Group theory, summer semester 2011

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1) Prove that every refinement of a normal series with abelian factor groups has abelian factor groups.

2) Let G be a finite group. Prove that the following conditions are equivalent:

(i) G is solvable.

(ii) G has a normal series whose factor groups are cyclic of prime order.

(iii) The factor groups of every normal series of G are cyclic of prime order.

3) Prove that the symmetric group S_n is solvable if $n \leq 4$. (Remark. The case $n \geq 5$ will be dealt with soon.)

4) Let K be a field. Prove that the group

$$G := \left\{ \left(\begin{smallmatrix} a & b \\ 0 & c \end{smallmatrix} \right) \middle| a, b, c \in K, ac \neq 0 \right\}$$

(with matrix multiplication) is solvable. Hint. Consider the homomorphism

$$\varphi : G \longrightarrow K^* \times K^*, \quad \left(\begin{smallmatrix} a & b \\ 0 & c \end{smallmatrix} \right) \mapsto (a, c).$$

5) Prove that the group $\mathrm{GL}_2(\mathbb{Z}_2)$ is solvable.

6) Prove that a solvable group that has a composition series must be finite.

7) a) Let G be a group and assume that both $N \trianglelefteq G$ and $H \leq G$ are solvable. Prove that NH is solvable.
b) Prove that every finite group G has a unique maximal normal solvable subgroup $\mathcal{S}(G)$.
c) Prove that under the assumptions of b) $G/\mathcal{S}(G)$ has no nontrivial normal solvable subgroups.

8) Prove that if p and q are primes with $p < q$ then every group G of order $|G| = pq^n$ (for some $n \in \mathbb{N}$) is solvable. (Hint. Use Sylow's theorems.)

9) Prove that if p and q are two distinct primes then every group G of order $|G| = p^2q$ is solvable.

10) W. Burnside proved that every group of order $p^m q^n$ is solvable (where p, q are two primes). Furthermore, one can show that every group of order pqr is solvable (where p, q, r are primes). Use these two results to check that every finite group G of order $|G| < 60$ is solvable. (Remark. However, there is a group of order 60 that is not solvable.)

11) Prove that for every group G its center $Z(G)$ is characteristic in G.

12) Let G be a group and $H \leq K \leq G$. Prove that if H is characteristic in G and K/H is characteristic in G/H then K is characteristic in G.

13) Give an example of a group G containing a normal subgroup $N \trianglelefteq G$ that is not a characteristic subgroup of G.

14) Let G be a finite group of order $|G| > 1$. Prove the following two assertions:
a) If G is solvable then G contains a nontrivial normal abelian subgroup N.
b) If G is not solvable then G contains a nontrivial normal subgroup N with $N' = N$.

15) a) Prove Theorem 1 using Theorem 11.
b) Prove Theorem 2 using Theorem 11. (Hint. If the homomorphism $\varphi : G \to H$ is surjective then $\varphi(G^{(i)}) = \varphi(G)^{(i)}$ for all $i \geq 0$.)
c) Prove Theorem 3 using Theorem 11.

16) Prove that every finite group G of order $|G| = pqr$ (with p, q, r primes) is solvable.

17) Determine $S_n{}'$ (for $n \geq 1$) and $A_n{}'$ (for $n \geq 3$).

18) Let $n \geq 3$ and let D_{2n} denote the dihedral group with $2n$ elements. I.e.,

$$D_{2n} = \langle a, b \rangle = \left\{ a^i b^j \mid 0 \leq i < n, j \in \{0, 1\} \right\} \text{ where } a^n = b^2 = 1 \text{ and } ba = a^{-1}b.$$

Prove that $D_{2n}{}' = \langle a^2 \rangle$. Use this to determine the structure of $D_{2n}{}'$. Is D_{2n} solvable?

Definition. Let G be a group. Then $G = G_1 \geq G_2 \geq \cdots \geq G_n = \{1\}$ is called a central series if $G_i \trianglelefteq G$ for $1 \leq i \leq n$ and $G_i/G_{i+1} \leq Z(G/G_{i+1})$ for $1 \leq i < n$.

19) Let G be a group. Prove the following:
a) If G is nilpotent the upper and lower central series of G are both central series.
b) If G has a central series $G = G_1 \geq G_2 \geq \cdots \geq G_n = \{1\}$ then $\gamma_i(G) \leq G_i$ for $1 \leq i \leq n$.
c) G is nilpotent if and only if G has a central series.

20) Let G be group that has a central series $\{1\} = G_0 \leq G_1 \leq \cdots \leq G_n = G$ (i.e., $G_i \trianglelefteq G$ for $0 \leq i \leq n$ and $G_{i+1}/G_i \leq Z(G/G_i)$ for $0 \leq i < n$). Prove that $G_i \leq \zeta^i(G)$ for $0 \leq i \leq n$.

Definition. Let G be a group. If $x, y \in G$ then $x^y := yxy^{-1}$.
If $n \geq 3$ and $x_1, \ldots, x_n \in G$ set $[x_1, \ldots, x_n] := [x_1, [x_2, \ldots, x_n]]$.

21) Let G be a group and $x, y, z \in G$. Prove the following commutator identities:

a) $[x, y] = [y, x]^{-1}$,
b) $[x, yz] = [x, y][x, z]^y$,
c) $[xy, z] = [y, z]^x[x, z]$,
d) $[x, y^{-1}] = \left([x, y]^{y^{-1}}\right)^{-1}$,
e) $[x^{-1}, y] = \left([x, y]^{x^{-1}}\right)^{-1}$,
f) $[x, y^{-1}, z]^y[y, z^{-1}, x]^z[z, x^{-1}, y]^x = 1$. Hint. Prove $[x, y^{-1}, z]^y = (yxy^{-1}zy)(zyz^{-1}xz)^{-1}$.

22) Let G be a nilpotent group of class 2 and $a \in G$. Prove that the function

$$\varphi_a : G \to G, \quad \varphi_a(x) = [a, x]$$

is a homomorphism. Deduce that $C_G(a) \trianglelefteq G$ (where $C_G(a) = \{x \in G \mid xa = ax\}$).

23) a) Let G and H be groups and $\varphi : G \to H$ an epimorphism. Prove that

$$\gamma_i(H) = \varphi(\gamma_i(G)) \text{ for } i \geq 1.$$

b) Prove that the group G is nilpotent of class $c \geq 2$ if and only if

$$\gamma_c(G) \leq Z(G) \text{ and } \gamma_{c-1}(G) \not\leq Z(G).$$

c) Prove that if G is nilpotent of class $c \geq 1$ then $G/Z(G)$ is nilpotent of class $c - 1$.

24) Let G be a group. Prove that G is nilpotent if and only if $G/Z(G)$ is nilpotent.

25) Let G be a group and $x, y, z \in G$. Prove the following commutator identities:

a) If x commutes with $[x, y]$ then $[x^n, y] = [x, y]^n$ for all $n \geq 0$,
b) If y commutes with $[x, y]$ then $[x, y^n] = [x, y]^n$ for all $n \geq 0$,
c) If both x and y commute with $[x, y]$ then

$$(xy)^n = [y, x]^{n(n-1)/2}x^ny^n = [y, x]^{\binom{n}{2}}x^ny^n \text{ for all } n \geq 0,$$

d) $[x, y]^z = [x^z, y^z]$,
e) $[x, y^n] = [x, y][x^y, y][x^{y^2}, y] \cdots [x^{y^{n-1}}, y]$ for all $n \geq 1$.

26) a) For which $n \geq 1$ is S_n nilpotent? Do not use $Z(S_3) = \{1\}$.
b) For which $n \geq 3$ is A_n nilpotent?

27) Let $n \geq 3$ and let D_{2n} denote the dihedral group with $2n$ elements as in Exercise 18.
a) Determine $Z(D_{2n})$.
b) Let n be even. Prove that

$$D_{2n}/Z(D_{2n}) \cong \begin{cases} V_4 & \text{if } n = 4, \\ D_n & \text{if } n \geq 6. \end{cases}$$

c) Prove that D_{2n} is nilpotent if and only if n is a power of 2.

28) Let K be a field and let

$$G = \left\{ \begin{pmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{pmatrix} \middle| a, b, c \in K \right\}.$$

Prove that G is a group and find $\gamma_i(G)$ for $i \geq 1$.

29) Let G be a finite nilpotent group of order $|G| = n$. Prove that if $m \mid n$ then G has a subgroup of order m.

30) Find three subgroups H, J, K of S_3 such that $[[H, J], K] \neq [H, [J, K]]$.

31) Let p be a prime and $n \geq 1$. Determine the Frattini subgroup $\Phi(\mathbb{Z}_{p^n})$.

32) Let $n \geq 3$. Prove that $\Phi(D_{2^n}) = \langle a^2 \rangle$ (where D_{2^n} denotes the dihedral group with 2^n elements and $a \in D_{2^n}$ is as in Exercise 18).

33) Let G be a group. Prove the following two assertions:
a) Let $M < G$ be a maximal subgroup. Then either $Z(G) \leq M$ or $G' \leq M$.
(Hint. Show that $Z(G) \not\leq M$ implies $M \trianglelefteq G$.)
b) $G' \cap Z(G) \leq \Phi(G)$.

34) Let G be a finite group and $N \trianglelefteq G$. Prove that $N \leq \Phi(G)$ if and only if there is no subgroup $H < G$ such that $HN = G$.

35) Let G be a finite group, $H \leq G$ and $N \trianglelefteq G$. Prove the following assertions:
a) If $N \leq \Phi(H)$ then $N \leq \Phi(G)$,
b) $\Phi(N) \leq \Phi(G)$.

36) Let G be a finite group and $N \trianglelefteq G$. Prove the following assertions:
a) $\Phi(G)N/N \leq \Phi(G/N)$,
b) If $N \leq \Phi(G)$ then $\Phi(G)/N = \Phi(G/N)$,
c) $\Phi\big(G/\Phi(G)\big) = \{1\}$.

37) Prove that $\Phi(S_n) = \{1\}$ for all $n \geq 1$.

38) Let p be a prime. Prove that the order of $\operatorname{Aut}(\underbrace{\mathbb{Z}_p \times \cdots \times \mathbb{Z}_p}_{n \text{ times}})$ is

$$|\operatorname{Aut}(\underbrace{\mathbb{Z}_p \times \cdots \times \mathbb{Z}_p}_{n \text{ times}})| = \prod_{i=1}^{n}(p^n - p^{i-1}) = (p^n - 1)(p^n - p)\cdots(p^n - p^{n-1}).$$

39) Prove that $\operatorname{Aut}(D_8) \cong D_8$ and yet $\operatorname{Inn}(D_8) < \operatorname{Aut}(D_8)$.

40) Let G be a finite nonabelian p-group. Prove that $p^2 \big| |\operatorname{Aut}(G)|$.

41) Let G and H be two finite groups such that $\gcd(|G|, |H|) = 1$.
Prove that $\operatorname{Aut}(G \times H) \cong \operatorname{Aut}(G) \times \operatorname{Aut}(H)$.

42) Prove that $\operatorname{Aut}(\mathbb{Z}_2 \times \mathbb{Z}_4) \cong D_8$.

43) Let G be a finite group and $\varphi, \psi \in \operatorname{Aut}(G)$. Prove the following two assertions:
a) If $|\{x \in G \mid \varphi(x) = x\}| > |G|/2$ then $\varphi(x) = x$ for all $x \in G$.
b) If $|\{x \in G \mid \varphi(x) = \psi(x)\}| > |G|/2$ then $\varphi = \psi$.

44) Let $n \geq 4$. Prove the following assertions:
a) $C_{S_n}(A_n) = \{1\}$, $Z(A_n) = \{1\}$ and $\operatorname{Inn}(A_n) \cong A_n$,
b) The map $S_n \to \operatorname{Aut}(A_n)$, $\omega \mapsto \varphi_\omega$ (where $\varphi_\omega(\sigma) = \omega \circ \sigma \circ \omega^{-1}$) is a monomorphism,
c) A_n is not complete.

45) Prove that $\operatorname{Aut}(A_4) \cong S_4$. (Hint. Prove that $A_4 = \langle (123), (124) \rangle$ and use this fact to show $|\operatorname{Aut}(A_4)| \leq 24$.)

46) Prove that $\operatorname{Hol}(\mathbb{Z}_2) \cong \mathbb{Z}_2$, $\operatorname{Hol}(\mathbb{Z}_3) \cong S_3$ and $\operatorname{Hol}(\mathbb{Z}_4) \cong D_8$.

47) a) Let G be a complete group. Prove that $\operatorname{Hol}(G) = G^\ell \times G^r$.
b) Prove that $\operatorname{Hol}(S_n) \cong S_n \times S_n$ for all $n \in \mathbb{N} \setminus \{2, 6\}$.

48) Prove that $\mathrm{Aut}(\mathbb{Z}_2 \times \mathbb{Z}_2) \cong S_3$ and $\mathrm{Hol}(\mathbb{Z}_2 \times \mathbb{Z}_2) \cong S_4$.

49) Let G be a group and $f : G \to G$ bijective. Prove that $f \in \mathrm{Hol}(G)$ if and only if $f(xy^{-1}z) = f(x)f(y)^{-1}f(z)$ for all $x, y, z \in G$.

50) a) Let G be a group and $N_i \trianglelefteq G$ for $1 \leq i \leq n$. Prove that $[N_1, \dots, N_n] \trianglelefteq G$.
b) Let G be a group, $H \trianglelefteq G$, $K \trianglelefteq G$ and $L \trianglelefteq G$. Prove that $[L, H, K] \leq [H, K, L][K, L, H]$. (Hint. Use the three subgroups lemma.)

51) Let G be a group with lower central series $G = \gamma_1(G) \geq \gamma_2(G) \geq \gamma_3(G) \geq \cdots$ and upper central series $\{1\} = \zeta^0(G) \leq \zeta^1(G) \leq \zeta^2(G) \leq \cdots$ and $i, j \geq 1$. Prove the following assertions. (Hint. Use induction and Exercise 50.)
a) $[\gamma_i(G), \gamma_j(G)] \leq \gamma_{i+j}(G)$,
b) $\gamma_i(\gamma_j(G)) \leq \gamma_{ij}(G)$,
c) $[\gamma_i(G), \zeta^j(G)] \leq \zeta^{j-i}(G)$ if $i \leq j$.

52) Let G be a group. Prove the following two assertions:
a) $G^{(i)} \leq \gamma_{2^i}(G)$ for all $i \geq 1$, (Remark. Compare this with the proof of Theorem 20.)
b) If G is nilpotent of class $c \geq 1$ then $G^{(i)} = \{1\}$ for all $i \geq \frac{\log(c+1)}{\log 2}$.

53) Let G be a group with $G' = G$. Prove that $Z(G/Z(G)) = \{1\}$.

54) Let G be a group and $H_i \trianglelefteq G$ for $1 \leq i \leq n$ (where $n \geq 2$). Prove that

$$[H_1, \dots, H_n] = \big\langle \big\{ [x_1, \dots, x_n] \,\big|\, x_i \in H_i \text{ for } 1 \leq i \leq n \big\} \big\rangle.$$

55) Let G be group. Prove that

$$\gamma_n(G) = [\underbrace{G, \dots, G}_{n \text{ times}}] = \big\langle \big\{ [x_1, \dots, x_n] \,\big|\, x_i \in G \text{ for } 1 \leq i \leq n \big\} \big\rangle$$

for all $n \geq 2$.

56) Let K be field. Prove that $\mathrm{GL}_n(K) = \mathrm{SL}_n(K) \rtimes K^*$ (where $K^* = K \setminus \{0\}$).

57) Let K and Q be solvable groups and $\theta : Q \to \mathrm{Aut}(K)$ a homomorphism. Prove that $K \rtimes_\theta Q$ is solvable.

58) Prove that the semidirect product $K \rtimes_\theta Q$ is the direct product $K \times Q$ if and only if θ is trivial (i.e., $\theta_x(a) = a$ for all $x \in Q$ and all $a \in K$).

59) Let $\mathbb{Z}_p = \langle a \rangle = \{1, a, a^2, \ldots, a^{p-1}\}$ and $\mathbb{Z}_q = \langle x \rangle = \{1, x, x^2, \ldots, x^{q-1}\}$. where p and q are distinct primes.

a) Suppose that $q \nmid (p-1)$. Prove that in this case $\mathbb{Z}_p \times \mathbb{Z}_q$ is the only possible semidirect product $\mathbb{Z}_p \rtimes_\theta \mathbb{Z}_q$.

b) Suppose that $q \mid (p-1)$. Prove that in this case there exists a semidirect product $\mathbb{Z}_p \rtimes_\theta \mathbb{Z}_q$ which is not isomorphic to $\mathbb{Z}_p \times \mathbb{Z}_q$. (Hint. Use the fact that $\mathrm{Aut}(\mathbb{Z}_p)$ is cyclic to prove the existence of a positive integer s with the properties $s^q \equiv 1 \pmod p$ and $s \not\equiv 1 \pmod p$. The homomorphism $\theta : \mathbb{Z}_q \to \mathrm{Aut}(\mathbb{Z}_p)$ can be defined by $\theta_x(a) = a^s$.)

60) Let $Q_8 = \{\pm 1, \pm i, \pm j, \pm k\}$ be the quaternion group. Here $1, i, j, k$ denote the standard basis of the Hamiltonian quaternions $\mathbb{H}$ with multiplication given by $i^2 = j^2 = k^2 = -1$, $ij = k$, $jk = i$, $ki = j$, $ji = -i$, $kj = -i$, $ik = -j$ and the usual rules for multiplying by ± 1. Those who do not know the Hamiltonian quaternions might prefer to use the ismorphic group $\{\pm E, \pm I, \pm J, \pm K\} \leq \mathrm{GL}_2(\mathbb{C})$ where

$$E = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad I = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \quad J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \quad \text{and} \quad K = \begin{pmatrix} 0 & -i \\ -i & 0 \end{pmatrix}.$$

This also has the advantage that associativity is very easy to prove.

a) Prove that Q_8 is a nonabelian group of order 8 that is not isomorphic to any of the groups $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$, $\mathbb{Z}_2 \times \mathbb{Z}_4$, $\mathbb{Z}_8$ and D_8.

b) Determine $Q_8{}'$, $Z(Q_8)$ and $Q_8/Z(Q_8)$.

c) Determine $\mathrm{Inn}(Q_8)$ and prove that Q_8 is nilpotent of class 2.

d) Determine all subgroups of Q_8 and prove that they are all normal.
(Remark. Nonabelian groups with this property are called hamiltonian. Dedekind proved that every finite hamiltonian group contains Q_8 as a direct factor.)

e) Prove that Q_8 is not a semidirect product.

61) Prove the Five Lemma: Let G_i and H_i be groups (for $1 \leq i \leq 5$) and $\alpha_i : G_i \to H_i$ (with $1 \leq i \leq 5$) and $\varphi_i : G_i \to G_{i+1}$ and $\psi_i : H_i \to H_{i+1}$ (with $1 \leq i \leq 4$) be homomorphisms. Furthermore, let the commutative diagram

$$\begin{array}{ccccccccc}
G_1 & \xrightarrow{\varphi_1} & G_2 & \xrightarrow{\varphi_2} & G_3 & \xrightarrow{\varphi_3} & G_4 & \xrightarrow{\varphi_4} & G_5 \\
\downarrow{\scriptstyle \alpha_1} & & \downarrow{\scriptstyle \alpha_2} & & \downarrow{\scriptstyle \alpha_3} & & \downarrow{\scriptstyle \alpha_4} & & \downarrow{\scriptstyle \alpha_5} \\
H_1 & \xrightarrow{\psi_1} & H_2 & \xrightarrow{\psi_2} & H_3 & \xrightarrow{\psi_3} & H_4 & \xrightarrow{\psi_4} & H_5
\end{array}$$

have exact rows. Prove the following assertions:

a) If α_1 is an epimorphism and α_2 and α_4 are monomorphisms, then α_3 is a monomorphism.

b) If α_5 is a monomorphism and α_2 and α_4 are epimorphisms, then α_3 is an epimorphism.

c) If α_1, α_2, α_4 and α_5 are isomorphisms, then α_3 is an isomorphism.