{dt}|_0F(\mathrm{Fl}^\xi_{-t})\circ s\circ\mathrm{Fl}^\xi_t\,.$$

This is a smooth section of FM in the case of a natural vector bundle and a smooth section of the vertical tangent bundle VFM in the case of an arbitrary natural fiber bundle.

An alternative way to view the Lie derivative is the following: Fixing a point $x\in M$, the flow Fl^ξ_t is defined locally around x for sufficiently small t. Hence, $F(\mathrm{Fl}^\xi_t)$ is a family of locally defined diffeomorphisms of FM, which depends smoothly on t by regularity. Differentiating at $t=0$, one obtains a vector field $\mathcal{F}\xi$ on FM, which is p_M–related to ξ, and $\mathcal{L}_\xi s = Ts\circ\xi-\mathcal{F}\xi\circ s$.

In the special case of the tangent bundle TM we obtain the standard Lie bracket, i.e. $\mathcal{L}_\xi\eta=[\xi,\eta]$ and for all tensor bundles one recovers the classical approach. In particular, it is easy to deduce the Leibniz rule for general tensor products of natural vector bundles

$$\mathcal{L}_\xi(s_1 \otimes s_2) = (\mathcal{L}_\xi s_1) \otimes s_2 + s_1 \otimes (\mathcal{L}_\xi s_2)$$

and compatibility with contractions. For example, for a k–times covariant tensor field τ and $\xi, \eta_1, \dots, \eta_k \in \mathfrak{X}(M)$, we obtain

$$(\mathcal{L}_\xi \tau)(\eta_1, \dots, \eta_k) = \mathcal{L}_\xi(\tau(\eta_1, \dots, \eta_k)) - \sum_{i=1}^{k} \tau(\eta_1, \dots, \mathcal{L}_\xi(\eta_i), \dots, \eta_k).$$

In the case of k–forms, i.e. antisymmetric k–times covariant tensor–fields, this formula easily leads to the formula $\mathcal{L}_\xi = i_\xi \circ d + d \circ i_\xi$ for the Lie derivative in terms of the exterior derivative d (see 1.2.1) and the insertion operator i_ξ defined by $(i_\xi \tau)(\eta_2, \dots, \eta_k) = \tau(\xi, \eta_2, \dots, \eta_k)$.

The Lie derivative $\mathcal{L}_\xi$ depends on derivatives of the vector field ξ up to the order of the natural bundle. Thus, the only case in which the Lie derivative is tensorial in the direction ξ are natural bundles of order zero, which are always trivial.

1.2.10. Complex manifolds and complex differential geometry. We conclude this section with a brief discussion of holomorphic aspects of differential geometry. More basic information on these issues can be found, for example, in [**KoNo69**].

A *complex manifold* M is defined similarly to the real case discussed in 1.2.1, but with charts having values in $\mathbb{C}^n$ and holomorphic transition functions. The number n is called the *complex dimension* of M, and of course we can view M also as a manifold of real dimension $2n$. For functions between complex manifolds (and in particular for functions with values in $\mathbb{C}^m$) one defines holomorphicity by requiring holomorphicity in some (or equivalently any) chart. A holomorphic diffeomorphism whose inverse is holomorphic, too, is called a *biholomorphism.* Two complex manifolds are called *biholomorphic* if there is a biholomorphism between them. It happens often that complex manifolds are diffeomorphic without being biholomorphic.

Of course, the product of two complex manifolds is canonically a complex manifold. On the one hand, this implies that there is a well–defined notion of a *complex Lie group* as a complex manifold endowed with a holomorphic group structure. In particular, the group $GL(n, \mathbb{C})$ is a complex Lie group and thus for any complex Lie group one can talk about *holomorphic representations* on complex vector spaces. On the other hand, given complex manifolds M and S, one can define *holomorphic fiber bundles* over M with standard fiber S similarly as in 1.2.6. One just has to require the total space to be a complex manifold and the fiber bundle charts to be holomorphic. In particular, one has the subclass of *holomorphic vector bundles* among complex vector bundles. For principal bundles with structure group a complex Lie group, there is the subclass of holomorphic principal bundles. Given any holomorphic fiber bundle, there is a natural notion of holomorphicity for sections via holomorphicity in some (or equivalently any) fiber bundle chart. One can then consider jets of holomorphic sections similarly as in 1.2.8, and so on.

Since functions between open subsets of $\mathbb{C}^n$ are holomorphic if and only if they have complex linear derivatives, one can use the charts of a complex atlas to make any tangent space of a complex manifold M into a complex vector space. In this way, the tangent bundle TM becomes a complex (and even a holomorphic) vector bundle. The most convenient way to encode this is to consider the linear maps on the tangent spaces of M given by multiplication by $i = \sqrt{-1}$. These fit together to define a smooth bundle map $J : TM \to TM$, which clearly satisfies $J^2 = J \circ J = -\mathrm{id}$. If M and $\tilde{M}$ are complex manifolds and J and $\tilde{J}$ are the corresponding bundle maps, then a smooth map $f : M \to \tilde{M}$ is holomorphic if and only if its tangent map is complex linear, i.e. $Tf \circ J = \tilde{J} \circ Tf$. In particular, the underlying real manifold M and the bundle map J encode the structure of a complex manifold on M, since one may characterize holomorphic charts $u : U \to \mathbb{C}^n$ using J.

Now one can turn the game around, starting with a real manifold M and a bundle map $J : TM \to TM$ such that $J^2 = -\mathrm{id}$. Such a bundle map is called an *almost complex structure* on M, and existence of such a structure implies that the dimension of M is even. A manifold M endowed with an almost complex structure J is called an *almost complex manifold*. The classical *Newlander–Nirenberg theorem* characterizes those almost structures which are integrable, i.e. which come from the structure of a complex manifold on M: For vector fields $\xi, \eta \in \mathfrak{X}(M)$ consider the expression

$$[\xi, \eta] - [J\xi, J\eta] + J([J\xi, \eta] + [\xi, J\eta]).$$

One immediately verifies that this expression is bilinear over smooth functions on M, so it defines a tensor field $N = N_J : TM \times TM \to TM$, called the *Nijenhuis tensor* of the almost complex structure J. From the definition one immediately verifies that N_J is skew symmetric and conjugate linear in both variables, i.e. $N_J(J\xi, \eta) = -J(N_J(\xi, \eta))$, and likewise in the other variable. In particular, N_J always vanishes if M is of real dimension two, since then a basis of each tangent space is given by $\{\xi, J\xi\}$ for some nonzero tangent vector ξ. The Newlander–Nirenberg theorem (see [**NeNi57**]) states that J is induced by a complex structure on M if and only if $N_J = 0$. It should be remarked that this theorem is not too difficult in the case that M and J are assumed to be real analytic; see [**KoNo69**]. The hard part is to show that vanishing of the Nijenhuis tensor implies real analyticity.

One of the basic features of almost complex and complex manifolds is a natural decomposition of the spaces of complex valued differential forms. Let (M, J) be an almost complex manifold and V a complex vector space, and for some $0 \leq k \leq \dim_{\mathbb{R}}(M)$ consider the space $\Omega^k(M, V)$ of real k–forms with values in V. For $\phi \in \Omega^k(M, V)$ and $x \in M$, the value $\phi(x)$ is a map $(T_xM)^k \to V$, which is $\mathbb{R}$–linear in each entry. The finer decomposition has the form $\Omega^k(M, V) = \bigoplus_{p+q=k} \Omega^{p,q}(M, V)$ with $p, q \geq 0$, and is often referred to as the decomposition into (p, q)–types. There are two ways to describe this decomposition: On the one hand, one can look at the action of multiplication by nonzero complex numbers in the real picture. From this point of view, the subspace $\Omega^{p,q}(M, V)$ is formed by all ϕ such that for each nonzero complex number λ and all vector fields $\xi_1, \dots, \xi_k$ on M, one has

$$\phi(\lambda\xi_1, \dots, \lambda\xi_k) = \lambda^p \bar{\lambda}^q \phi(\xi_1, \dots, \xi_k).$$

In particular, for $k = 1$, the subspaces $\Omega^{1,0}(M, V)$ and $\Omega^{0,1}(M, V)$ consist of those ϕ for which each of the maps $\phi(x) : T_xM \to V$ is complex linear, respectively, conjugate linear. For $k = 2$, the subspaces $\Omega^{2,0}(M, V)$ and $\Omega^{0,2}(M, V)$ consist of those forms whose values are complex linear, respectively, conjugate linear in both arguments. The subspace $\Omega^{1,1}(M, V)$ consists of forms ϕ whose values are totally real, i.e. such that $\phi(J\xi, J\eta) = \phi(\xi, \eta)$ for all vector fields ξ and η on M.

The second approach to obtain the decomposition into (p,q)–types is via complexification. Let (M,J) be an almost complex manifold and consider the complexified tangent bundle $T_{\mathbb{C}}M := TM \otimes_{\mathbb{R}} \mathbb{C}$. The bundle map $J : TM \to TM$ extends to a complex linear bundle map $J_{\mathbb{C}} : T_{\mathbb{C}}M \to T_{\mathbb{C}}M$ such that $J_{\mathbb{C}}^2 = -\mathrm{id}$. Since each tangent space is a complex vector space, it splits into the direct sum of eigenspaces for $J_{\mathbb{C}}$ with eigenvalues $\pm i$, and of course, this depends smoothly on the base point. Thus, one obtains a splitting $T_{\mathbb{C}}M = T^{1,0}M \oplus T^{0,1}M$ into a direct sum of smooth subbundles. The eigenspaces $T^{1,0}_xM$ (respectively $T^{0,1}_xM$) are spanned by all vectors of the form $\xi - iJ\xi$ (respectively $\xi + iJ\xi$) with $\xi \in T_xM$. Vanishing of the Nijenhuis tensor is easily seen to be equivalent to the fact that the sections of either (or equivalently both) these bundles are closed under the Lie bracket.

The dual space $T^*M \otimes \mathbb{C}$ of $T_{\mathbb{C}}M$ splits accordingly as $T^*_{1,0}M \oplus T^*_{0,1}M$, where $T^*_{1,0}$ is the annihilator of $T^{0,1}$. Next, we get

$$\Lambda^k(T^*M \otimes \mathbb{C}) = \bigoplus_{p+q=k} \Lambda^p T^*_{1,0}M \otimes \Lambda^q T^*_{0,1}M.$$

For a complex vector space V, one may then identify $L_{\mathbb{R}}(\Lambda^k T^*M, V)$ with $\Lambda^k(T^*M \otimes \mathbb{C}) \otimes_{\mathbb{C}} V$, which leads to the decomposition into (p,q)–types.

Let us next specialize to the case $V = \mathbb{C}$ of complex–valued forms. Then it is natural to look at the compatibility of the exterior derivative d (which is defined exactly as the real counterpart) with the decomposition into (p,q)–types. For a general almost complex structure, one can only show that for $\phi \in \Omega^{p,q}(M) := \Omega^{p,q}(M,\mathbb{C})$ the exterior derivative $d\phi$ can have nontrivial components only in bidegrees $(p+2,q-1)$, $(p+1,q)$, $(p,q+1)$, and $(p-1,q+2)$. In the case of a complex structure, the situation is much nicer, since in that case the nontrivial components can only lie in degrees $(p+1,q)$ and $(p,q+1)$. Decomposing $d\phi$ into these two parts leads to two natural first order operators $\partial : \Omega^{p,q}(M) \to \Omega^{p+1,q}(M)$ and $\bar\partial : \Omega^{p,q}(M) \to \Omega^{p,q+1}(M)$ such that $d = \partial + \bar\partial$. The fact that $d^2 = 0$ immediately implies that $\partial^2 = 0$, $\bar\partial^2 = 0$ and $\bar\partial \circ \partial = -\partial \circ \bar\partial$.

In the case of complex structures, the splitting into (p,q)–types and the operators ∂ and $\bar\partial$ can be nicely described in local coordinates. Suppose that one has local $\mathbb{C}^n$–values coordinates $(x^1 + iy^1, \dots, x^n + iy^n)$. Then from above we conclude that we obtain local frames $\{\frac{\partial}{\partial z^j} : j = 1, \dots, n\}$ for $T^{1,0}M$ and $\{\frac{\partial}{\partial \bar z^j} : j = 1, \dots, n\}$ for $T^{0,1}M$ by defining

$$\tfrac{\partial}{\partial z^j} := \tfrac{1}{2}(\tfrac{\partial}{\partial x^j} - i\tfrac{\partial}{\partial y^j}) \qquad \tfrac{\partial}{\partial \bar z^j} := \tfrac{1}{2}(\tfrac{\partial}{\partial x^j} + i\tfrac{\partial}{\partial y^j}).$$

(Observe that J maps $\frac{\partial}{\partial x^j}$ to $\frac{\partial}{\partial y^j}$.) The dual frames for $T^*_{1,0}M$ and $T^*_{0,1}M$ consist of the elements $dz^j = dx^j + idy^j$, respectively, $d\bar z^j = dx^j - idy^j$. Hence, the forms $dz^{a_1} \wedge \dots \wedge dz^{a_p} \wedge d\bar z^{b_1} \wedge \dots \wedge d\bar z^{b_q}$ with $p+q = k$, $a_1 < \dots < a_p$ and $b_1 < \dots < b_q$ form a local frame for $\Omega^k(M,\mathbb{C})$, and the splitting into (p,q)–types corresponds to the number of unbarred and barred factors. Using the sum convention, we get for $\phi = \phi_{a_1\dots a_p b_1\dots b_q} dz^{a_1} \wedge \dots \wedge d\bar z^{b_q}$ the formulae

$$\partial(\phi) = \frac{\partial \phi_{a_1\dots a_p b_1\dots b_q}}{\partial z^j} dz^j \wedge dz^{a_1} \wedge \dots \wedge d\bar z^{b_q},$$
$$\bar\partial(\phi) = \frac{\partial \phi_{a_1\dots a_p b_1\dots b_q}}{\partial \bar z^j} d\bar z^j \wedge dz^{a_1} \wedge \dots \wedge d\bar z^{b_q}.$$

Finally, observe that by the Cauchy–Riemann equations a complex–valued function $f : M \to \mathbb{C}$ is holomorphic if and only if $\frac{\partial f}{\partial \bar{z}^j} = 0$ for all $j = 1, \dots, n$. This implies that a (p, q)–form $\phi \in \Omega^{p,q}(M)$ is holomorphic if and only if $\bar{\partial}\phi = 0$, i.e. if and only if $d\phi \in \Omega^{p+1,q}(M)$. This works in the same way for forms with values in any finite–dimensional complex vector space.

1.3. A survey on connections

This section provides background on various versions of connections. We start from the simple idea of linear connections on vector bundles, then pass to general connections on fiber bundles, and specialize to principal and induced connections. Finally, we discuss affine connections and, more generally, connections on G–structures, which are the simplest special cases of Cartan connections.

1.3.1. Linear connections. The general idea of a connection is to provide a notion of directional derivatives for sections of bundles or fibered manifolds. The directions are given by vector fields on the base, and for real–valued smooth functions, the action of vector fields provides a natural operation of this type. We have seen two further instances of such operations already.

First, for smooth functions with values in a Lie group G, the trivialization of the tangent bundle defined by the Maurer–Cartan form allows us to define a derivative, which has values in the tangent space at the unit element. This is the left logarithmic derivative defined in 1.2.4. We shall see below how to link this derivative to the trivial principal connection on the trivial principal bundle $M \times G \to M$.

Second, on all natural bundles, there is the notion of Lie derivative of sections along vector fields; cf. 1.2.9. This is, however, not an analogue of a directional derivative, since the value of $\mathcal{L}_\xi s$ in a point does not only depend on the value of ξ in that point but on a higher jet of the vector field. Hence, a different concept is needed.

Let us consider an arbitrary vector bundle $\mathcal{V} \to M$ with standard fiber V. We wish to have derivatives of sections $s \in \Gamma(\mathcal{V})$ in the direction of a vector field $\xi \in \mathfrak{X}(M)$ which are tensorial in ξ. In the other variable, one requires a Leibniz rule with respect to the multiplication by smooth functions. Such an operation is usually called a "connection" or a *covariant derivative* on the vector bundle $\mathcal{V}$. It is well known (see e.g. [**KMS**, Lemma 7.3]) that the requirement to be tensorial in ξ can be equivalently formulated as being linear over $C^\infty(M)$. Thus, a *linear connection* on a vector bundle $\mathcal{V} \to M$ is often defined as a bilinear operator $\nabla : \mathfrak{X}(M) \times \Gamma(\mathcal{V}) \to \Gamma(\mathcal{V})$, written as $(\xi, s) \mapsto \nabla_\xi s$, such that for all $\xi \in \mathfrak{X}(M)$, $s \in \Gamma(\mathcal{V})$ and $f \in C^\infty(M, \mathbb{R})$ one has

$$\begin{aligned} \nabla_\xi f s &= (\mathcal{L}_\xi f)s + f\nabla_\xi s, \\ \nabla_{f\xi} s &= f\nabla_\xi s. \end{aligned} \tag{1.2}$$

Choosing local charts $\mathbb{R}^n \times \mathbb{R}^k \to \mathcal{V}$ for M and $\mathcal{V}$ and using the usual summation convention, we may write the vector field ξ as $\xi = \xi^i \frac{\partial}{\partial x^i}$, where x is the coordinate on $\mathbb{R}^n$, and the section s as $s^p e_p$, where $\{e_1, \dots, e_k\}$ is the the local frame of $\mathcal{V}$ corresponding to the chosen chart. From the defining properties (1.2) of ∇ we get

$$\nabla_\xi s = \xi^i \nabla_{\frac{\partial}{\partial x^i}} (s^p e_p) = \xi^i \frac{\partial s^p}{\partial x^i} e_p + B^p_{qi} s^q \xi^i e_p,$$

for smooth functions B^p_{qi} characterized by $\nabla_{\frac{\partial}{\partial x^i}} e_q = B^p_{qi} e_p$. This formula shows that $\nabla_\xi s(x)$ depends only on $j^1_x s$. Moreover, for each fixed target $y = s(x)$ there is a unique one–jet $j^1_x s$ such that $\nabla_{\xi_x} s = 0$ for all $\xi_x \in T_x M$. Conversely, knowing that one–jet is equivalent to knowing the functions B^p_{qi} and thus to knowing the linear connection ∇.

The last observation directly leads to the definition of the *horizontal distribution* $\mathcal{H} \subset T\mathcal{V}$ associated to ∇. Given $x \in M$ and $y \in \mathcal{V}_x$, we can find a smooth section $s \in \Gamma(V)$ such that $s(x) = y$ and $\nabla_\xi s(x) = 0$ for all $\xi \in \mathfrak{X}(M)$ and the one–jet $j^1_x s$ is uniquely determined by this condition. In particular, $T_x s$ is unambiguously defined, and we put $\mathcal{H}_y := T_x s(T_x M)$. Of course, $T_y p$ is inverse to $T_x s$, so it restricts to a linear isomorphism between $\mathcal{H}_y$ and $T_x M$. Hence, for each tangent vector ξ_x at x and each point y over x, we can find a unique lift of ξ_x in $\mathcal{H}_y$. This defines the *horizontal lift* of tangent vectors on M.

In local coordinates as above, the distinguished one–jet $j^1_x s$ is characterized by $\frac{\partial s^p}{\partial x^i}(x) = -B^p_{qi}(x) s^q(x)$. In particular, in a point x_0 the distinguished jet for $s(x_0) = y$ can be represented by the map $y - B^p_{qi}(x_0) y^q (x^i - x^i_0)$. This immediately shows that the horizontal lift of $\xi^i(x)\frac{\partial}{\partial x^i}$ is given by

$$\xi^i(x)\tfrac{\partial}{\partial x^i} - \xi^i(x) B^p_{qi}(x) y^q \tfrac{\partial}{\partial y^p}.$$

Starting from a smooth vector field $\xi \in \mathfrak{X}(M)$, the horizontal lifts fit together to define a smooth vector field $\xi^{\text{hor}} \in \mathfrak{X}(\mathcal{V})$, called the *horizontal lift* of ξ. It is the unique projectable vector field over ξ whose value in each point lies in the horizontal subspace. Since such horizontal lifts span the horizontal subspaces, we see that the horizontal distribution is smooth.

Let us put this into a broader perspective. Working in local coordinates as above, the fact that the horizontal lift in a point is a linear map implies that we may write

$$\xi^{\text{hor}}(x,y) = \xi^i(x)\tfrac{\partial}{\partial x^i} + \gamma^p_i(x,y)\xi^i(x)\tfrac{\partial}{\partial y^p},$$

for some functions γ^p_i. We have seen that in our case $\gamma^p_i(x,y) = -B^p_{qi}(x) y^q$, so this is linear in y. This is the origin of the term "linear connection".

Next, we show that the covariant derivative can be recovered from the horizontal lift map. This will lead to the notion of a general connection, which is based on the horizontal lift.

Lemma 1.3.1. *Let ∇ be a linear connection on a vector bundle $\mathcal{V} \to M$. Then for any vector field $\xi \in \mathfrak{X}(M)$ and any section $s \in \Gamma(\mathcal{V})$ we have*

$$\nabla_\xi s = Ts \circ \xi - \xi^{hor} \circ s = \tfrac{d}{dt}|_0 \left(\mathrm{Fl}^{\xi^{hor}}_{-t} \circ s \circ \mathrm{Fl}^\xi_t \right),$$

where we identify the vertical tangent space to $\mathcal{V}$ in the point $s(x)$ with the fiber $\mathcal{V}_x$.

Proof. The first equality follows immediately from the coordinate formulae of the covariant derivative and the horizontal lift, and the second equality is a direct computation. □

The *curvature of a linear connection* ∇ on $\mathcal{V}$ is defined by

$$R(\xi,\eta)(s) := \nabla_\xi \nabla_\eta s - \nabla_\eta \nabla_\xi s - \nabla_{[\xi,\eta]} s \tag{1.3}$$

for $\xi, \eta \in \mathfrak{X}(M)$. By construction, this is skew symmetric in ξ and η, and one easily verifies that it is linear over smooth functions in all three entries. Thus, we may view the curvature R as a section of the bundle $\Lambda^2 T^*M \otimes L(\mathcal{V},\mathcal{V})$, i.e. as a two–form with values in endomorphisms of $\mathcal{V}$.

For later use, let us note here that a linear connection ∇ on a vector bundle $\mathcal{V} \to M$ induces operators on $\mathcal{V}$–valued differential forms, called the *covariant exterior derivative*. By definition, the space $\Omega^k(M,\mathcal{V})$ of $\mathcal{V}$–valued k–forms is the space of smooth sections of the bundle $\Lambda^k T^*M \otimes \mathcal{V}$. Alternatively, $\Omega^k(M,\mathcal{V})$ is the space of all k–linear alternating maps $(\mathfrak{X}(M))^k \to \Gamma(\mathcal{V})$ which are linear over $C^\infty(M,\mathbb{R})$ in one (or equivalently any) variable. Now one defines the covariant exterior derivative $d^\nabla : \Omega^k(M,\mathcal{V}) \to \Omega^{k+1}(M,\mathcal{V})$ by taking the formula for the exterior derivative from 1.2.1 and replacing the action of a vector field on a smooth function by a covariant derivative, i.e.

$$\begin{aligned} d^\nabla\omega(\xi_0,\dots,\xi_k) = &\sum_{i=0}^{k}(-1)^i\nabla_{\xi_i}(\omega(\xi_0,\dots,\widehat{\xi_i},\dots,\xi_k)) \\ &+\sum_{i<j}(-1)^{i+j}\omega([\xi_i,\xi_j],\xi_0,\dots,\widehat{\xi_i},\dots,\widehat{\xi_j}\dots,\xi_k). \end{aligned}$$

The verification that this is alternating and linear over $C^\infty(M,\mathbb{R})$ is exactly as in the case of usual differential forms. In contrast to the exterior derivative, the covariant exterior derivative is, however, not a differential, i.e. $d^\nabla \circ d^\nabla \neq 0$ in general.

1.3.2. General connections. A *general connection* on an arbitrary fibered manifold $p : Y \to M$ is a smooth horizontal distribution $\mathcal{H} \subset TY$ which is complementary to the vertical tangent bundle VY. For each $y \in Y$, $\mathcal{H}_y \subset T_yY$ is called the *horizontal subspace* at y of the connection.

There are three further equivalent ways to view this: First, we can consider the induced horizontal lift of vector fields which associates $\xi^{\text{hor}} \in \mathfrak{X}(Y)$ to $\xi \in \mathfrak{X}(M)$. As in the case of linear connections, ξ^{hor} is the unique projectable vector field lying over ξ whose value in each point is horizontal. By construction, the horizontal lift map is linear over smooth functions, and conversely a lift map with this property comes from a smooth horizontal distribution. Second, we have $TY = VY \oplus \mathcal{H}$ and this decomposition can be equivalently described by the smooth *vertical projection* $\Psi : TY \to VY$ with kernel $\mathcal{H}$. Of course, the vertical projection also defines the *horizontal projection* $\chi = \text{id}_{TY} - \Psi$. Finally, as in the case of linear connections, one may view the connection as specifying a unique one–jet of sections $j^1_x s$ with $s(x) = y$ for each $y \in Y$, such that the horizontal lift $T_xM \to T_yY$ is given by T_xs. From that point of view, a connection can equivalently be viewed as a smooth section of the first jet prolongation $J^1Y \to Y$.

The *curvature of a general connection* is defined by $R(\xi,\eta) = -\Psi([\chi(\xi),\chi(\eta)])$ for $\xi,\eta \in \mathfrak{X}(Y)$, so $R(\xi,\eta)$ is minus the vertical projection of the brackets of the horizontal projections of the vector fields. (The sign is used to obtain the usual sign conventions for principal and linear connections.) By definition, R is horizontal, i.e. vanishes upon insertion of one vertical vector field, and has vertical values. Moreover, $\Psi \circ \chi = 0$ immediately implies that R is bilinear over $C^\infty(Y,\mathbb{R})$, so $R \in \Omega^2_{\text{hor}}(Y,VY)$. Moreover, $R(\xi,\eta) = 0$ for all ξ and η if and only if the Lie bracket of any two horizontal vector fields on Y is horizontal, too, i.e. if and only if the horizontal distribution is involutive. By the Frobenius theorem (see 1.2.2), a connection has vanishing curvature if and only if it is locally given by local trivializations $s : \mathbb{R}^m \times \mathbb{R}^n \to Y$. A more sophisticated definition of the curvature of general connections is to define R as the Frölicher–Nijenhuis bracket $[\Psi, \Psi]$ of the vertical projection Ψ; see [**KMS**, section 9].

Since one–dimensional distributions are always involutive, a general connection defines the so–called *parallel transport* along curves on $p : Y \to M$: For each interval $(a, b) \subset \mathbb{R}$ containing zero and each parametrized curve $c : (a, b) \to M$, with $x = c(0)$, and any $y \in Y$ with $p(y) = x$, there is a unique maximal subinterval $(a', b') \subset (a, b)$ containing 0 and a unique curve $\tilde{c}_y : (a', b') \to Y$ such that $\tilde{c}_y(0) = y$, $p \circ \tilde{c}_y = c$, and $T\tilde{c}_y$ has values in the horizontal distribution.

The *absolute derivative* ∇_ξ defined by a general connection is then defined by the formula from Lemma 1.3.1, i.e.

$$\nabla_\xi s = Ts \circ \xi - \xi^{\mathrm{hor}} \circ s = \tfrac{d}{dt}|_0 \left(\mathrm{Fl}^{\xi^{\mathrm{hor}}}_{-t} \circ s \circ \mathrm{Fl}^{\xi}_t \right).$$

For a section $s \in \Gamma(Y)$ and a vector field $\xi \in \mathfrak{X}(M)$, we have $\nabla_\xi s \in \Gamma(VY \to M)$. This formula may also be explained in terms of the parallel transport: Take a section s and a direction $\xi_x \in T_xM$. Extend ξ_x into a vector field ξ, consider its flow through x and the corresponding parallel transport. Then the derivative of the pullback of the section by the parallel transport along the flow lines of ξ is exactly given by our formula.

There is also another concept of derivative, the *exterior absolute differential* $d^\nabla : \Omega^k(Y, V) \to \Omega^{k+1}(Y, V)$ for any vector space V, defined by means of the pullback with respect to the horizontal projection, i.e. by the formula $d^\nabla = \chi^* d$. We shall see some explicit relation between the last two concepts under specific geometric assumptions, but they are completely independent in the general setting.

For fiber bundles, there is a nonlinear version of the Christoffel symbols. For a local trivialization $\mathbb{R}^m \times S \to Y$, the horizontal lift of vector fields is described uniquely by the mappings $\gamma_i : TM \to \mathfrak{X}(S)$, so that the horizontal lift of $\xi^i \frac{\partial}{\partial x^i}$ equals $\xi^i \frac{\partial}{\partial x^i} + \xi^i \gamma_i$. Similarly, we obtain the formula for the absolute derivative in a local trivialization.

Finally, in the special case of a vector bundle $Y \to M$, the first jet prolongation J^1Y carries a vector bundle structure too. It is a simple exercise to verify that a general connection $\sigma : Y \to J^1Y$ is linear if and only if the section σ is a vector bundle homomorphism.

1.3.3. Principal connections. In the special case of a principal bundle, it is natural to look at connections that are compatible with the principal right action. Let $p : \mathcal{P} \to M$ be a principal fiber bundle. A *principal connection* on $\mathcal{P}$ is a (general) connection whose horizontal distribution is invariant with respect to the principal action of the structure group G, i.e. $\mathcal{H}_{u \cdot g} = Tr^g(\mathcal{H}_u)$ for all $g \in G$. Notice that this immediately implies that the parallel transport along curves is also right invariant. This means that for any curve c, the corresponding parallel transport $\tilde{c}$ satisfies $\tilde{c}_{u \cdot g}(t) = \tilde{c}_u(t) \cdot g$ for all $u \in \mathcal{P}$ and $g \in G$.

There are two further equivalent ways to express the invariance condition:

(1) The horizontal lifts of vector fields are right invariant vector fields on the principal bundle $\mathcal{P}$, i.e. $(r^g)^* \xi^{\mathrm{hor}} = \xi^{\mathrm{hor}}$ for all $g \in G$ and $\xi \in \mathfrak{X}(M)$.
(2) The corresponding section $\sigma : \mathcal{P} \to J^1\mathcal{P}$ is G–equivariant with respect to the obvious induced G–action on $J^1\mathcal{P}$.

In the case of a principal bundle, one may reformulate the vertical projection defining a connection in a simple way, since the vertical bundle $V\mathcal{P}$ of a principal G–bundle $p : \mathcal{P} \to M$ is trivialized by the fundamental vector fields. This means that any vertical tangent vector $\xi \in T_u\mathcal{P}$ can be written as the value $\zeta_X(u)$ for a unique element $X \in \mathfrak{g}$, the Lie algebra of the structure group G. Consequently, we may specify the vertical projection of any general connection on $\mathcal{P}$ by a $\mathfrak{g}$–valued one form $\gamma \in \Omega^1(\mathcal{P}, \mathfrak{g})$, such that the vertical projection of $\xi \in T_u\mathcal{P}$ equals $\zeta_{\gamma(\xi)}(u)$. This is a projection if and only if γ reproduces the generators of fundamental vector fields, i.e. $\gamma(\zeta_X) = X$ for all $X \in \mathfrak{g}$. One easily verifies that γ corresponds to a principal connection if and only if $(r^g)^*\gamma = \mathrm{Ad}(g^{-1}) \circ \gamma$. We call the one–form γ the *principal connection form*, or briefly the principal connection, on $\mathcal{P}$.

The trivialization of the vertical bundle by fundamental vector fields also leads to a nice interpretation of the curvature of any connection on a principal fiber bundle. Since the curvature R has vertical values, there is a unique $\mathfrak{g}$–valued two–form $\rho \in \Omega^2(\mathcal{P}, \mathfrak{g})$ such that $R(\xi, \eta)(u) = \zeta_{\rho(\xi,\eta)(u)}(u)$ for any $u \in \mathcal{P}$. If we deal with a principal connection, then the definition of the curvature R by means of the horizontal lifts shows that R, viewed as an element in $\Omega^2(\mathcal{P}, V\mathcal{P})$, is G–equivariant, which in turn immediately implies that $(r^g)^*\rho = \mathrm{Ad}(g^{-1}) \circ \rho$. The form ρ is called the *curvature form* or often simply the curvature of the given principal connection. Since ρ is horizontal and equivariant, it may also be viewed as a two–form on M with values in the associated bundle $\mathcal{P} \times_G \mathfrak{g}$ corresponding to the adjoint action of G on $\mathfrak{g}$; see Corollary 1.2.7.

To compute the curvature ρ from the connection form γ, observe that by definition we have $\rho(\xi, \eta) = -\gamma([\xi - \zeta_{\gamma(\xi)}, \eta - \zeta_{\gamma(\eta)}])$. Since both fields on the right–hand side lie in the kernel of γ, this coincides with $d\gamma(\xi - \zeta_{\gamma(\xi)}, \eta - \zeta_{\gamma(\eta)})$. Note that this by definition means that $\rho = d^\nabla\gamma$, where d^∇ is the exterior absolute derivative from 1.3.2. Now for $X \in \mathfrak{g}$, the flow of the fundamental vector field ζ_X is $r^{\exp(tX)}$, so differentiating the equivariancy property $(r^{\exp(tX)})^*\gamma = \mathrm{Ad}(\exp(-tX)) \circ \gamma$ at $t = 0$, we obtain $\mathcal{L}_{\zeta_X}\gamma = -\mathrm{ad}(X) \circ \gamma$. Moreover, by definition $i_{\zeta_X}\gamma = X$ and thus $di_{\zeta_X}\gamma = 0$, whence we conclude that $d\gamma(\zeta_X, \eta) = -[X, \gamma(\eta)]$. Using this, we immediately conclude from above that

$$\rho(\xi, \eta) = d\gamma(\xi, \eta) + [\gamma(\xi), \gamma(\eta)],$$

which is the usual definition of the curvature of a principal connection.

On the trivial principal bundle $M \times G$ there is the trivial principal connection, whose horizontal subspaces are the kernels of $T\pi_G$, where $\pi_G : M \times G \to G$ is the projection. The connection form of this connection is the pullback of the Maurer–Cartan form by π_G, and for simplicity, we also denote this form by ω_G. Note that then the absolute differential of sections $M \to M \times G$, viewed as functions $M \to G$, is exactly the left logarithmic derivative. Let us further specialize to $M = \mathbb{R}^n$, which, via local trivializations, also describes the local situation for general principal bundles. A general connection form on $\mathbb{R}^n \times G$ can be written as

$$\gamma = \omega_G - \gamma_i dx^i,$$

for $\mathfrak{g}$–valued one forms γ_i. These are called the *Christoffel symbols*, and they are determined by the restriction to the distinguished section $(x, e) \subset \mathbb{R}^n \times G$. In particular, for principal connections on the linear frame bundle $\mathcal{P}^1M$ of a manifold M we obtain the usual Christoffel symbols γ^k_{ji}.

Obviously, the exterior differential of the connection form γ equals

$$d\gamma = d\omega_G - d\gamma_i \wedge dx^i$$

and the evaluation of $d\gamma + \frac{1}{2}[\gamma,\gamma]$ yields the usual coordinate expression for the curvature form. If we write $\gamma_i^p Y_p$ for the expression of $\gamma_i(x,e)$ in the basis Y_p of $\mathfrak{g}$, $\rho = \rho^p Y_p$, and c^p_{qr} for the corresponding structure constants of $\mathfrak{g}$, i.e. $[Y_q, Y_r] = c^p_{qr} Y_p$, we obtain the expression of ρ at the points (x,e) in $M \times G$,

$$\rho^p = \left(\frac{\partial \gamma_i^p}{\partial x^j} + c^p_{qr}\gamma_i^q \gamma_j^r \right) dx^i \wedge dx^j.$$

If the curvature of a principal connection γ vanishes, then locally γ is isomorphic to a trivial principal connection on a trivial principal bundle. There may appear global obstructions to the product structure, however. Next, let us note that the conditions on a general connection on a principal fiber bundle $\mathcal{P}$ to be a principal connection are of affine character. Thus, we may glue together the trivial connections with the help of a cocycle of local trivializations of $\mathcal{P}$ and a subordinated partition of unity, to obtain a globally defined principal connection. In particular, there are always smooth principal connections on any principal fiber bundle $\mathcal{P}$. Let us finally note that the difference of two principal connections γ and $\bar{\gamma}$ is a horizontal equivariant $\mathfrak{g}$–valued one–form. By Corollary 1.2.7 it may equivalently be viewed as a one–form on M with values in the bundle $\mathcal{P} \times_G \mathfrak{g}$.

1.3.4. Induced connections. The main advantage of principal connections is that a single principal connection on a principal bundle gives rise to a connection on any associated bundle, and the connections on various associated bundles are nicely compatible. There are at least two ways to see that this has to work: On the one hand, from 1.2.6 and 1.2.7 we know that the elements of a G–principal bundle $\mathcal{P} \to M$ may be viewed as frames, which give a coordinate–like description of elements of the associated bundles $\mathcal{P}\times_G S$. Given a principal connection on $\mathcal{P}$, we know which local frame fields are constant to first order. We can then define sections of $\mathcal{P}\times_G S$ to be constant to first order, if their coordinates in such frame fields have vanishing derivatives, and this suffices to define a connection on $\mathcal{P}\times_G S$. On the other hand, it is also easy to see that equivariancy of the horizontal distribution of a principal connection implies that it can be pushed down to a horizontal distribution on any associated bundle.

Recall from 1.2.7 that, given a principal G–bundle $p : \mathcal{P} \to M$ and a left action of G on a manifold S, we have the associated bundle $\pi : \mathcal{P}\times_G S \to M$ and a natural projection $q : \mathcal{P} \times S \to \mathcal{P}\times_G S$, which is a G–principal bundle and has the property that $\pi \circ q = p \circ \mathrm{pr}_1$. Now the tangent map of the multiplication makes TG into a Lie group, and the tangent map of the principal right action of G on $\mathcal{P}$ makes $T\mathcal{P}$ into a TG–principal bundle. Then Tq identifies $T(\mathcal{P}\times_G S)$ with $T\mathcal{P} \times_{TG} TS$. Moreover, embedding $\mathcal{P}$ into $T\mathcal{P}$ and G into TG as the zero sections, the restriction of Tq to $\mathcal{P} \times TS$ induces an identification of the vertical bundle $V(\mathcal{P}\times_G S)$ with $\mathcal{P}\times_G TS$.

Now assume that we have given a principal connection on $\mathcal{P}$ with horizontal distribution $\mathcal{H}$. Then one immediately verifies that for any $(u,s) \in \mathcal{P} \times S$ the restriction of $T_{(u,s)}q$ to $\mathcal{H}_u \times \{0_s\}$ is injective, so the image of this subspace defines a candidate for a horizontal subspace in $T_{[\![u,s]\!]}(\mathcal{P}\times_G S)$. Equivariancy of the horizontal distribution easily implies that this subspace is independent of the choice of the representative (u, s), so we get a well–defined horizontal distribution $\mathcal{H}^S$ on the associated bundle $\mathcal{P} \times_G S$. Viewing $T(\mathcal{P} \times_G S)$ as an associated bundle as above, we have

$$\mathcal{H}^S([\![u, s]\!]) = \{[\![\xi(u), 0_s]\!] \in T\mathcal{P} \times_{TG} TS, \xi(u) \in \mathcal{H}(u), u \in \mathcal{P}\}.$$

Thus, any fixed principal connection γ on $\mathcal{P}$ yields on each associated bundle $\mathcal{P} \times_G S$ a general connection γ^S, which is called the *induced connection*.

An induced connection is closely related to the principal connection it comes from. We will prove this only in the case of associated vector bundles, but with straightforward changes in the formulations and proofs, similar statements hold for general induced connections.

PROPOSITION 1.3.4. *Let $p : \mathcal{P} \to M$ be a G–principal bundle, λ a linear representation of G on a vector space V and $\mathcal{V} = \mathcal{P} \times_G V$ the corresponding induced vector bundle. Consider a principal connection on $\mathcal{P}$ with connection form $\gamma \in \Omega^1(\mathcal{P}, \mathfrak{g})$ and curvature $\rho \in \Omega^2(\mathcal{P}, \mathfrak{g})$, and let γ^V be the induced connection on $\mathcal{V}$.*

(1) γ^V is a linear connection on $\mathcal{V}$. Using $T\mathcal{V} = T\mathcal{P} \times_G TV$, its horizontal lift is given by $\xi \mapsto Tq \circ (\xi^{hor}, 0) = [\![\xi^{hor}, 0]\!]$, where $\xi^{hor} \in \mathfrak{X}(\mathcal{P})$ is the horizontal lift of ξ with respect to γ.

(2) Let $s \in \Gamma(\mathcal{V})$ be a section corresponding to the function $f \in C^\infty(\mathcal{P}, V)$, and let $\xi \in \mathfrak{X}(M)$ be a vector field. Then the covariant derivative $\nabla_\xi s \in \Gamma(\mathcal{V})$ corresponds to the function $\xi^{hor} \cdot f : \mathcal{P} \to V$.

(3) In a local trivialization $\mathbb{R}^n \times G \to \mathcal{P}$ with $\gamma = \omega_G - \gamma_i dx^i$ and the corresponding local trivialization $\mathbb{R}^n \times V \to \mathcal{V}$, the absolute derivative is given by

$$\nabla_\xi s = \xi^i \frac{\partial s}{\partial x^i} - \lambda'(\gamma_i \xi^i) \circ s,$$

where $\lambda' : \mathfrak{g} \to \mathfrak{gl}(V)$ is the Lie algebra representation corresponding to λ.

(4) The parallel transport on $\mathcal{V}$ along the curve c on M is given by $\tilde{c}^V_{[\![u,v]\!]}(t) = [\![\tilde{c}_u(t), v]\!]$, where $\tilde{c}_u$ is the parallel transport in $\mathcal{P}$ starting at u.

(5) Let $s \in \Gamma(\mathcal{V})$ be a section corresponding to the function $f : \mathcal{P} \to V$, and let $\xi, \eta \in \mathfrak{X}(M)$ be vector fields. Then the section $\nabla_\xi \nabla_\eta s - \nabla_\eta \nabla_\xi s - \nabla_{[\xi,\eta]} s$ corresponds to the function $\lambda'(\rho(\xi^{hor}, \eta^{hor})) \circ f$.

PROOF. The statement on the horizontal lift in (1) is obvious from the construction. Given a section s corresponding to the function $f : \mathcal{P} \to V$, let us choose a local smooth section σ of $\mathcal{P}$, so that we can locally write s as $q \circ (\sigma, f \circ \sigma)$. By definition of the absolute derivative associated to a general connection, $\nabla_\xi s(x)$ is given by

$$Tq \cdot (T_x\sigma \cdot \xi(x) - \xi^{\mathrm{hor}}(\sigma(x)), Tf \cdot T_x\sigma \cdot \xi(x)).$$

Now, we may fix a frame $u \in \mathcal{P}$ over x and choose σ in such a way that $\xi^{\mathrm{hor}}(u) = T_x\sigma \cdot \xi(x)$ for all ξ, i.e. a section σ which represents the defining jet of γ in x with target u. But then the above expression simplifies to $Tq \cdot (0_u, Tf \cdot \xi^{\mathrm{hor}}(\sigma(x)))$. Identifying the vertical tangent space in $[\![u, f(u)]\!]$ with the vector space V, this shows that the function $\mathcal{P} \to V$ corresponding to $\nabla_\xi s$ has value $\xi^{\mathrm{hor}}(\sigma(x)) \cdot f$ in the point $\sigma(x)$, which proves claim (2). Moreover, since the sum of two sections corresponds to the pointwise sum of the associated functions, this immediately implies that ∇ is actually a covariant derivative on $\mathcal{V}$, and thus defines a linear connection. By construction, this linear connection has associated horizontal distribution $\mathcal{H}^V$, so it coincides with γ^V, which completes the proof of (1).

(3) Let σ be the local smooth section of $\mathcal{P}$ corresponding to the chosen trivialization. Then the associated local trivialization of $\mathcal{V}$ is given by $(x, v) \mapsto [\![\sigma(x), v]\!]$. Consequently, viewing a section s as a V–valued function in this trivialization, one obtains exactly $f \circ \sigma$, where $f : \mathcal{P} \to V$ is the equivariant function corresponding to s. Now the horizontal lift of $\xi^i \frac{\partial}{\partial x^i}$ is given by $\xi^i \frac{\partial}{\partial x^i} + \zeta_{\gamma_i \xi^i}$, and the claim follows from (2) and equivariancy of f.

(4) The curve $[\![\tilde{c}_u(t), v]\!]$ obviously has horizontal derivatives, covers c, and starts at $[\![u, v]\!]$, so the claim follows from uniqueness of the parallel transport.

(5) By (2), the section $\nabla_\xi \nabla_\eta s - \nabla_\eta \nabla_\xi s - \nabla_{[\xi,\eta]} s$ corresponds to the function $([\xi^{\mathrm{hor}}, \eta^{\mathrm{hor}}] - [\xi, \eta]^{\mathrm{hor}}) \cdot f$ and the field in the bracket by definition equals $\zeta_{-\rho(\xi^{\mathrm{hor}}, \eta^{\mathrm{hor}})}$. Equivariancy of f now implies the claim. □

1.3.5. Affine connections on manifolds. The previous construction of induced connections can be often inverted. In particular, any linear connection on a vector bundle $\mathcal{V}$ over M with standard fiber V is induced by a unique principal connection on the frame bundle of $\mathcal{V}$, i.e. the principal bundle of all bases of the fibers of $\mathcal{V}$ with structure group $GL(V)$. (Since a local frame for a vector bundle is made up from local sections, we can simply declare a local frame to be constant to first order in a point, if all the sections that constitute the frame have this property.)

In particular, linear connections on the tangent bundle TM of a smooth manifold M are in one–to–one correspondence with principal connections on the linear frame bundle $\mathcal{P}^1 M$. The latter connections are briefly referred to as *linear connections on M*. Once we fix such a connection γ, then there are the induced connections on all tensor bundles (and more generally any associated bundle to $\mathcal{P}^1 M$). The coordinate formula from part (3) of Proposition 1.3.4 is just the classical formula for the covariant derivative with respect to a linear connection on a manifold.

The next ingredient which is specific to the case of the tangent bundle is the existence of the *canonical form* $\theta \in \Omega^1(\mathcal{P}^1 M, \mathbb{R}^m)$, $m = \dim(M)$. The value of this form in a frame $u \in \mathcal{P}^1 M$ on a tangent vector $\xi \in T_u \mathcal{P}^1 M$ is defined to be the coordinates of the projection $Tp \cdot \xi \in T_{p(u)} M$ in the frame u. Otherwise put, viewing u as linear isomorphism $\mathbb{R}^m \to T_x M$, we have $\theta(u)(\xi) := u^{-1}(T_u p \cdot \xi)$. The names *solder form* and *soldering form* for θ are often used in the literature.

By construction, the canonical form θ is $GL(m, \mathbb{R})$–equivariant with respect to the standard action of $GL(m, \mathbb{R})$ on $\mathbb{R}^m$ and it is strictly horizontal, i.e.

$$(r^g)^* \theta = g^{-1} \circ \theta \text{ for } g \in GL(m, \mathbb{R}), \tag{1.4}$$

$$\theta(\xi) = 0 \text{ if and only if } \xi \text{ is vertical.} \tag{1.5}$$

The second property is obvious from the definition of θ, the first one simply follows from the fact that viewing frames as linear isomorphisms $\mathbb{R}^m \to T_x M$, the principal right action of $GL(m, \mathbb{R})$ is given by composition from the right.

Given a principal connection $\gamma \in \Omega^1(\mathcal{P}^1 M, \mathfrak{gl}(m, \mathbb{R}))$, we may consider the one–form $\omega = \theta + \gamma \in \Omega^1(\mathcal{P}^1 M, \mathbb{R}^m \oplus \mathfrak{gl}(m, \mathbb{R}))$, which is equivariant with respect to the direct sum of the standard action and the adjoint action. Moreover, since the kernel of θ in a point is the vertical subbundle and γ is injective on the vertical bundle, we see that for each $u \in \mathcal{P}^1 M$, the restriction of ω to $T_u \mathcal{P}^1 M$ is injective and hence a linear isomorphism.

In this picture, we can nicely describe the affine m–space A^m as the homogeneous model of such a structure. As a set, $A^m = \mathbb{R}^m$ and the group $A(m, \mathbb{R})$ of affine motions is the group of all maps from $\mathbb{R}^m$ to itself, which are of the form

$x \mapsto Ax + b$ for $A \in GL(m, \mathbb{R})$ and $b \in \mathbb{R}^m$. Viewing A^m as the affine hyperplane $x_1 = 1$ in $\mathbb{R}^{m+1}$ the affine motions are exactly the elements of $GL(m+1, \mathbb{R})$ which map this affine hyperplane to itself, i.e.

$$A(m, \mathbb{R}) = \left\{ \begin{pmatrix} 1 & 0 \\ b & A \end{pmatrix},\ A \in GL(m, \mathbb{R}),\ b \in \mathbb{R}^m \right\} \subset GL(m+1, \mathbb{R}).$$

The group $A(m, \mathbb{R})$ obviously acts transitively on A^m and the isotropy subgroup of the first unit vector is just the subgroup of all elements with $b = 0$, so $A^m \cong A(m, \mathbb{R})/GL(m, \mathbb{R})$. On the Lie algebra level, we get

$$\mathfrak{a}(m, \mathbb{R}) = \left\{ \begin{pmatrix} 0 & 0 \\ X & B \end{pmatrix},\ B \in \mathfrak{gl}(m, \mathbb{R}),\ X \in \mathbb{R}^m \right\},$$

so as a vector space this is isomorphic to $\mathbb{R}^m \oplus \mathfrak{gl}(m, \mathbb{R})$. Moreover, since the adjoint action of a matrix group is given by conjugation, one immediately sees that this splitting is invariant under the restriction of the adjoint action to $GL(m, \mathbb{R})$, and the action of this group on $\mathfrak{a}(m, \mathbb{R})$ is the direct sum of the standard representation and the adjoint action.

The natural projection $p : A(m, \mathbb{R}) \to A(m, \mathbb{R})/GL(m, \mathbb{R})$ is a principal bundle with structure group $GL(m, \mathbb{R})$; see 1.2.6. In 1.2.4 we have introduced the Maurer–Cartan from $\omega \in \Omega^1(A(m, \mathbb{R}), \mathfrak{a}(m, \mathbb{R}))$. Now we may split $\omega = \theta + \gamma$ according to the splitting $\mathfrak{a}(m, \mathbb{R}) = \mathbb{R}^m \oplus \mathfrak{gl}(m, \mathbb{R})$, and since this splitting is $GL(m, \mathbb{R})$–invariant, both θ and γ are $GL(m, \mathbb{R})$–equivariant forms. The form θ associates to each element $g \in A(m, \mathbb{R})$ a linear isomorphism $T_{p(g)}A^m \to \mathbb{R}^m$, and thus identifies $A(m, \mathbb{R})$ with the frame bundle $\mathcal{P}^1 A^m$. Hence, the component γ may be viewed as a linear connection on A^m and one immediately sees that this gives the canonical flat connection on A^m.

Returning to a general manifold M, we may now view the data defining a linear connection on M as an analog of the homogeneous space $A(m, \mathbb{R})/GL(m, \mathbb{R}) = A^m$. Indeed, the analog of the principal $GL(m, \mathbb{R})$–bundle $A(m, \mathbb{R}) \to A^m$ is the frame bundle $\mathcal{P}^1 M \to M$. On the other hand, the form $\omega = \theta + \gamma$ as considered above may be viewed as an element of $\Omega^1(\mathcal{P}^1 M, \mathfrak{a}(m, \mathbb{R}))$, which is an analog of the Maurer–Cartan form on $A(m, \mathbb{R})$. In fact, we have already noted above that ω is $GL(m, \mathbb{R})$–equivariant, it reproduces generators of fundamental vector fields, and defines a trivialization of the tangent bundle. More formally, we have

$$(r^g)^* \omega = \mathrm{Ad}(g^{-1}) \circ \omega \text{ for all } g \in GL(m, \mathbb{R}), \tag{1.6}$$

$$\omega(\zeta_X) = X \text{ for all } X \in \mathfrak{gl}(m, \mathbb{R}) \subset \mathfrak{a}(m, \mathbb{R}), \tag{1.7}$$

$$\omega(u) : T_u \mathcal{P}^1 M \to \mathfrak{a}(m, \mathbb{R}) \text{ is a linear isomorphism for all } u \in \mathcal{P}^1 M. \tag{1.8}$$

Here Ad denotes the adjoint action of $A(m, \mathbb{R})$. These are exactly the strongest analogs of the properties of the Maurer–Cartan form that make sense on a general principal $GL(m, \mathbb{R})$–bundle.

To complete this interpretation of linear connections on the tangent bundle, we need one more observation: Suppose that M is an m–dimensional manifold, $p : \mathcal{P} \to M$ is a principal $GL(m, \mathbb{R})$–bundle and $\omega \in \Omega^1(\mathcal{P}, \mathfrak{a}(m, \mathbb{R}))$ is a one–form which satisfies (1.6)–(1.8). Then (1.8) and (1.7) make sure that the kernel of the $\mathbb{R}^m$–component of ω in each point $u \in \mathcal{P}$ is exactly the vertical subspace $V_u \mathcal{P}$. Hence, this component descends to an injective linear map $T_u \mathcal{P}/V_u \mathcal{P} \cong T_{p(u)} M \to \mathbb{R}^m$ which must be an isomorphism by dimensional reasons. We can view this as defining a smooth fiber bundle map $\mathcal{P} \to \mathcal{P}^1M$, and condition (1.6) implies that this must be a homomorphism and thus an isomorphism of principal bundles. By construction, the pullback of the soldering form under this isomorphism is exactly the $\mathbb{R}^m$–component of ω. As before, we can then view the $\mathfrak{gl}(m,\mathbb{R})$–component of ω as a principal connection on $\mathcal{P}$.

This shows that a linear connection on an m–dimensional manifold M can be equivalently described as an *affine structure*, i.e. a principal $GL(m,\mathbb{R})$–bundle $p : \mathcal{P} \to M$, together with a one–form $\omega \in \Omega^1(\mathcal{P}, \mathfrak{a}(m,\mathbb{R}))$ which has the properties (1.6)–(1.8) from above. The basic features of linear connections on manifolds can be nicely phrased in this picture, which motivates many developments for more general Cartan geometries. We continue to write $\omega = \theta + \gamma$ for the splitting into the $\mathbb{R}^m$– and the $\mathfrak{gl}(m,\mathbb{R})$–component.

There is another feature of linear connections on manifolds that we have not discussed yet. For a linear connection γ on TM with covariant derivative ∇, one can define the *torsion* T by $T(\xi,\eta) = \nabla_\xi\eta - \nabla_\eta\xi - [\xi,\eta]$. This expression is visibly skew symmetric and easily seen to be bilinear over $C^\infty(M,\mathbb{R})$, so it defines a tensor field $T \in \Gamma(\Lambda^2T^*M \otimes TM)$. In local coordinates we can describe the linear connection ∇ by the Christoffel symbols γ^i_{jk} (see 1.3.1), and we compute

$$\begin{aligned} T(\xi,\eta) &= \left(\frac{\partial\eta^j}{\partial x^i}\xi^i - \gamma^j_{ik}\eta^i\xi^k - \frac{\partial\xi^j}{\partial x^i}\eta^i + \gamma^j_{ik}\xi^i\eta^k - [\xi,\eta]\right)\tfrac{\partial}{\partial x^j} \\ &= \left((\gamma^j_{ik} - \gamma^j_{ki})\xi^i\eta^k\right)\tfrac{\partial}{\partial x^j}. \end{aligned}$$

Hence, the torsion is given by the antisymmetrization of the Christoffel symbols, $T^j_{ik} = \gamma^j_{ik} - \gamma^j_{ki}$.

To compute the torsion in terms of ω, we observe that by definition of the canonical form θ, for a vector field $\xi \in \mathfrak{X}(M)$ the corresponding function $f : \mathcal{P}^1M \to \mathbb{R}^m$ can be written as $\theta(\tilde{\xi})$, where $\tilde{\xi}$ is any lift of ξ to a vector field on $\mathcal{P}^1M$. In particular, we may use the horizontal lift, and then the definition of the torsion together with part (2) of Proposition 1.3.4 implies that the function $\mathcal{P}^1M \to \mathbb{R}^m$ corresponding to $T(\xi,\eta)$ is given by

$$\xi^{\text{hor}} \cdot \theta(\eta^{\text{hor}}) - \eta^{\text{hor}} \cdot \theta(\xi^{\text{hor}}) - \theta([\xi^{\text{hor}},\eta^{\text{hor}}]) = d\theta(\xi^{\text{hor}},\eta^{\text{hor}}).$$

By definition, this is the absolute exterior derivative $d^\nabla\theta$. As in the case of the curvature dealt with in 1.3.3, one next verifies that by equivariancy of θ, for arbitrary vector fields ξ and η on $\mathcal{P}^1M$, we get

$$d\theta(\xi - \zeta_{\gamma(\xi)}, \eta - \zeta_{\gamma(\eta)}) = d\theta(\xi,\eta) + \gamma(\xi)(\theta(\eta)) - \gamma(\eta)(\theta(\xi)). \tag{1.9}$$

The Lie bracket on $\mathfrak{a}(m,\mathbb{R})$ is given by $[(X,B),(X',B')] = (BX' - B'X, BB' - B'B)$, so (1.9) turns out to be exactly the $\mathbb{R}^m$–component of $d\omega(\xi,\eta) + [\omega(\xi),\omega(\eta)]$. The $\mathfrak{gl}(m,\mathbb{R})$–component of the latter expression is $d\gamma(\xi,\eta) + [\gamma(\xi),\gamma(\eta)]$, and we know from 1.3.4 that this represents the curvature of our connection. Thus, we see that the extent to which ω fails to satisfy the Maurer–Cartan equation is exactly measured by the torsion and the curvature of the corresponding linear connection. In the special case of the homogeneous model A^m the Maurer–Cartan equation expresses the fact that the canonical connection on A^m is torsion free and flat.

Another feature of linear connections on TM that relates nicely to the affine point of view is the concept of geodesics which generalize the straight lines in A^m and the related concept of normal coordinates. Given a linear connection ∇ on M

the geodesics are characterized by the fact that $\nabla_{c'}c' = 0$. (From Lemma 1.3.1 it follows that $\nabla_\xi s$ depends only on the restriction of s to flow lines of ξ, which implies that the above expression makes sense although c' is only defined along c.) In the case of A^m, the geodesics are the (linearly parametrized) straight lines, and it is easy to see that they are exactly the projections of flows of left invariant vector fields on $A(m,\mathbb{R})$. Similarly, one may describe the geodesics for arbitrary affine structures. Indeed, for a point $x \in M$ and a tangent vector $\xi \in T_xM$ choose a point $u \in \mathcal{P}_x$. Then there is a unique element $X \in \mathbb{R}^m$ such that $\xi = T_up\cdot\omega_u^{-1}(X,0)$, and we consider the constant vector field $\tilde{X} := \omega^{-1}(X,0) \in \mathfrak{X}(\mathcal{P})$. Then the curve $\tilde{c}(t) := \mathrm{Fl}_t^{\tilde{X}}(u)$ is defined for $|t|$ sufficiently small, and $c := p \circ \tilde{c}$ satisfies $c(0) = x$, and $c'(0) = \xi$. Moreover, by construction $\tilde{c}$ has horizontal derivatives, so $\tilde{c}'(t)$ is the horizontal lift of $c'(t)$, and $\theta(\tilde{c}'(t)) = X$ for all t, which easily implies that c is a geodesic.

Fixing again $u \in \mathcal{P}$ with $p(u) = x$, we can now consider the mapping $\phi_u(X) := p(\mathrm{Fl}_1^{\tilde{X}}(u))$, which is defined on a neighborhood of $0 \in \mathbb{R}^m$. Moreover, the derivative of ϕ_u at 0 is a linear isomorphism $\mathbb{R}^m \to T_xM$, so possibly restricting to a smaller neighborhood of zero, ϕ_u is a diffeomorphism onto an open neighborhood of x in M. We can view this as defining a local coordinate system around x, which has the property that the straight lines through 0 exactly correspond to the geodesics through x, whence these are exactly the *normal coordinates* associated to the connection γ. Changing from u to $u \cdot g$, the normal coordinates change as $\phi_{u\cdot g} = \phi_u \circ \mathrm{Ad}(g)$, so they are unique up to linear transformations. Alternatively, the last fact may be interpreted in such a way that the normal coordinates define a unique diffeomorphism from an open neighborhood of 0 in T_xM to an open neighborhood of $x \in M$, the affine exponential map.

1.3.6. Connections on G–structures. First order G–structures are among the simplest examples of geometric structures. For any such structure, there is an obvious notion of compatible connections, and these can be interpreted very similarly to affine connections on manifolds. Let $H \subset GL(m,\mathbb{R})$ be a closed subgroup and M an m–dimensional manifold. A *G–structure* with structure group H (or briefly an H–structure) on M is a reduction $i : \mathcal{P} \to \mathcal{P}^1M$ of the frame bundle of M to the structure group H. This means that $\mathcal{P}$ is a principal H–bundle and i is a morphism of principal bundles over the inclusion $H \hookrightarrow GL(m,\mathbb{R})$ which covers the identity on M. Equivalently, one may characterize this as a principal H–bundle $p : \mathcal{P} \to M$ together with a form $\Theta \in \Omega^1(\mathcal{P},\mathbb{R}^m)$ which is strictly horizontal and H–equivariant, i.e. satisfies the analogs of (1.4) and (1.5) from 1.3.5 for elements $g \in H$. Given the reduction $i : \mathcal{P} \to \mathcal{P}^1M$, we obtain the form Θ as the pullback $i^*\theta$ of the canonical form, while in the other direction for $u \in \mathcal{P}$ the isomorphism $T_xM \to \mathbb{R}^m$ corresponding to the frame $i(u)$ is $\Theta(u) : T_u\mathcal{P}/V_u\mathcal{P} \cong T_xM \to \mathbb{R}^m$.

Slightly more generally, we also use the term G–structure with structure group H in the case where H is not a closed subgroup of $GL(m,\mathbb{R})$ but a covering of a virtual subgroup, i.e. if there is a given homomorphism $j : H \to GL(m,\mathbb{R})$ such that the derivative $j' : \mathfrak{h} \to \mathfrak{gl}(m,\mathbb{R})$ is injective. A well–known example of this situation is Riemannian spin structures, corresponding to the universal covering $\mathrm{Spin}(m) \to SO(m) \subset GL(m,\mathbb{R})$. As before, the structure may be either characterized as a reduction of structure group or via an $\mathbb{R}^m$–valued one–form Θ. Since this one form is essentially the same object as the canonical one–form, we will also denote it by

θ and call it the canonical form or the soldering form of the G–structure in the sequel.

G–structures corresponding to subgroups in $GL(m,\mathbb{R})$ allow yet another simple description. From Lemma 1.2.6 we know that reductions of the principal $GL(m,\mathbb{R})$ bundle $\mathcal{P}^1M$ to the structure group $H \subset GL(m,\mathbb{R})$ are in bijective correspondence with smooth sections σ of the bundle $\mathcal{P}^1M/H \to M$. Explicitly, the subbundle $\mathcal{Q} \subset \mathcal{P}^1M$ associated to a section $\sigma \in \Gamma(\mathcal{P}^1M/H)$ is just the preimage of $\sigma(M)$ under the natural projection $\mathcal{P}^1M \to \mathcal{P}^1M/H$. Conversely, the images of local smooth sections of $\mathcal{Q}$ under that projection are immediately seen to piece together a global smooth section of $\mathcal{P}^1M/H$. Finally, the bundle $\mathcal{P}^1M/H$ can be nicely viewed as the associated bundle $\mathcal{P}^1M \times_{GL(m,\mathbb{R})} (GL(m,\mathbb{R})/H)$: Consider the map from $\mathcal{P}^1M$ to this associated bundle that maps u to $[\![u, eH]\!]$. This is clearly smooth and it is surjective since $[\![u, gH]\!] = [\![u \cdot g^{-1}, eH]\!]$. On the other hand, $[\![u, eH]\!] = [\![u', eH]\!]$ if and only if $u' = u \cdot h$ for some $h \in H$, so our map factors to a bijection from $\mathcal{P}^1M/H$ to the associated bundle and one easily verifies that this actually is a diffeomorphism.

Example 1.3.6. There are many well–known G–structures, let us name just a few:

(1) A Riemannian structure is the reduction of $\mathcal{P}^1M$ to orthonormal frames, i.e. to $H = O(m,\mathbb{R}) \subset GL(m,\mathbb{R})$. The group $GL(m,\mathbb{R})$ acts transitively on the space of all inner products on $\mathbb{R}^m$ and the isotropy group of the standard inner product is $O(m)$, so $GL(m,\mathbb{R})/O(m)$ is the space of all inner products on $\mathbb{R}^m$. Hence, the associated bundle $\mathcal{P}^1M/H$ is the bundle of all inner products on the tangent spaces of M, i.e. smooth sections of this bundle are exactly Riemannian metrics on M.

Riemannian spin structures may be similarly interpreted as G–structures corresponding to the homomorphism $j : Spin(m,\mathbb{R}) \to GL(m,\mathbb{R})$. This is an important example in which the structure group is not a subgroup of $GL(m,\mathbb{R})$ but a covering of such a subgroup.

(2) An almost symplectic structure on a smooth manifold M of dimension $2n$ is a reduction of $\mathcal{P}^1M$ to the symplectic group $H = Sp(2n,\mathbb{R})$. As in (1), the homogeneous space $GL(2n,\mathbb{R})/Sp(2n,\mathbb{R})$ is the space of all non–degenerate skew symmetric bilinear maps on $\mathbb{R}^{2n}$. The bundle $\mathcal{P}^1M/H$ is the bundle of non–degenerate 2–forms, and its closed sections are called symplectic forms.

(3) An absolute parallelism is a reduction to the trivial subgroup $\{\mathrm{id}\} \subset GL(m,\mathbb{R})$. In this case, the G–structures are the global trivializations of TM.

A *connection on a G-structure* $\mathcal{P}$ with structure group H is a principal connection on the H–principal bundle $\mathcal{P}$. Notice that any such connection extends to a principal connection on $\mathcal{P}^1M$ by equivariancy. Given $i : \mathcal{P} \to \mathcal{P}^1M$ and $j' : \mathfrak{h} \hookrightarrow \mathfrak{gl}(m,\mathbb{R})$ and a principal connection $\gamma \in \Omega^1(\mathcal{P},\mathfrak{h})$, consider a point $i(u) \in \mathcal{P}^1M$ for $u \in \mathcal{P}$. Since γ reproduces the generators of fundamental vector fields, we get a well–defined linear map $\tilde{\gamma} : T_{i(u)}\mathcal{P}^1M \to \mathfrak{gl}(m,\mathbb{R})$ which coincides with $j' \circ \gamma(u)$ on $T_u i(T_u\mathcal{P})$ and reproduces the generators of fundamental vector fields. Now we can extend $\tilde{\gamma}$ to an element of $\Omega^1(\mathcal{P}^1M, \mathfrak{gl}(m,\mathbb{R}))$ by requiring equivariancy, i.e $(r^g)^*\tilde{\gamma} = \mathrm{Ad}(g^{-1}) \circ \tilde{\gamma}$ for all $g \in GL(m,\mathbb{R})$. This is well–defined by equivariancy of γ.

Conversely, let us assume that γ is a principal connection on $\mathcal{P}^1M$ and we have given a reduction $i : \mathcal{P} \to \mathcal{P}^1M$ corresponding to $j : H \to GL(m,\mathbb{R})$. Then visibly γ is an equivariant extension of a principal connection as constructed above if and only if $\gamma(i(u))(T_u i(T_u\mathcal{P})) \subset j'(\mathfrak{h})$. In the case where H is a closed subgroup of $GL(m,\mathbb{R})$, there is a simple necessary condition. Namely, consider the section $\sigma \in \Gamma(\mathcal{P}^1M/H)$ describing the reduction and the corresponding equivariant function $f : \mathcal{P}^1M \to GL(m,\mathbb{R})/H$. From above we see that by construction this function is constantly equal to eH along the subbundle $\mathcal{P} \subset \mathcal{P}^1M$. For a tangent vector ξ on M and a point $u \in \mathcal{P}$, choose a lift $\tilde{\xi} \in T_u\mathcal{P}$. Then the condition on γ above ensures that $\xi^{\mathrm{hor}}(u) \in T_u\mathcal{P}$, whence we conclude that $\xi^{\mathrm{hor}} \cdot f$ vanishes identically on $\mathcal{P}$ and thus on $\mathcal{P}^1M$ by equivariancy. By the analog of part (2) of Proposition 1.3.4, this means that $\nabla\sigma$ has to vanish identically, i.e. the section defining the reduction must be covariantly constant. In many cases, this condition is also sufficient, e.g. for Riemannian and almost symplectic structures.

We should also remark at this point that there are also general results concerning the question whether a connection on $\mathcal{P}^1M$ comes from some reduction to the structure group H. These results are based on the notion of the holonomy of a connection and are known as the Ambrose–Singer Theorem. A version of this theorem for connections on general fiber bundles can be found in [**KMS**, 9.11].

Connections on G–structures can be treated in a similar style as the affine picture for linear connections on the tangent bundle. Indeed, given $j : H \to GL(m,\mathbb{R})$ such that the infinitesimal homomorphism j' is injective, consider the *affine extension* $B := \mathbb{R}^m \rtimes H$ of H. This means that $B = \mathbb{R}^m \times H$ as a set and the multiplication is given by $(X,g)(Y,h) = (X+j(g)(Y), gh)$. If H is a closed subgroup of $GL(m,\mathbb{R})$, then B is a closed subgroup of $A(m,\mathbb{R})$, in general there is an obvious homomorphism $\tilde{j} : B \to A(m,\mathbb{R})$ such that $\tilde{j}'$ is injective. Now we can equivalently view connections on G–structures with structure group H as principal H–bundles $\mathcal{P} \to M$ endowed with a one–form $\omega \in \Omega^1(\mathcal{P},\mathfrak{b})$ which satisfy the analogs of (1.6)–(1.8) from 1.3.5 with respect to elements $g \in H$. The interpretations of torsion and curvature as well as geodesics and normal coordinates works exactly in the same way as in 1.3.5.

As in the case of the affine space A^m discussed in 1.3.5, one may look at the homogeneous space $B/H \cong \mathbb{R}^m$, and view the Maurer–Cartan form on B as a connection on the G–structure $B \to \mathbb{R}^m$ with structure group H. Connections on G–structures with structure group H can thus be thought of as “curved analogs” of this homogeneous space. In particular, in the case $H = O(m) \subset GL(m,\mathbb{R})$ the affine extension B is exactly the *Euclidean group* $\mathrm{Euc}(n)$ as discussed in 1.1.2, and from example (1) above, we see that this picture leads to viewing m–dimensional Riemannian manifolds as curved analogs of the Euclidean space E^m. This is one of the motivating examples for the concept of Cartan connections.

1.3.7. Partial connections. The starting point for our development of connections was a linear connection on a vector bundle, viewed as an analog of directional derivatives. In this picture, partial connections are operators with similar properties, except that one may only differentiate in directions lying in a fixed distribution $\mathcal{D} \subset TM$. Given a vector bundle $p : \mathcal{V} \to M$ one defines a *partial linear connection* on $\mathcal{V}$ corresponding to the distribution $\mathcal{D}$ as a bilinear operator $\Gamma(\mathcal{D}) \times \Gamma(\mathcal{V}) \to \Gamma(\mathcal{V})$ written as $(\xi, s) \mapsto \nabla_\xi s$ which is linear over $C^\infty(M,\mathbb{R})$ in the first variable and satisfies the Leibniz rule in the second variable.

Most of the developments in this section can be also carried out for partial connections. There are two ways to do this. Either one repeats the development outlined so far, taking into account the obvious changes, or one views a partial connection as an equivalence class of true connections. Here two connections are equivalent if they coincide on $\Gamma(\mathcal{D}) \times \Gamma(\mathcal{V}) \subset \mathfrak{X}(M) \times \Gamma(\mathcal{V})$. In both ways, the development is basically straightforward, so we only state a few main points.

Partial linear connections can be equivalently described by a horizontal lift, which is now only defined on the subbundle $\mathcal{D} \subset TM$. This leads to the notion of a general partial connection on a fibered manifold $p : Y \to M$, which is easiest described as a direct sum decomposition $(Tp)^{-1}(\mathcal{D}) \cong VY \oplus \mathcal{D}^{\mathrm{hor}}$, or equivalently by a vertical projection defined on $(Tp)^{-1}(\mathcal{D})$ only. In the case of a principal bundle $p : \mathcal{P} \to M$ with structure group G, the subbundle $(Tp)^{-1}(\mathcal{D}) \subset T\mathcal{P}$ is G–invariant, so the definition of a partial principal connection is obvious. Such a partial principal connection may be described by a connection form γ which is now a smooth section of the bundle $L((Tp)^{-1}(\mathcal{D}), \mathfrak{g})$. The concept of induced partial connections poses no problems and these behave similarly to true induced connections.

The notion of curvature for partial connection is slightly subtle. The problem is that the bracket of two sections of the distribution $\mathcal{D}$ is not a section of $\mathcal{D}$ in general. Hence, the covariant derivative in the direction of a bracket and the vertical projection of a bracket are not defined in general. The Lie bracket of vector fields induces a bundle map $\Lambda^2\mathcal{D} \to TM/\mathcal{D}$. If we assume that this bundle map has constant rank, then its kernel is a subbundle $\Lambda^2_0\mathcal{D} \subset \Lambda^2\mathcal{D}$. In this case, the curvature of a partial linear connection can be defined by the usual formula (equation (1.3) from 1.3.1) as a section of the bundle $L(\Lambda^2_0\mathcal{D}, L(\mathcal{V}, \mathcal{V}))$. The curvature of partial general connections and partial principal connections can be defined similarly.

A partial linear connection on TM is called a *partial affine connection on M*. In this case, there is at least a well–defined concept of the torsion of the partial connection which is defined by the standard formula $T(\xi, \eta) = \nabla_\xi \eta - \nabla_\eta \xi - [\xi, \eta]$ for all $\xi, \eta \in \Gamma(\mathcal{D})$. This means that the torsion is a bundle map $\Lambda^2\mathcal{D} \to TM$. One can also use this definition of the torsion for partial connections on $\mathcal{D}$.

It is worth mentioning at this point, that a part of the torsion of any (partial) connection which preserves a distribution $\mathcal{D}$ depends only on $\mathcal{D}$ and not on the connection. The Lie bracket of vector fields induces a tensorial map $\Lambda^2\mathcal{D} \to TM/\mathcal{D}$. For any connection preserving $\mathcal{D}$, the composition of the quotient projection $TM \to TM/\mathcal{D}$ with the restriction of the torsion to $\Lambda^2\mathcal{D}$ is evidently given by this map. Notice, that this tensorial map is the obstruction against integrability of $\mathcal{D}$.

1.3.8. Remark. Affine connections on manifolds as treated in 1.3.5 and more generally connections on G–structures as discussed in 1.3.6 are the simplest examples of Cartan connections. In all of these cases, there is a homogeneous space in the background, which is simply $\mathbb{R}^m$ as a homogeneous space of the affine extension $B = \mathbb{R}^m \rtimes H$ of H in the case of G–structures with structure group H and, in particular, affine space A^m in the case of affine connections. The developments in 1.3.5 and 1.3.6 give a first glance on the passage from the homogeneous model to arbitrary Cartan connections, which is fundamental in the theory of Cartan geometries. Any reasonable concept for a Cartan geometry must first of all work for the homogeneous model and many constructions for the homogeneous model carry over to general Cartan connections. In view of this fact, we do not directly move on to general Cartan connections, which would be the next natural item in our chain of various notions of connections, but first study the invariant geometry of homogeneous spaces.

1.4. Geometry of homogeneous spaces

In F. Klein's Erlangen program, homogeneous spaces are the basic setting for classical geometry. Using Cartan connections, one may associate to any homogeneous space a differential geometric structure, called the corresponding Cartan geometry. The given homogeneous space is then called the *homogeneous model* of the Cartan geometry, and it plays a central role in the theory. On the one hand, it provides a distinguished basic object. On the other hand, surprisingly many general geometric properties can be read off directly from the homogeneous model. Otherwise put, many questions about Cartan geometries can be answered by looking at the homogeneous model only. Finally, there is always the subclass of locally flat geometries, in which the relation to the homogeneous model is even closer.

The point of view we take in this chapter is that a homogeneous space G/H carries a geometric structure, whose automorphisms are exactly the actions of elements of G. There are three main directions in our study of these geometric structures in this section. First we study homogeneous bundles and invariant sections of such bundles, which may be viewed as simpler geometric structures underlying the given one. Second, we take some basic steps towards the study of differential operators which are intrinsic to such a structure, which in this setting are G–invariant differential operators. In particular, we discuss the question of existence of invariant connections of various types. Finally, we briefly discuss distinguished curves in homogeneous spaces, which provide generalizations of geodesics of affine connections.

1.4.1. Klein geometries. Let G be a Lie group and let $H \subset G$ be a closed subgroup. Then H is a submanifold and thus a Lie subgroup and the set G/H of all cosets gH is canonically a smooth manifold endowed with a transitive left action of G. Up to the choice of a base point, any transitive action is of this form; see 1.2.5. Here we want to view G/H as carrying a geometric structure whose automorphisms are exactly the left actions ℓ_g for $g \in G$. In this context, the pair (G, H) is referred to as a *Klein geometry.* A careful geometric study of Klein geometries is available in [**Sh97**, Chapter 4].

Given a Klein geometry (G, H) we may first ask whether all of G is "visible" on G/H, i.e. whether the action ℓ of G on G/H is effective. In this case, we call the Klein geometry *effective.* The *kernel* $K \subset G$ of the Klein geometry is defined as the set of all elements $g \in G$ such that $\ell_g = \mathrm{id}_{G/H}$. Since this is the intersection of all isotropy subgroups, it is a closed subgroup of G and since H is the isotropy subgroup of eH, we see that $K \subset H$. Moreover, for $k \in K$ and $g \in G$, we have $\ell_{gkg^{-1}} = \ell_g \circ \ell_k \circ \ell_{g^{-1}}$ and thus $gkg^{-1} \in K$, so the subgroup K is normal in G. On the other hand, suppose that H' is a virtual Lie subgroup of H which is normal in G. Then for $h \in H'$ and $g \in G$ we get $g^{-1}hgH = eH$ and thus $hgH = gH$, so $\ell_h = \mathrm{id}_{G/H}$ and $H' \subset K$. Hence, the kernel K is the maximal normal subgroup of G which is contained in H, and a Klein geometry is effective if and only if there is no non–trivial normal subgroup of G which is contained in H.

Given an arbitrary Klein geometry (G, H) with kernel K, one may pass to the effective quotient $(G/K, H/K)$, i.e. view G/H as a homogeneous space of G/K rather than G. Since the kernel K is a normal subgroup of G its Lie algebra $\mathfrak{k}$ is an ideal in $\mathfrak{g}$ which is contained in $\mathfrak{h}$. If $\mathfrak{h}' \leq \mathfrak{g}$ is an ideal contained in $\mathfrak{h}$, then the corresponding virtual Lie subgroup is normal in G and contained in H, which implies that $\mathfrak{h}' \subset \mathfrak{k}$, so $\mathfrak{k}$ is the maximal ideal in $\mathfrak{g}$ that is contained in $\mathfrak{h}$.

In several situations (for example to treat Spin structures), requiring effectivity of a Klein geometry would be too much. However, one usually wants the geometry to be *infinitesimally effective*, which means that the kernel K has to be discrete. This may be equivalently characterized as the fact that there is no non–zero ideal in the Lie algebra $\mathfrak{g}$ which is contained in the subalgebra $\mathfrak{h}$.

Motivated by the examples from 1.3.5 and 1.3.6, we next define two important subclasses of Klein geometries. Note that the adjoint action Ad of G on its Lie algebra $\mathfrak{g}$ may be restricted to a representation of the subgroup H on $\mathfrak{g}$. Of course, the Lie subalgebra $\mathfrak{h}$ is an H–invariant subspace of $\mathfrak{g}$.

(1) The Klein geometry (G, H) is called *reductive* if there is an H–invariant subspace $\mathfrak{n} \subset \mathfrak{g}$ which is complementary to $\mathfrak{h}$, i.e. such that $\mathfrak{g} = \mathfrak{n} \oplus \mathfrak{h}$ as an H–module.

(2) The Klein geometry (G, H) is called *split* if there is a Lie subalgebra $\mathfrak{g}_- \subset \mathfrak{g}$, which is complementary to $\mathfrak{h}$ as a vector space, i.e. such that $\mathfrak{g} = \mathfrak{g}_- \oplus \mathfrak{h}$ as a vector space.

When dealing with reductive or split Klein geometries, we will usually assume that the complementary space $\mathfrak{n}$, respectively, $\mathfrak{g}_-$ is fixed as part of the geometry.

Example 1.4.1. Let $H \subset GL(m, \mathbb{R})$ be a closed subgroup and consider the affine extension $B := \mathbb{R}^m \rtimes H \subset A(m, \mathbb{R})$ as defined in 1.3.6. Representing $A(m, \mathbb{R})$ as a matrix group as in 1.3.5 we see that we may view B as the closed subgroup

$$\left\{ \begin{pmatrix} 1 & 0 \\ X & A \end{pmatrix} : X \in \mathbb{R}^m, A \in H \right\} \subset GL(m+1, \mathbb{R}),$$

with the subgroup H corresponding to elements with $X = 0$. Correspondingly, the Lie algebra $\mathfrak{b}$ is the subalgebra of $\mathfrak{gl}(m+1, \mathbb{R})$ of all elements of the form $\left(\begin{smallmatrix} 0 & 0 \\ Y & C \end{smallmatrix}\right)$ with $Y \in \mathbb{R}^m$ and $C \in \mathfrak{h}$. In particular, there is an obvious decomposition $\mathfrak{b} = \mathbb{R}^m \oplus \mathfrak{h}$ as a vector space, and using that the adjoint action of a matrix group is given by conjugation, one immediately verifies that this decomposition is H–invariant. Thus, any Klein geometry of the form (B, H) is reductive. On the other hand, one immediately verifies that the subspace $\mathbb{R}^m \subset \mathfrak{b}$ is not only a Lie subalgebra but even an abelian ideal in $\mathfrak{b}$, so these geometries are naturally split as well.

1.4.2. Homogeneous bundles. Let G be a Lie group, $H \subset G$ a closed subgroup, $M := G/H$ the corresponding homogeneous space and $p : G \to G/H$ the canonical projection, which is an H–principal bundle; see 1.2.5. By $\ell : G \times M \to M$ we denote the canonical left action $\ell(g, g'H) := gg'H$. The first step towards G–invariant geometric objects on M is to get an appropriate class of fiber bundles over the homogeneous space.

Definition 1.4.2. (1) A *homogeneous bundle* over $M = G/H$ is a locally trivial fiber bundle $\pi : E \to M$ together with a left G–action $\tilde{\ell} : G \times E \to E$, which lifts the action on M, i.e. which satisfies $\pi(\tilde{\ell}(g, y)) = \ell(g, \pi(y))$ for all $g \in G$ and $y \in E$.

(2) A *homogeneous vector bundle* over M is a homogeneous bundle $\pi : E \to M$, which is a vector bundle and such that for each element $g \in G$ the bundle map $\tilde{\ell}_g : E \to E$ is a vector bundle homomorphism, i.e. linear in each fiber.

(3) A *homogeneous principal bundle* is a homogeneous bundle $\pi : E \to M$ which is a principal bundle and such that for each $g \in G$ the bundle map $\tilde{\ell}_g$ is a homomorphism of principal bundles, i.e. equivariant for the principal right action of the structure group.

(4) A *morphism* of homogeneous bundles (respectively homogeneous vector bundles or principal bundles) is a G–equivariant bundle map (respectively G–equivariant homomorphism of vector bundles or principal bundles) which covers the identity on M.

If there is no risk of confusion, we will always denote all actions simply by dots, so the definition of a homogeneous bundle simply reads as $\pi(g \cdot y) = g \cdot \pi(y)$. From the definitions it is also clear that homogeneous fiber bundles (respectively homogeneous vector bundles) form a category $\mathrm{Fib}_G(G/H)$ (respectively $\mathrm{Vect}_G(G/H)$).

Example 1.4.2. There are two obvious sources of homogeneous bundles. We shall soon see that the first one is a special case of the second one and that there are no other homogeneous bundles, up to isomorphisms.

(1) Let n be the dimension of the homogeneous space $M = G/H$ and suppose that F is a natural bundle on the category $\mathcal{M}_n$ of n–dimensional manifolds and local diffeomorphisms; see 1.2.8 or [**KMS**, section 14]. Now consider the bundle $p_M : FM \to M$. For any $g \in G$, the left action $\ell_g : M \to M$ of g on M is a diffeomorphism (with inverse $\ell_{g^{-1}}$), so we have the induced bundle map $F(\ell_g) : FM \to FM$ which covers $\ell_g : M \to M$. Since F is a functor, $F(\ell_e) = \mathrm{id}_{FM}$ and $F(\ell_{gg'}) = F(\ell_g \circ \ell_{g'}) = F(\ell_g) \circ F(\ell_{g'})$. Hence, the map $\tilde{\ell} : G \times FM \to FM$, $\tilde{\ell}(g, u) := F(\ell_g)(u)$ defines a left action of G on FM, which is smooth by regularity of F. Thus, any natural bundle over a homogeneous space is a homogeneous bundle.

In particular, we can apply this line of argument to natural vector bundles, for which each of the bundle maps $F(f)$ is a vector bundle homomorphism, to obtain homogeneous vector bundles. In particular, all tensor bundles over G/H are homogeneous vector bundles.

(2) The second basic source of homogeneous bundles is the canonical H–principal bundle $p : G \to G/H$. By definition of the action of G on G/H, this is a homogeneous principal bundle under the action $\tilde{\ell} : G \times G \to G$ which is just given by the multiplication map. Now assume that S is any smooth manifold with a smooth left action $H \times S \to S$. Then we can form the associated (or induced) bundle $E := G \times_H S \to M$. By definition (see 1.2.7) this is the space of orbits in $G \times S$ of the right action $(g, s) \cdot h = (gh, h^{-1} \cdot s)$ of H. We continue to denote the orbit of (g, s) by $[\![g, s]\!]$. Since left and right translations on a Lie group always commute, the map $G \times G \times S \to G \times_H S$ defined by $(g, g', s) \mapsto [\![gg', s]\!]$ descends to a map $G \times (G \times_H S) \to G \times_H S$ which is smooth since $G \times G \times S \to G \times (G \times_H S)$ is a surjective submersion (even an H–principal bundle). From the construction it is clear that this defines a smooth left action of G on E, and since the projection $E \to M$ is simply given by $[\![g, s]\!] \mapsto gH$, this action extends the canonical action of G on M. Thus, any associated bundle to the principal bundle $G \to G/H$ is canonically a homogeneous bundle.

If we start with a representation of H on a vector space V, then the induced bundle $G \times_H V$ is a vector bundle, and the linear structure is determined by

$$[\![g, v]\!] + t[\![g, w]\!] = [\![g, v + tw]\!].$$

Thus, the above construction gives rise to an action of G on $G \times_H V$ by vector bundle homomorphisms, and $G \times_H V \to G/H$ is a homogeneous vector bundle. This construction evidently generalizes to arbitrary homogeneous principal bundles. Any associated bundle (respectively vector bundle) to a homogeneous principal bundle is canonically a homogeneous bundle (respectively homogeneous vector bundle).

1.4.3. Classification of homogeneous bundles. Suppose that $p_E : E \to M$ is a homogeneous fiber bundle over $M = G/H$. Consider the base point $o = eH \in G/H$ and the fiber $S := E_o$ of E over o. For $g \in G$ and $s \in S$ we get $p_E(g \cdot s) = g \cdot o = gH$ by definition of a homogeneous bundle. In particular, $p_E(h \cdot s) = o$ for all $h \in H$, so the left G–action on E restricts to a left action of H on S. Now consider the associated bundle $G \times_H S \to M$, and consider the map $G \times S \to E$ defined by $(g, s) \mapsto g \cdot s$, where we use the action of G on E, and view s as an element of $E_o \subset E$. Since the action of H on S is just the restriction of the action of G on E, this mapping factorizes to a map $\Phi : G \times_H S \to E$ which is smooth since the projection $G \times S \to G \times_H S$ is a surjective submersion. Moreover, by definition $\Phi([\![g, s]\!]) = g \cdot s$, which, on one hand, shows that Φ is a bundle map covering the identity. On the other hand, it implies that Φ is G–equivariant, so $\Phi : G \times_H S \to E$ is a morphism of homogeneous bundles.

If $e \in E$ is any element, then choose an element $g \in G$ such that $g \cdot p_E(e) = o$. Then $p_E(g \cdot e) = o$, so $g \cdot e \in S = E_o$, and we may consider $[\![g^{-1}, g \cdot e]\!] \in G \times_H S$. This is independent of the choice of g, since for another choice $g' \in G$ we must have $(g'g^{-1}) \cdot o = o$, so $g'g^{-1} \in H$, and thus

$$[\![(g')^{-1}, g' \cdot e]\!] = [\![(g')^{-1}g'g^{-1}, g(g')^{-1}g' \cdot e]\!] = [\![g^{-1}, g \cdot e]\!].$$

Thus, we get a well–defined map $\Psi : E \to G \times_H S$. Choosing a local smooth section σ of $G \to G/H$ we can locally write $\Psi(e) = [\![\sigma(p_E(e))^{-1}, \sigma(p_E(e)) \cdot e]\!]$, which shows that Ψ is smooth. One immediately verifies that Ψ and Φ are inverse isomorphisms of homogeneous bundles.

If $f : E \to E'$ is a morphism of homogeneous bundles, then $f(E_o) \subset E'_o$ and the restriction $f|_{E_o} : E_o \to E'_o$ is H–equivariant. Equivariancy of f implies that for $g \in G$ and $s \in S = E_o$, we get $f(g \cdot s) = g \cdot f(s)$ which means that $f(\Phi([\![g, s]\!])) = \Phi'([\![g, f|_{E_o}(s)]\!])$, so we see that identifying E and E' with $G \times_H E_o$, respectively, $G \times_H E'_o$, the map f is induced by $\mathrm{id} \times f|_{E_o}$. Conversely, any H–equivariant map $S \to S'$ clearly induces a morphism $G \times_H S \to G \times_H S'$ of homogeneous bundles.

Finally, note that for a homogeneous vector bundle, the above construction clearly produces an isomorphism $G \times_H E_o \to E$ of homogeneous vector bundles, and homomorphisms of homogeneous vector bundles correspond to linear H–equivariant maps between their standard fibers. Thus, we have proved

PROPOSITION 1.4.3. *The mapping $E \mapsto E_o$ and $f \mapsto f|_{E_o}$ induces equivalences between the category $\mathrm{Fib}_G(G/H)$ and the category of manifolds endowed with left H–actions and H–equivariant smooth maps, as well as between the category $\mathrm{Vect}_G(G/H)$ and the category of finite–dimensional representations of H.*

REMARK 1.4.3. The proposition also implies that these equivalences of categories are compatible with various constructions. For example, the fibered product of homogeneous fiber bundles corresponds to the product of left H–spaces, the Whitney sum of homogeneous vector bundles corresponds to the direct sum of representations and the tensor product of homogeneous vector bundles corresponds to the tensor product of representations. This also works the other way around. If, for example, we start with two homogeneous vector bundles corresponding to indecomposable representations of H, then the decomposition of the tensor product of these two representations into a direct sum of indecomposable representations induces a decomposition of the tensor product of the two homogeneous vector bundles into a Whitney sum.

Example 1.4.3. Let us determine the H–representations corresponding to tensor bundles. We start by identifying the tangent bundle, so we have to determine $T_o(G/H)$ as an H–module. The projection map $p : G \to G/H$ is a submersion and thus for each $g \in G$ the tangent map $T_g p : T_g G \to T_{p(g)}(G/H)$ is surjective. In particular, we get a surjection $T_e p : \mathfrak{g} = T_e G \to T_o(G/H)$. Moreover, the representation of H on $T_o(G/H)$ is just obtained as the restriction of the action of G on the homogeneous vector bundle $T(G/H)$, so the action of $h \in H$ on $T_o(G/H)$ is just $T_o\ell_h$. By definition, the action on G/H is induced by left translations in G, so $\ell_h \circ p = p \circ \lambda_h$ and thus $T_o\ell_h \circ T_e p = T_h p \circ T_e\lambda_h$. On the other hand, p commutes with right multiplications by elements from h, and differentiating this, we see that $T_h p = T_e p \circ T_h \rho^{h^{-1}}$. Since $\rho^{h^{-1}} \circ \lambda_h$ is the conjugation by h, whose derivative is the adjoint action, we get $T_o\ell_h \circ T_e p = T_e p \circ \mathrm{Ad}(h)$. Hence, $T_o(G/H)$ is a quotient of $\mathfrak{g}$, with the H–module structure defined by the restriction of the adjoint representation Ad to the subgroup $H \subset G$. On the other hand, the kernel of $T_e p$ is simply the subalgebra $\mathfrak{h}$, which is also an H–submodule by naturality of the adjoint action.

Thus, the homogeneous vector bundle $T(G/H)$ corresponds to the H–representation on $\mathfrak{g}/\mathfrak{h}$ coming from the restriction to H of the adjoint representation of G. By the proof of Proposition 1.4.3, the isomorphism $G \times_H (\mathfrak{g}/\mathfrak{h}) \to T(G/H)$ maps $[\![g, X+\mathfrak{h}]\!]$ to $T_g p \cdot T_e\lambda_g \cdot X$. The opposite isomorphism can be conveniently described using the Maurer–Cartan form $\omega \in \Omega^1(G, \mathfrak{g})$ of the group G. For a tangent vector $\xi_x \in T_x(G/H)$, we have to choose an element $g \in G$ such that $x = g^{-1}H$ and then view $T\ell_g \cdot \xi_x \in T_o(G/H)$ as an element of $\mathfrak{g}/\mathfrak{h}$. Having chosen g, we know that $T_{g^{-1}}p : T_{g^{-1}}G \to T_x(G/H)$ is surjective, so we can choose a tangent vector $\tilde{\xi} \in T_g G$ such that $\xi_x = Tp \cdot \tilde{\xi}$. But then $T\ell_g \cdot \xi_x = Tp \cdot T\lambda_g \cdot \tilde{\xi}$, and identifying $T_e G$ with $\mathfrak{g}$ and $T_o(G/H)$ with $\mathfrak{g}/\mathfrak{h}$, the projection Tp becomes just the canonical projection to the quotient. Since $T\lambda_g \cdot \tilde{\xi}$ by definition equals $\omega(\tilde{\xi})$, we see that the isomorphism $T(G/H) \to G \times_H (\mathfrak{g}/\mathfrak{h})$ maps ξ_x to $[\![g^{-1}, \omega(\tilde{\xi}) + \mathfrak{h}]\!]$.

By the naturality of the correspondence between homogeneous vector bundles and H–representations, this implies that the cotangent bundle $T^*(G/H)$ corresponds to the dual representation $(\mathfrak{g}/\mathfrak{h})^*$. Note that $(\mathfrak{g}/\mathfrak{h})^*$ is just the annihilator of $\mathfrak{h}$ in $\mathfrak{g}^*$, so this is a subrepresentation of the restriction of the coadjoint representation to H. Again, by naturality, the tensor bundle $\otimes^k T(G/H) \otimes \otimes^\ell T^*(G/H)$ corresponds to the representation $\otimes^k(\mathfrak{g}/\mathfrak{h}) \otimes \otimes^\ell(\mathfrak{g}/\mathfrak{h})^*$.

There is an interesting way to rephrase the description of the tangent bundle as an associated bundle, namely as a first order G–structure underlying a Klein geometry. The representation of H on $\mathfrak{g}/\mathfrak{h}$ induced by the restriction of the adjoint action of G may be viewed as a homomorphism $H \to GL(\mathfrak{g}/\mathfrak{h})$. The kernel of this homomorphism is a closed subgroup $K_1 \subset H$ and we define $H_1 := H/K_1$. Then we may consider the homogeneous space G/K_1 which clearly is a principal H_1–bundle over G/H. Viewing the manifold G/H as being modelled on the vector space $\mathfrak{g}/\mathfrak{h}$, we may interpret the frame bundle $\mathcal{P}^1(G/H)$ as the bundle of linear isomorphisms between $\mathfrak{g}/\mathfrak{h}$ and tangent spaces of G/H. Now consider $\pi \circ \omega \in \Omega^1(G, \mathfrak{g}/\mathfrak{h})$, where ω is the Maurer–Cartan form of G and $\pi : \mathfrak{g} \to \mathfrak{g}/\mathfrak{h}$ is the canonical surjection. This one–form is clearly H–equivariant (since the Maurer–Cartan form is even G–equivariant) and its kernel in any point of G is exactly the vertical tangent space in that point. Hence, from 1.3.6 we conclude that there is a unique homomorphism $G \to \mathcal{P}^1(G/H)$ of principal bundles such that $\pi \circ \omega$ is the pullback of the soldering form. This homomorphism factors to a homomorphism $G/K \to \mathcal{P}^1(G/H)$ of principal bundles, which defines a first order H_1–structure on G/H.

There is a natural higher order analog of this construction. We can view $GL(\mathfrak{g}/\mathfrak{h})$ as the group of one–jets at $o = eH \in G/H$ of diffeomorphisms fixing o. The homomorphism $H \to GL(\mathfrak{g}/\mathfrak{h})$ from above is then given by mapping $h \in H$ to the one–jet of ℓ_h in o. Now we can replace $GL(\mathfrak{g}/\mathfrak{h})$ by the group of invertible r–jets in o of diffeomorphisms fixing o. This group is clearly isomorphic to the rth jet group G^r_n introduced in 1.2.8, where $n = \dim(G/H)$. Mapping $h \in H$ to the r–jet of ℓ_h in o defines a homomorphism. Defining K_r to be the kernel of this homomorphism and putting $H_r = H/K_r$, the obvious projection $G/K_r \to G/H$ is a principal bundle with structure group H_r. By construction, this can be viewed as a reduction of the rth order frame bundle $\mathcal{P}^r(G/H)$ (see 1.2.8), i.e. as an rth order G–structure. This point of view has been explored in [**Sl96**].

1.4.4. Sections of homogeneous bundles. Let $p_E : E \to G/H$ be a homogeneous bundle over a homogeneous space G/H, and let $\Gamma(E)$ be the set of all sections of E, i.e. $\Gamma(E) = \{\sigma : G/H \to E : p_E \circ \sigma = \mathrm{id}\}$. Then we get a left action of G on $\Gamma(E)$ defined by $g \cdot \sigma := \tilde{\ell}_g \circ \sigma \circ \ell_{g^{-1}}$, where $\tilde{\ell}$ and ℓ are the actions on E and G/H, respectively. Equivalently, this can be written as $(g \cdot \sigma)(x) = g \cdot (\sigma(g^{-1} \cdot x))$. In particular, if E is a homogeneous vector bundle, then $\Gamma(E)$ is a vector space, and the action of G on $\Gamma(E)$ is clearly linear, and thus we get a representation of G on the space $\Gamma(E)$. In view of Proposition 1.4.3, the homogeneous vector bundle E is completely determined by the representation of H on its standard fiber $V = E_o$. The representation of G on $\Gamma(E)$ is then called the *induced representation* of G corresponding to the representation V of H, and denoted by $\mathrm{Ind}_H^G(V)$.

The correspondence between sections of an associated bundle and functions of the total space of the principal bundle (see 1.2.7) gives rise to another interpretation of the action of G on the set of sections of a homogeneous bundle. This picture is (in the case of homogeneous vector bundles) frequently used in representation theory. Namely, via the isomorphism $E \cong G \times_H E_o$ from Proposition 1.4.3, we can identify $\Gamma(E)$ with $\Gamma(G \times_H E_o)$, which in turn by Proposition 1.2.7 is in bijective correspondence with the set $C^\infty(G, E_o)^H = \{f : G \to E_o : f(gh) = h^{-1} \cdot f(g)\}$. Explicitly, the correspondence between a section σ and a function f is characterized by $\sigma(gH) = [\![g, f(g)]\!] = g \cdot f(g)$ or by $f(g) = g^{-1} \cdot \sigma(gH)$. In the case of a natural vector bundle, this bijection clearly is a linear isomorphism.

To get the action of G on $\Gamma(E)$ in the picture of equivariant functions, we only have to notice that by definition $(g \cdot \sigma)(g'H) = g \cdot \sigma(g^{-1}g'H)$. Consequently, the map $g \cdot f : G \to E_o$ corresponding to $g \cdot \sigma \in \Gamma(E)$ is given by

$$(g \cdot f)(g') = (g')^{-1} \cdot (g \cdot \sigma(g^{-1}g'H)) = (g^{-1}g')^{-1} \cdot \sigma(g^{-1}g'H) = f(g^{-1}g').$$

Our next result is a first version of Frobenius reciprocity, which states that in certain situations one can reduce questions about the (often infinite–dimensional) space $\Gamma(E)$ to questions about the finite–dimensional manifold E_o. In particular, determining G–invariant sections of a homogeneous bundle always reduces to a finite–dimensional problem.

THEOREM 1.4.4 (Geometric version of Frobenius reciprocity). *Let $E \to G/H$ be a homogeneous bundle with standard fiber E_o (viewed as an H–space), and let X be a smooth manifold endowed with a smooth left G–action. Then the evaluation* *at o induces a natural bijection between the set of G–equivariant maps $X \to \Gamma(E)$ and the set of H–equivariant maps $X \to E_o$. In particular, there is a natural bijection between the set $\Gamma(E)^G$ of G–invariant sections of E and the set $(E_o)^H$ of H–invariant elements in the standard fiber.*

If E is the natural vector bundle induced by an H–representation W, and V is a representation of G, then the bijections above are linear and respect the subspaces of linear equivariant maps. Denoting by $\text{Res}^G_H(V)$ the space V viewed as an H–representation, this implies that we get a linear isomorphism

$$\text{Hom}_G(V, \text{Ind}^G_H(W)) \cong \text{Hom}_H(\text{Res}^G_H(V), W),$$

i.e. Res^G_H and Ind^G_H are adjoint functors.

Proof. Suppose we have given a map $X \to \Gamma(E)$, which we write as $x \mapsto \sigma_x$. This map is G–equivariant if and only if $\sigma_{g\cdot x}(g'H) = g \cdot \sigma_x(g^{-1}g'H)$. In particular, if we consider the map $X \to E_o$ defined by $x \mapsto \sigma_x(o)$, then for $h \in H$, we get $\sigma_{h\cdot x}(o) = h \cdot \sigma_x(h^{-1} \cdot o) = h \cdot \sigma_x(o)$, so this is H–equivariant. Conversely, if $f : X \to E_o$ is H–equivariant, then for $x \in X$ we define $\sigma_x : G/H \to E$ by $\sigma_x(gH) = g \cdot f(g^{-1} \cdot x)$. This is well defined since $f(h^{-1}g^{-1} \cdot x) = h^{-1} \cdot f(g^{-1} \cdot x)$, and thus $gh \cdot f(h^{-1}g^{-1} \cdot x) = g \cdot f(g^{-1} \cdot x)$. Moreover, $\sigma_x(o) = f(x)$ and

$$\sigma_{g\cdot x}(g'H) = g' \cdot f((g')^{-1}g \cdot x) = g \cdot g^{-1}g' \cdot f((g')^{-1}g \cdot x) = g \cdot \sigma_x(g^{-1}g'H),$$

so $x \mapsto \sigma_x$ is G–equivariant. Since for any G–equivariant map $x \mapsto \sigma_x$ we must have $\sigma_x(gH) = g \cdot g^{-1} \cdot \sigma_x(g^{-1}gH) = g \cdot \sigma_{g\cdot x}(o)$, the two constructions are inverse bijections between the set of G–equivariant maps $X \to \Gamma(E)$ and the set of H–equivariant maps $X \to E_o$.

Taking $X = \{pt\}$ a single point space with the trivial G–action, a G–equivariant map $X \to \Gamma(E)$ is the same thing as a section $\sigma_{pt} \in \Gamma(E)$ such that $g \cdot \sigma_{pt} = \sigma_{pt}$, i.e. a G–invariant section, and similarly an H–equivariant map $X \to E_o$ is an H–invariant element in E_o. Thus, we get a bijection $\Gamma(E)^G \to (E_o)^H$.

Finally, if E is a natural vector bundle induced by an H–representation W and X is a G–representation V, then $L(V, \Gamma(E))$ and $L(V, E_o)$ are vector spaces under the pointwise operations, and the evaluation in o induces a linear map $L(V, \Gamma(E)) \to L(V, E_o)$. If we start from a linear map $f : V \to E_o$, the construction above produces $\sigma_v(gH) = g \cdot f(g^{-1} \cdot v)$. Since f is linear and G acts by linear maps, this is linear in v. □

Example 1.4.4. The geometric version of Frobenius reciprocity immediately allows us to reduce questions about the existence of invariant Riemannian metrics and other invariant tensor fields to problems of finite–dimensional representation theory: If $M = G/H$ is a homogeneous space, then from 1.4.3 we know that the tensor bundle $\otimes^k TM \otimes \otimes^\ell T^*M$ is the associated bundle to $p : G \to M$ corresponding to the representation $\otimes^k(\mathfrak{g}/\mathfrak{h}) \otimes \otimes^\ell(\mathfrak{g}/\mathfrak{h})^*$. By Frobenius reciprocity, G–invariant sections of this bundle are in bijective correspondence with H–invariant elements in the representation. Since invariant elements in a representation are the same thing as homomorphisms from the trivial representation to the given representation, this can be rephrased in such a way that the dimension of invariant tensor fields of type $\binom{k}{\ell}$ on G/H equals the multiplicity of the trivial representation in $\otimes^k(\mathfrak{g}/\mathfrak{h}) \otimes \otimes^\ell(\mathfrak{g}/\mathfrak{h})^*$. Moreover, the proof of Theorem 1.4.4 gives an explicit construction of the invariant tensor field from the invariant element in the representation. Clearly, the whole construction respects symmetries of any kind, so invariant tensor fields having certain symmetries are in bijective correspondence to invariant elements in the representations having the same symmetries. Finally, if we consider (pointwise) questions of non–degeneracy they clearly reduce to the analogous non–degeneracy properties in the representation. In particular, the set of G–invariant Riemannian metrics on G/H is in bijective correspondence with the set of H–invariant positive definite inner products on the vector space $\mathfrak{g}/\mathfrak{h}$. Hence, we get

Corollary 1.4.4. *A homogeneous space G/H admits a G–invariant Riemannian metric if and only if the image $H_1 \subset GL(\mathfrak{g}/\mathfrak{h})$ of H under the map induced by the restriction to H of the adjoint representation of G has compact closure in $GL(\mathfrak{g}/\mathfrak{h})$.*

Proof. From above we know that existence of an invariant Riemannian metric is equivalent to existence of an H_1–invariant positive definite inner product on $\mathfrak{g}/\mathfrak{h}$. If such an inner product exists, then H_1 is contained in the orthogonal group of this inner product, which is isomorphic to $O(n)$ for some n and thus compact. Hence, H_1 has compact closure.

Conversely, if $K \subset GL(\mathfrak{g}/\mathfrak{h})$ is a compact subgroup containing H_1, then averaging any inner product on $\mathfrak{g}/\mathfrak{h}$ over K gives a K–invariant and thus an H_1–invariant inner product. □

Consider the examples from 1.1 of viewing spheres as homogeneous spaces. In the case of the Riemannian sphere 1.1.1, we have $H = O(n)$ and $\mathfrak{g}/\mathfrak{h} \cong \mathbb{R}^n$ as an H–representation, so the set of $O(n+1)$–invariant Riemannian metrics on S^n consists exactly of all constant positive multiples of the standard metric. The description of invariant Riemannian metrics on $\mathbb{R}^n$, viewed as a homogeneous space of the Euclidean group $\mathrm{Euc}(n)$ is exactly the same. For the projective sphere from 1.1.3, we get $H_1 = GL(\mathfrak{g}/\mathfrak{h})$, so there certainly is no $SL(n+1,\mathbb{R})$–invariant metric on S^n. Similarly, one shows that in the other examples from 1.1.4–1.1.6 there are no invariant Riemannian metrics.

Finally, we remark that invariance often allows us to analyze the situation further in a pretty elementary way. For example, applying natural operators to invariant objects, we get invariant objects. Thus, if ϕ is an invariant differential form on G/H, then the exterior derivative $d\phi$ is invariant as well. Similarly, the curvature of an invariant Riemannian metric has to be an invariant tensor field, and so on.

1.4.5. Homogeneous principal bundles and invariant principal connections. Next we switch to the problem of invariant differential operators on a homogeneous space G/H. A simple special case of this problems is the question whether a given homogeneous bundle admits a connection which is compatible with the action of G. Since connections can be viewed as sections of natural bundles, we could reduce this to the determination of invariant sections. A direct discussion will be easier, however. First, we classify homogeneous principal bundles as defined in 1.4.2.

Lemma 1.4.5. *Let G and K be Lie groups and let $H \subset G$ be a closed subgroup. Let $\mathcal{P} \to G/H$ be a homogeneous principal bundle with structure group K. Then there is a smooth homomorphism $i : H \to K$ such that $\mathcal{P} \cong G \times_H K$, where the action of H on K is given by $h \cdot k = i(h)k$ for $h \in H$ and $k \in K$.*

The bundles corresponding to two homomorphism $i, \hat{i} : H \to K$ *are isomorphic (over* $\mathrm{id}_{G/H}$*) if and only if* i *and* $\hat{i}$ *are conjugate, i.e.* $\hat{i}(h) = ki(h)k^{-1}$ *for some fixed* $k \in K$ *and all* $h \in H$*.*

Proof. Let $\mathcal{P}_o$ be the fiber of $\mathcal{P}$ over the base point $o = eH \in G/H$. As discussed in 1.4.3 the left action of G on $\mathcal{P}$ restricts to a left action of H on $\mathcal{P}_o$, and $\mathcal{P} \cong G \times_H \mathcal{P}_o$ as a homogeneous bundle. Fixing a point $u_0 \in \mathcal{P}_o$, the map $k \mapsto u_0 \cdot k$ is a diffeomorphism $K \to \mathcal{P}_0$, so it remains to describe the H–action in this picture.

For $h \in H$, we have $h \cdot u_0 \in \mathcal{P}_o$, so there is a unique element $i(h) \in K$ such that $h \cdot u_0 = u_0 \cdot i(h)$. By smoothness of the two actions, the map $i : H \to K$ is smooth, and by definition $i(e) = e$. Since H acts by principal bundle maps, we see that $h \cdot (u_0 \cdot k) = (h \cdot u_0) \cdot k = u_0 \cdot i(h)k$. Using this, one immediately concludes that $i(h_1h_2) = i(h_1)i(h_2)$, so i is a homomorphism.

Suppose that we have given an isomorphism $G \times_i K \to G \times_{\hat{i}} K$ of homogeneous principal bundles. Then the restriction to the fibers over o is a diffeomorphism $\phi : K \to K$ which commutes with the principal right action of K and is equivariant for the two left actions of H. By the first property, $\phi(k) = k_0k$, where $k_0 = \phi(e)$. But then the second property reads as follows:

$$k_0i(h)k = \phi(i(h)k) = \hat{i}(h)\phi(k) = \hat{i}(h)k_0k.$$

In particular, $\hat{i}(h) = k_0i(h)k_0^{-1}$. Conversely, if i and $\hat{i}$ are related in this way, right multiplication by k_0 induces a diffeomorphism with the two equivariancy properties. From 1.4.3 we conclude that this gives rise to an isomorphism of homogeneous principal bundles. □

For a homogeneous principal bundle $\mathcal{P} \to G/H$ with structure group K, we can next study invariant principal connections. Recall from 1.3.3 that a principal connection on $\mathcal{P}$ can be described by a one–form $\gamma \in \Omega^1(\mathcal{P}, \mathfrak{k})$, where $\mathfrak{k}$ is the Lie algebra of K. This form has to be K–equivariant and it has to reproduce the generators of fundamental vector fields. Denoting by r^k the principal right action by $k \in K$ and by ζ_A the fundamental vector field generated by $A \in \mathfrak{k}$, these conditions explicitly read as $(r^k)^*\gamma = \mathrm{Ad}(k^{-1}) \circ \gamma$ and $\gamma(\zeta_A) = A$. Denoting by ℓ_g the left action of $g \in G$ on $\mathcal{P}$, we can consider the pullback $(\ell_g)^*\gamma$. Since ℓ_g is a principal bundle automorphism, this is again a principal connection. We call γ an *invariant principal connection* if and only if $(\ell_g)^*\gamma = \gamma$ for all $g \in G$.

Theorem 1.4.5. *Let* $i : H \to K$ *be a homomorphism and consider the corresponding homogeneous bundle* $\mathcal{P} := G \times_i K \to G/H$*. Then invariant principal connections on* $\mathcal{P}$ *are in bijective correspondence with linear maps* $\alpha : \mathfrak{g} \to \mathfrak{k}$ *such that:*

(i) $\alpha|_{\mathfrak{h}} = i' : \mathfrak{h} \to \mathfrak{k}$*, the derivative of* i*.*
(ii) $\alpha \circ \mathrm{Ad}(h) = \mathrm{Ad}(i(h)) \circ \alpha$ *for all* $h \in H$*.*

Putting $\hat{i}(h) = k_0i(h)k_0^{-1}$*, then under the isomorphism* $G \times_i K \to G \times_{\hat{i}} K$ *from the lemma, the homogeneous principal connection on* $G \times_i K$ *induced by* α *corresponds to the one on* $G \times_{\hat{i}} K$ *induced by* $\hat{\alpha} = \mathrm{Ad}(k_0) \circ \alpha$*.*

Proof. Consider the point $u_0 = [\![e, e]\!] \in G \times_i K$. Given an invariant principal connection γ on $\mathcal{P}$, consider its value at u_0, which is a linear map $\gamma(u_0) : T_{u_0}\mathcal{P} \to \mathfrak{k}$. Recall from 1.2.7 that the natural projection $q : G \times K \to G \times_i K$ is an H–principal bundle and, in particular, a surjective submersion. By definition $u_0 = q(e, e)$, so we may consider $\gamma(u_0) \circ T_{(e,e)}q : \mathfrak{g} \times \mathfrak{k} \to \mathfrak{k}$. Now we define $\alpha : \mathfrak{g} \to \mathfrak{k}$ by

$$\alpha(X) := \gamma(u_0) \circ T_{(e,e)}q \cdot (X, 0).$$

First, observe that for $k \in K$ we have $q(e, k) = u_0 \cdot k$. Putting $k = \exp(tA)$ for $A \in \mathfrak{k}$ and differentiating in $t = 0$, we see that $T_{(e,e)}q \cdot (0, A) = \zeta_A(u_0)$. This shows that $\gamma(u_0) \circ T_{(e,e)}q \cdot (X, A) = \alpha(X) + A$, and hence α determines $\gamma(u_0)$. Equivariancy under the principal right action then implies that $\gamma(u_0 \cdot k) = \mathrm{Ad}(k^{-1}) \circ \gamma(u_0) \circ Tr^{k^{-1}}$, so γ is determined along the fiber over o. Further, equivariancy under the left action of G implies

$$\gamma(g \cdot u_0 \cdot k) = \mathrm{Ad}(k^{-1}) \circ \gamma(u_0) \circ Tr^{k^{-1}} \circ T\ell_{g^{-1}}, \tag{1.10}$$

which shows that γ is completely determined by α.

On the other hand, for $h \in H$ we have $q(h, e) = q(e, i(h))$. Putting $h = \exp(tX)$ for $X \in \mathfrak{h}$ and differentiating in $t = 0$ we obtain

$$T_{(e,e)}q \cdot (X, 0) = T_{(e,e)}q \cdot (0, i'(X)),$$

which shows that $\alpha(X) = i'(X)$ for $X \in \mathfrak{h}$.

For $h \in H$ and $g \in G$, we next have

$$q(hgh^{-1}, e) = q(hg, i(h^{-1})) = \ell_h \circ r^{i(h^{-1})} q(g, e).$$

Putting $g = \exp(tX)$ for $X \in \mathfrak{g}$ and differentiating at $t = 0$, we obtain

$$T_{(e,e)}q \cdot (\mathrm{Ad}(h)(X), 0) = T\ell_h \circ Tr^{i(h^{-1})} \cdot T_{(e,e)}q \cdot (X, 0).$$

Applying $\gamma(u_0)$ to the left–hand side, we obtain $\alpha(\mathrm{Ad}(h)(X))$. For the right–hand side, we first note that G–invariance of γ implies that $\gamma(h \cdot u_0) \circ T\ell_h = \gamma(u_0)$. Then K–equivariancy implies that composing with $Tr^{i(h^{-1})}$, one obtains

$$\mathrm{Ad}(i(h)) \circ \gamma(h \cdot u_0 \cdot i(h^{-1})) = \mathrm{Ad}(i(h)) \circ \gamma(u_0).$$

Therefore, the right–hand side evaluates to $\mathrm{Ad}(i(h))(\alpha(X))$ which proves property (ii).

It remains to conversely construct an invariant principal connection γ from a linear map α with properties (i) and (ii). The obvious way is to require

$$\gamma(u_0)(T_{(e,e)}q \cdot (X, A)) := \alpha(X) + A.$$

This is well defined if and only if $(X, A) \mapsto \alpha(X) + A$ vanishes on the kernel of $T_{(e,e)}q$. Since $q^{-1}(u_0) = \{(h, i(h^{-1})) : h \in H\} \subset G \times K$, this kernel evidently consists of all elements of the form $(X, -i'(X))$ for $X \in \mathfrak{h}$. By property (i) we conclude that the above formula uniquely defines $\gamma(u_0) : T_{u_0}\mathcal{P} \to \mathfrak{k}$. We have seen above that $\zeta_A(u_0) = T_{(e,e)}q \cdot (0, A)$, so $\gamma(u_0)(\zeta_A(u_0)) = A$. Now one extends γ to all points using formula (1.10) from above as a definition. Since

$$q(g, k) = g \cdot u_0 \cdot k = \tilde{g} \cdot u_0 \cdot \tilde{k} = q(\tilde{g}, \tilde{k})$$

if and only if $\tilde{g} = gh$ and $\tilde{k} = i(h^{-1})k$ for some $h \in H$, one easily verifies that this is well defined using property (ii). Next, one immediately concludes that the resulting form γ is G–invariant and K–equivariant. Finally, from the definition one verifies that

$$\zeta_A(g \cdot u_0 \cdot k) = T\ell_g \cdot Tr^k \cdot \zeta_{\mathrm{Ad}(k)(A)}(u_0)$$

for all $A \in \mathfrak{k}$. Then $\gamma(\zeta_A) = A$ follows from the definition, whence γ is an invariant principal connection.

For the last statement, observe that the isomorphism $\phi : G \times_{\hat{i}} K \to G \times_i K$ from the proof of the lemma is characterized $\phi(\hat{q}(g,k)) = q(g, k_0^{-1}k)$. In particular,

$$\phi(\hat{q}(g,e)) = q(g, k_0^{-1}) = r^{k_0^{-1}}(q(g,e)).$$

Putting $g = \exp(tX)$ and differentiating at $t = 0$, we obtain $T_{(e,e)}(\phi \circ \hat{q}) \cdot (X, 0) = Tr^{k_0^{-1}} \cdot T_{(e,e)}q \cdot (X,0)$. Using this, we compute

$$\begin{aligned}\hat{\alpha}(X) &= (\phi^*\gamma)(\hat{u}_0) \circ T_{(e,e)}\hat{q} \cdot (X,0) = \gamma(u_0 \cdot k_0^{-1})(T_{(e,e)}(\phi \circ \hat{q}) \cdot (X,0)) \\ &= \gamma(u_0 \cdot k_0^{-1})(Tr^{k_0^{-1}} \cdot T_{(e,e)}q \cdot (X,0)) = ((r^{k_0^{-1}})^*\gamma)(T_{(e,e)}q \cdot (X,0)) \\ &= \mathrm{Ad}(k_0)(\alpha(X)).\end{aligned}$$

□

Notice that in general, a linear map $\alpha : \mathfrak{g} \to \mathfrak{k}$ with properties (i) and (ii) need not exist. (We shall soon see this in an example.) If there is one such map, however, then for any other map with properties (i) and (ii), the difference vanishes on $\mathfrak{h}$, and hence defines a linear map $\beta : \mathfrak{g}/\mathfrak{h} \to \mathfrak{k}$. By construction, β is equivariant for the actions $h \cdot (X + \mathfrak{h}) := \mathrm{Ad}(h)(X) + \mathfrak{h}$ on the left–hand side and $h \cdot A = \mathrm{Ad}(i(h))(A)$ on the right–hand side. Conversely, given α with properties (i) and (ii) and an equivariant map β, then $\hat{\alpha}(X) = \alpha(X) + \beta(X + \mathfrak{h})$ also has properties (i) and (ii). Hence, we conclude that if one invariant principal connection exists, then the space of all such connections is an affine space modelled on the vector space $\mathrm{Hom}_H(\mathfrak{g}/\mathfrak{h}, \mathfrak{k})$ with actions as above.

1.4.6. The curvature of an invariant principal connection. In 1.3.3 we have seen that the curvature ρ of a principal connection γ on a principal K–bundle $\mathcal{P} \to M$ can be interpreted as a two–form on M with values in the bundle $\mathcal{P} \times_K \mathfrak{k}$. Here K acts on its Lie algebra $\mathfrak{k}$ via the adjoint action. Moreover, we have seen there that ρ is induced by the $\mathfrak{k}$–valued two–form $(\xi, \eta) \mapsto d\gamma(\xi, \eta) + [\gamma(\xi), \gamma(\eta)]$ on $\mathcal{P}$.

Now suppose that $\mathcal{P}$ is a homogeneous principal K–bundle over G/H. From 1.4.5 we know that there is a homomorphism $i : H \to K$ such that $\mathcal{P} = G \times_i K$. Hence, we can identify the bundle $\mathcal{P} \times_K \mathfrak{k}$ with $G \times_H \mathfrak{k}$ where the action of H on $\mathfrak{k}$ is given by $h \cdot A := \mathrm{Ad}(i(h))(A)$. From Example 1.4.3 we know that $T^*(G/H) \cong G \times_H (\mathfrak{g}/\mathfrak{h})^*$, with the action induced by the adjoint action. Hence, the curvature of any principal connection γ on $\mathcal{P}$ has values in the homogeneous vector bundle corresponding to the representation $\Lambda^2(\mathfrak{g}/\mathfrak{h})^* \otimes \mathfrak{k}$, which consists of all skew symmetric bilinear maps $\mathfrak{g}/\mathfrak{h} \times \mathfrak{g}/\mathfrak{h} \to \mathfrak{k}$.

Suppose further that γ is an invariant principal connection on $\mathcal{P}$. Denoting by $\ell_g : \mathcal{P} \to \mathcal{P}$ the action of $g \in G$, by definition we have $(\ell_g)^*\gamma = \gamma$. This immediately implies that the $\mathfrak{k}$–valued two–form on $\mathcal{P}$ which induces the curvature ρ is preserved by each ℓ_g. Hence, $\rho \in \Omega^2(G/H, G \times_H \mathfrak{k})$ is an invariant section. By Theorem 1.4.4, such a section is equivalent to a bilinear map $\mathfrak{g}/\mathfrak{h} \times \mathfrak{g}/\mathfrak{h} \to \mathfrak{k}$, which is H–equivariant. We can determine this bilinear map explicitly:

PROPOSITION 1.4.6. *Consider a homogeneous space G/H and the homogeneous principal K–bundle $\mathcal{P} = G \times_i K \to G/H$ corresponding to a homomorphism $i : H \to K$. Let γ be an invariant principal connection on $\mathcal{P}$ and let $\alpha : \mathfrak{g} \to \mathfrak{k}$ be the corresponding linear map as in Theorem 1.4.5.*

Then the map $\mathfrak{g}\times\mathfrak{g}\to\mathfrak{k}$ *defined by* $(X,Y)\mapsto[\alpha(X),\alpha(Y)]-\alpha([X,Y])$ *descends to an* H*–equivariant map* $(\mathfrak{g}/\mathfrak{h})\times(\mathfrak{g}/\mathfrak{h})\to\mathfrak{k}$. *The corresponding invariant section of the homogeneous bundle* $G\times_H(\Lambda^2(\mathfrak{g}/\mathfrak{h})^*\otimes\mathfrak{k})$ *is exactly the curvature of* γ.

Proof. Property (ii) of α from Theorem 1.4.5 reads as $\alpha\circ\mathrm{Ad}(h)=\mathrm{Ad}(i(h))\circ\alpha$ for all $h\in H$. Using this, we compute

$$\begin{aligned}[\alpha(\mathrm{Ad}(h)(X)),\alpha(\mathrm{Ad}(h)(Y))]&=[\mathrm{Ad}(i(h))(\alpha(X)),\mathrm{Ad}(i(h))(\alpha(Y))]\\&=\mathrm{Ad}(i(h))\left([\alpha(X),\alpha(Y)]\right).\end{aligned}$$

In the same way, one verifies that $(X,Y)\mapsto\alpha([X,Y])$ is equivariant. Further, for $X\in\mathfrak{h}$ we can apply the equation $\alpha\circ\mathrm{Ad}(h)=\mathrm{Ad}(i(h))\circ\alpha$ to $h=\exp(tX)$ and differentiate at $t=0$, to get $\alpha\circ\mathrm{ad}(X)=\mathrm{ad}(i'(X))\circ\alpha$. By property (i) of α from Theorem 1.4.5 we have $\alpha(X)=i'(X)$ for $X\in\mathfrak{h}$, and we get

$$[\alpha(X),\alpha(Y)]-\alpha([X,Y])=\mathrm{ad}(i'(X))(\alpha(Y))-\alpha(\mathrm{ad}(X)(Y))=0.$$

This implies that our bilinear map descends to an H–equivariant mapping as required. It remains to verify that the corresponding invariant section equals the curvature of γ.

To verify this, we have to recall the definition of α. Let $q:G\times K\to G\times_i K$ be the canonical projection. Then we can form the pullback $q^*\gamma\in\Omega^1(G\times K,\mathfrak{k})$, and from the proof of Theorem 1.4.5 we see that $q^*\gamma(e,e)(X,A)=\alpha(X)+A$ for $X\in\mathfrak{g}=T_eG$ and $A\in\mathfrak{k}=T_eK$. Our strategy will be to compute

$$dq^*\gamma(\xi,\eta)+[q^*\gamma(\xi),q^*\gamma(\eta)](e,e). \tag{1.11}$$

Now by definition $q(g,kk')=q(g,k)\cdot k'$. Taking $A\in\mathfrak{k}$, putting $k'=\exp(tA)$ and differentiating at $t=0$, we see that Tq maps $(0,L_A)$ to the (vertical) fundamental vector field ζ_A. Since $d\gamma(\zeta_A,\_)+[A,\gamma(\_)]=0$, we conclude that it suffices to compute (1.11) for $\xi=(L_X,0)$ and $\eta=(L_Y,0)$ with $X,Y\in\mathfrak{g}$.

By definition $q(gg',k)=\ell_g\circ r^k(q(g',e))$. Putting $g'=\exp(tX)$ and differentiating at $t=0$, we see that $T_{(g,k)}q\cdot(L_X,0)=T\ell_g\circ Tr^k\cdot T_{(e,e)}q\cdot(X,0)$. Applying γ and taking into account G–invariance and K–equivariance (see formula (1.10) in 1.4.5), we conclude that $q^*\gamma(g,k)(L_X,0)=\mathrm{Ad}(k^{-1})(\alpha(X))$. In particular, this function is independent of g, so acting on it with a vector field of the form $(L_Y,0)$ we get zero. This shows that

$$\begin{aligned}dq^*\gamma(e,e)((L_X,0),(L_Y,0))&=-q^*\gamma(e,e)([L_X,L_Y],0)\\&=-q^*\gamma(e,e)([X,Y])=-\alpha([X,Y]).\end{aligned}$$

This implies that (1.11) is given by $((X,A),(Y,B))\mapsto-\alpha([X,Y])+[\alpha(X),\alpha(Y)]$. Since the composition $G\times K\to\mathcal{P}\to G/H$ is given by $(g,k)\mapsto gH$, we conclude that (X,A) represents the tangent vector $X+\mathfrak{h}\in\mathfrak{g}/\mathfrak{h}\cong T_oG/H$, and the result follows. □

The most fundamental example of a homogeneous principal bundle is of course the bundle $G\to G/H$ itself. As we shall see below, the existence of an invariant principal connection on this bundle has far reaching consequences.

Corollary 1.4.6. *There exists a* G*–invariant principal connection on the* H*–principal bundle* $p:G\to G/H$*, if and only if the Klein geometry* (G,H) *is reductive. If this is the case and* $\mathfrak{n}\subset\mathfrak{g}$ *is a fixed* H*–invariant complement to* $\mathfrak{h}$*, then the set of all such connections is an affine space modelled on the vector space* $\mathrm{Hom}_H(\mathfrak{n},\mathfrak{h})$.

The curvature of the invariant connection defined by a complement $\mathfrak{n}$ *is induced by the map* $\Lambda^2\mathfrak{n} \to \mathfrak{h}$ *given by* $(X,Y) \mapsto -[X,Y]_{\mathfrak{h}}$*. Here the subscript denotes the* $\mathfrak{h}$*–component with respect to the decomposition* $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{n}$*.*

Proof. The bundle $G \to G/H$ of course corresponds to $i = \mathrm{id}_H : H \to H$. By Theorem 1.4.5 existence of an invariant principal connection is equivalent to existence of an H–equivariant map $\alpha : \mathfrak{g} \to \mathfrak{h}$ such that $\alpha|_{\mathfrak{h}} = i' = \mathrm{id}_{\mathfrak{h}}$. Putting $\mathfrak{n} = \ker(\alpha)$ this condition is equivalent to $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{n}$ as an H–module. Fixing the choice of α (respectively $\mathfrak{n}$), we have $\mathfrak{n} \cong \mathfrak{g}/\mathfrak{h}$ as an H–module, and the remaining claims follow from Theorem 1.4.5 and Proposition 1.4.6. □

Example 1.4.6. (1) As a more specific example, suppose that the subgroup $H \subset G$ is compact. Then averaging any positive definite inner product on $\mathfrak{g}$ over H, we obtain an H–invariant positive definite inner product on $\mathfrak{g}$. Now, the orthogonal projection onto $\mathfrak{h}$ with respect to such an inner product is H–equivariant, thus giving a G–invariant principal connection on $p : G \to G/H$. The complementary subset $\mathfrak{n}$ is then $\mathfrak{h}^\perp$.

In particular, consider the case where $H = O(n)$ and $\mathfrak{g}/\mathfrak{h} \cong \mathfrak{n} \cong \mathbb{R}^n$ as an H–module, which occurs both in the case of the Riemannian sphere from 1.1.1 (with $G = O(n+1)$) and the Euclidean space, viewed as a homogeneous space of the group of Euclidean motions from 1.1.2. In both cases, we not only have a G–invariant principal connection on $p : G \to G/H$ but it is also uniquely determined. This is due to the fact that both $\mathfrak{n} \cong \mathbb{R}^n$ and $\mathfrak{h} \cong \mathfrak{o}(n)$ are irreducible $O(n)$–modules and thus the zero map is the only H–homomorphism $\mathfrak{n} \to \mathfrak{h}$. In the case of the Euclidean space 1.1.2, the complement $\mathfrak{n}$ is an abelian subalgebra, so $[\mathfrak{n},\mathfrak{n}] = \{0\}$ and the principal connection is flat.

On the other hand, in the case of the Riemannian sphere, the decomposition $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{n}$ derived in 1.1.1 is the unique H–invariant decomposition, and one immediately verifies that $[\mathfrak{n},\mathfrak{n}] \subset \mathfrak{h}$, but the bracket of two elements of $\mathfrak{n}$ is nonzero in general, so we get nonzero curvature. More precisely, one easily computes that the curvature is induced by the map $\mathbb{R}^n \times \mathbb{R}^n \to \mathfrak{o}(n)$ defined by $(v,w) \mapsto vw^t - wv^t$. Otherwise put, identifying $\mathfrak{o}(n)$ with $\Lambda^2\mathbb{R}^n$, the curvature is simply induced by $(v,w) \mapsto v \wedge w$. Notice that up to scale this is the unique possibility for an invariant tensor field of this type.

(2) Consider the example of the projective sphere from 1.1.3. In this case $G = SL(n+1,\mathbb{R})$ and H is the stabilizer of the line through the first unit vector. The subalgebra $\mathfrak{h} \subset \mathfrak{g}$ is given by

$$\mathfrak{h} = \left\{ \begin{pmatrix} -\operatorname{tr}(A) & Z \\ 0 & A \end{pmatrix} : A \in \mathfrak{gl}(n,\mathbb{R}), Z \in \mathbb{R}^{n*} \right\}.$$

If there were an H–invariant complement $\mathfrak{n}$ to $\mathfrak{h}$ in $\mathfrak{g}$, then, in particular, $[\mathfrak{h},\mathfrak{n}] \subset \mathfrak{n}$. On the other hand, in order to be a vector space complement, for any $X \in \mathbb{R}^n$, $\mathfrak{n}$ would have to contain a unique element of the form $\begin{pmatrix} * & * \\ X & * \end{pmatrix}$. In particular, the zero matrix would be the only element of $\mathfrak{n}$, whose entry in the left lower block is zero. But one immediately computes that for $Z, W \in \mathbb{R}^{n*}$, $X \in \mathbb{R}^n$ and $A \in \mathfrak{gl}(n,\mathbb{R})$, we get

$$\left[\begin{pmatrix} 0 & Z \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} -\operatorname{tr}(A) & W \\ X & A \end{pmatrix} \right] = \begin{pmatrix} ZX & Z(A + \operatorname{tr}(A)\mathrm{id}) \\ 0 & -XZ \end{pmatrix}.$$

This clearly is incompatible with the existence of an H–invariant complement $\mathfrak{n}$. In particular, for the case of the projective sphere, there is no $SL(n+1,\mathbb{R})$–invariant connection on the H–principal bundle $SL(n+1,\mathbb{R}) \to S^n$. Similarly, one can show that for the examples of the projective contact sphere, the conformal sphere and the CR–sphere from 1.1.4–1.1.6 there never exists a G–invariant principal connection.

1.4.7. Invariant linear connections on homogeneous vector bundles. We have already noted in Example 1.4.2 that any associated (vector) bundle to a homogenous principal bundle is a homogeneous (vector) bundle. It is easy to describe this correspondence in the picture of associate bundles to $G \to G/H$ developed in 1.4.3. We restrict to the case of vector bundles here, general bundles can be dealt with similarly. If $\mathcal{P} \to G/H$ is a homogeneous principal K–bundle, then $\mathcal{P} = G \times_i K$ for some homomorphism $i : H \to K$. If $\rho : K \to GL(V)$ is a representation of K on a vector space V, then of course $\rho \circ i$ is a representation of H on V. From the definitions, one immediately concludes that $\mathcal{P} \times_\rho V \cong G \times_{\rho \circ i} V$.

Conversely, assume that $E \to G/H$ is a homogeneous vector bundle. Then we have a representation $\rho : H \to GL(E_o)$ such that $E \cong G \times_H E_o$. Now we can also use the homomorphism ρ to obtain a homogeneous principal bundle $\mathcal{P} \to G/H$ with structure group $GL(E_o)$. From above, we see that $\mathcal{P} \times_{GL(E_o)} E_o \cong G \times_H E_o \cong E$, so this is exactly the frame bundle of E. Hence, the frame bundle of a homogeneous vector bundle is a homogeneous principal bundle.

The construction of induced connections on associated bundles as described in 1.3.4 is of functorial nature, too. In particular, starting with an invariant principal connection on a homogeneous principal bundle, all induced connections on associated bundles (which we have just seen are homogeneous) are automatically invariant. In particular, we get

Observation 1.4.7. An invariant principal connection on the H–principal bundle $G \to G/H$ induces an invariant linear connection on *any* homogeneous vector bundle $E \to G/P$.

Conversely, an invariant linear connection on a homogeneous vector bundle induces a unique principal connection on the frame bundle, which is invariant, too. Thus, we see that there is a bijective correspondence between invariant linear connections on a homogeneous vector bundle and invariant principal connections on its frame bundle. Using this, we can now give a complete classification of invariant linear connections.

Theorem 1.4.7. *Let G be a Lie group, $H \subset G$ a closed subgroup, $\rho : H \to GL(V)$ a representation of H, and consider the corresponding homogeneous vector bundle $E = G \times_H V \to G/H$.*

Then G–invariant linear connections on E are in bijective correspondence with linear maps $\alpha : \mathfrak{g} \to L(V,V)$ such that

(i) $\alpha|_{\mathfrak{h}} = \rho'$*, the derivative of the representation* ρ*,*
(ii) $\alpha(\mathrm{Ad}(h)(X)) = \rho(h) \circ \alpha(X) \circ \rho(h^{-1})$ *for all* $X \in \mathfrak{g}$ *and* $h \in H$*.*

In particular, if there is one such connection, then the space of all of them is an affine space modelled on the vector space $\mathrm{Hom}_H(\mathfrak{g}/\mathfrak{h}, L(V,V))$*.*

The curvature of the invariant linear connection corresponding to α is the invariant section of $\Omega^2(G/H, L(E,E))$ induced by the map $\Lambda^2(\mathfrak{g}/\mathfrak{h}) \to L(V,V)$ defined by

$$(X+\mathfrak{h}, Y+\mathfrak{h}) \mapsto \alpha(X)\circ\alpha(Y) - \alpha(Y)\circ\alpha(X) - \alpha([X,Y]).$$

Proof. From above we know that we have to determine all invariant principal connections on the principal $GL(V)$–bundle $G\times_\rho GL(V)$. The Lie algebra of $GL(V)$ is $L(V,V)$ with the commutator of endomorphisms as the Lie bracket. Moreover, the adjoint action of $GL(V)$ on $L(V,V)$ is just given by conjugation. Now all the claims follow from Theorem 1.4.5, Proposition 1.4.6, and the fact that the curvature of an induced connection is induced by the principal curvature. □

Having solved the problem of existence of invariant linear connections, we want to describe them by an explicit formula. We do this in terms of equivariant functions. A linear connection on a vector bundle $E\to G/H$ can be considered as an operator ∇ which maps sections of E to sections of $T^*(G/H)\otimes E$. If E is the homogeneous vector bundle corresponding to a representation V of H, then the target bundle corresponds to $L(\mathfrak{g}/\mathfrak{h},V)$. Using the correspondence between sections and equivariant functions from Proposition 1.2.7, we can therefore view ∇ as an operator $C^\infty(G,V)^H\to C^\infty(G,L(\mathfrak{g}/\mathfrak{h},V))^H$. In these terms, there are two nice descriptions:

Proposition 1.4.7. *Let $E\to G/H$ be the homogeneous vector bundle corresponding to a representation $\rho: H\to GL(V)$, and let ∇ be the invariant linear connection on E corresponding to a linear map $\alpha:\mathfrak{g}\to L(V,V)$ as in the theorem. Let $s\in\Gamma(E)$ be a section and let $f: G\to V$ be the corresponding equivariant function.*

(1) The equivariant function $\phi: G\to L(\mathfrak{g}/\mathfrak{h},V)$ corresponding to ∇s is explicitly given by

$$\phi(g)(X+\mathfrak{h})=(L_X\cdot f)(g)+\alpha(X)(f(g))$$

for $X\in\mathfrak{g}$ with corresponding left invariant vector field L_X and $g\in G$.

(2) For a vector field $\xi\in\mathfrak{X}(G/H)$ let $\tilde{\xi}$ be a local lift to a vector field on G and let $\omega\in\Omega^1(G,\mathfrak{g})$ be the Maurer–Cartan form of G. Then on the domain of $\tilde{\xi}$ the equivariant function $G\to V$ corresponding to $\nabla_\xi s$ is given by

$$g\mapsto(\tilde{\xi}\cdot f)(g)+\alpha(\omega(\tilde{\xi})(g))(f(g)).$$

Proof. (1) First note that $\phi(g)$ is well defined, since for $X\in\mathfrak{h}$, equivariancy of f implies that $L_X\cdot f=-\rho'(X)\circ f$ while $\alpha(X)=\rho'(X)$ by assumption on α. Let $\mathcal{P}=G\times_\rho GL(V)$ be the frame bundle of E and let $q: G\times GL(V)\to\mathcal{P}$ be the natural projection. Recall that we can identify E either with $G\times_H V$ or with $\mathcal{P}\times_{GL(V)}V$. The isomorphism between these two bundles is obtained in such a way that the class of $(q(g,A),v)$ in $\mathcal{P}\times_{GL(V)}V$ corresponds to the class of $(g,A(v))$ in $G\times_H V$. This immediately implies that if $F:\mathcal{P}\to V$ is the equivariant function representing a section $s\in\Gamma(E)$, then $f(g)=F(q(g,e))$ is the equivariant function $G\to V$ representing the same section.

In the proof of Proposition 1.4.6, we have verified that for the principal connection γ determined by α, $X\in\mathfrak{g}$ and the left invariant vector field L_X, we have $\gamma(T_{(g,e)}q\cdot(L_X,0))=\alpha(X)$. Denoting by $p: G\to G/H$ the projection, this implies that taking the horizontal lift of $T_gp\cdot L_X\in T_{gH}(G/H)$ and using it to differentiate the equivariant function F, we obtain

$$(T_{(g,e)}q\cdot(L_X,0))\cdot F+\alpha(X)(F(q(g,e)))=(L_X\cdot f)(g)+\alpha(X)(f(g)).$$

Using the left action by g to transport $T_gp \cdot L_X$ back to o, we see that this vector corresponds to $X + \mathfrak{h} \in \mathfrak{g}/\mathfrak{p}$. Together with the description of the covariant derivative induced by a principal connection from part (2) of Proposition 1.3.4, this shows that the above formula computes $\phi(g)(X + \mathfrak{h})$.

(2) In Example 1.4.3 we have seen that $\xi(gH) = [\![g, \omega(\tilde{\xi}(g)) + \mathfrak{h}]\!]$. This implies that the equivariant function $G \to \mathfrak{g}/\mathfrak{h}$ corresponding to ξ is given by $g \mapsto \omega(\tilde{\xi}(g)) + \mathfrak{h}$. On the other hand, it says that $\tilde{\xi}(g) = L_{\omega(\tilde{\xi})(g)}(g)$, and this projects onto $\xi(gH)$. Now the result follows from part (1). □

1.4.8. Invariant affine connections. The question of existence of invariant linear connections is particularly important in the case of the tangent bundle $T(G/H)$ of a homogeneous space G/H. From Example 1.4.3 we know that the tangent bundle corresponds to the representation $\underline{\mathrm{Ad}} : H \to GL(\mathfrak{g}/\mathfrak{h})$ induced by the adjoint representation. By Theorem 1.4.7, invariant linear connections on $T(G/H)$ are in bijective correspondence with linear maps $\alpha : \mathfrak{g} \to L(\mathfrak{g}/\mathfrak{h}, \mathfrak{g}/\mathfrak{h})$ such that $\alpha|_{\mathfrak{h}} = \underline{\mathrm{ad}}$ and $\alpha(\mathrm{Ad}(h)(X)) = \underline{\mathrm{Ad}}(h) \circ \alpha(X) \circ \underline{\mathrm{Ad}}(h^{-1})$ for all $X \in \mathfrak{g}$ and $h \in H$.

Theorem 1.4.7 also describes the map $\Lambda^2(\mathfrak{g}/\mathfrak{h}) \to L(\mathfrak{g}/\mathfrak{h}, \mathfrak{g}/\mathfrak{h})$ which induces the curvature of the linear connection determined by a linear map α. For linear connections on the tangent bundle, there is another invariant, the torsion. Recall from 1.3.5 that this is a bundle map $T : \Lambda^2 T(G/H) \to T(G/H)$ characterized by $T(\xi, \eta) = \nabla_\xi \eta - \nabla_\eta \xi - [\xi, \eta]$. For an invariant linear connection ∇, the torsion is an invariant section, so it corresponds to an H–equivariant map $\Lambda^2(\mathfrak{g}/\mathfrak{h}) \to \mathfrak{g}/\mathfrak{h}$. We can easily determine this map explicitly.

Proposition 1.4.8. *Let ∇ be the linear connection on the tangent bundle $T(G/H)$ detemined by a linear map $\alpha : \mathfrak{g} \to L(\mathfrak{g}/\mathfrak{h}, \mathfrak{g}/\mathfrak{h})$. Then the map $\tilde{\tau} : \Lambda^2\mathfrak{g} \to \mathfrak{g}/\mathfrak{h}$ defined by*

$$\tilde{\tau}(X, Y) := \alpha(X)(Y + \mathfrak{h}) - \alpha(Y)(X + \mathfrak{h}) - [X, Y] + \mathfrak{h}$$

factors to an H–invariant map $\tau : \Lambda^2\mathfrak{g}/\mathfrak{h} \to \mathfrak{g}/\mathfrak{h}$. The corresponding invariant section of $\Omega^2(M, TM)$ is the torsion of ∇.

Proof. From equivariancy of α it follows immediately that $\tilde{\tau}$ is H–equivariant. For $X \in \mathfrak{h}$ we have $\alpha(X) = \underline{\mathrm{ad}}(X)$, which immediately implies that $\tilde{\tau}(X, Y) = 0$ for all $Y \in \mathfrak{g}$. Hence, $\tilde{\tau}$ factors to an H–equivariant map τ as claimed. For $\xi, \eta \in \mathfrak{X}(G/H)$ choose local lifts $\tilde{\xi}$ and $\tilde{\eta}$ on G defined around e. Then locally around e, the function $G \to \mathfrak{g}/\mathfrak{h}$ representing $\nabla_\xi \eta$ is given by $\tilde{\xi} \cdot \omega(\tilde{\eta}) + \mathfrak{h} + \alpha(\omega(\tilde{\xi}))(\omega(\tilde{\eta}) + \mathfrak{h})$. Since $[\tilde{\xi}, \tilde{\eta}]$ is a local lift of $[\xi, \eta]$, the function describing this Lie bracket is, locally around e, given by $\omega([\tilde{\xi}, \tilde{\eta}]) + \mathfrak{h}$. By definition of the exterior derivative, the section $\nabla_\xi \eta - \nabla_\eta \xi - [\xi, \eta]$ corresponds to the function

$$\alpha(\omega(\tilde{\xi}))(\omega(\tilde{\eta}) + \mathfrak{h}) - \alpha(\omega(\tilde{\eta}))(\omega(\tilde{\xi}) + \mathfrak{h}) + d\omega(\tilde{\xi}, \tilde{\eta}) + \mathfrak{h}.$$

Expressing $d\omega$ using the Maurer–Cartan equation, we obtain $\tilde{\tau}(\omega(\tilde{\xi}), \omega(\tilde{\eta}))$. □

Even more special, suppose that the invariant linear connection on $T(G/H)$ is induced by the invariant principal connection on $G \to G/H$ corresponding to an H–equivariant projection $\pi : \mathfrak{g} \to \mathfrak{h}$; see Corollary 1.4.6. Then the corresponding map $\alpha : \mathfrak{g} \to L(\mathfrak{g}/\mathfrak{h}, \mathfrak{g}/\mathfrak{h})$ is simply given by $\alpha(X) = \underline{\mathrm{ad}}(\pi(X))$. Hence, $\alpha(X) = 0$ for $X \in \mathfrak{n} = \ker(\pi)$. Identifying $\mathfrak{g}/\mathfrak{h}$ with $\mathfrak{n}$, the formulae for curvature and torsion simplify considerably. The curvature is induced by the map $\Lambda^2\mathfrak{n} \to \mathfrak{h}$ given by

$(X,Y) \mapsto -\pi([X,Y])$. The torsion is induced by $\tau : \Lambda^2\mathfrak{n} \to \mathfrak{n}$, where $\tau(X,Y) = \pi([X,Y]) - [X,Y] \in \mathfrak{n}$. Hence, the curvature and the torsion are induced by the $\mathfrak{h}$–component, respectively, the $\mathfrak{n}$–component of the negative of the Lie bracket $\Lambda^2\mathfrak{n} \to \mathfrak{g}$.

Example 1.4.8. By way of an example, we show that in general there is no invariant linear connection on the tangent bundle of a homogeneous space. Consider the projective sphere from 1.1.3. From there we know that for $\phi \in \mathbb{R}^{n*}$, the element $h := \begin{pmatrix} 1 & \phi \\ 0 & \mathbb{I} \end{pmatrix}$ lies in H, where $\mathbb{I}$ denotes the $n \times n$ identity matrix. For $X \in \mathbb{R}^n$ one gets

$$\mathrm{Ad}(h) \cdot \begin{pmatrix} 0 & 0 \\ X & 0 \end{pmatrix} = \begin{pmatrix} \phi(X) & -\phi(X)\phi \\ X & -X \otimes \phi \end{pmatrix}.$$

But in view of the presentation of the Lie algebras in example (2) of 1.4.6, this also implies that $\underline{\mathrm{Ad}}(h) = \mathrm{id}_{\mathfrak{g}/\mathfrak{h}}$. In particular, if $\alpha : \mathfrak{g} \to L(\mathfrak{g}/\mathfrak{h}, \mathfrak{g}/\mathfrak{h})$ is H–equivariant, then we obtain

$$\alpha \begin{pmatrix} 0 & 0 \\ X & 0 \end{pmatrix} = \alpha \begin{pmatrix} \phi(X) & -\phi(X)\phi \\ X & -X \otimes \phi \end{pmatrix}.$$

But if α corresponds to an invariant connection, then the difference of these two elements must be $\underline{\mathrm{ad}} \begin{pmatrix} \phi(X) & -\phi(X)\phi \\ 0 & -X\otimes\phi \end{pmatrix}$. If $Y \in \mathbb{R}^n$ is another element, then we get

$$\left[\begin{pmatrix} \phi(X) & -\phi(X)\phi \\ 0 & -X \otimes \phi \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ Y & 0 \end{pmatrix}\right] = \begin{pmatrix} -\phi(X)\phi(Y) & 0 \\ -\phi(Y)X - \phi(X)Y & \phi(X)Y \otimes \phi \end{pmatrix}.$$

This shows that the difference is nonzero, which contradicts the existence of a $SL(n+1,\mathbb{R})$–invariant affine connection on S^n. We shall see in 3.1.4 that this argument extends systematically to the homogeneous models of all parabolic geometries. In particular, also the projective contact, conformal, and CR–spheres from 1.1.4–1.1.6 do not admit invariant affine connections.

1.4.9. Invariant differential operators. From the point of view of geometry, invariant differential operators are the differential operators intrinsic to the given geometric structure, so these provide basic tools for working with such structures. Questions on invariant differential operators will be the main topic of volume two. This and the next subsection, which develop the basic background on homogeneous spaces, should be considered rather as a teaser for volume two, and can be skipped at the first reading.

Consider a homogeneous space G/H and two homogeneous vector bundles E, F over G/H. From 1.4.4 we get G–actions on the spaces $\Gamma(E)$ and $\Gamma(F)$ of sections, and an *invariant differential operator* is a differential operator $D : \Gamma(E) \to \Gamma(F)$ which is equivariant for the G actions, i.e. such that $D(g \cdot s) = g \cdot D(s)$ for all $s \in \Gamma(E)$ and $g \in G$.

The first step towards an algebraic description of invariant differential operators is to pass to jet prolongations. If M is a smooth manifold and $E \to M$ is any vector bundle, then for $k \in \mathbb{N}$ we have the k–jet prolongation J^kE. The fiber of J^kE at $x \in M$ is exactly the vector space of all k–jets at x of sections of E. From 1.2.8 we know that J^kE is a vector bundle over M. If F is another vector bundle over M, then a differential operator $D : \Gamma(E) \to \Gamma(F)$ is of order $\leq k$ if and only if for any two sections $s, t \in \Gamma(E)$ and any point $x \in M$, the equation $j^k_x s = j^k_x t$ implies $D(s)(x) = D(t)(x)$. If D is such an operator, then we get an induced bundle map $\tilde{D} : J^kE \to F$ over the identity on M, defined by $\tilde{D}(j^k_x s) := D(s)(x)$. Conversely, this formula associates to any bundle map $\tilde{D}$ a differential operator D. Clearly, D is linear if and only if $\tilde{D}$ is a homomorphism of vector bundles.

In the special case of a homogeneous vector bundle $E \to G/H$, functoriality of the construction of jets immediately implies that each J^kE is again a homogeneous vector bundle. More precisely, the action is defined by $g \cdot (j^k_x s) := j^k_x(g \cdot s)$. By construction, a differential operator D corresponding to the bundle map $\tilde{D} : J^kE \to F$ is invariant if and only if $\tilde{D}$ is a morphism of homogeneous fiber bundles, i.e. G–equivariant. Hence, we have reduced the determination of linear invariant differential operators to the determination of homomorphisms between homogeneous vector bundles. Since a homomorphism $J^kE \to F$ of homogeneous vector bundles is the same thing as a G–invariant section of the homogeneous vector bundle $L(J^kE, F)$, this reduces to representation theory of the group H by Theorem 1.4.4.

The problem with this is that even if the H–representation V which induces the homogeneous bundle E is very simple, the representations inducing J^kE tend to become very complicated, thus making the problem unmanageable. In any case, there is one possible simple step, namely look at the symbol of the operator. Again, we digress and consider an arbitrary vector bundle $p : E \to M$ over a smooth manifold M with k–jet prolongation $J^kE \to M$. Recall from 1.2.8 that for $\ell < k$ there is an obvious projection $\pi^k_\ell : J^kE \to J^\ell E$, defined by $\pi^k_\ell(j^k_x s) = j^\ell_x s$. Obviously, this is a vector bundle homomorphism and for $\ell = 0$ one simply gets the projection $J^kE \to E$ onto the target of a jet. The symbol of an operator of order $\leq k$ is then the restriction of the corresponding vector bundle homomorphism to the kernel of π^k_{k-1}. It is easy to see in local coordinates that this kernel is isomorphic to $S^kT^*M \otimes E$.

There is a nice description in terms of linear connections. Take a linear connection ∇ on E. For a section $s \in \Gamma(E)$, consider $\nabla s(x) \in T^*_xM \otimes E_x$. This functional depends only on $j^1_x s$ and if, in addition, $s(x) = 0$, then it is independent of ∇. Hence, it identifies $\ker \pi^1_0$ with $T^*M \otimes E$. The identification of $\ker \pi^k_{k-1}$ for $k > 1$ admits a similar description in terms of arbitrary linear connections on E and on the tangent bundle TM. Given these, one can define the k–fold covariant derivative $\nabla^k s$ for each $s \in \Gamma(E)$. For $x \in M$ the value $\nabla^k s(x)$ depends only on $j^k_x s$, and if $j^{k-1}_x s = 0$, then $\nabla^k s(x)$ is totally symmetric and independent of ∇. Hence, it induces the required isomorphism.

If F is another vector bundle over M and $D : \Gamma(E) \to \Gamma(F)$ is a differential operator of order $\leq k$ corresponding to a bundle map $\tilde{D} : J^kE \to F$, then the kth order *symbol* of D is the vector bundle map $\sigma(D) : S^kT^*M \otimes E \to F$ given by the restriction of $\tilde{D}$ to the kernel of π^k_{k-1}.

Returning to the case of a homogeneous vector bundle $E \to G/H$, the projections π^k_ℓ are by construction homomorphisms of homogeneous vector bundles, and a moment of thought shows that also the inclusions $S^k(T^*M) \otimes E \to J^kE$ are homomorphisms of homogeneous vector bundles. In particular, for any invariant linear differential operator D the symbol $\sigma(D)$ is a homomorphism of homogeneous vector bundles. Hence, it corresponds to a G–invariant section of the bundle $L(S^kT^*M \otimes E, F)$, which immediately restricts the possibilities for the existence of invariant differential operators. It is crucial that the representation corresponding to the bundle $L(S^kT^*M \otimes E, F)$ is as manageable as $\mathfrak{g}/\mathfrak{h}$ and the representations V and W inducing E and F. If there are invariant linear connections on E and $T(G/H)$, then things become easy:

PROPOSITION 1.4.9. *Let E and F be homogeneous vector bundles over a homogeneous space G/H, and suppose that there are G–invariant linear connections on E and $T(G/H)$. Fixing such connections, we can construct for each invariant bundle map $\Phi : S^kT^*M \otimes E \to F$ a canonical invariant differential operator $D_\Phi : \Gamma(E) \to \Gamma(F)$ with symbol $\sigma(D_\Phi) = \Phi$. Any G–invariant differential operator $D : \Gamma(E) \to \Gamma(F)$ can be written as a finite sum of operators obtained in that way.*

PROOF. Let us denote the invariant linear connections on E and $T(G/H)$ by ∇. Then for each $k \geq 0$ we get an induced invariant linear connection on the bundle $\otimes^kT^*(G/H) \otimes E$, so we can define iterated covariant derivatives $\nabla^k s \in \Gamma(\otimes^iT^*(G/H) \otimes E)$. We can then symmetrize this section, to obtain a section $\nabla^{(k)}s \in \Gamma(S^kT^*(G/H) \otimes E)$.

By construction, $s \mapsto \nabla^{(k)}s$ is an invariant differential operator. Given a homomorphism $\Phi : S^kT^*(G/H) \otimes E \to F$ of homogeneous vector bundles we put $D_\Phi(s) := \Phi(\nabla^{(k)}s)$. By construction, this is an invariant differential operator with symbol $\sigma(D_\Phi) = \Phi$.

Given a general invariant differential operator D, assume that k is the order of D, i.e. that D has order $\leq k$ and the kth order symbol of D is nonzero. Putting $\Phi = \sigma(D)$ we see that $D - D_\Phi$ has vanishing kth order symbol and thus is of order $\leq k - 1$. Now the result follows by induction. □

In the presence of invariant linear connections on E and $T(G/H)$ we see that each invariant symbol is realized as the symbol of an invariant differential operator. The proposition also reduces the problem of finding invariant differential operators to the description of possible symbols. This is as manageable as the representations $\mathfrak{g}/\mathfrak{h}$, V, and W. Note that the required invariant connections always exist for reductive Klein geometries. More generally, they also exist if there exists an invariant affine connection on G/H and E is a tensor bundle.

1.4.10. Invariant differential operators and homomorphisms of induced modules. Let us now switch to a homogeneous bundle E which does not admit an invariant linear connection. Then the considerations of the last subsection show that the H–representations inducing J^kE will be complicated, and this already occurs for $k = 1$: A linear connection ∇ on E is equivalent to a splitting $J^1E \cong E \oplus (T^*(G/H) \otimes E)$ via $j^1_x s \mapsto (s(x), \nabla s(x))$. The connection ∇ is invariant, if and only if this is an isomorphism of homogenous vector bundles. Passing to the corresponding representations, let V and W be the representations inducing E and J^1E. Then π^1_0 corresponds to an H–equivariant surjection $W \to V$ whose kernel is isomorphic to $(\mathfrak{g}/\mathfrak{h})^* \otimes V$. If this H–invariant subspace would admit an H–invariant complement, this would give an isomorphism $W \cong V \oplus (\mathfrak{g}/\mathfrak{h})^* \otimes V$ of H–modules. From above we know that this would give rise to an invariant linear connection on E. In Example 1.4.8 we have seen that in some cases, there are very simple representations V for which there is no invariant linear connection on the corresponding associated bundle, so in these cases W cannot be completely reducible.

There still is a way to reformulate the classification of linear invariant differential operators as an algebraic problem, which can be solved in some cases. In particular, this is true for the Klein geometries underlying parabolic geometries. In these cases, the Lie group G is semisimple and the subgroup H is a parabolic subgroup. In particular, the representation theory of the Lie algebra $\mathfrak{g}$ is well understood, while the situation with representations of H is much more complicated.

Completely reducible representations of H are manageable since they come from a reductive subgroup, but from above we see that passing to jet prolongation one leaves the realm of completely reducible representations.

The algebraic reformulation that can be successfully used in the parabolic case is in principle available in general. However, since it is technically rather demanding and the parabolic case is the main application, we prefer to only briefly sketch this approach here and give a detailed treatment in the parabolic case in volume two.

The starting point is to look at the universal enveloping algebra $\mathcal{U}(\mathfrak{g})$ of the Lie algebra $\mathfrak{g}$; see 2.1.10. This is a unital associative algebra, endowed with an inclusion $i : \mathfrak{g} \to \mathcal{U}(\mathfrak{g})$ such that $i([X,Y]) = i(X)i(Y) - i(Y)i(X)$. It turns out that representations of $\mathfrak{g}$ are equivalent to representations of $\mathcal{U}(\mathfrak{g})$. The universal enveloping algebra of the subalgebra $\mathfrak{h} \subset \mathfrak{g}$ naturally is a subalgebra in $\mathcal{U}(\mathfrak{g})$. In particular, $\mathcal{U}(\mathfrak{g})$ is naturally a right $\mathcal{U}(\mathfrak{h})$–module.

Given a representation of $\mathfrak{h}$ on a vector space V, this space is a left $\mathcal{U}(\mathfrak{h})$–module, and one can form the *induced module* $\mathcal{U}(\mathfrak{g}) \otimes_{\mathcal{U}(\mathfrak{h})} V$, which is a $\mathcal{U}(\mathfrak{g})$–module under multiplication from the left, and thus a representation of $\mathfrak{g}$ (of infinite dimension). If one wants to take into account the group H, one may consider the induced module as a $(\mathfrak{g}, H)$–module.

The relation to differential operators comes from looking at infinite jets. Given a homogeneous vector bundle $E \to G/H$, one looks at the infinite jet prolongation $J^\infty E$, which is defined as the direct limit of the system $\dots \to J^kE \to J^{k-1}E \to \dots$. By construction, this is an infinite–dimensional homogeneous vector bundle, so we are naturally led to consider its fiber $J^\infty E_o$ over the point $o = eH$, as an H–module. The advantage of passing to infinite jets is that there is a canonical action of the Lie algebra $\mathfrak{g}$ on this fiber, which comes from differentiation by right invariant vector fields. This makes $J^\infty E_o$ into a $(\mathfrak{g}, H)$–module.

Next, the identification of the Lie algebra $\mathfrak{g}$ as left invariant vector fields on G induces an identification of the universal enveloping algebra $\mathcal{U}(\mathfrak{g})$ with the space of left invariant differential operators on $C^\infty(G, \mathbb{R})$. This gives rise to a pairing between $J^\infty_e(G, \mathbb{R})$ and $\mathcal{U}(\mathfrak{g})$ induced by applying a left invariant differential operator to the representative of an infinite jet and evaluating the result in e. One shows that this induces a linear isomorphism between $\mathcal{U}(\mathfrak{g})$ and the set of those linear maps $J^\infty_e(G, \mathbb{R}) \to \mathbb{R}$ which factor over some $J^k_e(G, \mathbb{R})$.

More generally, looking at functions with values in a finite–dimensional vector space V, one gets an identification of $\mathcal{U}(\mathfrak{g}) \otimes V^*$ and the space of those maps $J^\infty_e(G, V) \to \mathbb{R}$ which factor over some $J^k_e(G, V)$. Analyzing the action on jets of equivariant maps, one finally obtains an identification between the induced module $\mathcal{U}(\mathfrak{g}) \otimes_{\mathcal{U}(\mathfrak{h})} V^*$ and the space of those linear maps $J^\infty_o E \to \mathbb{R}$ which factor over some $J^k_o E$. One verifies directly that this pairing is compatible with the $(\mathfrak{g}, H)$–module structures on both spaces.

Now consider two homogeneous vector bundles E and F corresponding to H–representations V and W. From the discussion in 1.4.9 we conclude that the space of all linear invariant differential operators $D : \Gamma(E) \to \Gamma(F)$ is isomorphic to the space of all H–module maps $J^\infty_o E \to F$ which factor over some $J^k_o E$. Using the above pairing, such maps may be interpreted as H–invariant elements in $(\mathcal{U}(\mathfrak{g}) \otimes_{\mathcal{U}(\mathfrak{h})} V^*) \otimes W$, or equivalently as H–module homomorphisms $W^* \to \mathcal{U}(\mathfrak{g}) \otimes_{\mathcal{U}(\mathfrak{h})} V^*$. This also works the other way around, so one gets an isomorphism between invariant linear differential operators and H–homomorphisms of the above type.

The final ingredient then is an algebraic version of Frobenius reciprocity from 1.4.4, which we will prove in 2.1.10. This leads to a bijective correspondence between H–homomorphisms $W^* \to \mathcal{U}(\mathfrak{g}) \otimes_{\mathcal{U}(\mathfrak{h})} V^*$ and $(\mathfrak{g}, H)$–module homomorphisms $\mathcal{U}(\mathfrak{g}) \otimes_{\mathcal{U}(\mathfrak{h})} W^* \to \mathcal{U}(\mathfrak{g}) \otimes_{\mathcal{U}(\mathfrak{h})} V^*$. Thus, we get the following description of linear invariant differential operators:

THEOREM 1.4.10. *Let G be a Lie group, $H \subset G$ a closed subgroup, and V and W finite–dimensional representations of H. Let E and F denote the homogeneous vector bundles $G \times_H V$ and $G \times_H W$, respectively.*

Then the space of linear invariant differential operators $D : \Gamma(E) \to \Gamma(F)$ is isomorphic to the space $\mathrm{Hom}_{(\mathfrak{g},H)}(\mathcal{U}(\mathfrak{g}) \otimes_{\mathcal{U}(\mathfrak{h})} W^*, \mathcal{U}(\mathfrak{g}) \otimes_{\mathcal{U}(\mathfrak{h})} V^*)$ of homomorphisms of $(\mathfrak{g}, H)$–modules.

In the case of a semisimple Lie algebra $\mathfrak{g}$ and a parabolic subalgebra $\mathfrak{h}$ the induced modules in question are called generalized Verma modules, and they have been intensively studied in representation theory. In particular, a good amount of results on homomorphisms between such modules is available. In the special case where $\mathfrak{h}$ is even a Borel subalgebra, one obtains Verma modules, and there is a complete classification of homomorphisms between such modules. We will discuss these results in more detail in volume two.

1.4.11. Distinguished curves. The final notion that we want to discuss in this section are preferred families of curves, which lead to analogs of geodesics and normal coordinates. Depending on the choice of homogenous space, there may be various interesting families of distinguished curves. We will always require that a family of distinguished curves on G/H is stable under the action of G. This implies that the whole family is determined by the curves through the origin $o = eH$. For this section, we will restrict to a specific concept of distinguished curves which is always available.

For the purpose of motivation, suppose that there is an invariant principal connection on the bundle $p : G \to G/H$. Then we get an induced linear connection on $T(G/H)$ and we can use the geodesics of this linear connection as the distinguished curves. The principal connection on G/H is determined by the choice of an H–invariant complement $\mathfrak{n}$ to $\mathfrak{h}$ in $\mathfrak{g}$; see Corollary 1.4.6. Now for $X \in \mathfrak{n}$ we can consider the curve $t \mapsto p(\exp(tX))$ in G/H. By definition $t \mapsto \exp(tX)$ is a lift of this curve on which the Maurer–Cartan form is constant, so our curve is a geodesic through o.

This suggests a generalization. Assume that we have given a Lie group G, a closed subgroup $H \subset G$, and fix a linear complement $\mathfrak{n}$ of $\mathfrak{h}$ in $\mathfrak{g}$. For any $g \in G$ we then have the notion of horizontal tangent vectors at g, namely $\xi \in T_gG$ is horizontal if and only if $\omega(\xi) = T\lambda_{g^{-1}} \cdot \xi \in \mathfrak{n}$, where ω is the Maurer–Cartan form on G. In particular, for $X \in \mathfrak{n}$ we have the left invariant vector field $L_X \in \mathfrak{X}(G)$ which is horizontal. The projections of the flow lines of these fields are then the basic distinguished curves.

A smooth curve $c : I \to G/H$ defined on an open interval $I \subset \mathbb{R}$ is called distinguished if and only if there are a point $t_0 \in I$ and elements $g \in G$ with $p(g) = c(t_0)$ and $X \in \mathfrak{n}$ such that $c(t) = p(\mathrm{Fl}^{L_X}_{t-t_0}(g)) = p(g\exp((t-t_0)X))$ for all $t \in I$. Note that since left invariant vector fields are always complete, any distinguished curve can be canonically extended to a curve defined on $\mathbb{R}$. Note further, that by the flow property, if $c : I \to G/H$ is a distinguished curve, then for any $t_0 \in I$ there are $g \in G$ and $X \in \mathfrak{n}$ such that $c(t) = p(g \exp((t - t_0)X))$.

PROPOSITION 1.4.11. *Let G be a Lie group, $H \subset G$ a closed subgroup and $\mathfrak{n} \subset \mathfrak{g}$ a linear subspace complementary to $\mathfrak{h}$. Then we have:*

(1) *If $c : I \to M$ is a distinguished curve and $g \in G$ is any element, then $\ell_g \circ c : I \to M$ is a distinguished curve, too.*
(2) *For any point $x \in G/H$ and any tangent vector $\xi \in T_xG/H$ there is at least one distinguished curve $c : \mathbb{R} \to G/H$ such that $c(0) = x$ and $c'(0) = \xi$.*
(3) *If the complement $\mathfrak{n}$ is H–invariant, then the curve c in (2) is uniquely determined. It coincides with the geodesic of the linear connection on $T(G/H)$ induced by $\mathfrak{n}$.*
(4) *For any $g \in G$, the mapping $\mathfrak{n} \to G/H$, $X \mapsto p(g \exp X)$ defines local coordinates around $p(g)$ in which the straight lines through the origin in $\mathfrak{n}$ map to distinguished curves through $p(g)$.*

PROOF. (1) By definition, there are elements $t_0 \in I$, $g_0 \in G$ and $X \in \mathfrak{n}$ such that $c(t) = p(g_0 \exp((t - t_0)X))$. But then by definition of the action ℓ_g, we get $\ell_g(c(t)) = p(gg_0 \exp((t - t_0)X))$, so $\ell_g \circ c$ is distinguished, too.

(2) By (1) it suffices to consider the case $x = o$. Then for $\xi \in T_oG/H$ there is a unique element $X \in \mathfrak{n}$ such that $\xi = T_ep \cdot X$. Thus, $p(\exp(tX))$ is a distinguished curve as required.

(3) Again, we may confine ourselves to the case $x = o$. As above, take $X \in \mathfrak{n}$ such that $T_ep \cdot X = \xi$. Since $p^{-1}(o) = H$, any other distinguished curve through o in direction ξ is of the form $p(h \exp(tY))$ for h in H, with $Y \in \mathfrak{n}$ the unique element such that $T_hp \cdot L_Y(h) = \xi$. But then

$$\xi = T_hp \circ T_e\lambda_h \cdot Y = T_ep \circ T_h\rho^{h^{-1}} \circ T_e\lambda_h \cdot Y = T_ep \cdot \mathrm{Ad}(h)Y.$$

Thus, $X - \mathrm{Ad}(h)Y \in \mathfrak{h}$, but since $\mathfrak{n}$ is H–invariant, we have $\mathrm{Ad}(h)Y \in \mathfrak{n}$ and thus $Y = \mathrm{Ad}(h^{-1})X$. But then $h \exp(t\,\mathrm{Ad}(h^{-1})X) = hh^{-1} \exp(tX)h$, so $p(h \exp(tY)) = p(\exp(tX))$.

(4) is obvious from the definitions. □

REMARK 1.4.11. We may equivalently define the distinguished curves by H–invariant data at the origin $o \in G/H$. The distinguished curves $c(t)$ with $c(0) = o$ form an H–invariant set, and the entire set of the distinguished curves is obtained from them by the left shifts.

More generally, we may fix any H-invariant subset A of curves $\alpha(t)$, $\alpha(0) = o$ and to define the A–distinguished curves as all curves of the form $\ell_g \circ \alpha$ for $g \in G$ and $\alpha \in A$. In particular, each choice of an H–invariant subspace $\mathfrak{a} \subset \mathfrak{n}$ in the complement $\mathfrak{n}$ to $\mathfrak{h}$ with respect to the induced adjoint action leads to a subclass of distinguished curves emanating in directions contained in the distribution $\mathcal{A} \subset T(G/H)$ determined by the subspace $\mathfrak{a}$.

1.5. Cartan connections

Having the necessary background at hand, we can now start to investigate Cartan geometries. Throughout this section we will take Cartan geometries as a given input and develop basic tools for the analysis of such structures. We will look for simpler structures underlying a Cartan geometry, but we will not touch the question