# Wallis' Product and Stirling's Formula

**Theorem 1.** *The Wallis'*[1] *product* $W_n := \frac{2^2 \cdot 4^2 \cdots (2n)^2}{1 \cdot 3^2 \cdots (2n-1)^2 \cdot (2n+1)}$ *tends to* $\frac{\pi}{2}$ *as* $n \to \infty$.

*Proof.* Let $S_n := \int_0^\pi (\sin x)^n \, dx$. Then $S_0 = \pi$, $S_1 = 2$ and in general $S_{n+1} < S_n < S_{n-1}$.

Integration by parts gives:

$$\begin{aligned} S_n &= \int_0^\pi \sin x \cdot (\sin x)^{n-1} \, dx = \left[-\cos x (\sin x)^{n-1}\right]_0^\pi + \int_0^\pi (\cos x)^2 (n-1) \sin x)^{n-2} \, dx \\ &= (n-1) \int_0^\pi (1 - (\sin x)^2)(\sin x)^{n-2} \, dx = (n-1) S_{n-2} - (n-1) S_n. \end{aligned}$$

Therefore $S_n = \frac{n-1}{n} S_{n-2}$.

We get by induction

$$\begin{aligned} S_{2n+1} &= \frac{2n}{2n+1} \cdot \frac{2n-2}{2n-1} \cdots \frac{2}{3} \cdot S_1 \\ &< S_{2n} = \frac{2n-1}{2n} \cdot \frac{2n-3}{2n-2} \cdots \frac{1}{2} \cdot S_0 \\ &< S_{2n-1} = \frac{2n-2}{2n-1} \cdots \frac{2}{3} \cdot S_1. \end{aligned}$$

Therefore

$$W_n = \frac{2^2 \cdot 4^2 \cdots (2n)^2}{1^2 \cdot 3^3 \cdots (2n-1)^2 \cdot (2n+1)} < \frac{S_0}{S_1} = \frac{\pi}{2},$$

and also

$$\frac{\pi}{2} = \frac{S_0}{S_1} < \frac{2n}{2n-1} \cdot W_{n-1} = \frac{2^2 \cdot 4^2 \cdots (2n-2)^2}{1^2 \cdot 3^3 \cdots (2n-3)^2 \cdot (2n-1)} \cdot \frac{2n}{2n-1}.$$

So we have

$$W_n < \frac{\pi}{2} < \frac{2n}{2n-1} \cdot W_{n-1} = \frac{2n+1}{2n} \cdot W_n.$$

Taking the limit $n \to \infty$ gives the result. □

**Theorem 2.** *Stirling*[2]*'s formula:* $n! \sim n^n e^{-n} \sqrt{2\pi n}$.

*Proof.* First take the logarithm:

$$\log n! = \log \prod_{k=1}^n k = \sum_{k=1}^n \log k =_{\text{because } \log 1 = 0} \sum_{k=2}^n \log k.$$

By the trapezium rule of integration, see Figure 1:

$$\log k = \int_{k-1}^k \log x \, dx + \frac{\log k - \log(k-1)}{2} - E_k$$

where the error terms $E_k \in (0, C/k^2)$, in fact $E_k < \frac{1}{2} \frac{1}{(k-1)^2}$. In particular, the error terms are summable.

[1] John Wallis (1606-1703, British mathematician and clergyman, also gave us the symbol $\infty$

[2] James Stirling (1692-1770) was a Scottish mathematician, working in Venice, London and eventually becoming an industrial in Glasgow.

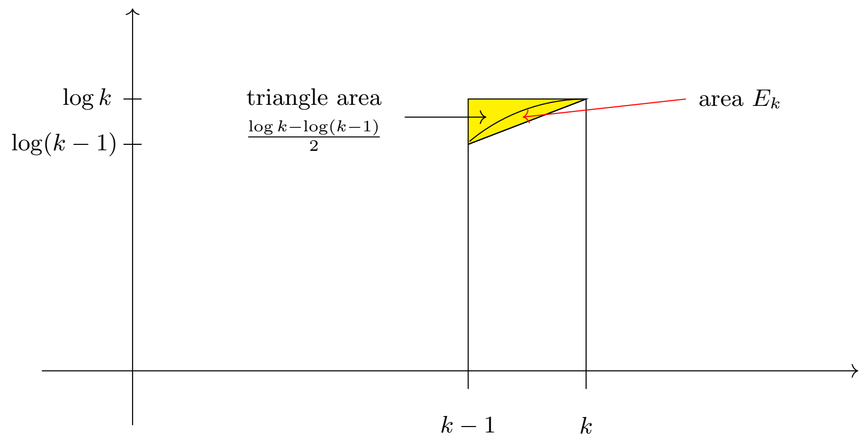


Figure 1: Approximating $\log k$ by an integral + error terms

Now

$$\begin{aligned}\sum_{k=2}^{n}\log k &= \sum_{k=2}^{n}\left(\int_{k-1}^{k}\log x\,dx+\frac{\log k-\lg k-1}{2}-E_k\right)\\ &= \int_{1}^{n}\log x\,dx+\log n-\sum_{k=2}^{n}E_k\\ &= n\log n-n+1+\frac{1}{2}\log n-\sum_{k=2}^{n}E_k.\end{aligned}$$

Therefore, if we take the $e$-power again:

$$u_n := \frac{n!e^n}{n^n\sqrt{n}} = e^{1-\sum_{k=2}^{n}E_k}$$

and this converges because the $E_k$'s are a summable sequence of positive terms.

Set $\alpha := \lim_{n\to\infty} u_n$. Then

$$\begin{aligned}\alpha &= \frac{\alpha^2}{\alpha}=\frac{\lim_{n\to\infty}u_n^2}{\lim_{n\to\infty}u_{2n}}=\lim_{n\to\infty}\frac{u_n^2}{u_{2n}}\\ &= \lim_{n\to\infty}\frac{(n!)^2}{(2n)!}\frac{(2n)^{2n}}{n^{2n}}\cdot\frac{2}{\sqrt{2n}}\\ &= \lim_{n\to\infty}\frac{2^2\cdot 4^2\cdots(2n)^2}{1\cdot 2\cdot 3\cdots(2n)}\cdot\frac{2}{\sqrt{2n}}\\ &= \lim_{n\to\infty}\frac{2\cdot 4\cdots(2n)}{1\cdot 3\cdots(2n-1)}\cdot\frac{1}{\sqrt{2n+1}}\cdot\frac{2\sqrt{2n+1}}{\sqrt{2n}}=\lim_{n\to\infty}\sqrt{W_n}\cdot 2\cdot\frac{\sqrt{2n+1}}{\sqrt{2n}}=\sqrt{2\pi},\end{aligned}$$

because the square-root of the Wallis product tends to $\sqrt{\frac{\pi}{2}}$. □